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Common Fixed Point Theorems on Orthogonal Branciari Metric Spaces with an Application

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Abstract: In this article, we modify the symmetry of orthogonal metric spaces and we prove common fixed point theorems via simulation functions in orthogonal Rectangular metric spaces. We also provide an illustrative example to support our results. The derived results have been applied to find analytical solutions to integral equations. The analytical solutions are verified with a numerical simulation.

Keywords: fixed point; simulation functions; orthogonal Rectangular metric space; orthogonal α-admissible functions

MSC: 47H10; 54H25; 54C30

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Citation: Mani, G.; Prakasam, S.K.; Gnanaprakasam, A.J.; Ramaswamy, R.; Abdelnaby, O.A.A.; Khan, K.H.; Radenović, S. Common Fixed Point Theorems on Orthogonal Branciari Metric Spaces with an Application. *Symmetry* **2022**, *14*, 2420. https://doi.org/10.3390/sym14112420

Academic Editor: Hüseyin Budak

Received: 26 October 2022
Accepted: 9 November 2022
Published: 15 November 2022

1. Introduction

The French mathematician M. Frechet [1] introduced the notion of metric space. The contraction mapping theorem (CMT) of Banach [2], establishing the existence of unique fixed points via auxiliary functions in complete metric spaces, laid the foundation stone for the metric fixed point theory. Many generalizations of the CMT were reported by mathematicians under different contractive conditions in the setting of metric and metric-like spaces. In the sequel, in 2000, Branciari [3] introduced the notion of Rectangular metric spaces, by replacing the right-hand side of the triangular inequality with a three-term expression and proving an analog of CMT. Many researchers extended the fixed point results in Rectangular metric spaces (see [4–22]). In 2017, Aydi et al. [23] introduced the notion of (α, ψ)—Meir–Keeler contractions in Rectangular metric spaces and obtained some common fixed point theorems involving these contractions. In 2019, Abodayeh et al. [24] initiated hybrid contractions and proved the fixed point theorem in Branciari-type distance spaces.

The concept of the simulation function was introduced by Khojasteh et al. [25]. Many authors developed the fixed point theorems via simulation function in Rectangular metric spaces, (see [26–32]). Shatanawi and Postolache [33] proved common fixed point results via nonlinear contractions of cyclic form in ordered metric spaces, and applied the result to find unique common fixed point to integral type contractions.

Gordji et al. [34] initiated the notion of orthogonality in metric spaces. The fixed point results in generalized orthogonal metric space and various metric spaces were proven by many researchers, see [35–41]. More recently, in 2022, Aiman et al. [42] initiated an orthogonal Branciari metric space and proved the fixed point results thereon.

Inspired, we introduce the new notion of an orthogonal generalized Λ-contraction pair of maps with respect to a simulation function, and establish fixed point results in the setting of complete orthogonal Rectangular metric spaces using these contractions. Suitable numerical examples and an application to find the analytical solution of integral equations are provided to supplement the derived results. The analytical solutions are compared with numerical solutions. The rest of the paper is organized as follows:

In Section 2, we review and present some preliminaries and monographs required in the sequel. In Section 3, we establish fixed point results in the setting of Orthogonal Branciari Metric Spaces using the orthogonal generalized Λ-contractions, and supplement the derived results with nontrivial numerical examples. In Section 4, we present an application to find the analytical solution to integral equations. The analytical solutions are compared with numerical solutions.

2. Preliminaries

The following are required in the sequel.

In 2000, Branciari introduced the concept of generalized metric space (or Rectangular metric space), defined as follows:

Definition 1 ([3])**.** *Let $\digamma$ be a set and $\delta\colon \digamma \times \digamma \to [0,\infty)$ a mapping, such that for all $\beta,\gamma \in \digamma$ and for all distinct point $\mathfrak{a},\mathfrak{b} \in \digamma$, each distinct β and γ:*

1. $\delta(\beta,\gamma) = 0 \iff \beta = \gamma$;
2. $\delta(\beta,\gamma) = \delta(\gamma,\beta)$;
3. $\delta(\beta,\gamma) \leq \delta(\beta,\mathfrak{a}) + \delta(\mathfrak{a},\mathfrak{b}) + \delta(\mathfrak{b},\gamma)$ *(rectangular inequality).*

Then, we will say that $(\digamma,\delta)$ is a Branciari (or Rectangular) metric space.

The following proposition proved by Kirk and Shahzad [43] will be required in the sequel.

Proposition 1 ([43])**.** *Suppose that $\{\beta_\vartheta\}$ is a Cauchy sequence in a Rectangular metric space, such that*

$$\lim_{\vartheta\to\infty} \delta(\beta_\vartheta,\mathfrak{u}) = \lim_{\vartheta\to\infty} \delta(\beta_\vartheta,\mathfrak{z}) = 0, \quad \textit{where} \;\; \mathfrak{u},\mathfrak{z} \in \digamma.$$

Then, $\mathfrak{u} = \mathfrak{z}$.

The concept of a simulation function was introduced by Khojasteh et al. [25] in 2015, as follows;

Definition 2 ([25])**.** *A map $\eta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is said to be a simulation function if the following conditions are satisfied;*

$(\eta 1)$ $\eta(0,0) = 0$;
$(\eta 2)$ $\eta(\mathfrak{t},\mathfrak{s}) < \mathfrak{t} - \mathfrak{s}$ *for each* $\mathfrak{s},\mathfrak{t} \in [0,\infty)$;
$(\eta 3)$ *for any two sequences* $(\mathfrak{s}_\vartheta)$ *and* $(\mathfrak{t}_\vartheta)$ *in* $[0,\infty)$, *such that* $\lim_{\vartheta\to\infty} \mathfrak{s}_\vartheta = \lim_{\vartheta\to\infty} \mathfrak{t}_\vartheta > 0$, *we have* $\limsup_{\vartheta\to\infty} \eta(\mathfrak{t}_\vartheta,\mathfrak{s}_\vartheta) < 0$.

Now, we recall the idea of (c)-comparison functions. Let us consider the set of functions $\psi : [0,\infty) \to [0,\infty)$, such that

(Ψ_1) ψ is non-decreasing;
(Ψ_2) $\sum_{\vartheta=1}^{+\infty} \psi^{\vartheta}(\mathfrak{t}) <\in$ for all $\mathfrak{t} > 0$, where ψ^{ϑ} is the ϑ^{th}-iterate of ψ.

These functions are known in the literature as (c)-comparison functions. The family of such functions are denoted by Ψ. Also, it can be easily proven that if ψ is a (c)-comparison function, then $\psi(\mathfrak{t}) < \mathfrak{t}$ for any $\mathfrak{t} > 0$.

Now, we recollect the notion of α-admissible mappings defined by Aydi et al. [23] as follows:

Definition 3 ([23]). *Given that $\mathfrak{f}, \mathfrak{g} : \digamma \to \digamma$ are two self-maps and $\alpha : \digamma \times \digamma \to [0, \infty)$. Then, the pair $(\mathfrak{f}, \mathfrak{g})$ is said to be α-admissible if*

$$\beta, \gamma \in \digamma, \;\; \alpha(\beta, \gamma) \geq 1 \implies \min\{\alpha(\mathfrak{f}\beta, \mathfrak{f}\gamma), \alpha(\mathfrak{g}\beta, \mathfrak{f}\gamma), \alpha(\mathfrak{f}\beta, \mathfrak{g}\gamma), \alpha(\mathfrak{g}\beta, \mathfrak{g}\gamma)\} \geq 1.$$

If $\mathfrak{f} = \mathfrak{g}$, then $\mathfrak{f}$ is called α-admissible.

Further, Aydi et al. [23] also introduced the concept of a generalized (α, ψ)-contractive pair of mappings as follows:

Definition 4 ([23]). *Let $(\digamma, \delta)$ be a Rectangular metric space and $\mathfrak{f}, \mathfrak{g} \colon \digamma \to \digamma$ be two given mappings. We say that $(\mathfrak{f}, \mathfrak{g})$ is a generalized (α, ψ)-contractive pair of mappings if there are two functions $\alpha \colon \digamma \times \digamma \to [0, \infty)$ and $\psi \in \Psi$, such that*

$$\alpha(\beta, \gamma)\delta(\mathfrak{f}\beta, \mathfrak{g}\gamma) \leq \psi(\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta, \gamma)) \;\; \textit{and}$$
$$\alpha(\beta, \gamma)\delta(\mathfrak{g}\beta, \mathfrak{f}\gamma) \leq \psi(\mathtt{M}_{\mathfrak{g},\mathfrak{f}}(\beta, \gamma)), \;\; \forall\, \beta, \gamma \in \digamma,$$

where

$$(\mathtt{M}_{\mathfrak{h},\mathfrak{k}}(\beta, \gamma)) = \max\{\delta(\beta, \gamma), \delta(\beta, \mathfrak{h}\beta), \delta(\gamma, \mathfrak{k}\gamma)\},$$

for $\mathfrak{h}, \mathfrak{k} \colon \digamma \to \digamma$.

Gordji et al. [34] proposed orthogonal sets and generalized Banach fixed point theorems in 2017. He describes the following definitions as follows:

Definition 5 ([34]). *Let $\digamma$ be a non-void set, and let $\perp$ be a binary relation defined on $\digamma \times \digamma$. If $(\digamma, \perp)$ is called an orthogonal set, then*

$$\exists\; \beta_0 \in \digamma : (\forall\, \beta \in \digamma,\; \beta \perp \beta_0) \quad \textit{or} \quad (\forall\, \beta \in \digamma, \beta_0 \perp \beta).$$

Definition 6 ([34]). *Let $(\digamma, \perp)$ be an orthogonal set. A sequence $\{\beta_\vartheta\}_{\vartheta \in \mathbb{N}}$ is called an orthogonal sequence if*

$$(\forall\, \vartheta,\; \beta_\vartheta \perp \beta_{\vartheta+1}) \;\; \textit{or} \;\; (\forall\, \vartheta,\; \beta_{\vartheta+1} \perp \beta_\vartheta).$$

Definition 7 ([34]). *Let $(\digamma, \delta)$ be an orthogonal metric space. Then, $\mathfrak{f} \colon \digamma \to \digamma$ is called orthogonal-continuous at $\beta \in \digamma$ if for each orthogonal sequence, $\{\beta_\vartheta\} \in \digamma$ with $\delta(\beta_\vartheta, \beta) \to 0$, we obtain $\delta(\mathfrak{f}\beta_\vartheta, \mathfrak{f}\beta) \to 0$.*

Definition 8 ([34]). *Let $(\digamma, \perp)$ be an orthogonal set with the metric δ. Then $(\digamma, \perp, \delta)$ is said to be an orthogonal-complete if each orthogonal Cauchy sequence is convergent.*

Definition 9 ([34]). *Let $(\digamma, \perp \delta)$ be an orthogonal metric space and $0 < \lambda < 1$. A map $\mathfrak{f} : \digamma \to \digamma$ is said to be an orthogonal contraction with Lipschitz constant λ if $\forall\;\; \beta, \gamma \in \digamma$ with $\beta \perp \gamma$,*

$$\delta(\mathfrak{f}\beta, \mathfrak{f}\gamma) \leq \lambda\delta(\beta, \gamma).$$

Definition 10 ([34]). *Let $(\digamma, \perp)$ be an orthogonal set. A mapping $\mathfrak{f} : \digamma \to \digamma$ is said to be orthogonal-preserving if $\beta \perp \gamma$ implies $\mathfrak{f}\beta \perp \mathfrak{f}\gamma$.*

Ramezani [36] introduced the concept of $\boldsymbol{\alpha}$-admissible in the following way:

Definition 11 ([36]). *Let $(\digamma, \perp)$ be an orthogonal set and δ be a metric on $\digamma$, $\mathfrak{f} : \digamma \to \digamma$ be a map, and let $\boldsymbol{\alpha} : \digamma \times \digamma \to \mathbb{R}^+$ be a function. Then, map $\mathfrak{f}$, say that orthogonal-$\boldsymbol{\alpha}$-admissible, if $\forall\ \beta, \gamma \in \digamma$ with $\beta \perp \gamma$*

$$\boldsymbol{\alpha}(\beta, \gamma) \geq 1 \implies \boldsymbol{\alpha}(\mathfrak{f}\beta, \mathfrak{f}\gamma) \geq 1.$$

In 2022, Aiman et al. [42] defined the orthogonal Branciari (Rectangular) metric spaces as follows:

Definition 12 ([42]). *The triplet $(\digamma, \perp, \delta)$ is said to be an orthogonal Rectangular metric space if $(\digamma, \perp)$ is an orthogonal set and $(\digamma, \delta)$ is a Rectangular metric space.*

In the next section, we present our main results.

3. Main Results

We commence this section by introducing the concept of an orthogonal generalized Λ-contraction mapping. Then, we prove a couple of common fixed point results in an orthogonal complete orthogonal Rectangular metric space.

Definition 13. *Let $(\digamma, \perp, \delta)$ be an orthogonal complete orthogonal Rectangular metric space, and $\mathfrak{f}, \mathfrak{g} \colon \digamma \to \digamma$ be a two maps. We say that $(\mathfrak{f}, \mathfrak{g})$ is an orthogonal generalized Λ-contraction pair of maps with respect to a simulation function η, if there are two functions $\alpha \colon \digamma \times \digamma \to [0, \infty)$ and $\psi \in \Psi$, such that for all $\beta, \gamma \in \digamma$ with $\beta \perp \gamma$ or $\gamma \perp \beta$,*

$$\begin{aligned} &(a)\ \eta\Big(\alpha(\beta,\gamma)\delta(\mathfrak{f}\beta,\mathfrak{g}\gamma), \mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta,\gamma)\Big) \geq 0 \ \text{and} \\ &(b)\ \eta\Big(\alpha(\beta,\gamma)\delta(\mathfrak{g}\beta,\mathfrak{f}\gamma), \mathtt{M}_{\mathfrak{g},\mathfrak{f}}(\beta,\gamma)\Big) \geq 0. \end{aligned} \tag{1}$$

Whenever $\mathfrak{f} = \mathfrak{g}$, the mapping $\mathfrak{f}$ is said to be an orthogonal generalized Λ-contraction with respect to η.

In case when either (a) or (b) holds, $(\mathfrak{f}, \mathfrak{g})$ is called an orthogonal semi-generalized Λ-contraction pair of maps with respect to η.

Now, we prove the common fixed point via orthogonal generalized Λ-contraction.

Theorem 1. *Let $(\digamma, \perp, \delta)$ be an orthogonal complete orthogonal Rectangular metric space, $\mathfrak{f}, \mathfrak{g} \colon \digamma \to \digamma$ be a two maps and $\eta \in \mathtt{F}$. Suppose that*

(i) *$(\mathfrak{f}, \mathfrak{g})$ is an orthogonal preserving;*
(ii) *$(\mathfrak{f}, \mathfrak{g})$ is an orthogonal generalized Λ-contraction pair of mappings with respect to η;*
(iii) *There exists $\beta_0 \in \digamma$, such that $\alpha(\beta_0, \mathfrak{f}\beta_0) \geq 1, \alpha(\beta_0, \mathfrak{g}\beta_0) \geq 1$ and $\alpha(\beta_0, \mathfrak{f}\mathfrak{g}\beta_0) \geq 1$;*
(iv) *Both $\mathfrak{f}$ and $\mathfrak{g}$ are orthogonal continuous, and for any sufficiently large $\vartheta \in \mathbb{Z}^+$, $(\mathfrak{f}\mathfrak{g})^\vartheta \beta_0 = (\mathfrak{g}\mathfrak{f})^\vartheta \beta_0$.*
Then, there exists a unique common fixed point $\mathfrak{u} \in \digamma$ of $\mathfrak{f}$ and $\mathfrak{g}$.

Proof. Since $(\digamma, \perp)$ is an orthogonal set,

$$\exists\ \beta \in \digamma : \forall\ \beta_0 \in \digamma,\ \ \beta \perp \beta_0 \ \text{ or } \ \forall\ \beta_0 \in \digamma,\ \ \beta_0 \perp \beta.$$

It follows that $\beta_0 \perp \mathfrak{f}\beta_0$ or $\mathfrak{f}\beta_0 \perp \beta_0$, and $\beta_0 \perp \mathfrak{g}\beta_0$ or $\mathfrak{g}\beta_0 \perp \beta_0$.
Let

$$\begin{aligned} &\beta_1 = \mathfrak{f}(\beta_0) = \mathfrak{g}\beta_0; \ \ \beta_2 = \mathfrak{f}\beta_1 = \mathfrak{g}\beta_1 = \mathfrak{f}^2\beta_0 = \mathfrak{g}^2\beta_0; \ldots \\ &\beta_\vartheta = \mathfrak{f}\beta_{\vartheta-1} = \mathfrak{g}\beta_{\vartheta-1} = \mathfrak{f}^\vartheta\beta_0 = \mathfrak{g}^\vartheta\beta_0, \ \ \forall\ \vartheta \in \mathbb{N}. \end{aligned}$$

Firstly, we prove the common fixed points of $\mathfrak{f}$ and $\mathfrak{g}$. If $\beta_\vartheta = \mathfrak{f}\beta_{\vartheta-1} = \mathfrak{g}\beta_{\vartheta-1}, \ \forall\ \vartheta \in \mathbb{N}$. Now, we assume the below cases:

(i) If there exists $\beta_0 \in \mathbb{N} \cup \{0\}$ such that $\beta_{\mathfrak{n}_o} = \beta_{\mathfrak{n}_o+1}$, then we have $\mathfrak{f}\beta_{\mathfrak{n}_o} = \mathfrak{g}\beta_{\mathfrak{n}_o} = \beta_{\mathfrak{n}_o}$. It is clear that $\beta_{\mathfrak{n}_o}$ is a common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$. Therefore, the proof is completed.

(ii) If $\beta_\vartheta \neq \beta_{\vartheta+1}$ for any $\vartheta \in \mathbb{N} \cup \{0\}$, then we have $\vartheta > 0$ for each $\vartheta \in \mathbb{N}$. Since $(\mathfrak{f}, \mathfrak{g})$ is an orthogonal preserving such that $\delta(\beta_\vartheta, \beta_{\vartheta+1}) > 0$, we have

$$\beta_\vartheta \perp \beta_{\vartheta+1} \text{ or } \beta_{\vartheta+1} \perp \beta_\vartheta.$$

From assumption (iii), $\exists\ \beta_0 \in \digamma$, such that

$$\alpha(\beta_0, \mathfrak{f}\beta_0) \geq 1,\ \ \alpha(\beta_0, \mathfrak{g}\beta_0) \geq 1 \text{ and } \alpha(\beta_0, \mathfrak{f}\mathfrak{g}\beta_0) \geq 1.$$

We construct an orthogonal sequence $\{\beta_\vartheta\} \in \digamma$ as follows:

$$\beta_\vartheta = \begin{cases} \mathfrak{g}\beta_{\vartheta-1}, & \text{if } \vartheta \textit{ is even}, \\ \mathfrak{f}\beta_{\vartheta-1}, & \text{if } \vartheta \textit{ is odd}, \end{cases}$$

for all $\vartheta \in \mathbb{N}$. So $\beta_1 = \mathfrak{f}\beta_0$ and $\beta_2 = \mathfrak{g}\beta_1, \ \forall\ \vartheta \in \mathbb{N}_0$. Since the pair $(\mathfrak{f}, \mathfrak{g})$ is an orthogonal α-admissible, we have

$$\alpha(\beta_0, \beta_1) = \alpha(\beta_0, \mathfrak{f}\beta_0) \geq 1 \implies \alpha(\beta_1, \beta_2) = \alpha(\mathfrak{f}\beta_0, \mathfrak{g}\beta_1) = \alpha(\mathfrak{f}\beta_0, \mathfrak{g}\mathfrak{f}\beta_0) \geq 1.$$

By induction, we obtain

$$\alpha(\beta_\vartheta, \beta_{\vartheta+1}) \geq 1, \ \forall\ \vartheta \in \mathbb{N}_0. \tag{2}$$

Starting with

$$\alpha(\beta_0, \beta_2) = \alpha(\beta_0, \mathfrak{g}\mathfrak{f}\beta_0) \geq 1 \implies \alpha(\beta_1, \beta_3) = \alpha(\mathfrak{f}\beta_0, \mathfrak{f}\beta_2) = \alpha(\mathfrak{f}\beta_0, \mathfrak{f}(\mathfrak{g}\mathfrak{f}\beta_0)) \geq 1,$$

and so,

$$\alpha(\beta_\vartheta, \beta_{\vartheta+2}) \geq 1, \ \forall\ \vartheta \in \mathbb{N}_o.$$

Suppose that there exists ϑ_0 such that $\beta_{2\mathfrak{n}_o} = \beta_{2\mathfrak{n}_o+1}$ for some $\vartheta_0 \in \mathbb{N}$.
Then, $\mathfrak{u} = \beta_{2\mathfrak{n}_o}$ is a common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$. Indeed,

$$\mathfrak{u} = \beta_{2\mathfrak{n}_o} = \beta_{2\mathfrak{n}_o+1} = \mathfrak{f}\beta_{2\mathfrak{n}_o} = \mathfrak{f}\mathfrak{u}.$$

Now, we show that $\delta(\beta_{2\mathfrak{n}_o+1}, \beta_{2\mathfrak{n}_o+2}) = 0$. Since

$$\begin{aligned} 0 &\leq \eta(\alpha(\beta_{2\mathfrak{n}_o}, \beta_{2\mathfrak{n}_o+1})\delta(\beta_{2\mathfrak{n}_o+1}, \beta_{2\mathfrak{n}_o+2}), \mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta_{2\mathfrak{n}_o}, \beta_{2\mathfrak{n}_o+1})) \\ &= \eta(\alpha(\beta_{2\mathfrak{n}_o}, \beta_{2\mathfrak{n}_o+1})\delta(\beta_{2\mathfrak{n}_o+1}, \beta_{2\mathfrak{n}_o+2}), \max\{\delta(\beta_{2\mathfrak{n}_o}, \beta_{2\mathfrak{n}_o+1}), \delta(\beta_{2\mathfrak{n}_o+1}, \beta_{2\mathfrak{n}_o+2})\}) \\ &= \eta(\alpha(\beta_{2\mathfrak{n}_o}, \beta_{2\mathfrak{n}_o+1})\delta(\beta_{2\mathfrak{n}_o+1}, \beta_{2\mathfrak{n}_o+2}), \delta(\beta_{2\mathfrak{n}_o+1}, \beta_{2\mathfrak{n}_o+2})) \\ &< \delta(\beta_{2\mathfrak{n}_o+1}, \beta_{2\mathfrak{n}_o+2}) - \alpha(\beta_{2\mathfrak{n}_o}, \beta_{2\mathfrak{n}_o+1})\delta(\beta_{2\mathfrak{n}_o+1}, \beta_{2\mathfrak{n}_o+2}). \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha(\beta_{2\mathfrak{n}_o}, \beta_{2\mathfrak{n}_o+1})\delta(\beta_{2\mathfrak{n}_o+1}, \beta_{2\mathfrak{n}_o+2}) &< \delta(\beta_{2\mathfrak{n}_o+1}, \beta_{2\mathfrak{n}_o+2}) \\ \implies \alpha(\beta_{2\mathfrak{n}_o}, \beta_{2\mathfrak{n}_o+1}) &< 1, \end{aligned}$$

which is a contradiction. Hence,

$$\mathfrak{u} = \beta_{2\mathfrak{n}_o+1} = \beta_{2\mathfrak{n}_o+2} = \mathfrak{g}\beta_{2\mathfrak{n}_o+1} = \mathfrak{g}\mathfrak{u}.$$

Therefore, $\mathfrak{u}$ is a common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$. Similarly, when $\beta_{2\mathfrak{n}_o-1} = \beta_{2\mathfrak{n}_o}$ for some $\vartheta_0 \in \mathbb{N}$, then also we can deduce that $\mathfrak{u}$ is a common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$. For the rest of the proof, we can assume that

$$\beta_\vartheta \neq \beta_{\vartheta+1}, \ \forall \ \vartheta \in \mathbb{N}.$$

Set

$$\mathtt{M}(\beta_\vartheta, \beta_\mathfrak{m}) = \begin{cases} \mathtt{M}_{\mathfrak{g},\mathfrak{f}}(\beta_\vartheta, \beta_\mathfrak{m}), & \text{if } \vartheta \text{ is odd and if } \mathfrak{m} \text{ is even,} \\ \mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta_\vartheta, \beta_\mathfrak{m}), & \text{if } \vartheta \text{ is even and if } \mathfrak{m} \text{ is odd,} \end{cases}$$

for all $\mathfrak{m}, \vartheta \in \mathbb{N}$.

Step A:
We prove that

$$\lim_{\vartheta\to\infty} \delta(\beta_\vartheta, \beta_{\vartheta+1}) = 0.$$

First, we claim that

$$\max\{\delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}), \delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2})\} = \delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}), \ \forall \ \vartheta \in \mathbb{N}_0.$$

We argue by a contradiction. Suppose that for some $\vartheta \in \mathbb{N}_0$,

$$\max\{\delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}), \delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2})\} = \delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}).$$

For such $\vartheta \in \mathbb{N}_0$, we have

$$\begin{aligned} 0 &\leq \eta(\alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}), \mathtt{M}(\beta_{2\vartheta}, \beta_{2\vartheta+1})) \\ &= \eta(\alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}), \delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2})) \\ &< \delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}) - \alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}). \end{aligned}$$

Hence,

$$\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}) > \alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2})$$

and so $\alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1}) < 1$, which is a contradiction with respect to (2). Thus,

$$\max\{\delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}), \delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2})\} = \delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}), \ \forall \ \vartheta \in \mathbb{N}_0.$$

Using (2) and Definition 13, it follows that

$$\begin{aligned} 0 &\leq \eta(\alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}), \mathtt{M}(\beta_{2\vartheta}, \beta_{2\vartheta+1})) \\ &= \eta(\alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}), \mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta_{2\vartheta}, \beta_{2\vartheta+1})) \\ &= \eta(\alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}), \max\{\delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}), \delta(\beta_{2\vartheta}, \mathfrak{f}\beta_{2\vartheta}), \delta(\beta_{2\vartheta+1}, \mathfrak{g}\beta_{2\vartheta+1})\}) \\ &= \eta(\alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}), \max\{\delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}), \delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}), \delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2})\}) \\ &= \eta(\alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}), \max\{\delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}), \delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2})\}) \\ &= \eta(\alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}), \delta(\beta_{2\vartheta}, \beta_{2\vartheta+1})) \\ &< \delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}) - \alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}), \ \forall \ \vartheta \in \mathbb{N}_0. \end{aligned}$$

Thus, we have

$$\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}) \leq \alpha(\beta_{2\vartheta}, \beta_{2\vartheta+1})\delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+2}) < \delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}), \ \forall \ \vartheta \in \mathbb{N}_0. \quad (3)$$

Similarly, we can obtain that

$$\max\{\delta(\beta_{2\vartheta-1},\beta_{2\vartheta}),\delta(\beta_{2\vartheta},\beta_{2\vartheta+1})\} = \delta(\beta_{2\vartheta-1},\beta_{2\vartheta}), \ \forall \ \vartheta \in \mathbb{N}.$$

Therefore,

$$\begin{aligned} 0 &\leq \eta(\alpha(\beta_{2\vartheta-1},\beta_{2\vartheta})\delta(\beta_{2\vartheta},\beta_{2\vartheta+1}),\mathtt{M}(\beta_{2\vartheta-1},\beta_{2\vartheta})) \\ &= \eta(\alpha(\beta_{2\vartheta-1},\beta_{2\vartheta})\delta(\beta_{2\vartheta},\beta_{2\vartheta+1}),\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta_{2\vartheta-1},\beta_{2\vartheta})) \\ &= \eta(\alpha(\beta_{2\vartheta-1},\beta_{2\vartheta})\delta(\beta_{2\vartheta},\beta_{2\vartheta+1}),\max\{\delta(\beta_{2\vartheta-1},\beta_{2\vartheta}),\delta(\beta_{2\vartheta-1},\mathfrak{g}\beta_{2\vartheta-1}),\delta(\beta_{2\vartheta},\mathfrak{f}\beta_{2\vartheta})\}) \\ &= \eta(\alpha(\beta_{2\vartheta-1},\beta_{2\vartheta})\delta(\beta_{2\vartheta},\beta_{2\vartheta+1}),\max\{\delta(\beta_{2\vartheta-1},\beta_{2\vartheta}),\delta(\beta_{2\vartheta-1},\beta_{2\vartheta}),\delta(\beta_{2\vartheta},\beta_{2\vartheta+1})\}) \\ &= \eta(\alpha(\beta_{2\vartheta-1},\beta_{2\vartheta})\delta(\beta_{2\vartheta},\beta_{2\vartheta+1}),\max\{\delta(\beta_{2\vartheta-1},\beta_{2\vartheta}),\delta(\beta_{2\vartheta},\beta_{2\vartheta+1})\}) \\ &= \eta(\alpha(\beta_{2\vartheta-1},\beta_{2\vartheta})\delta(\beta_{2\vartheta},\beta_{2\vartheta+1}),\delta(\beta_{2\vartheta-1},\beta_{2\vartheta})) \\ &< \delta(\beta_{2\vartheta-1},\beta_{2\vartheta}) - \alpha(\beta_{2\vartheta-1},\beta_{2\vartheta})\delta(\beta_{2\vartheta},\beta_{2\vartheta+1}), \ \forall \ \vartheta \in \mathbb{N}_0. \end{aligned}$$

Thus, we have

$$\delta(\beta_{2\vartheta},\beta_{2\vartheta+1}) \leq \alpha(\beta_{2\vartheta-1},\beta_{2\vartheta})\delta(\beta_{2\vartheta},\beta_{2\vartheta+1}) < \delta(\beta_{2\vartheta-1},\beta_{2\vartheta}), \ \forall \ \vartheta \geq 1. \tag{4}$$

From (3) and (4), we have

$$\delta(\beta_{\vartheta},\beta_{\vartheta+1}) < \delta(\beta_{\vartheta-1},\beta_{\vartheta}), \ \forall \ \vartheta \geq 1.$$

So, there exists some $\epsilon \geq 0$, such that $\lim_{\vartheta\to\infty} \delta(\beta_{\vartheta},\beta_{\vartheta+1}) = \epsilon$. We shall prove that $\epsilon = 0$. Suppose $\epsilon > 0$. From (3) and (4), we have

$$\lim_{\vartheta\to\infty} \alpha(\beta_{\vartheta-1},\beta_{\vartheta+})\delta(\beta_{\vartheta},\beta_{\vartheta+1}) = \epsilon.$$

Set

$$\mathfrak{s}_{\vartheta} = \alpha(\beta_{\vartheta-1},\beta_{\vartheta+})\delta(\beta_{\vartheta},\beta_{\vartheta+1})$$

and

$$\mathfrak{t}_{\vartheta} = \delta(\beta_{\vartheta},\beta_{\vartheta+1}).$$

By the Definition 2-(η3), we have

$$0 \leq \limsup_{\vartheta\to\infty} \eta(\delta(\beta_{\vartheta},\beta_{\vartheta+1}),\alpha(\beta_{\vartheta-1},\beta_{\vartheta+})\delta(\beta_{\vartheta},\beta_{\vartheta+1})) = \limsup_{\vartheta\to\infty} \eta(\mathfrak{s}_{\vartheta},\mathfrak{t}_{\vartheta}) < 0,$$

which is a contradiction. Therefore, $\epsilon = 0$.

Step B:

We prove

$$\lim_{\vartheta\to\infty} \delta(\beta_{\vartheta},\beta_{\vartheta+2}) = 0.$$

We consider that

$$\lim_{\vartheta\to\infty} \delta(\beta_{\vartheta},\beta_{\vartheta+2}) = \mathfrak{a} > 0.$$

Also, we construct another sequence $\{\gamma_{\vartheta}\}$ defined as

$$\gamma_0 = \beta_0, \gamma_1 = \mathfrak{g}\gamma_0, \gamma_2 = \mathfrak{f}\gamma_1, \ldots, \gamma_{2\vartheta} = \mathfrak{f}\beta_{2\vartheta-1} \ \text{and} \ \gamma_{2\vartheta+1} = \mathfrak{g}\gamma_{2\vartheta} \ldots, \ \forall \ \vartheta \in \mathbb{N}.$$

Now by (iii), we can derive that

$$\beta_{2\vartheta} = (\mathfrak{g}\mathfrak{f})^{\vartheta}\beta_0 = (\mathfrak{f}\mathfrak{g})^{\vartheta}\beta_0 = (\mathfrak{f}\mathfrak{g})^{\vartheta}\gamma_0 = \gamma_{2\vartheta},$$

for a sufficiently large positive integer ϑ. Also, using similar calculations as in the proof of

$$\lim_{\vartheta\to\infty} \delta(\beta_\vartheta, \beta_{\vartheta+1}) = 0,$$

we can obtain

$$\lim_{\vartheta\to\infty} \delta(\beta_\vartheta, \beta_{\vartheta+1}) = 0.$$

Further, from (2) we have, $\alpha(\beta_{2\vartheta-1}, \beta_{2\vartheta+1}) \geq 1$, and hence,

$$\alpha(\beta_{2\vartheta-1}, \beta_{2\vartheta+1}) = \alpha(\beta_{2\vartheta-1}, \gamma_{2\vartheta+1}) = \alpha(\mathfrak{f}\beta_{2\vartheta-2}, \mathfrak{g}\gamma_{2\vartheta}) = \alpha(\mathfrak{f}\beta_{2\vartheta-2}, \mathfrak{g}\beta_{2\vartheta}) \geq 1.$$

On the other hand, we obtain

$$\begin{aligned} 0 &\leq \eta\big(\alpha(\beta_{2\vartheta-1}, \beta_{2\vartheta+1})\delta(\mathfrak{g}\beta_{2\vartheta-1}, \mathfrak{f}\beta_{2\vartheta+1}), \mathtt{M}_{\mathfrak{g},\mathfrak{f}}(\beta_{2\vartheta-1}, \gamma_{2\vartheta+1})\big) \\ &< \mathtt{M}_{\mathfrak{g},\mathfrak{f}}(\beta_{2\vartheta-1}, \gamma_{2\vartheta+1}) - \alpha(\beta_{2\vartheta-1}, \gamma_{2\vartheta+1})\delta(\mathfrak{g}\beta_{2\vartheta-1}, \mathfrak{f}\beta_{2\vartheta+1}). \end{aligned}$$

This implies

$$\alpha(\beta_{2\vartheta-1}, \beta_{2\vartheta+1})\delta(\mathfrak{g}\beta_{2\vartheta-1}, \mathfrak{f}\beta_{2\vartheta+1}) < \mathtt{M}_{\mathfrak{g},\mathfrak{f}}(\beta_{2\vartheta-1}, \gamma_{2\vartheta+1}). \tag{5}$$

Now, using (5), we obtain

$$\begin{aligned} \delta(\beta_{2\vartheta}, \beta_{2\vartheta+2}) &= \delta(\beta_{2\vartheta}, \gamma_{2\vartheta+2}) \\ &= \delta(\mathfrak{g}\beta_{2\vartheta-1}, \mathfrak{f}\gamma_{2\vartheta+1}) \\ &\leq \alpha(\beta_{2\vartheta-1}, \gamma_{2\vartheta+1})\delta(\mathfrak{g}\beta_{2\vartheta-1}, \mathfrak{f}\gamma_{2\vartheta+1}) \\ &\leq \mathtt{M}_{\mathfrak{g},\mathfrak{f}}(\beta_{2\vartheta-1}, \gamma_{2\vartheta+1}) \\ &= \max\{\delta(\beta_{2\vartheta-1}, \beta_{2\vartheta+1}), \delta(\beta_{2\vartheta-1}, \mathfrak{g}\beta_{2\vartheta-1}), \delta(\gamma_{2\vartheta+1}, \mathfrak{f}\gamma_{2\vartheta+1})\} \\ &= \max\{\delta(\beta_{2\vartheta-1}, \beta_{2\vartheta+1}), \delta(\beta_{2\vartheta-1}, \beta_{2\vartheta}), \delta(\gamma_{2\vartheta+1}, \gamma_{2\vartheta+2})\} \\ &= \delta(\beta_{2\vartheta-1}, \beta_{2\vartheta+1}) \\ &= \delta(\mathfrak{f}\beta_{2\vartheta-2}, \mathfrak{g}\beta_{2\vartheta}) \\ &\leq \alpha(\beta_{2\vartheta-2}, \beta_{2\vartheta})\delta(\mathfrak{f}\beta_{2\vartheta-2}, \mathfrak{g}\beta_{2\vartheta}) \\ &\leq \mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta_{2\vartheta-2}, \beta_{2\vartheta}) \\ &= \max\{\delta(\beta_{2\vartheta-2}, \beta_{2\vartheta}), \delta(\beta_{2\vartheta-2}, \mathfrak{f}\beta_{2\vartheta-2}), \delta(\beta_{2\vartheta+1}, \mathfrak{g}\beta_{2\vartheta})\} \\ &= \max\{\delta(\beta_{2\vartheta-2}, \beta_{2\vartheta}), \delta(\beta_{2\vartheta-2}, \beta_{2\vartheta-1}), \delta(\beta_{2\vartheta+1}, \beta_{2\vartheta+1})\} \\ &= \delta(\beta_{2\vartheta-2}, \beta_{2\vartheta}), \quad \forall\ \vartheta \in \mathbb{N}_0. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \delta(\beta_{2\vartheta}, \beta_{2\vartheta+2}) &\leq \alpha(\beta_{2\vartheta-2}, \beta_{2\vartheta})\delta(\mathfrak{f}\beta_{2\vartheta-2}, \mathfrak{g}\beta_{2\vartheta}) \\ &= \alpha(\beta_{2\vartheta-2}, \beta_{2\vartheta})\delta(\beta_{2\vartheta-1}, \beta_{2\vartheta+1}) \\ &\leq \delta(\beta_{2\vartheta-2}, \beta_{2\vartheta}), \quad \forall\ \vartheta \in \mathbb{N}_0. \end{aligned} \tag{6}$$

From (6), we obtain

$$\lim_{\vartheta\to\infty} \alpha(\beta_{2\vartheta-2}, \beta_{2\vartheta})\delta(\beta_{2\vartheta-1}, \beta_{2\vartheta+1}) = \mathfrak{a}.$$

Set $\mathfrak{s}_\vartheta = \alpha(\beta_{2\vartheta-2}, \beta_{2\vartheta})\delta(\beta_{2\vartheta-1}, \beta_{2\vartheta+1})$ and $\mathfrak{t}_\vartheta = \delta(\beta_{2\vartheta}, \beta_{2\vartheta+2})$. By Definition 2–($\eta$3), we obtain

$$0 \leq \limsup_{\vartheta\to\infty} \eta(\delta(\beta_{2\vartheta}, \beta_{2\vartheta+2}), \alpha(\beta_{2\vartheta-2}, \beta_{2\vartheta})\delta(\beta_{2\vartheta-1}, \beta_{2\vartheta+1})) = \limsup_{\vartheta\to\infty} \eta(\mathfrak{s}_\vartheta, \mathfrak{t}_\vartheta) < 0,$$

which is a contradiction. Therefore, $\mathfrak{a} = 0$.

Step C:

Here, we prove that

$$\beta_{2\vartheta+1} \neq \beta_{2\mathfrak{m}+1} \text{ and } \beta_{2\vartheta} \neq \beta_{2\mathfrak{m}}, \ \forall\, \vartheta \neq \mathfrak{m}.$$

The discussion naturally splits into the following two cases:

Case 1: If for some $\mathfrak{m}, \vartheta \in \mathbb{N}_0$, with $\mathfrak{m} > \vartheta, \beta_{2\vartheta} = \beta_{2\mathfrak{m}}$;

Case 2: If for some $\mathfrak{m}, \vartheta \in \mathbb{N}_0$, with $\mathfrak{m} > \vartheta, \beta_{2\vartheta+1} = \beta_{2\mathfrak{m}+1}$.

In Case 1, by Step A, an orthogonal sequence $(\delta(\beta_\vartheta, \beta_{\vartheta+1}))$ is decreasing, so we obtain,

$$\begin{aligned}\delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}) &= \delta(\beta_{2\vartheta}, \mathfrak{f}\beta_{2\vartheta}) \\ &= \delta(\beta_{2\mathfrak{m}}, \mathfrak{f}\beta_{2\mathfrak{m}}) \\ &= \delta(\beta_{2\mathfrak{m}}, \beta_{2\mathfrak{m}+1}) \\ &< \delta(\beta_{2\vartheta}, \beta_{2\vartheta+1}),\end{aligned}$$

which is a contradiction. In Case 2, via Step A, an orthogonal sequence $\{\delta(\beta_\vartheta, \beta_{\vartheta+1})\}$ is decreasing; thus, we have,

$$\begin{aligned}\delta(\beta_{2\vartheta+2}, \beta_{2\vartheta+1}) &= \delta(\mathfrak{g}\beta_{2\vartheta+1}, \beta_{2\vartheta+1}) \\ &= \delta(\mathfrak{g}\beta_{2\mathfrak{m}+1}, \beta_{2\mathfrak{m}+1}) \\ &= \delta(\beta_{2\mathfrak{m}+2}, \beta_{2\mathfrak{m}+1}) \\ &< \delta(\beta_{2\vartheta+2}, \beta_{2\vartheta+1}),\end{aligned}$$

which is a contradiction. Thus, we can assume that $\beta_\vartheta \neq \beta_\mathfrak{m}$ for all $\vartheta \neq \mathfrak{m}$.

Step D:

We now prove that $\{\beta_\vartheta\}$ is an orthogonal Cauchy sequence.

Assume that $\{\beta_\vartheta\}$ is not an orthogonal Cauchy sequence.

Since $\{\beta_\vartheta\} \in \digamma$ is an orthogonal sequence with distinct elements, and since from Step A and Step B,

$$\delta(\beta_\vartheta, \beta_{\vartheta+1}) \to 0 \text{ and } \delta(\beta_\vartheta, \beta_{\vartheta+2}) \to 0 \text{ as } \vartheta \to \infty,$$

using Lemma 3.3 from [15], $\exists\, \epsilon > 0$ and two orthogonal sub-sequences $\{\mathfrak{m}_\mathfrak{k}\}$ and $\{\vartheta_\mathfrak{k}\}$ of positive integers such that $\vartheta_\mathfrak{k} > \mathfrak{m}_\mathfrak{k} > \mathfrak{k}$ and the following orthogonal sequences go to ϵ as $\vartheta \to \infty$

$$\delta(\beta_{\mathfrak{m}_\mathfrak{k}}, \beta_{\vartheta_\mathfrak{k}}), \delta(\beta_{\mathfrak{m}_\mathfrak{k}}, \beta_{\vartheta_{\mathfrak{k}+1}}), \delta(\beta_{\mathfrak{m}_{\mathfrak{k}-1}}, \beta_{\vartheta_\mathfrak{k}}), \delta(\beta_{\mathfrak{m}_{\mathfrak{k}-1}}, \beta_{\vartheta_{\mathfrak{k}+1}}). \tag{7}$$

Hence, using Step A, Step B, and (7), we have

$$\limsup_{\mathfrak{k}\to\infty} \mathtt{M}(\beta_{\mathfrak{m}_\mathfrak{k}}, \beta_{\vartheta_\mathfrak{k}}) = \epsilon. \tag{8}$$

Since the pair $(\mathfrak{f}, \mathfrak{g})$ is an orthogonal α-admissible, we have $\alpha(\beta_{\mathfrak{m}_\mathfrak{k}}, \beta_{\vartheta_\mathfrak{k}}) \geq 1$. Regarding $(\mathfrak{f}, \mathfrak{g})$ is an orthogonal generalized Λ-contraction pair of maps with respect to η, and considering $\mathfrak{m}_\mathfrak{k}$ as an odd number and $\vartheta_\mathfrak{k}$ as an even number, we have

$$
\begin{aligned}
0 &\le \eta\big(\alpha(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\delta(\beta_{\vartheta_{\mathfrak{k}+1}},\beta_{\mathfrak{m}_{\mathfrak{k}+1}}),\mathtt{M}(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\big)\\
&= \eta\big(\alpha(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\delta(\beta_{\vartheta_{\mathfrak{k}+1}},\beta_{\mathfrak{m}_{\mathfrak{k}+1}}),\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\big)\\
&= \eta\big(\alpha(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\delta(\beta_{\vartheta_{\mathfrak{k}+1}},\beta_{\mathfrak{m}_{\mathfrak{k}+1}}),\max\{\delta(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}}),\delta(\beta_{\mathfrak{m}_\mathfrak{k}},\mathfrak{f}\beta_{\mathfrak{m}_\mathfrak{k}}),\delta(\beta_{\vartheta_\mathfrak{k}},\mathfrak{g}\beta_{\vartheta_\mathfrak{k}})\}\big)\\
&= \eta\big(\alpha(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\delta(\beta_{\vartheta_{\mathfrak{k}+1}},\beta_{\mathfrak{m}_{\mathfrak{k}+1}}),\max\{\delta(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}}),\delta(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\mathfrak{m}_{\mathfrak{k}+1}}),\delta(\beta_{\vartheta_\mathfrak{k}},\beta_{\vartheta_{\mathfrak{k}+1}})\}\big)\\
&< \max\{\delta(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}}),\delta(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\mathfrak{m}_{\mathfrak{k}+1}}),\delta(\beta_{\vartheta_\mathfrak{k}},\beta_{\vartheta_{\mathfrak{k}+1}})\} - \alpha(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\delta(\beta_{\vartheta_{\mathfrak{k}+1}},\beta_{\mathfrak{m}_{\mathfrak{k}+1}}),
\end{aligned}
$$

for all $\mathfrak{k} \in \mathbb{N}$. Consequently, we obtain

$$
\begin{aligned}
0 &< \delta(\beta_{\mathfrak{m}_{\mathfrak{k}+1}},\beta_{\vartheta_{\mathfrak{k}+1}}) \le \alpha(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\delta(\beta_{\mathfrak{m}_{\mathfrak{k}+1}},\beta_{\vartheta_{\mathfrak{k}+1}})\\
&< \max\{\delta(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}}),\delta(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\mathfrak{m}_{\mathfrak{k}+1}}),\delta(\beta_{\vartheta_\mathfrak{k}},\beta_{\vartheta_{\mathfrak{k}+1}})\},\ \forall\, \mathfrak{k} \in \mathbb{N}.
\end{aligned} \tag{9}
$$

From (9), together with (7) and (8), we have

$$0 \le \lim_{\mathfrak{k}\to\infty} \alpha(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\delta(\beta_{\mathfrak{m}_{\mathfrak{k}+1}},\beta_{\vartheta_{\mathfrak{k}+1}}) = \epsilon.$$

Set $\mathfrak{s}_\vartheta = \mathtt{M}(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})$ and $\mathfrak{t}_\vartheta = \alpha(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\delta(\beta_{\mathfrak{m}_{\mathfrak{k}+1}},\beta_{\vartheta_{\mathfrak{k}+1}})$. By the Definition 2 ($\eta 3$) and the relation (8), we obtain the result as follows

$$0 \le \limsup_{\mathfrak{k}\to\infty} \eta\big(\alpha(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\delta(\beta_{\vartheta_{\mathfrak{k}+1}},\beta_{\mathfrak{m}_{\mathfrak{k}+1}}),\mathtt{M}(\beta_{\mathfrak{m}_\mathfrak{k}},\beta_{\vartheta_\mathfrak{k}})\big) < 0,$$

which is a contradiction. Therefore, $\{\beta_\vartheta\}$ is an orthogonal Cauchy sequence. Since $(\mathcal{F},\perp,\delta)$ is an orthogonal complete orthogonal Rectangular metric space, there exists $\mathfrak{u} \in \mathcal{F}$, such that $\{\beta_\vartheta\}$ converges to $\mathfrak{u}$.

Hence,

$$\lim_{\vartheta\to\infty} \delta(\beta_\vartheta,\mathfrak{u}) = 0. \tag{10}$$

Step E:

We claim that $\mathfrak{u}$ is a common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$. Since $\mathfrak{f}$ and $\mathfrak{g}$ are orthogonal continuous, by (10), we obtain

$$\lim_{\vartheta\to\infty} \delta(\beta_{2\vartheta+1},\mathfrak{f}\mathfrak{u}) = \lim_{\vartheta\to\infty} \delta(\mathfrak{f}\beta_{2\vartheta},\mathfrak{f}\mathfrak{u}) = 0,$$

and

$$\lim_{\vartheta\to\infty} \delta(\beta_{2\vartheta+1},\mathfrak{g}\mathfrak{u}) = \lim_{\vartheta\to\infty} \delta(\mathfrak{g}\beta_{2\vartheta-1},\mathfrak{g}\mathfrak{u}) = 0.$$

By Proposition 1, we conclude that $\mathfrak{f}\mathfrak{u} = \mathfrak{u} = \mathfrak{g}\mathfrak{u}$. Hence, $\mathfrak{u}$ is a common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$.

Now, we prove a unique common fixed point. Consider that $\mathfrak{f}\mathfrak{p} = \mathfrak{p} = \mathfrak{g}\mathfrak{p}$ is another common fixed point for $\mathfrak{f}$ and $\mathfrak{g}$. By the choice of $\mathfrak{u}^*$, we have

$$\mathfrak{u}^* \perp \mathfrak{p} \text{ (or) } \mathfrak{p} \perp \mathfrak{u}^*.$$

Since $\mathfrak{f}$ and $\mathfrak{g}$ is orthogonal preserving, we obtain

$$(\mathfrak{f}\mathfrak{u}^* \perp \mathfrak{f}\mathfrak{p} \text{ and } \mathfrak{g}\mathfrak{u}^* \perp \mathfrak{g}\mathfrak{p}) \text{ or } (\mathfrak{f}\mathfrak{p} \perp \mathfrak{f}\mathfrak{u}^* \text{ and } \mathfrak{g}\mathfrak{p} \perp \mathfrak{g}\mathfrak{u}^*).$$

From Equation (1), we have

$$\eta\Big(\alpha(\mathfrak{u}^*,\mathfrak{p})\delta(\mathfrak{f}\mathfrak{u}^*,\mathfrak{g}\mathfrak{p}),\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\mathfrak{u}^*,\mathfrak{p})\Big) \ge 0 \ \text{ and } \ \eta\Big(\alpha(\mathfrak{u}^*,\mathfrak{p})\delta(\mathfrak{g}\mathfrak{u}^*,\mathfrak{f}\mathfrak{p}),\mathtt{M}_{\mathfrak{g},\mathfrak{f}}(\mathfrak{u}^*,\mathfrak{p})\Big) \ge 0.$$

Whenever $\mathfrak{f} = \mathfrak{g}$, the mapping $\mathfrak{f}$ is said to be an orthogonal generalized Λ-contraction with respect to η. Therefore, $\mathfrak{u}^* = \mathfrak{p}$ and the common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$ are unique. □

Our next result involves an orthogonal semi-generalized Λ-contraction pair of mappings.

Theorem 2. *Let $(\digamma, \perp, \delta)$ be an orthogonal complete orthogonal Rectangular metric space, $\mathfrak{f}, \mathfrak{g} \colon \digamma \to \digamma$ be two mappings and $\eta \in \Lambda$. Suppose that*

(i) $(\mathfrak{f}, \mathfrak{g})$ *is an orthogonal semi-generalized Λ-contraction pair of mappings with respect to η;*
(ii) *There exists $\beta_0 \in \digamma$, such that $\alpha(\beta_0, \mathfrak{f}\beta_0) \geq 1, \alpha(\beta_0, \mathfrak{g}\beta_0) \geq 1$ and $\alpha(\beta_0, \mathfrak{f}\mathfrak{g}\beta_0) \geq 1$;*
(iii) *For every $\beta, \gamma \in \digamma$, $\alpha(\beta, \gamma) = \alpha(\gamma, \beta)$;*
(iv) *Both $\mathfrak{f}$ and $\mathfrak{g}$ are orthogonal continuous and for any sufficiently large $\vartheta \in \mathbb{Z}^+, (\mathfrak{f}\mathfrak{g})^\vartheta \beta_0 = (\mathfrak{g}\mathfrak{f})^\vartheta \beta_0$;*
(v) $(\mathfrak{f}, \mathfrak{g})$ *is orthogonal preserving.*

Then, there exists a common fixed point in $\digamma$ of $\mathfrak{f}$ and $\mathfrak{g}$.

Proof. We omit the proof. It is similar to the proof of Theorem 1. □

We constructive examples authenticate our obtained Theorem 1 concerning an orthogonal generalized Λ-contraction pair of self-maps.

Example 1. *We consider an orthogonal complete metric space $\digamma = \{0, \frac{1}{\vartheta} \colon \vartheta \in \mathbb{N},\ \vartheta \geq 2\}$ endowed with the orthogonal Rectangular metric,*

$$\delta(\beta, \gamma) = \begin{cases} 0, & \text{if } \beta = \gamma; \\ 2, & \text{if } \beta, \gamma \in \{\frac{1}{\vartheta} \colon \vartheta \in \mathbb{N}, \vartheta \geq 2\}; \\ \frac{1}{2\vartheta}, & \text{if } \beta = \frac{1}{\vartheta}, \gamma = 0, \text{ or } \beta = 0, \gamma = \frac{1}{\vartheta}. \end{cases}$$

It can be easily verified that $\digamma$ is not a complete metric, but it is an orthogonal Rectangular metric. Define a relation $\perp$ on $\digamma$ by which there exists $\beta \in \digamma$,

$$\forall\, \gamma \in \digamma,\ \ \beta \perp \gamma \iff \beta\gamma \in \{\beta, \gamma\}.$$

Now, we define the mappings $\mathfrak{f} \colon \digamma \to \digamma$ such that

$$\mathfrak{f}\beta = \begin{cases} 0, & \text{if } \beta = 0; \\ \frac{1}{\vartheta}, & \text{if } \beta = \frac{1}{\vartheta}, \end{cases}$$

and $\mathfrak{g} \colon \digamma \to \digamma$ such that $\mathfrak{g}\beta = 0$, $\forall\, \beta \in \digamma$. We also consider $\eta(\mathfrak{t}, \mathfrak{s}) = \lambda\mathfrak{s} - \mathfrak{t}$, $\ \mathfrak{t}, \mathfrak{s} \in [0, \infty)$ as

$$\alpha(\beta, \gamma) = \begin{cases} \frac{1}{2}, & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta = \frac{1}{\vartheta}, \gamma = \frac{1}{\mathtt{m}}, \\ \frac{1}{\vartheta}, & \text{if } \beta = \frac{1}{\vartheta}, \gamma = 0, \text{ or } \ \beta = 0, \gamma = \frac{1}{\vartheta}. \end{cases}$$

Therefore, we obtain

$$\eta(\alpha(\beta, \gamma)\delta(\mathfrak{f}\beta, \mathfrak{g}\gamma), \mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta, \gamma)) = \lambda \mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta, \gamma) - \alpha(\beta, \gamma)\delta(\mathfrak{f}\beta, \mathfrak{g}\gamma). \tag{11}$$

Here, we have three cases:
Case 1: *When $\beta = \gamma$;*

Sub-case 1a: *$\beta = 0 = \gamma$;*
In this case, we obtain

$$
\begin{aligned}
\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta,\gamma) &= \max\{\delta(\beta,\gamma),\delta(\beta,\mathfrak{f}\beta),\delta(\gamma,\mathfrak{g}\gamma)\} \\
&= \max\{\delta(0,0),\delta(0,\mathfrak{f}0),\delta(0,\mathfrak{g}0)\} \\
&= \max\{\delta(0,0),\delta(0,0),\delta(0,0)\} \\
&= 0,
\end{aligned}
$$

and

$$\delta(\mathfrak{f}0,\mathfrak{g}0) = 0.$$

Putting these values in (11), we have

$$
\begin{aligned}
\eta(\alpha(\beta,\gamma)\delta(\mathfrak{f}\beta,\mathfrak{g}\gamma),\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta,\gamma)) = \eta(0,0) &= 0 \\
&\geq 0.
\end{aligned}
$$

This is the trivial case.

Sub-case 1b: $\beta = \frac{1}{\vartheta} = \gamma$;
clearly β *and* γ *have the same as shown in Figure 1,*

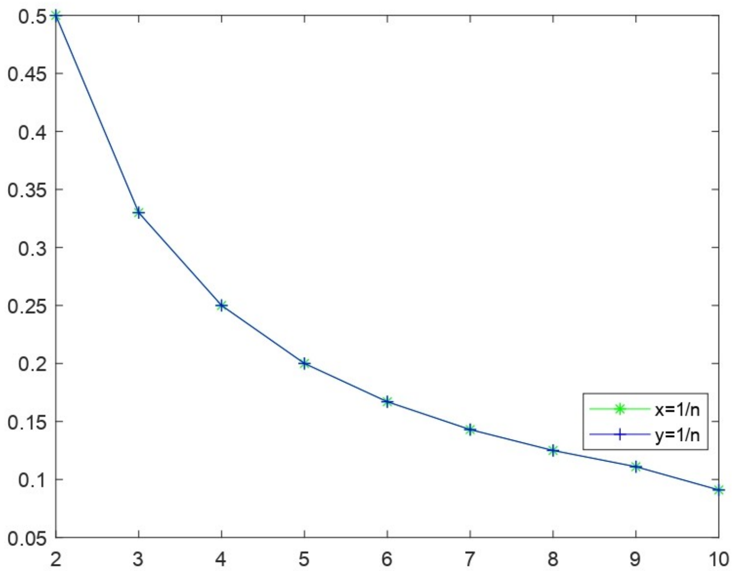

Figure 1. Figure shows that $\beta = \gamma = \frac{1}{\vartheta}$ with h = 1.

$$
\begin{aligned}
\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta,\gamma) &= \max\left\{\delta\left(\frac{1}{\vartheta},\frac{1}{\vartheta}\right),\delta\left(\frac{1}{\vartheta},\mathfrak{f}\frac{1}{\vartheta}\right),\left(\frac{1}{\vartheta},\mathfrak{g}\frac{1}{\vartheta}\right)\right\} \\
&= \max\left\{\delta\left(\frac{1}{\vartheta},\frac{1}{\vartheta}\right),\delta\left(\frac{1}{\vartheta},\frac{1}{\vartheta}\right),\left(\frac{1}{\vartheta},0\right)\right\} \\
&= \max\left\{0,\frac{1}{2\vartheta}\right\} \\
&= \frac{1}{2\vartheta},
\end{aligned}
$$

and

$$\delta\left(\mathfrak{f}\frac{1}{\vartheta},\mathfrak{g}\frac{1}{\vartheta}\right) = \delta\left(\frac{1}{\vartheta},0\right) = \frac{1}{2\vartheta}.$$

Putting these values in (11), we have

$$\begin{aligned}
\eta(\alpha(\beta,\gamma)\delta(\mathfrak{f}\beta,\mathfrak{g}\gamma),\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta,\gamma)) &= \eta\left(\frac{1}{4\vartheta},\frac{1}{2\vartheta}\right)\\
&= \frac{9}{10}\frac{1}{2\vartheta}-\frac{1}{4\vartheta}\\
&= \frac{1}{4\vartheta}\left(\frac{9}{5}-1\right)\\
&\geq 0.
\end{aligned}$$

Case 2: $\beta=\frac{1}{\vartheta},\ \gamma=0$ *or* $\beta=0,\ \gamma=\frac{1}{\vartheta}$;
Sub-case 2a: $\beta=\frac{1}{\vartheta},\ \gamma=0$;
For this case, we obtain,

$$\begin{aligned}
\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta,\gamma) &= \left\{\delta\left(\frac{1}{\vartheta},\frac{1}{\vartheta}\right),\delta\left(\frac{1}{\vartheta},\mathfrak{f}\frac{1}{\vartheta}\right),\left(\frac{1}{\vartheta},\mathfrak{g}\frac{1}{\vartheta}\right)\right\}\\
&= \max\left\{\delta\left(\frac{1}{\vartheta},0\right),\delta\left(\frac{1}{\vartheta},\mathfrak{f}\frac{1}{\vartheta}\right),\left(0,\mathfrak{g}0\right)\right\}\\
&= \max\left\{\frac{1}{2\vartheta},\delta\left(\frac{1}{\vartheta},\frac{1}{\vartheta}\right),\delta(0,0)\right\}\\
&= \frac{1}{2\vartheta},
\end{aligned}$$

and

$$\delta\left(\mathfrak{f}\frac{1}{\vartheta},\mathfrak{g}0\right)=\delta\left(\frac{1}{\vartheta},0\right)=\frac{1}{2\vartheta}.$$

From (11), we have

$$\begin{aligned}
\eta(\alpha(\beta,\gamma)\delta(\mathfrak{f}\beta,\mathfrak{g}\gamma),\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta,\gamma)) &= \eta\left(\frac{1}{\vartheta},\frac{1}{2\vartheta},\frac{1}{2\vartheta}\right)\\
&= \frac{9}{10}\cdot\frac{1}{2\vartheta}-\frac{1}{2\vartheta^2}\\
&= \frac{1}{2\vartheta}\left(\frac{9}{10}-\frac{1}{\vartheta}\right)\\
&\geq 0,[as\ \ \vartheta\geq 2].
\end{aligned}$$

Sub-case 2b:
$\beta=0,\ \gamma=\frac{1}{\vartheta}$;
we obtain,

$$\begin{aligned}
\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta,\gamma) &= \max\{\delta(\beta,\gamma),\delta(\beta,\mathfrak{f}\beta),\delta(\gamma,\mathfrak{g}\gamma)\}\\
&= \max\left\{\delta\left(0,\frac{1}{\vartheta}\right),\delta\left(0,\mathfrak{f}0\right),\left(\frac{1}{\vartheta},\mathfrak{g}\frac{1}{\vartheta}\right)\right\}\\
&= \max\left\{\delta\left(\frac{1}{2\vartheta},0\right),\delta(0,0),\left(\frac{1}{\vartheta},0\right)\right\}\\
&= \max\left\{\frac{1}{2\vartheta},0\right\}\\
&= \frac{1}{2\vartheta},
\end{aligned}$$

and

$$\delta\left(\mathfrak{f}0, \mathfrak{g}\frac{1}{\vartheta}\right) = \delta(0,0) = 0.$$

Hence, taking care of (11), we obtain,

$$\begin{aligned}\eta(\alpha(\beta,\gamma)\delta(\mathfrak{f}\beta,\mathfrak{g}\gamma), \mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta,\gamma)) &= \eta\left(0, \frac{1}{2\vartheta}\right)\\ &= \frac{9}{10} \cdot \frac{1}{2\vartheta}\\ &\geq 0.\end{aligned}$$

Case 3:
$\beta = \frac{1}{\vartheta}$, $\gamma = \frac{1}{\mathfrak{m}}$ *with* $\vartheta \neq \mathfrak{m}$*;*
So, we find to differentiate β *and* γ *as following Figure 2,*

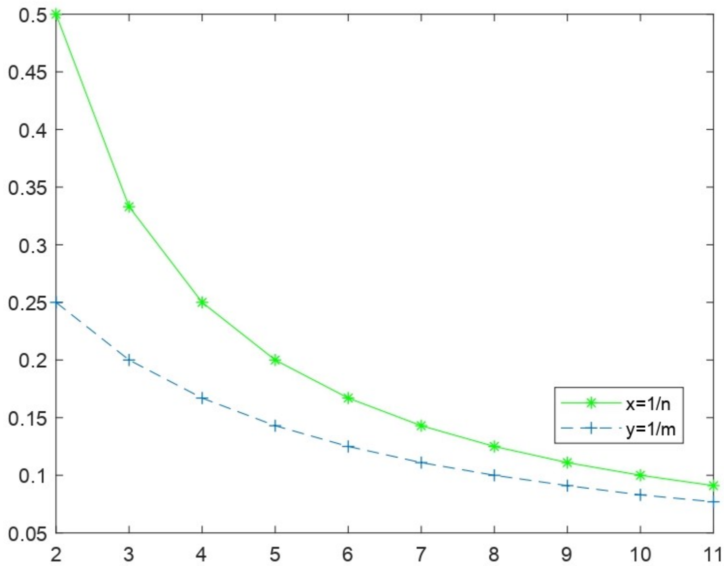

Figure 2. Comparison of $\beta = \frac{1}{\vartheta}$ and $\gamma = \frac{1}{\mathfrak{m}}$ with $\vartheta \neq \mathfrak{m}$.

$$\begin{aligned}\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta,\gamma) &= \max\left\{\delta\left(\tfrac{1}{\vartheta}, \tfrac{1}{\mathfrak{m}}\right), \delta\left(\tfrac{1}{\vartheta}, \mathfrak{f}\tfrac{1}{\vartheta}\right), \delta\left(\tfrac{1}{\mathfrak{m}}, \mathfrak{g}\tfrac{1}{\mathfrak{m}}\right)\right.\\ &= \max\left\{2, \delta\left(\tfrac{1}{\vartheta}, \tfrac{1}{\vartheta}\right), \delta\left(\tfrac{1}{\mathfrak{m}}, 0\right)\right.\\ &= \max\{2, 0, \frac{1}{2\mathfrak{m}}\}\\ &= 2,\end{aligned}$$

and

$$\delta\left(\mathfrak{f}\frac{1}{\vartheta}, \mathfrak{g}\frac{1}{\mathfrak{m}}\right) = \delta\left(\frac{1}{\vartheta}, 0\right) = \frac{1}{2\vartheta}.$$

Putting the values in (11), we obtain,

$$\begin{aligned}\eta(\alpha(\beta,\gamma)\delta(\mathfrak{f}\beta,\mathfrak{g}\gamma),\mathtt{M}_{\mathfrak{f},\mathfrak{g}}(\beta,\gamma)) &= \eta(0,2)\\ &= \frac{9}{10}\cdot 2\\ &\geq 0.\end{aligned}$$

So, the condition of (a) of Definition 13 is satisfied. Similarly, one can check for condition (b). Therefore, $\mathfrak{f}$ and $\mathfrak{g}$ satisfy both the hypotheses of Theorem 1, and using the theorem, $\mathfrak{f}$ and $\mathfrak{g}$ have a common fixed point.

4. Application

Theorem 3. *Consider the integral equations:*

$$\beta(\mathfrak{t}) = \mathfrak{g}(\mathfrak{t}) + \int_0^1 \mathtt{K}_1(\mathfrak{t},\mathfrak{s},\beta(\mathfrak{s}))\delta\mathfrak{s},\ \mathfrak{t}\in[0,1], \tag{12}$$

$$\beta(\mathfrak{t}) = \mathfrak{g}(\mathfrak{t}) + \int_0^1 \mathtt{K}_2(\mathfrak{t},\mathfrak{s},\beta(\mathfrak{s}))\delta\mathfrak{s},\ \mathfrak{t}\in[0,1]. \tag{13}$$

Suppose that

(1) $\mathtt{K}_1,\mathtt{K}_2 : [0,1]^2\times\mathbb{R}\to\mathbb{R}$ *and* $\mathfrak{g} : [0,1]\to\mathbb{R}$ *are members of* $\mathtt{L}^1([0,1])$*;*

(2) There exists $\lambda\in[0,1)$*, such that for* $\mathfrak{t},\mathfrak{s}\in[0,1]$ *and* $\mathfrak{u},\mathfrak{v}\in\mathbb{R}$*,*

$$|\mathtt{K}_1(\mathfrak{t},\mathfrak{s},\mathfrak{u}) - \mathtt{K}_2(\mathfrak{t},\mathfrak{s},\mathfrak{v})| \leq \lambda|\mathfrak{u}-\mathfrak{v}|.$$

Then, the integral Equations (12) and (13) have a unique solution in $\mathtt{C}([0,1])$*. Proof. Let* $\digamma = \mathtt{C}([0,1])$*. We define the orthogonal relation* $\perp$ *on* $\digamma$ *by*

$$\beta\perp\gamma \iff (\mathfrak{f}\beta\perp\mathfrak{f}\gamma \text{ and } \mathfrak{g}\beta\perp\mathfrak{g}\gamma) \text{ or } (\mathfrak{f}\gamma\perp\mathfrak{f}\beta \text{ and } \mathfrak{g}\gamma\perp\mathfrak{g}\beta).$$

We define $\delta : \digamma\times\digamma\to[0,\infty)$ *by*

$$\delta(\mathfrak{f},\mathfrak{g}) = ||\mathfrak{f}-\mathfrak{g}||_\infty = \max_{\mathfrak{s}\in[0,1]}|\mathfrak{f}(\mathfrak{s})-\mathfrak{g}(\mathfrak{s})|.$$

Then, $(\digamma,\perp,\delta)$ *is an orthogonal metric space, and hence,* $(\digamma,\perp,\delta)$ *is an orthogonal Rectangular metric space. We define* $\alpha : \digamma\times\digamma\to[0,\infty)$ *by*

$$\alpha(\beta,\gamma) = \begin{cases}1, & if\ \ \beta,\gamma\in[0,1],\\ 0, & otherwise.\end{cases}$$

Let $\mathtt{S},\mathtt{T} : \digamma\to\digamma$ *with* $\mathfrak{s}\perp\mathfrak{t}$

$$\mathtt{T}(\beta(\mathfrak{t})) = \mathfrak{g}(\mathfrak{t}) + \int_0^1 \mathtt{K}_1(\mathfrak{t},\mathfrak{s},\beta(\mathfrak{s}))\delta\mathfrak{s},\ \mathfrak{s},\mathfrak{t}\in[0,1],$$

$$\mathtt{S}(\beta(\mathfrak{t})) = \mathfrak{g}(\mathfrak{t}) + \int_0^1 \mathtt{K}_2(\mathfrak{t},\mathfrak{s},\beta(\mathfrak{s}))\delta\mathfrak{s},\ \mathfrak{s},\mathfrak{t}\in[0,1].$$

We mention that the integral Equations (12) *and* (13) *have a unique common solution if and only if the operators* T *and* S *have a common fixed point. Thus, we have,*

$$\begin{aligned}
\delta(\mathtt{T},\mathtt{S}) = ||\mathtt{T}\beta(\mathfrak{t}) - \mathtt{S}\gamma(\mathfrak{t})|| &= |\int_0^1 (\mathtt{K}_1(\mathfrak{t},\mathfrak{s},\beta(\mathfrak{s})) - \mathtt{K}_2(\mathfrak{t},\mathfrak{s},\gamma(\mathfrak{s}))\delta\mathfrak{s})| \\
&\leq \int_0^1 |\mathtt{K}_1(\mathfrak{t},\mathfrak{s},\beta(\mathfrak{s})) - \mathtt{K}_2(\mathfrak{t},\mathfrak{s},\gamma(\mathfrak{s}))|\delta\mathfrak{s} \\
&\leq \int_0^1 \lambda|\beta(\mathfrak{s}) - \gamma(\mathfrak{s}|\delta\mathfrak{s} \\
&= \lambda||\beta-\gamma||\infty \\
&= \lambda\delta(\beta,\gamma) \\
&\leq \lambda\mathtt{M}_{\mathtt{T},\mathtt{S}}(\beta,\gamma), \\
&\implies \lambda\mathtt{M}_{\mathtt{T}},\mathtt{S}(\beta,\gamma) - \delta(\mathtt{T},\mathtt{S}) \geq 0.
\end{aligned} \tag{14}$$

Again,

$$\begin{aligned}
\delta(\mathtt{S},\mathtt{T}) = ||\mathtt{S}\beta(\mathfrak{t}) - \mathtt{T}\gamma(\mathfrak{t})|| &= |\int_0^1 (\mathtt{K}_2(\mathfrak{t},\mathfrak{s},\beta(\mathfrak{s})) - \mathtt{K}_1(\mathfrak{t},\mathfrak{s},\gamma(\mathfrak{s}))\delta\mathfrak{s})| \\
&\leq \int_0^1 |\mathtt{K}_2(\mathfrak{t},\mathfrak{s},\beta(\mathfrak{s})) - \mathtt{K}_1(\mathfrak{t},\mathfrak{s},\gamma(\mathfrak{s}))|\delta\mathfrak{s} \\
&= \int_0^1 |\mathtt{K}_1(\mathfrak{t},\mathfrak{s},\gamma(\mathfrak{s})) - \mathtt{K}_2(\mathfrak{t},\mathfrak{s},\beta(\mathfrak{s}))|\delta\mathfrak{s} \\
&\leq \int_0^1 \lambda|\gamma(\mathfrak{s}) - \beta(\mathfrak{s})|\delta\mathfrak{s} \\
&\leq \int_0^1 \lambda|\beta(\mathfrak{s}) - \gamma(\mathfrak{s})|\delta\mathfrak{s} \\
&= \lambda||\beta-\gamma||\infty \\
&= \lambda\delta(\beta,\gamma) \\
&\leq \lambda\mathtt{M}_{\mathtt{S},\mathtt{T}}(\beta,\gamma), \\
&\implies \lambda\mathtt{M}_{\mathtt{S}},\mathtt{T}(\beta,\gamma) - \delta(\mathtt{S},\mathtt{T}) \geq 0.
\end{aligned} \tag{15}$$

We consider the simulation function as $\eta(\mathfrak{t},\mathfrak{s}) = \lambda\mathfrak{s} - \mathfrak{t}$*. Then, from* (14) *and* (15)*, and considering* $\alpha(\beta,\gamma) = 1$ *we have, for all* $\mathtt{T},\mathtt{S} \in \digamma$

$$\eta(\alpha(\beta,\gamma)\delta(\mathtt{T},\mathtt{S}),\mathtt{M}_{\mathtt{T},\mathtt{S}}(\beta,\gamma)) \geq 0 \;\; \textit{and} \;\; \eta(\alpha(\beta,\gamma)\delta(\mathtt{S},\mathtt{T}),\mathtt{M}_{\mathtt{S},\mathtt{T}}(\beta,\gamma)) \geq 0.$$

Then, by Theorem 1, the integral Equations (12) *and* (13) *have a unique solution.*

Example 2. *Given that the Volterra integral equation is as follows:*

$$\beta(\mathfrak{t}) = 1 - \beta - \frac{\beta^2}{2} + \int_0^1 (\beta - \mathfrak{s})\beta(\mathfrak{s})\delta\mathfrak{s},$$

Proof. Here, $1 - \beta - \dfrac{\beta^2}{2}$ is not an orthogonal continuous function on $[0,1]$.

Kernel $\mathtt{K}(\beta,\gamma)$ is an orthogonal continuous on $\mathbb{R} = \{(\beta,\gamma), 0 < \beta,\gamma < 1\}$. Below Figure 3 is the comparison of numerical results with analytic results.

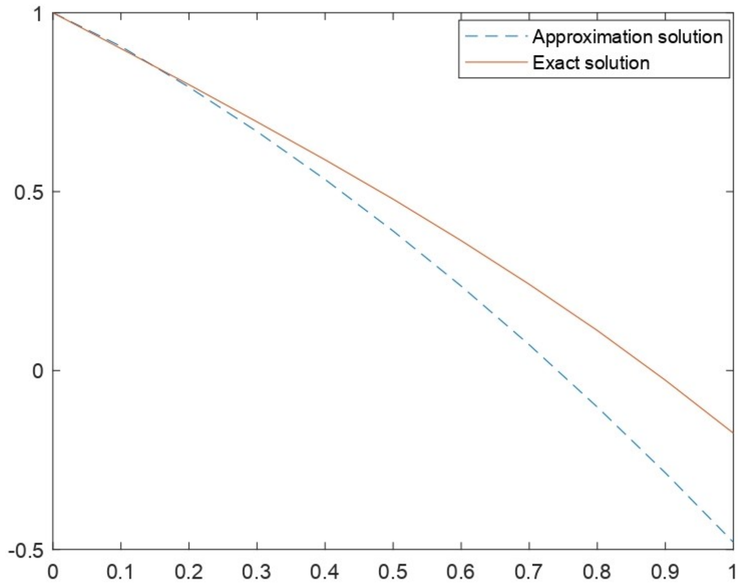

Figure 3. Figure shows the approximation solution compared to the exact solution with $h = 0.1$ for Example 2.

The error calculation of the approximation solution compared to the exact solution for Example 2 is given in Table 1 below.

Table 1. Comparison of approximation solution and exact solution.

β_j	Approximation Solution	Exact Solution	Error
0.000	1.000	1.000	0.000
0.100	0.906	0.900	0.006
0.200	0.792	0.799	−0.007
0.300	0.668	0.695	−0.028
0.400	0.534	0.589	−0.055
0.500	0.390	0.479	−0.089
0.600	0.236	0.363	−0.127
0.700	0.072	0.241	−0.169
0.800	−0.102	0.122	−0.214
0.900	−0.286	−0.027	−0.260
1.000	−0.480	−0.175	−0.305

The table shows that the error of the approximation solution compared to the exact solution is also relatively small. □

5. Conclusions

In this article, we proved the common fixed point theorem for an orthogonal generalized Λ-contraction in an orthogonal complete orthogonal Rectangular metric space. The derived results have been supplemented with suitable nontrivial examples. We have also provided an application to find the solution of the integral equation. The derived analytical results have been compared with the numerical results. It is an open problem to extend and to generalize the derived results using other contractive conditions.

Author Contributions: Investigation: G.M., R.R. and A.J.G.; Methodology: R.R., G.M. and S.K.P.; Project administration: R.R. and S.R.; Software: A.J.G. and O.A.A.A.; Supervision: R.R. and S.R.; Writing—original draft: G.M., R.R., S.K.P. and K.H.K.; Writing—review and editing: R.R., G.M., K.H.K. and O.A.A.A. All authors have read and agreed to the published version of the manuscript.

Funding: The research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: 1. The authors are thankful for the support by the Deanship of Scientific Research, Prince Sattam Bin Abulaziz University, Alkharj. 2. The authors are thankful to the anonymous reviewers for their valuable comments, which has helped in bringing the manuscript to the present form.

Conflicts of Interest: The authors declare no conflict of interest.

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