symmetry

MDPI

Article

Extraction of Exact Solutions of Higher Order Sasa-Satsuma Equation in the Sense of Beta Derivative

Emad Fadhal [1], Arzu Akbulut [2], Melike Kaplan [3], Muath Awadalla [1,*] and Kinda Abuasbeh [1]

[1] Department of Mathematics and Statics, Collage of Science, King Faisal University, Hafuf 31982, Al Ahsa, Saudi Arabia
[2] Department of Mathematics, Art-Science Faculty, Uludag University, 16059 Bursa, Turkey
[3] Department of Computer Engineering, Faculty of Engineering and Architecture, Kastamonu University, 37150 Kastamonu, Turkey
* Correspondence: mawadalla@kfu.edu.sa

Abstract: Nearly every area of mathematics, natural, social, and engineering now includes research into finding exact answers to nonlinear fractional differential equations (NFDES). In order to discover the exact solutions to the higher order Sasa-Satsuma equation in the sense of the beta derivative, the paper will discuss the modified simple equation (MSE) and exponential rational function (ERF) approaches. In general, symmetry and travelling wave solutions of the Sasa-Satsuma equation have a common correlation with each other, thus we reduce equations from wave transformations to ordinary differential equations with the help of Lie symmetries. Actually, we can say that wave moves are symmetrical. The considered procedures are effective, accurate, simple, and straightforward to compute. In order to highlight the physical characteristics of the solutions, we also provide 2D and 3D plots of the results.

Keywords: beta derivative; sasa satsuma equation; wave transformations; exact solutions

PACS: 02.30 Jr; 02.70.Wz; 04.20.Jb

Citation: Fadhal, E.; Akbulut, A.; Kaplan, M.; Awadalla, M.; Abuasbeh, K. Extraction of Exact Solutions of Higher Order Sasa-Satsuma Equation in the Sense of Beta Derivative. *Symmetry* **2022**, *14*, 2390. https://doi.org/10.3390/sym14112390

Academic Editors: Ji-Huan He and Muhammad Nadeem

Received: 10 October 2022
Accepted: 5 November 2022
Published: 11 November 2022

1. Introduction

Nowadays, academic researchers deal with many physical phenomena in plasma physics, physical chemistry, geophysics, fluid mechanics, nonlinear optics, electromagnetic theory, and fluid motion, and their mathematical models are expressed by NFDEs [1,2]. These equations are commonly used in various scientific disciplines and have been investigated from different viewpoints [3]. The exact solutions of these equations have gained more and more interest. For this reason, a lot of different techniques have been dealt with by researchers. Among this research, the following ones can be listed, for instance, Hosseini et al. derived new exact solutions of a new (4+1)-dimensional Burgers equation [4]. Dehingia et al. applied the Hopf-bifurcation [5]. Mirzazadeh et al. worked on the second-order nonlinear Schrödinger equation with weakly nonlocal and parabolic laws [6]. Aktar et al. obtained the soliton solutions of a biological model [7]. Mathanaranjan et al. applied the sinh-Gordon procedure [8]. Jannat et al. constructed different types of travelling wave solutions for the considered equation by employing the Auto-Backlund transformations [9]. Ala et al. applied the improved Bernoulli sub-equation function procedure [10]. Kaplan and Akbulut utilized the modified Kudryashov approach [11], Hosseini et al. used the homotopy analysis method (HAM) [12], Mahmood et al. utilized discrete fractional operators [13], Mohammed et al. employed discrete CF-fractional operators [14], Arshed et al. utilized the modified auxiliary equation (MAE) method [15].

The limit relationships between solitary wave solutions and periodic wave solutions, which correspond to the outer and inner trajectories of asymmetric homoclinic trajectories, were explored by the authors in [16]. We deal with the wave transformations, which are connected to the Lie symmetries of the equations, as a symmetry of an equation is a transformation that does not change the set of solutions to the equation [17].

In the current work, we will take into consideration the generalized (3+1)-dimensional nonlinear Sasa-Satsuma (3DNSS) equation in the sense of the beta derivative,

$$\begin{aligned} i\frac{\partial^\alpha q(x,y,z,t)}{\partial t^\alpha} &+ \alpha_1(q_{xx}+q_{yy}+q_{zz}) + \alpha_2|q|^2 q + i(\alpha_3(q_{xxx}+q_{yyy}+q_{zzz}) \\ &+\alpha_4|q|^2(q_x+q_y+q_z) + \alpha_5((|q|^2)_x + (|q|^2)_y + (|q|^2)_z)q) = 0, \end{aligned} \tag{1}$$

where $i=\sqrt{-1}, q=q(x,y,z,t)$ is the soliton profile, and $\alpha_\vartheta, \vartheta = 1,\ldots,5$ are real constants. This equation includes self-steepening, third order dispersion, and self-frequency shift, and is an integrable generalization of the NFDE. Yao et al. applied the mapping method, extended hyperbolic function technique, Bäcklund transformation, and improved generalized Riccati equation to find optical solitons of this equation [18]. Wazwaz and Mehanna obtained exponential, bright, singular, and dark soliton solutions of this equation [19].

This work is arranged as follows. We provide some fundamental details and certain features of the beta derivative in Section 2. Then, in Section 3, the preliminary information is given. In Section 4, we describe the MSE technique. In Section 5, we present the ERF technique. Then, in Section 6, we represent the applications of the adopted procedure. Finally, we provide a discussion on the results.

2. Some Useful Properties of the Beta Derivative

In the current part of the paper, we are going to give some definitions and basic results:

Definition 1. *Let $\epsilon(t)$ be a function defined for all non-negative t. The β derivative of $f(t)$ of order β is given by [20,21]:*

$$T^\beta(\epsilon(t)) = \lim_{\varepsilon\to 0}\frac{\epsilon\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-\epsilon(t)}{\varepsilon}, 0<\beta\le 1,$$

and $T^\beta(\epsilon(t)) = \frac{d^\beta\epsilon(t)}{dt^\beta}$.

Some properties are given for β derivative in the following theorem [22–24]:

Theorem 1. *Let $\epsilon(t)$ and $\zeta(t)$ be β-differentiable functions for all $t>0$ and $\beta\,\epsilon\,(0,1]$. Some basic properties are discussed as foloows:*

1. $T^\beta(\gamma\epsilon(t)+\delta\zeta(t)) = \gamma T^\beta(\epsilon(t)) + \delta T^\beta(\zeta(t)), \forall\gamma,\delta\in R$
2. $T^\beta(\epsilon(t)\zeta(t)) = \zeta(t)T^\beta(\epsilon(t)) + \epsilon(t)T^\beta(\zeta(t))$,
3. $T^\beta(\frac{\epsilon(t)}{\zeta(t)}) = \frac{\zeta(t)T^\beta(\epsilon(t))-\epsilon(t)T^\beta(\zeta(t))}{\zeta(t)^2}$,
4. $T^\beta(\epsilon(t)) = \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{d\epsilon(t)}{dt}$.

3. Preliminary Information

In this piece of the work, we first introduce the background data for employing to convert a nonlinear partial differential problem to an ordinary differential equation (ODE). Therefore, the following time fractional NFDE will be taken into account:

$$P(q, D_t^\beta q, D_x q, D_y q, D_z q, D_t^\beta D_t^\beta q, D_t^\beta D_x q, D_x D_x q, \ldots) = 0, \quad 0<\beta\le 1, \tag{2}$$

where P inherits q and its various partial derivatives include β fractional derivatives with respect to t and classical derivatives with respect to x,y,z.

The wave transformation is defined as:

$$q(x,y,z,t) = u(\varepsilon)e^{i(\tau_1 x+\tau_2 y+\tau_3 z+\tau_4 t)}, \varepsilon = \varrho_1 x + \varrho_2 y + \varrho_3 z + \frac{\varrho_4}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta, \tag{3}$$

where ϱ_i and $\tau_i (i = 0, 1, \ldots, 4)$ are constants to be calculated. Then, (3) is used in Equation (2) and equates the imaginary and real parts to zero; we discover an equation system. After that, we solve the resulting system to determine the parameters' conditions and use those results. As a result, we obtain the following ODE, as follows, which is going to be integrated with respect to the ε possible times.

$$Q(u, u', u'', u''', \ldots) = 0, \tag{4}$$

where the partial derivative is given with respect to ε.

4. Modified Simple Equation Technique

We can summarize the MSE technique, as follows [25,26]:

Firstly, the exact solution of Equation (4) can used to demonstrate a polynomial in $\left(\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\right)$ as follows

$$u(\varepsilon) = \sum_{j=0}^{m} \sigma_j \left[\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\right]^j, \ \sigma_j = \text{const.}, \sigma_m \neq 0. \tag{5}$$

Here, m is a positive integer, and it is the balancing number. It can be evaluated by applying the homogeneous balance principle between the highest order derivative term with the highest order nonlinear term, which is shown in Equation (4). Moreover, σ_j are arbitrary real constants to be calculated.

We obtain a polynomial of order $\varsigma^{-j}(\varepsilon)$ with the derivatives of $\varsigma(\varepsilon)$ by inserting Equation (5) into Equation (4). After that, we obtain an algebraic equation system to be solved for finding $\sigma_j (j = 0, 1, 2, \ldots, m)$, ϱ_i and $\tau_i (i = 0, 1, \ldots, 4)$ and $\varsigma(\varepsilon)$, by equating all the coefficients of $\varsigma^{-j}(\varepsilon)$ to zero $(j \geq 0)$. Finally, we substitute the values of σ_j, c and $\varsigma(\varepsilon)$ into Equation (5) and we obtain the exact solutions of Equation (1).

5. The Exponential Rational Function Technique

Assume that Equation (4) has a solution of the following type [27,28]:

$$u(\varepsilon) = \sum_{j=0}^{m} \frac{\sigma_j}{(1 + e^{\varepsilon})^j}, \rho_m \neq 0 \tag{6}$$

where $\sigma_j (j = 0, 1, 2, \ldots, m)$ are constants to be evaluated, and m is the balancing number. If we substitute Equation (6) into Equation (4), we discover a collection of algebraic equations in parameters $\sigma_j (j = 0, 1, 2, \ldots, m)$, ϱ_i and $\tau_i (i = 0, 1, \ldots, 4)$. Hence, we can derive fresh answers to Equation (4).

6. Application of the Given Methods from the Solutions of This System

We shall look for the novel traveling wave solutions to Equation (1) in this piece of the work. Due to this, we first provide a mathematical examination of the problem under consideration.

6.1. Mathematical Analysis

We split Equation (1) into its real and imaginary components using the transformation specified in Equation (3):

$$\begin{gathered} \left(\left(\varrho_1^2 + \varrho_2^2 + \varrho_3^2\right)\alpha_1 - 3\alpha_3\left(\varrho_1^2\tau_1 + \varrho_2^2\tau_2 + \varrho_3^2\tau_3\right)\right)u'' \\ -\left(\tau_4 + \left(\tau_1^2 + \tau_2^2 + \tau_3^2\right)\alpha_1 - \left(\tau_1^3 + \tau_2^3 + \tau_3^3\right)\alpha_3\right)u + \left(\alpha_2 - (\tau_1 + \tau_2 + \tau_3)\alpha_4\right)u^3 = 0, \end{gathered} \tag{7}$$

and

$$\begin{gathered} \left(\varrho_4 + 2\alpha_1(\varrho_1\tau_1 + \varrho_2\tau_2 + \varrho_3\tau_3) - 3\alpha_3\left(\varrho_1\tau_1^2 + \varrho_2\tau_2^2 + \varrho_3\tau_3^2\right)\right)u' \\ +(\varrho_1 + \varrho_2 + \varrho_3)(\alpha_4 + 2\alpha_5)u^2u' + \alpha_3\left(\varrho_1^3 + \varrho_2^3 + \varrho_3^3\right)u''' = 0. \end{gathered} \tag{8}$$

If we integrate Equation (8) and take the integration constant as zero, we find the resultant equation as follows:

$$\begin{aligned} &\alpha_3\big(\varrho_1^3+\varrho_2^3+\varrho_3^3\big)u''+\big(\varrho_4+\varrho_3\tau_3^2\big)+2\alpha_1\big(\varrho_1\tau_1+\varrho_2\tau_2+\varrho_3\tau_3\big)\\ &-3\alpha_3\big(\varrho_1\tau_1^2+\varrho_2\tau_2^2\big)u+\tfrac{1}{3}(\varrho_1+\varrho_2+\varrho_3)(\alpha_4+2\alpha_5)u^3=0. \end{aligned} \tag{9}$$

If we compare the coefficients of Equations (7) and (9), we obtain the following relations:

$$\alpha_1=\frac{\alpha_3\big(3\big(\varrho_1^2\tau_1+\varrho_2^2\tau_2+\varrho_3^2\tau_3\big)+\big(\varrho_1^3+\varrho_2^3+\varrho_3^3\big)\big)}{\varrho_1^2+\varrho_2^2+\varrho_3^2}, \tag{10}$$

$$\alpha_5=\frac{3\alpha_2-\alpha_4((\varrho_1+\varrho_2+\varrho_3)+3(\tau_1+\tau_2+\tau_3))}{2(\varrho_1+\varrho_2+\varrho_3)}, \tag{11}$$

and

$$\begin{aligned} \varrho_4=\ &-\tau_4-\alpha_1\big(2(\varrho_1\tau_1+\varrho_2\tau_2+\varrho_3\tau_3)+\tau_1^2+\tau_2^2+\tau_3^2\big)\\ &+\alpha_3\big(3\big(\varrho_1\tau_1^2+\varrho_2\tau_2^2+\varrho_3\tau_3^2\big)+\tau_1^3+\tau_2^3+\tau_3^3\big). \end{aligned} \tag{12}$$

If we balance u'' and u^3 Equation (7), we obtain $m=1$.

6.2. MSE Technique

According to the adopted technique, the exact solution of the Equation (7) is given by:

$$u(\varepsilon)=\sigma_0+\sigma_1\left[\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\right],\sigma_1\neq 0. \tag{13}$$

Here, we substiture the Equation (13) into Equation (7), and we obtain:

$$\begin{aligned} \varsigma^0(\varepsilon):\ &\alpha_3\tau_2^3\sigma_0-\alpha_1\tau_3^2\sigma_0-\alpha_1\tau_1^2\sigma_0-\tau_4\sigma_0+\alpha_2\sigma_0^3+\alpha_3\tau_1^3\sigma_0\\ &-\alpha_1\tau_2^2\sigma_0-\alpha_4\tau_2\sigma_0^3-\alpha_4\tau_3\sigma_0^3+\alpha_3\tau_3^3\sigma_0-\alpha_4\tau_1\sigma_0^3=0, \end{aligned} \tag{14}$$

$$\begin{aligned} \varsigma^1(\varepsilon):\ &\alpha_1\varrho_1^2\sigma_1\frac{\varsigma'''(\varepsilon)}{\varsigma(\varepsilon)}-\alpha_1\tau_3^2\sigma_1\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}-3\alpha_3\varrho_3^2\tau_3\sigma_1\frac{\varsigma'''(\varepsilon)}{\varsigma(\varepsilon)}+3\alpha_2\sigma_0^2\sigma_1\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\\ &-\alpha_1\tau_2^2\sigma_1\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}+\alpha_1\varrho_3^2\sigma_1\frac{\varsigma'''(\varepsilon)}{\varsigma(\varepsilon)}-\alpha_1\tau_1^2\sigma_1\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}-\tau_4\sigma_1\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\\ &-3\alpha_3\varrho_2^2\tau_2\sigma_1\frac{\varsigma'''(\varepsilon)}{\varsigma(\varepsilon)}-3\alpha_4\tau_1\sigma_0^2\sigma_1\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}+\alpha_3\tau_1^3\sigma_1\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\\ &+\alpha_3\tau_2^3\sigma_1\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}-3\alpha_4\tau_2\sigma_0^2\sigma_1\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}-3\alpha_3\varrho_1^2\tau_1\sigma_1\frac{\varsigma'''(\varepsilon)}{\varsigma(\varepsilon)}\\ &+\alpha_1\varrho_2^2\sigma_1\frac{\varsigma'''(\varepsilon)}{\varsigma(\varepsilon)}+\alpha_3\tau_3^3\sigma_1\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}-3\alpha_4\tau_3\sigma_0^2\sigma_1\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}=0 \end{aligned} \tag{15}$$

$$\begin{aligned} \varsigma^2(\varepsilon):\ &9\alpha_3\varrho_1^2\tau_1\sigma_1\frac{\varsigma''(\varepsilon)}{\varsigma(\varepsilon)}\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}-3\alpha_4\tau_3\sigma_0\sigma_1^2\left(\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\right)^2-3\alpha_4\tau_1\sigma_0\sigma_1^2\left(\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\right)^2\\ &+3\alpha_2\sigma_0\sigma_1^2\left(\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\right)^2-3\alpha_1\varrho_3^2\sigma_1\frac{\varsigma''(\varepsilon)}{\varsigma(\varepsilon)}\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}+9\alpha_3\varrho_2^2\tau_2\sigma_1\frac{\varsigma''(\varepsilon)}{\varsigma(\varepsilon)}\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\\ &-3\alpha_1\varrho_1^2\sigma_1\frac{\varsigma''(\varepsilon)}{\varsigma(\varepsilon)}\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}-3\alpha_4\tau_2\sigma_0\sigma_1^2\left(\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\right)^2-3\alpha_1\varrho_2^2\sigma_1\frac{\varsigma''(\varepsilon)}{\varsigma(\varepsilon)}\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\\ &+9\alpha_3\varrho_3^2\tau_3\sigma_1\frac{\varsigma''(\varepsilon)}{\varsigma(\varepsilon)}\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}=0, \end{aligned} \tag{16}$$

$$\begin{aligned} \varsigma^3(\varepsilon):\ &(-6\alpha_3\varrho_2^2\tau_2\sigma_1-6\alpha_3\varrho_3^2\tau_3\sigma_1-\alpha_4\tau_1\sigma_1^3-\alpha_4\tau_2\sigma_1^3-\alpha_4\tau_3\sigma_1^3+2\alpha_1\varrho_1^2\sigma_1\\ &+\alpha_2\sigma_1^3-6\alpha_3\varrho_1^2\tau_1\sigma_1+2\alpha_1\varrho_2^2\sigma_1+2\alpha_1\varrho_3^2\sigma_1)\left(\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\right)^3=0 \end{aligned} \tag{17}$$

From the solutions of Equations (14) and (17), we find the following results for σ_0 and σ_1 :

$$\sigma_0=\frac{\sqrt{\varpi(-\tau_4-\alpha_1\tau_1^2-\alpha_1\tau_2^2-\alpha_1\tau_3^2+\alpha_3\tau_1^3+\alpha_3\tau_2^3+\alpha_3\tau_3^3)}}{\varpi}, \tag{18}$$

and

$$\sigma_1=\frac{\sqrt{-2\varpi(-\alpha_1\varrho_2^2+3\alpha_3\varrho_2^2\tau_2+3\alpha_3\varrho_1^2\tau_1-\alpha_1\varrho_3^2+3\alpha_3\varrho_3^2\tau_3-\alpha_1\varrho_1^2)}}{\varpi} \tag{19}$$

where $\varpi = \alpha_4\tau_3 + \alpha_4\tau_1 + \alpha_4\tau_2 - \alpha_2$. Then, if we substitute Equations (18) and (19) into Equations (15) and (16), we find a differential equation system for $\varsigma(\varepsilon)$. After that, we solve this differential equation system and find the resultant equation

$$\begin{aligned}\varsigma(\varepsilon) = \;& c_1 \\ & + c_2 \exp\Big(\sqrt{(+\alpha_4(\alpha_3-\alpha_1)(\tau_2\pi_4+\tau_1\pi_5) - \alpha_4\tau_3\alpha_1(\tau_1^2+\tau_2^2) + \alpha_4\tau_3\alpha_3\pi_4 + \alpha_2\tau_4 + \pi_7} \\ & \sqrt{-\sqrt{2}((\alpha_4\pi_2-\alpha_2))(-\alpha_1\pi_3 + 3\alpha_3(\varrho_3^2\tau_3 + \varrho_1^2\tau_1 + \varrho_2^2\tau_2))} \\ & \varepsilon/(\alpha_4\pi_2-\alpha_2)(-\alpha_1\pi_3 + 3\alpha_3(\varrho_3^2\tau_3 + \varrho_1^2\tau_1 + \varrho_2^2\tau_2))\end{aligned}$$

where $\pi_1 = \tau_1^2+\tau_2^2+\tau_3^2, \pi_2 = \tau_1+\tau_2+\tau_3, \pi_3 = \varrho_1^2+\varrho_2^2+\varrho_3^2, \pi_4 = \tau_1^3+\tau_3^3, \pi_5 = \tau_2^3+\tau_3^3, \pi_6 = \tau_1^3+\tau_2^3+\tau_3^3, \pi_7 = -\alpha_4\tau_1(\tau_4+\tau_1^2\alpha_1+\tau_1^3\alpha_3) - \alpha_4\tau_4(\tau_2+\tau_3) - \alpha_4\alpha_1\pi_5 + \alpha_4\alpha_3(\tau_2^4+\tau_3^4) + \alpha_2\alpha_1\pi_1 - \alpha_2\alpha_3\pi_6$

Therefore, the solution of $u(\varepsilon) = \sigma_0 + \sigma_1\left[\frac{\varsigma'(\varepsilon)}{\varsigma(\varepsilon)}\right]$.

We plotted the above result in Figure 1.

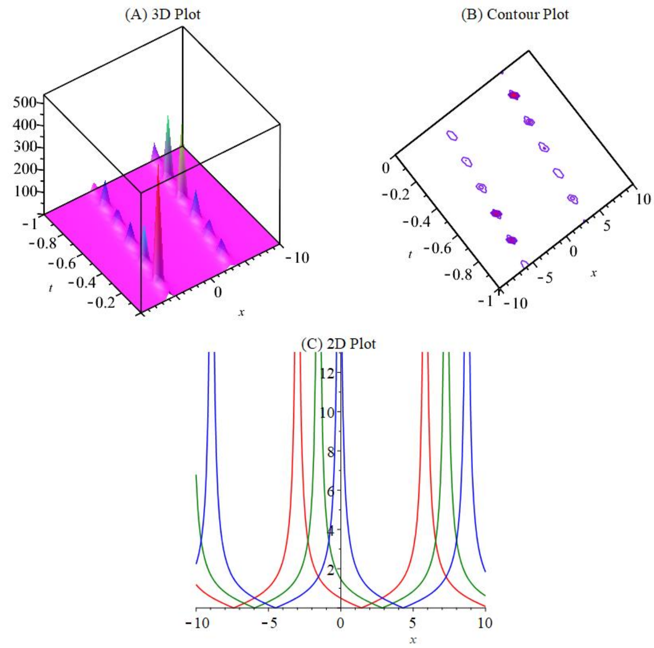

Figure 1. Figures plotted when $\tau_1 = 0.3$, $\tau_2 = 0.1$, $\tau_3 = 0.2$, $\tau_4 = 0.1$, $\varrho_1 = 0.2$, $\varrho_2 = 0.2$, $\varrho_3 = 0.1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 3$, $\alpha_4 = 1$, y = z = $c_1 = c_2 = 1$, $\beta = 0.99$.

6.3. ERF Technique

We will use the ERF method to solve Equation (1) in this subsection. The preferred response for $u(\varepsilon)$ has the form:

$$u(\varepsilon) = \sigma_0 + \frac{\sigma_1}{1+e^{\varepsilon}}. \tag{20}$$

Substituting the Equation (20) into Equation (7), then gathering all of the coefficients of $(e^{\varepsilon})^k$ to zero, we obtain:

$$\begin{aligned}&-\sigma_0\big(((\tau_1+\tau_2+\tau_3)\alpha_4-\alpha_2)\sigma_0^2 + (\tau_1^2+\tau_2^2+\tau_3^2)\alpha_1\big) \\ &-\sigma_0\big((-\tau_1^3-\tau_2^3-\tau_3^3)\alpha_3+\tau_4\big) = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}&\sigma_0((-3\alpha_4\tau_1 - 3\alpha_4\tau_2 - 3\alpha_4\tau_3 + 3\alpha_2)\sigma_0^2 + (\varrho_1^2+\varrho_2^2+\varrho_3^2-\tau_1^2-\tau_2^2-\tau_3^2)\alpha_1 \\ &+(-3\varrho_1^2\tau_1 - 3\varrho_2^2\tau_2 - 3\varrho_3^2\tau_3 + \tau_1^3+\tau_2^3+\tau_3^3)\alpha_3 - \tau_4)\sigma_1 + 3((-\alpha_4\tau_1 - \alpha_4\tau_2 \\ &-\alpha_4\tau_3+\alpha_2)\sigma_0^2 + (-\tau_1^2-\tau_2^2-\tau_3^2)\alpha_1 + (\tau_1^3+\tau_2^3+\tau_3^3)\alpha_3 - \tau_4) = 0,\end{aligned} \tag{22}$$

$$\begin{gathered}3\sigma_0((-\tau_1-\tau_2-\tau_3)\alpha_4+\alpha_2)\sigma_0\sigma_1^2+(((-6\tau_1-6\tau_2-6\tau_3)\alpha_4+6\alpha_2)\sigma_0^2\\+(-\varrho_1^2-\varrho_2^2-\varrho_3^2-2\tau_1^2-2\tau_2^2-2\tau_3^2)\alpha_1+(3\varrho_1^2\tau_1+3\varrho_2^2\tau_2+3\varrho_3^2\tau_3+2\tau_1^3\\+2\tau_2^3+2\tau_3^3)\alpha_3-2\tau_4)\sigma_1+3(((-\tau_1-\tau_2-\tau_3)\alpha_4+\alpha_2)\sigma_0^2+(-\tau_1^2-\tau_2^2-\tau_3^2)\alpha_1\\+(\tau_1^3+\tau_2^3+\tau_3^3)\alpha_3-\tau_4)=0,\end{gathered} \tag{23}$$

$$\begin{gathered}-(\sigma_0+\sigma_1)((\sigma_0+\sigma_1)^2(\tau_1+\tau_2+\tau_3)\alpha_4-\tau_1^3\alpha_3-\tau_2^3\alpha_3-\tau_3^3\alpha_3\\-\sigma_0^2\alpha_2-2\sigma_0\sigma_1\alpha_2-\sigma_1^2\alpha_2+\tau_1^2\alpha_1+\tau_2^2\alpha_1+\tau_3^2\alpha_1+\tau_4)=0.\end{gathered} \tag{24}$$

If we solve Equations (21)–(24), we obtain values of constants, as follows:

$$\sigma_0=\pm\frac{\sqrt{2}\sqrt{\frac{(\varrho_1^3+\varrho_2^3+\varrho_3^3)\alpha_3}{(\tau_1+\tau_2+\tau_3)\alpha_4-\alpha_2}}}{2},\sigma_1=\mp\frac{\sqrt{2}(\varrho_1^3+\varrho_2^3+\varrho_3^3)\alpha_3}{\sqrt{\frac{(\varrho_1^3+\varrho_2^3+\varrho_3^3)\alpha_3}{(\tau_1+\tau_2+\tau_3)\alpha_4-\alpha_2}}((\tau_1+\tau_2+\tau_3)\alpha_4-\alpha_2)},$$

$$\tau_4=\frac{\alpha_1(-\varrho_1^2-\varrho_2^2-\varrho_3^2-2\tau_1^2-2\tau_2^2-2\tau_3^2)}{2}+\frac{3\alpha_3(\varrho_1^2\tau_1+\varrho_2^2\tau_2+\varrho_3^2\tau_3+\frac{2}{3}(\tau_1^3+\tau_2^3+\tau_3^3))}{2}$$

As a result, we may verify the following solutions:

$$\begin{aligned}q(x,y,z,t)=\;&\pm\left(\frac{\sqrt{2}\alpha_3(\varrho_1^3+\varrho_2^3+\varrho_3^3)\left(e^{\left(\varrho_1x+\varrho_2y+\varrho_3z+\frac{\varrho_4t^\alpha}{\Gamma(1+\alpha)}\right)}-1\right)}{2\sqrt{\frac{(\varrho_1^3+\varrho_2^3+\varrho_3^3)\alpha_3}{(\tau_1+\tau_2+\tau_3)\alpha_4-\alpha_2}}((\tau_1+\tau_2+\tau_3)\alpha_4-\alpha_2)\left(1+e^{\left(\varrho_1x+\varrho_2y+\varrho_3z+\frac{\varrho_4}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta\right)}\right)}\right)\\&\times e^{i(\tau_1x+\tau_2y+\tau_3z+\tau_4t)}\end{aligned}$$

In Figure 2, we gave figure of the above solution.

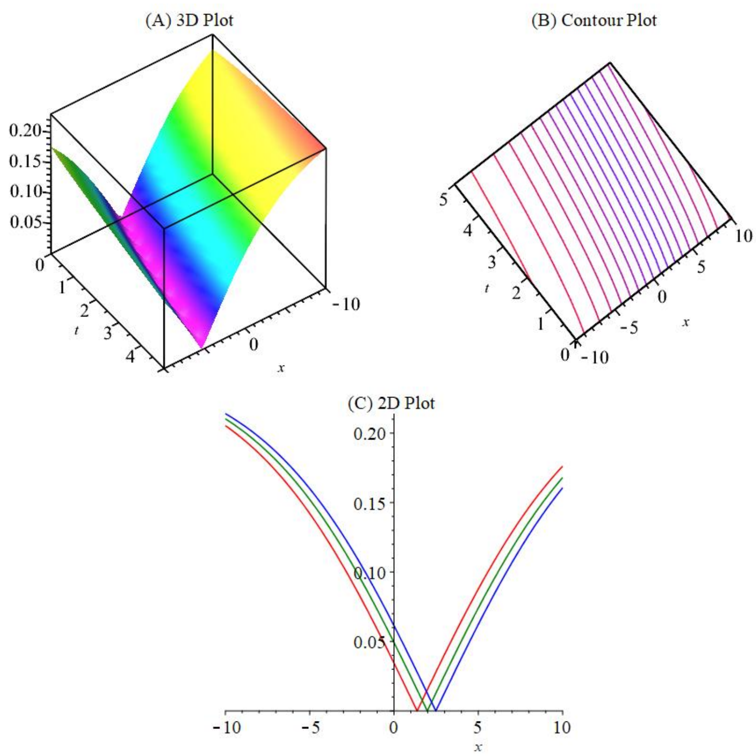

Figure 2. Figures plotted when τ_1 = 0.3, τ_2 = 0.1, τ_3 = 0.2, ϱ_1 = 0.2, ϱ_2 = 0.2, ϱ_3 = 0.1, α_0 = 1, α_2 = 1, α_3 = 3, α_4 = 1, y = z = 0.1, β = 0.6.

In Figure 3, we gave figures of the above solution.

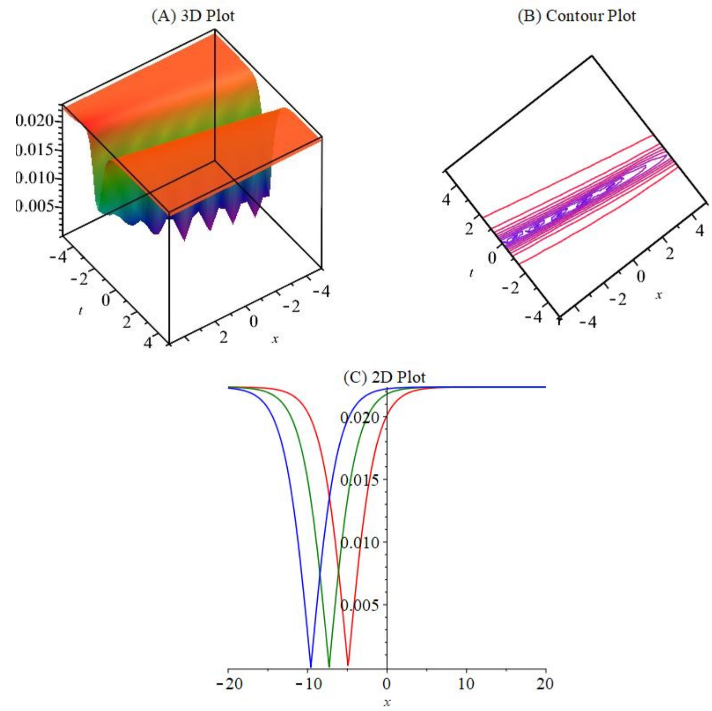

Figure 3. Figures plotted when $\tau_1 = 3$, $\tau_2 = 2$, $\tau_3 = -3$, $\varrho_1 = -0.6$, $\varrho_2 = -0.8$, $\varrho_3 = 0.9$, $\alpha_0 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, y = z = 1, $\beta = 0.9$.

7. Discussion on the Results

In this work, the higher order Sasa-Satsuma problem with the β derivative is solved by the MSE and ERF method, leading to novel and important results. The periodic wave solutions that correspond to periodic trajectories surrounded by symmetric homogeneous orbits satisfy the concept of symmetry, making it likely that these solutions will be essential for understanding the dynamics of this equation and pinpointing some of its physical characteristics. In a variety of situations where the equation in question is pertinent, the validated solutions could be useful for gaining a greater knowledge of the interacting wave phenomena. In order to verify the accuracy of the recovered solution, it is also helpful to know that the exact answers retrieved are fed back into the original equation. In order to highlight the physical characteristics of the solutions, we also provide 3D and 2D plots of the obtained results. To the best of our understanding, the results offer a lot of promise for mathematical physics. To ensure that the outcomes were accurate, we employed Maple software.

Author Contributions: Funding acquisition, Emad Fadhal; Project administration, Muath Awadalla; Writing—original draft, Arzu Akbulut; Writing—review and editing, Emad Fadhal, Arzu Akbulut, Melike Kaplan, Muath Awadalla and Kinda Abuasbeh. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. 1701], through its KFU Research Summer initiative.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zhou, Y.; Vijayakumar, V.; Murugesu, R. Controllability for fractional evolution inclusions without compactness. *Evol. Equ. Control Theory* **2015**, *4*, 507–524. [CrossRef]
2. Kavitha, K.; Vijayakumar, V.; Udhayakumar, R.; Ravichandran, C. Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness. *Asian J. Control* **2022**, *24*, 1406–1415. [CrossRef]
3. Ma, Y.K.; Vijayakumar, V.; Shukla, A.; Nisar, K.S.; Thilagavathi, K.; Nashine, H.K.; Singh, A.K.; Zakarya, M. Discussion on the existence of mild solution for fractional derivative by Mittag-Leffler kernel to fractional stochastic neutral differential inclusions. *Alex. Eng. J.* **2023**, *63*, 271–282 [CrossRef]
4. Hosseini, K.; Samavat, M.; Mirzazadeh, M.; Salahshour, S.; Baleanu, D. A New (4+1)-Dimensional Burgers Equation: Its Bäcklund Transformation and Real and Complex N-Kink Solitons. *Int. J. Appl. Comput. Math.* **2022**, *8*, 172. [CrossRef]
5. Dehingia, K.; Das, P.; Kumar, R.; Upadhyay, R.K.; Kumar, A.; Rihane, M.F.; Hosseini, K. Modelling and analysis of delayed tumour-immune system with hunting T-cells. *Math. Comput. Simul.* **2023**, *203*, 669–684. [CrossRef]
6. Mirzazadeh, M.; Hosseini, K.; Dehingi, K.; Salahshour, S. A second-order nonlinear Schrödinger equation with weakly nonlocal and parabolic laws and its optical solitons. *Optik* **2021**, *242*, 166911. [CrossRef]
7. Aktar, M.S.; Akbar, M.A.; Osman, M.S. Spatio-temporal dynamic solitary wave solutions and diffusion effects to the nonlinear diffusive predator-prey system and the diffusion-reaction equations. *Chaos Solitons Fractals* **2022**, *160*, 112212. [CrossRef]
8. Mathanaranjan, T.; Kumar, D.; Rezazadeh, H.; Akinyemi, L. Optical solitons in metamaterials with third and fourth order dispersions. *Opt. Quantum Electron.* **2022**, *54*, 271. [CrossRef]
9. Jannat, N.; Kaplan, M.; Raza, N. Abundant soliton-type solutions to the new generalized KdV equation via auto-Bäcklund transformations and extended transformed rational function technique. *Opt. Quantum Electron.* **2022**, *54*, 466. [CrossRef]
10. Ala, V.; Demirbilek, U.; Mamedo, K.R. An application of improved Bernoulli sub-equation function method to the nonlinear conformable time-fractional SRLW equation. *AIMS Math.* **2020**, *5*, 3751–3761. [CrossRef]
11. Kaplan, M.; Akbulut, A. A novel exploration for traveling wave solutions to the integrable equation of wave packet envelope. *Partial. Equations Appl. Math.* **2022**, *5*, 100312. [CrossRef]
12. Hosseini, K.; Ilie, M.; Mirzazadeh, M.; Baleanu, D.; Park, C.; Salahshour, S. The Caputo-Fabrizio time-fractional Sharma-Tasso-Olver-Burgers equation and its valid approximations. *Commun. Theor. Phys.* **2022**, *74*, 075003. [CrossRef]
13. Mahmood, S.A.; Mohammed, P.O.; Baleanu, D.; Aydi, H.; Hamed, Y.S. Analysing discrete fractional operators with exponential kernel for positivity in lower boundedness. *AIMS Math.* **2022**, *7*, 10387–10399. [CrossRef]
14. Mohammed, P.O.; O'Regan, D.; Brzo, A.B.; Abualnaja, K.M.; Baleanu, D. Analysis of positivity results for discrete fractional operators by means of exponential kernels. *AIMS Math.* **2022**, *7*, 15812–15823. [CrossRef]
15. Arshed, S.; Raza, N.; Butt, A.R.; Akgül, A. Exact solutions for Kraenkel-Manna-Merle model in saturated ferromagnetic materials using β-derivative. *Phys. Scr.* **2021**, *96*, 124018. [CrossRef]
16. Zhang, W.; Ling, X.; Wang, B.B.; Li, S. Solitary and Periodic Wave Solutions of Sasa-Satsuma Equation and Their Relationship with Hamilton Energy. *Complexity* **2020**, *2020*, 8760179. [CrossRef]
17. Mitofsky, A.M.; Carney, P.S. Symmetries and conservation laws for the wave equations of scalar statistical optics. *J. Phys. A Math. Theoratical* **2008**, *41*, 415207. [CrossRef]
18. Yao, S.W.; Akinyemi, L.; Mirzazadeh, M.; Inc, M.; Hosseini, K.; Senol, M. Dynamics of optical solitons in higher-order Sasa-Satsuma equation. *Results Phys.* **2021**, *30*, 104825. [CrossRef]
19. Wazwaz, A.M.; Mehanna, M. Higher-order Sasa-Satsuma equation: Bright and dark optical solitons. *Optik* **2021**, *243*, 167421. [CrossRef]
20. Yue, X.G.; Zhang, Z.; Akbulut, A.; Kaabar, M.K.A.; Kaplan, M. A new computational approach to the fractional-order Liouville equation arising from mechanics of water waves and meteorological forecasts. *J. Ocean. Eng. Sci.* 2022, *in press*. [CrossRef]
21. Martinez, Y.P.; Gomez-Aguilar, J.F.; Baleanu, D. Beta-derivative and sub-equation method applied to the optical solitons in medium with parabolic law nonlinearity and higher order dispersion. *Optik* **2018**, *155*, 357–365. [CrossRef]
22. Atangana, A.; Alqahtani, R.T. Modelling the spread of river blindness disease via the Caputo fractional derivative and the beta-derivative. *Entropy* **2016**, *18*, 40. [CrossRef]
23. Atangana, A.; Baleanu, D.; Alsaedi, A. Analysis of time-fractional Hunter-Saxton equation: A model of neumatic liquid crystal. *Open Phys.* **2016**, *14*, 145–149. [CrossRef]
24. Ganji, D.D.; Sadighi, A. Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations. *J. Comput. Appl. Math.* **2007**, *207*, 24–34. [CrossRef]
25. Khan, K.; Akbar, M.A.; Ali, N.H.M. The modified simple equation method for exact and solitary wave solutions of nonlinear evolution equation: The GZK-BBM equation and right-handed noncommutative Burgers equations. *ISRN Math. Phys.* **2013**, *2013*, 146704. [CrossRef]
26. Bekir, A.; Akbulut, A.; Kaplan, M. Exact Solutions of Nonlinear Evolution Equations by Using Modified Simple Equation Method. *Int. J. Nonlinear Sci.* **2015**, *19*, 159–164.
27. Kaplan, M.; Akbulut, A. A Mathematical Analysis of a Model Involving an Integrable Equation for Wave Packet Envelope. *J. Math.* **2022**, 2022, 3486780. [CrossRef]
28. Mohyud-Din, S.T.; Bibi, S. Exact solutions for nonlinear fractional differential equations using exponential rational function method. *Opt. Quantum Electron.* **2017**, *49*, 64. [CrossRef]