symmetry

MDPI

Article

On q-Limaçon Functions

Afis Saliu [1,2], Kanwal Jabeen [3], Isra Al-Shbeil [4,*], Najla Aloraini [5] and Sarfraz Nawaz Malik [6]

1 Department of Mathematics, University of the Gambia, Birkama Campus, MDI Road, Kanifing, Serrekunda P.O. Box 3530, The Gambia
2 Department of Mathematics, Gombe State University, Tudun Wada, Gombe 760253, Nigeria
3 Bright Future International School, Doha P.O. Box 8884, Qatar
4 Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan
5 Department of Mathematics, College of Arts and Sciences Onaizah, Qassim University, Buraidah 51452, Saudi Arabia
6 Department of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt 47040, Pakistan
* Correspondence: i.shbeil@ju.edu.jo

Abstract: Very recently, functions that map the open unit disc U onto a limaçon domain, which is symmetric with respect to the real axis in the right-half plane, were initiated in the literature. The analytic characterization, geometric properties, and Hankel determinants of these families of functions were also demonstrated. In this article, we present a q-analogue of these functions and use it to establish the classes of starlike and convex limaçon functions that are correlated with q-calculus. Furthermore, the coefficient bounds, as well as the third Hankel determinants, for these novel classes are established. Moreover, at some stages, the radius of the inclusion relationship for a particular case of these subclasses with the Janowski families of functions are obtained. It is worth noting that many of our results are sharp.

Keywords: univalent functions; Schwarz functions; limaçon domain; subordination; Hankel determinant

Citation: Saliu, A.; Jabeen, K.; Al-Shbeil, I.; Aloraini, N.; Malik, S.N. On q-Limaçon Functions. *Symmetry* **2022**, *14*, 2422. https://doi.org/10.3390/sym14112422

Academic Editor: Hüseyin Budak

Received: 3 October 2022
Accepted: 8 November 2022
Published: 15 November 2022

1. Introduction and Preliminaries

The notion of q-calculus (known as Quantum calculus) is a part of mathematics that deals with calculus without the concept of limits. This field of study has motivated researchers in recent times because of its numerous applications and importance in many areas of science, such as Geometry Function Theory (GFT), Quantum mechanics, cosmology, particle physics, and statistics. The development of this area began from the work of Jackson [1,2]. The idea was first used in GFT by Ismail et al. [3], where the concept of a q-extension of the class S^* of starlike functions was presented. As a result, various q-subclasses of univalent functions have been receiving attention in this area (see [4–15]).

The study of univalent functions that map the open unit disc onto a domain symmetric with respect to the real axis in the right-half plane is one of the fundamental aspects of GFT. On this note, examinations of its subclasses have gained momentum in recent times. To this end, Ma and Minda [16] provided a generalized classification of these subclasses; for more details, see [17–26].

Recently, Kanas and Masih [22] initiated a subfamily of univalent functions that were characterized by limaçon domains. The geometric properties of this class of functions were examined and used to present convex and starlike limaçon classes denoted by $CV_{\mathcal{L}}(s)$ and $ST_{\mathcal{L}}(s)$, respectively. Furthermore, Saliu et al. [26] continued with the investigation of these classes and proved many interesting results associated with them.

Motivated by these new works, our interest in this paper is to present a q-analogue of the analytic classification of the limaçon functions and use it to introduce the classes q-starlike limaçon (denoted by $ST_{\mathcal{L}_q}(s)$) and q-convex limaçon (depicted by $CV_{\mathcal{L}_q}(s)$). Furthermore, the coefficient bounds, third Hankel determinant, coefficient estimate, and radius results (of a particular case) for these novel classes are investigated.

To put our findings into a clear perspective, we present the following preliminaries and definitions:

Let $U = \{\xi \in \mathbb{C}\colon |\xi| < 1\}$ and $\mathcal{A}_n$ denote the class of normalized analytic functions $f(\xi)$ of the form

$$f(\xi) = \xi + \sum_{k=1+n}^{\infty} a_k \xi^k, \quad \xi \in U \tag{1}$$

and $\mathcal{A}_1 = \mathcal{A}$. Then, the subclass of $\mathcal{A}$, which is univalent in U, is depicted by S. Let $\mathcal{W}$ be the class of analytic functions

$$w(\xi) = \sum_{n=1}^{\infty} w_n \xi^n, \quad \xi \in U \tag{2}$$

such that $w(0) = 0$ and $|w(\xi)| < 1$. These functions are known as Schwarz functions. If $f(\xi)$ and $g(\xi)$ are analytic functions in U, then $f(\xi)$ is subordinate to $g(\xi)$ (written as $f(\xi) \prec g(\xi)$) if there exists $w \in \mathcal{W}$ such that $f(\xi) = g(w(\xi))$, $\xi \in U$.

Recall that $f \in \mathcal{A}$ is starlike if $f(U)$ is starlike with respect to origin. In addition, $f \in \mathcal{A}$ is convex if $f(U)$ is a convex domain. Analytically, $f \in \mathcal{A}$ is starlike or convex if and only if

$$\operatorname{Re} \frac{\xi f'(\xi)}{f(\xi)} > 0 \quad \text{or} \quad \operatorname{Re} \frac{(\xi f'(\xi))'}{f'(\xi)} > 0, \quad \xi \in U.$$

An analytic function

$$p(\xi) = 1 + \sum_{k=n}^{\infty} c_k \xi^k, \tag{3}$$

is a function with positive real part if Re $(p(\xi)) > 0$, $\xi \in U$. The class of all such functions is denoted by P_n with $P_1 = P$. We also symbolized the subclass of P_n satisfying Re $(p(\xi)) > \alpha$, $0 \le \alpha < 1$, by $P_n(\alpha)$. In particular, $P_1(\alpha) = P(\alpha)$ [27]. More generally, for $-1 \le B < A \le 1$, the class $P_n(A, B)$ consists of function $p(\xi)$ of the form (3) satisfying the subordination condition

$$p(\xi) \prec \frac{1 + A\xi}{1 + B\xi}, \quad \xi \in U.$$

We note that $P_1(A, B) = P(A, B)$ [28]. If we choose $p(\xi) = \frac{\xi f'(\xi)}{f(\xi)}$ or $p(\xi) = \frac{(\xi f'(\xi))'}{f'(\xi)}$, then $P_n(A, B)$ becomes $S_n^*(A, B)$ or $C_n(A, B)$.

Definition 1 ([29])**.** *Let $q \in (0, 1)$. Then, the q-number $[n]_q$ is given as*

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & n \in \mathbb{C}, \\ \sum_{\iota=0}^{n-1} q^{\iota} = 1 + q + q^2 + \dots q^{n-1}, & n \in \mathbb{N}, \\ n, \quad \text{as} \quad q \to 1^-. \end{cases} \tag{4}$$

and the q-derivative of a complex valued function $f(\xi)$ in U is given by

$$D_q f(\xi) = \begin{cases} \frac{f(q\xi) - f(\xi)}{(q-1)\xi}, & \xi \neq 0 \\ f'(0), & \xi = 0, \\ f'(\xi), \quad \text{as} \quad q \to 1^-. \end{cases} \tag{5}$$

From the above explanation, it is easy to see that for $f(\xi)$ given by (1),

$$D_q f(\xi) = 1 + \sum_{n=2}^{\infty} [n]_q a_n \xi^n. \tag{6}$$

Let $f, g \in \mathcal{A}$, we have the following rules for q-difference operator D_q.

(i) $D_q(f(\xi)g(\xi)) = f(qz)D_q g(\xi) + g(\xi)D_q f(\xi)$;

(ii) $D_q(\sigma f(\xi) \pm \delta g(\xi)) = \sigma D_q f(\xi) \pm \delta D_q g(\xi)$, for $\sigma, \delta \in \mathbb{C} \setminus \{0\}$;

(iii)

$$D_q\left(\frac{f(\xi)}{g(\xi)}\right) = \frac{g(\xi)D_q f(\xi) - f(\xi)D_q g(\xi)}{g(\xi)g(q\xi)}, \quad g(\xi)g(q\xi) \neq 0;$$

(iv) $D_q(\log f(\xi)) = \log q^{\frac{1}{q-1}} \frac{D_q f(\xi)}{f(\xi)}$, where the principal branch of the logarithm is chosen.

As a right inverse, Jackson [2] presented the q-integral of the analytic function $f(\xi)$ as

$$\int f(\xi) d_q \xi = (1-q)\xi \sum_{n=0}^{\infty} q^n f(q^n \xi).$$

For example, $f(\xi) = \xi^n$ has a q-antiderivative as

$$\int \xi^n d_q \xi = \frac{\xi^{n+1}}{[n+1]_q}, n \neq -1.$$

Definition 2. *Noonan and Thomas [30] defined for $k \geq 1, n \geq 1$, the kth Hankel determinant of $f \in S$ of the form (10) as follows:*

$$\mathcal{H}_k(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+k-1} & a_{n+k-2} & \cdots & a_{n+2k-2} \end{vmatrix}. \tag{7}$$

This determinant has been studied by many researchers. In particular Babalola [31] obtained the sharp bounds of $\mathcal{H}_3(1)$ for the classes S_{ST} and C_{CV}. By this definition, $\mathcal{H}_3(1)$ is given as

$$\begin{aligned} \mathcal{H}_3(1) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2), \quad a_1 = 1, \end{aligned}$$

and by triangle inequality,

$$|\mathcal{H}_3(1)| \leq |a_3|\,|a_2a_4 - a_3^2| + |a_4|\,|a_4 - a_2a_3| + |a_5|\,|a_3 - a_2^2|. \tag{8}$$

Clearly, one can see that $\mathcal{H}_2(1) = |a_3 - a_2^2|$ is a particular instance of the well-known Fekete Szegö functional $|a_3 - \mu a_2^2|$, where μ is a real number.

Definition 3 ([22])**.** *Let $p(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n$. Then, $p \in P(\mathbb{L}_s)$ if and only if*

$$p(\xi) \prec (1 + s\xi)^2, \quad 0 < s \leq \frac{1}{\sqrt{2}}, \; \xi \in U,$$

or if p satisfies the inequality

$$|p(\xi) - 1| < 1 - (1-s)^2, \quad 0 < s \le \frac{1}{\sqrt{2}}, \ \xi \in U.$$

Presented in [22] was the inclusion relation

$$\{w \in \mathbb{C} \colon |w-1| < 1-(1-s)^2\} \subset \mathbb{L}_s(U) \subset \{w \in \mathbb{C} \colon |w-1| < (1+s)^2 - 1\}, \qquad (9)$$

where

$$\mathbb{L}_s(U) = \left\{ u + iv \colon \left[(u-1)^2 + v^2 - s^4\right]^2 < \left[(u-1+s^2)^2 + v^2\right] \right\}.$$

Definition 4 ([22])**.** *Let $f \in \mathcal{A}$. Then, $f \in ST_{\mathcal{L}}(s)$ if and only if*

$$\frac{\xi f'(\xi)}{f(\xi)} \in P(\mathbb{L}_s), \quad 0 < s \le \frac{1}{\sqrt{2}}.$$

In addition, $f \in CV_{\mathcal{L}}(s)$ if and only if

$$\xi f'(\xi) \in ST_{\mathcal{L}}(s), \quad 0 < s \le \frac{1}{\sqrt{2}}.$$

Let $ST_{\mathcal{L}_n}(s) := \mathcal{A}_n \cap ST_{\mathcal{L}}(s)$ and $CV_{\mathcal{L}_n}(s) := \mathcal{A}_n \cap CV_{\mathcal{L}}(s)$.

Inspired by these definitions and the notion of q-calculus, we introduce the following novel classes of functions.

Definition 5. *Let $p(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n$. Then, $p \in P(\mathbb{L}_s(q))$ if and only if*

$$p(\xi) \prec \left(\frac{2(1+s\xi)}{2+s(1-q)\xi} \right)^2 := \mathbb{L}_{q,s}(\xi), \quad 0 < q < 1, \ 0 < s \le \frac{1}{\sqrt{2}}, \ \xi \in U.$$

Definition 6. *Let $f \in \mathcal{A}$. Then, $f \in ST_{\mathcal{L}_q}(s)$ if and only if*

$$\frac{\xi D_q f(\xi)}{f(\xi)} \in P(\mathbb{L}_s(q)), \quad 0 < q < 1, 0 < s \le \frac{1}{\sqrt{2}}.$$

In addition, $f \in CV_{\mathcal{L}_q}(s)$ if and only if

$$\xi D_q f \in ST_{\mathcal{L}_q}(s), \quad 0 < q < 1, 0 < s \le \frac{1}{\sqrt{2}}.$$

In particular, as $q \to 1^-$, we are back to Definitions 3 and 4. The integral representation of functions $f \in ST_{\mathcal{L}_q}(s)$ is given by

$$f(\xi) = \xi \exp\left(\frac{1}{q} \int_0^{\xi} \frac{p(t)-1}{t} d_q t \right), \quad p \in P(\mathbb{L}_s(q)),$$

and for $g \in CV_{\mathcal{L}_q}(s)$, we have

$$g(\xi) = \int_0^{\xi} \frac{f(t)}{t} d_q t, \quad f \in ST_{\mathcal{L}_q}(s).$$

More so, for $n \in \mathbb{N}$, we have the extremal functions for many problems in $ST_{\mathcal{L}_q}(s)$ as

$$\begin{aligned}
\Phi_n(q,s;\xi) =& \xi \exp\left(\frac{1}{q}\int_0^{\xi} \frac{\mathbb{L}_{q,s}(t^n)-1}{t}\, d_q t\right) \\
=& \xi \exp\left\{ \left(\frac{A-B}{q}\right)\left[\frac{2\xi^n}{[n]_q} + \frac{(A-3B)\xi^{2n}}{[2n]_q} - \sum_{k=0}^{\infty}\frac{(-B)^{k+1}(2B-(k+2)(A-B))\xi^{(k+3)n}}{[(k+3)n]_q}\right]\right\}, \\
& A = s,\ B = \frac{s(1-q)}{2} \\
=& \xi + \frac{s(1+q)}{q[n]_q}\xi^{n+1} + \frac{s^2(1+q)}{2q}\left(\frac{3q-1}{2[2n]_q} + \frac{1+q}{q[n]_q^2}\right)\xi^{2n+1} \\
& + \frac{s^3(1+q)}{q}\left(\frac{(1+q)^2}{3q[n]_q^3} - \frac{q(1-q)}{[3n]_q} + \frac{(3q-1)(1+q)}{2q[2n]_q[n]_q}\right)\xi^{2n+1} + \dots .
\end{aligned}$$

Similarly, the extremal function for various problems in $CV_{\mathcal{L}_q}(s)$ is given as

$$\Psi_n(q,s;\xi) = \int_0^z \frac{\Phi_n(q,s;t)}{t}\, d_q t.$$

We note that as $q \to 1^-$, $\Phi_n(q,s;\xi) = \Phi_n(s;\xi)$ and $\Psi_n(q,s;\xi) = \Psi_n(s;\xi)$.

2. Preliminary Lemmas

Lemma 1 ([32]). *If $w \in \mathcal{W}$ is of the form (2), then for a real number σ,*

$$|w_2 - \sigma w_1^2| \le \begin{cases} -\sigma, & \text{for } \sigma \le -1, \\ 1, & \text{for } -1 \le \sigma \le 1, \\ \sigma & \text{for } \sigma \ge 1. \end{cases}$$

When $\sigma < -1$ or $\sigma > 1$, equality holds if and only if $w(\xi) = \xi$ or one of its rotations. If $-1 < \sigma < 1$, then equality holds if and only if $w(\xi) = \xi^2$ or one of its rotations. Equality holds for $\sigma = -1$ if and only if $w(\xi) = \frac{\xi(x+\xi)}{1+x\xi}$ $(0 \le x \le 1)$ or one of its rotations, whereas for $\sigma = 1$, equality holds if and only if $w(\xi) = -\frac{\xi(x+\xi)}{1+x\xi}$ $(0 \le x \le 1)$ or one of its rotations.

In addition, the sharp upper bound above can be improved as follows when $-1 \le \sigma \le 1$:

$$|w_2 - \sigma w_1^2| + (1+\sigma)|w_1|^2 \le 1 \ \ (-1 < \sigma \le 0)$$

and

$$|w_2 - \sigma w_1^2| + (1-\sigma)|w_1|^2 \le 1 \ \ (0 < \sigma < 1).$$

Lemma 2 ([20]). *If $w \in \mathcal{W}$ is of the form in (2), then for some complex numbers ς and η such that $|\varsigma| \le 1$ and $|\eta| \le 1$,*

$$w_2 = \varsigma(1 - w_1^2)$$

and

$$w_3 = (1-w_1^2)(1-|\varsigma|^2)\eta - w_1(1-w_1^2)\varsigma^2.$$

Lemma 3 ([33]). *If $p \in P_n(A,B)$, then for $|\xi| = r$,*

$$\left| p(\xi) - \frac{1-ABr^{2n}}{1-B^2r^{2n}} \right| \le \frac{(A-B)r^n}{1-B^2r^{2n}}.$$

In particular, if $p \in P_n(\alpha)$, then for $|\xi| = r$,

$$\left| p(\xi) - \frac{1 + (1 - 2\alpha) r^{2n}}{1 - r^{2n}} \right| \leq \frac{2(1 - \alpha) r^n}{1 - r^{2n}}. \tag{10}$$

Lemma 4 ([34])**.** *If $p \in P_n(\alpha)$, then for $|\xi| = r$,*

$$\left| \frac{\xi p'(\xi)}{p(\xi)} \right| \leq \frac{2(1 - \alpha) n r^n}{(1 - r^n)(1 + (1 - 2\alpha) r^n)}.$$

Lemma 5 ([35])**.** *Let $h(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n$, $G(\xi) = 1 + \sum_{n=1}^{\infty} d_n \xi^n$ and $h(\xi) \prec G(\xi)$. If $G(\xi)$ is univalent in U and $G(U)$ is convex, then $|c_n| \leq |d_1|$ for all $n \geq 1$.*

The main results of this manuscript are presented in the subsequent sections with the assumption that the analytic function $f \in \mathcal{A}$ is of the form in (1) unless otherwise stated and $w \in \mathcal{W}$ has the series representation from (2) throughout.

3. Coefficient Bounds

Theorem 1. *Let $f \in ST\mathcal{L}_q(s)$. Then,*

$$|a_n| \leq \prod_{k=0}^{n-2} \frac{\left([k]_q + \frac{s(1+q)}{q}\right)}{[k+1]_q}, \quad [0]_q = 0. \tag{11}$$

Proof. For $f \in ST\mathcal{L}_q(s)$, we have

$$\frac{\xi D_q f(\xi)}{f(\xi)} = p(\xi), \quad p \in P(\mathbb{L}_s(q)),$$

where $p(\xi)$ is of the form in (3). Upon comparing the coefficients of ξ^n, we arrive at

$$q[n-1]_q |a_n| \leq |c_{n-1}| + |a_2 c_{n-2}| + |a_3 c_{n-3}| + \cdots + |a_{n-1} c_1|. \tag{12}$$

Since $p(\xi) \prec \mathbb{L}_{q,s}(\xi)$, then it is easy to see that $\mathbb{L}_{q,s}(\xi)$ is a convex of order β, where

$$\beta = \frac{(1-q)s^2 + 4s + 2}{(2 - s(1-q))(1+s)}.$$

Thus, $\mathbb{L}_{q,s}(\xi)$ is convex univalent in U. By Lemma 5, we have

$$|c_n| \leq s(1+q).$$

Therefore, (12) becomes

$$q[n-1]_q |a_n| \leq s(1+q) \sum_{k=1}^{n-1} |a_k|, \tag{13}$$

and from which, we have

$$
\begin{aligned}
|a_2| \leq& \frac{s(1+q)}{q[1]_q} = \prod_{k=0}^{2-2} \frac{\left([k]_q + \frac{s(1+q)}{q}\right)}{[k+1]_q}; \\
|a_3| \leq& \frac{s(1+q)}{q[2]_q}(|a_1| + |a_2|) \leq \frac{s(1+q)}{q[2]_q}\left(1 + \frac{s(1+q)}{q[1]_q}\right) \\
=& \prod_{k=0}^{3-2} \frac{\left([k]_q + \frac{s(1+q)}{q}\right)}{[k+1]_q}.
\end{aligned}
$$

Suppose (11) holds for $n = m$, we find that

$$
|a_m| \leq \prod_{k=0}^{m-2} \frac{\left([k]_q + \frac{s(1+q)}{q}\right)}{[k+1]_q}.
$$

On the other hand, from (13), we obtain

$$
|a_m| \leq \frac{s(1+q)}{q[m-1]_q} \sum_{k=1}^{m-1} |a_k| \,. \tag{14}
$$

By the induction hypothesis of (14), we have

$$
\prod_{k=0}^{m-2} \frac{\left([k]_q + \frac{s(1+q)}{q}\right)}{[k+1]_q} \geq \frac{s(1+q)}{q[m-1]_q} \sum_{k=1}^{m-1} |a_k|. \tag{15}
$$

Multiplying both sides of (15) by $\frac{\left([m-1]_q + \frac{s(1+q)}{q}\right)}{[m]_q}$, we obtain

$$
\frac{\left([m-1]_q + \frac{s(1+q)}{q}\right)}{[m]_q} \prod_{k=0}^{m-2} \frac{\left([k]_q + \frac{s(1+q)}{q}\right)}{[k+1]_q} \geq \frac{s(1+q)}{q[m-1]_q} \frac{\left([m-1]_q + \frac{s(1+q)}{q}\right)}{[m]_q} \sum_{k=1}^{m-1} |a_k|,
$$

which implies that

$$
\begin{aligned}
\prod_{k=0}^{m-1} \frac{\left([k]_q + \frac{s(1+q)}{q}\right)}{[k+1]_q} \geq& \frac{s(1+q)}{q[m]_q}\left(\sum_{k=1}^{m-1} |a_k| + \frac{s(1+q)}{q[m-1]_q} \sum_{k=1}^{m-1} |a_k|\right) \\
=& \frac{s(1+q)}{q[m]_q} \sum_{k=1}^{m} |a_k|, \quad a_1 = 1.
\end{aligned}
$$

This shows that the inequality of (15) is true for $n = m+1$. Hence, by the principle of mathematical induction on n, we complete the proof. □

Corollary 1. *Let $f \in \mathcal{A}$ be of the form in (1) and $f \in CV\mathcal{L}_q(s)$. Then,*

$$
|a_n| \leq \prod_{k=0}^{n-2} \frac{\left([k]_q + \frac{s(1+q)}{q}\right)}{[k+2]_q}. \tag{16}
$$

Upon letting $q \to 1^-$ in Theorem 1 and Corollary 1, our assertions scale down to those obtained by Saliu et al. in [26].

Theorem 2. *Let $f \in ST\mathcal{L}_q(s)$. Then,*

$$\left|a_2a_4 - a_3^2\right| \le \left(\frac{s}{q}\right)^2.$$

The bound is sharp for the function

$$\Phi_2(q,s;\xi) = \xi + \frac{s}{q}\xi^3 + \frac{s^2(5q^2-q+2)}{4q^2(1+q^2)}\xi^5 + \dots.$$

Proof. Since $f \in ST\mathcal{L}_q(s)$, then by the subordination property, we have

$$\frac{\xi D_q f(\xi)}{f(\xi)} = \left(\frac{2(1+sw(\xi))}{2+s(1-q)w(\xi)}\right),$$

where $w \in \mathcal{W}$. Then,

$$1 + qa_2\xi + q\left([2]_q a_3 - a_2^2\right)\xi^2 + q\left([3]_q a_4 - (2+q)a_2a_3 + a_2^3\right)\xi^3 + \dots = 1 + sw_1(1+q)\xi$$
$$+ \left(\frac{(3q-1)}{2}w_1^2 s + w_2\right)s(1+q)\xi^2 + \left(\frac{q(q-1)s^2w_1^3}{2} + \frac{(3q-1)}{2}w_1w_2s + w_3\right)s(1+q)\xi^3 + \dots.$$

Comparing the coefficients of ξ, ξ^2, and ξ^3, we obtain

$$\begin{cases} a_2 = \frac{(1+q)}{q}w_1 s \\ a_3 = \frac{s}{q}\left(\frac{(3q^2+3q+4)}{4q}sw_1^2 + w_2\right) \\ a_4 = \frac{s(1+q)}{2q^3(q^2+q+1)}\left[q^4s^2w_1^3 + \frac{q^3sw_1(sw_1^2+6w_2)}{2} + q^2\left(\frac{5s^2w_1^3}{2} + 2w_3 + sw_1w_2\right) + qsw_1(sw_1^2+4w_2)\right. \\ \qquad \left. +2s^2w_1^3\right]. \end{cases} \tag{17}$$

Then, by Lemma 2, we obtain

$$a_2a_4 - a_3^2 = \frac{s^2}{q^2(q^2+q+1)}\left[(1+q)^2w_1(1-w_1^2)(1-|\varsigma|^2)\eta - (1-w_1^2)(q^2+(1+w_1^2)q+1)\varsigma^2\right.$$
$$\left. + \frac{sw_1^2(1-w_1^2)(q^2-q+2)\varsigma}{2} - \frac{s^2w_1^4(q^4+7q^3+24q^2+23q+21)}{16}\right].$$

Let $x = w_1$ with $0 \le x \le 1$ and $\varsigma = y$ with $|y| \le 1$. Then,

$$\left|a_2a_4 - a_3^2\right| \le \frac{s^2}{q^2(q^2+q+1)}\left[(1+q)^2x(1-x^2)(1-|y|^2) + (1-x^2)(q^2+(1+x^2)q+1)|y|^2\right.$$
$$\left. + \frac{sx^2(1-x^2)(q^2-q+2)|y|}{2} + \frac{s^2x^4(q^4+7q^3+24q^2+23q+21)}{16}\right]$$
$$:= \frac{s^2}{q^2(q^2+q+1)}f_{q,s}(x,|y|),$$

where

$$\frac{\partial f_{q,s}(x,|y|)}{\partial |y|} = 2(1-x^2)\left[(1+x^2)q + (1-x)(1+q^2)\right]|y| + \frac{sx^2(q^2-q+2)}{4} > 0.$$

This means that $f_{q,s}(x,|y|)$ is an increasing function of $|y|$ on $[0,1]$. Thus,

$$\begin{aligned} f_{q,s}(x,|y|) \leq & f_{q,s}(x,1) \\ = & (1-x^2)(q^2+(1+x^2)q+1) + \frac{s(1-x^2)(q^2-q+2)x^2}{2} \\ & + \frac{s^2(q^4+7q^3+24q^2+23q+21)x^4}{16} \\ := & f_{q,s}(x), \end{aligned}$$

where

$$\frac{df_{q,s}(x)}{dx} = \left[\frac{(q^4+7q^3+24q^2+23q+21)s^2}{4} - 2(q^2-q+2)s - 4q\right]x^3 + [s(q^2-q+2) - 2(1+q^2)]x$$

and

$$\frac{d^2f_{q,s}(x)}{dx^2} = 3\left[\frac{(q^4+7q^3+24q^2+23q+21)s^2}{4} - 2(q^2-q+2)s - 4q\right]x^2 + s(q^2-q+2) - 2(1+q^2).$$

For $x=0$, $\frac{d^2f_{q,s}(x)}{dx^2} < 0$. Therefore, $f_{q,s}(x)$ has a maximum value at $x=0$. Thus,

$$f_{q,s}(x) \leq f_{q,s}(0) = q^2+q+1.$$

Hence, we have the thesis. □

As $q \to 1^-$ in Theorem 2, we have the following corollary:

Corollary 2. *Let $ST_{\mathcal{L}}(s)$. Then,*

$$\left|a_2a_4 - a_3^2\right| \leq s^2.$$

The bound is sharp for the function

$$\Phi_2(s;\xi) = \xi + s^2\xi^3 + \frac{3s^3}{4}\xi^5 + \ldots.$$

Remark 1. *It is worth noting that this bound is different from the one obtained in Lemma 4.3 of [26]. This variation is due to the computational error therein.*

Theorem 3. *Let $f \in CV\mathcal{L}_q(s)$. Then, for $s < \min\left\{\frac{\sqrt{2}}{2}, \frac{2(q^2-q+1)}{q^2+q+2}\right\}$*

$$\left|a_2a_4 - a_3^2\right| \leq \left(\frac{s}{q(q^2+q+1)}\right)^2.$$

The inequality cannot be improved due to the function

$$\Psi_2(q,s;\xi) = \xi + \frac{s}{q(q^2+q+1)}\xi^3 + \ldots.$$

Proof. From the definition of the class $CV\mathcal{L}_q(s)$ and (17), we easily have

$$\begin{cases} a_2 = \frac{w_1 s}{q} \\ a_3 = \frac{s}{q(q^2+q+1)}\left(\frac{(3q^2+3q+4)}{4q}sw_1^2 + w_2\right) \\ a_4 = \frac{s}{2q^3(q^2+q+1)(1+q^2)}\left[q^4s^2w_1^3 + \frac{q^3sw_1(sw_1^2+6w_2)}{2} + q^2\left(\frac{5s^2w_1^3}{2} + 2w_3 + sw_1w_2\right) + qsw_1(sw_1^2+4w_2) \right. \\ \qquad \left. +2s^2w_1^3\right]. \end{cases} \tag{18}$$

Then, by Lemma 2, we obtain

$$a_2a_4 - a_3^2 = \frac{s^2}{q^2(q^2+q+1)(q^2+1)}\left[w_1(1-w_1^2)(1-|\varsigma|^2)\eta + \frac{(1-w_1^2)(q^2+q-2)sw_1^2\varsigma}{2(q^2+q+1)} \right.$$
$$\left. - \frac{(1-w_1^2)(q^2+w_1^2q+1)\varsigma^2}{(q^2+q+1)} - \frac{s^2w_1^4(1+q)(q^3+5q^2+5q^2+5)}{16(q^2+q+1)}\right].$$

Therefore, reasoning along the same line as in the proof of Theorem 2, we arrive at the desired result. □

As $q \to 1^-$ in Theorem 3, we have the following corollary:

Corollary 3 ([26])**.** *Let $CV_{\mathcal{L}}(s)$. Then, for $s < \frac{1}{2}$,*

$$\left|a_2a_4 - a_3^2\right| \le \frac{s^2}{9}.$$

The bound is sharp for the function

$$\Psi_2(s;\xi) = \xi + \frac{s}{3}\xi^3 + \dots.$$

Theorem 4. *Let $f \in ST_{\mathcal{L}_q}(s)$. Then,*

$$|a_2a_3 - a_4| \le \frac{s(1+q)}{q(q^2+q+1)}.$$

The bound is sharp for the function

$$\Phi_3(q,s;\xi) = \xi + \frac{s(1+q)}{q(q^2+q+1)}\xi^4 + \dots.$$

Proof. Using (17) and Lemma 2, it follows that

$$a_2a_3 - a_4 = \frac{s(1+q)}{q(q^2+q+1)}\left[(1-w_1^2)(1-|\varsigma|^2)\eta + w_1(1-w_1^2)\varsigma^2 - \frac{sw_1(1-w_1^2)(q^2+q+2)\varsigma}{2q} \right.$$
$$\left. + \frac{s^2w_1^3(q^3+5q^2+5q+5)}{4q}\right].$$

Let $x = w_1$ with $0 \leq x \leq 1$ and $\varsigma = y$ with $|y| \leq 1$. Then, by triangle inequality, we have

$$\begin{aligned}|a_2a_3 - a_4| =& \frac{s(1+q)}{q(q^2+q+1)}\left[(1-x^2)(1-|y|^2) + x(1-x^2)|y|^2 + \frac{sx(1-x^2)(q^2+q+2)|y|}{2q}\right.\\ &\left. + \frac{s^2x^3(q^3+5q^2+5q+5)}{4q}\right]\\ :=& \frac{s(1+q)}{q(q^2+q+1)} g_{q,s}(x,|y|),\end{aligned}$$

where

$$\frac{\partial g_{q,s}(x,|y|)}{\partial |y|} = -2(x-1)^2(x+1) - \frac{x(1-x^2)(q^2-q+2)}{2q} < 0.$$

This implies that $g_{q,s}(x,|y|)$ is decreasing on [0,1]. Thus,

$$\begin{aligned}g_{q,s}(x,|y|) \leq& g_{q,s}(x,0)\\ =& (1-x^2) + \frac{s^2x^3(q^3+5q^2+5q+5)}{4q}\\ :=& g_{q,s}(x),\end{aligned}$$

where

$$\frac{dg_{q,s}(x)}{dx} = \frac{x}{4q}[3s^2(q^3+5q^2+5q+5)x - 8q]$$

and

$$\frac{d^2g_{q,s}(x)}{dx^2} = -2 + \frac{3xs^2(q^3+5q^2+5q+5)}{2q}.$$

For $x = 0$, the function $g_{q,s}(x)$ assumes its maximum value. Therefore,

$$g_{q,s}(x) \leq g_{q,s}(0) = 1.$$

Hence, we have the result. □

As $q \to 1^-$ in Theorem 4, we obtain the following results due to Saliu et al. [26].

Corollary 4. *Let $f \in ST_{\mathcal{L}}(s)$. Then,*

$$|a_2a_3 - a_4| \leq \frac{2s}{3}.$$

The bound is sharp for the function

$$\Phi_3(s;\xi) = \xi + \frac{2s}{3}\xi^4 + \dots.$$

Theorem 5. *Let $f \in CV_{\mathcal{L}_q}(s)$. Then, for $s < \min\left\{\frac{\sqrt{2}}{2}, \frac{4q}{q^2+q+1}\right\}$,*

$$|a_2a_3 - a_4| \leq \frac{s}{q(1+q^2)(q^2+q+1)}.$$

The bound is sharp for the function

$$\Psi_3(q,s;\xi) = \xi + \frac{s}{q(1+q^2)(q^2+q+1)}\xi^4 + \dots.$$

Proof. From (18) and Lemma 2, it follows that

$$a_2a_3 - a_4 = \frac{w_1 s(1-w_1^2)\varsigma^2}{q(1+q^2)(q^2+q+1)} - \frac{s^2 w_1(1-w_1^2)(q^2+q+2)\varsigma}{2q^2(1+q^2)(q^2+q+1)} - \frac{s(1-w_1^2)(1-|\varsigma|^2)\eta}{q(1+q^2)(q^2+q+1)} + \\ + \frac{s^3 w_1^3(1+q)}{4q^2(1+q^2)}.$$

Let $x = w_1$ with $0 \le x \le 1$ and $\varsigma = y$ with $|y| \le 1$. Then, by triangle inequality, we have

$$|a_2a_3 - a_4| = \frac{s}{q(q^2+q+1)(1+q^2)}\left[x(1-x^2)|y|^2 + \frac{sx(1-x^2)(q^2+q+2)|y|}{2q} \right. \\ \left. + (1-x^2)(1-|y|^2) + \frac{s^2x^3(1+q)}{4q}\right].$$

Continuing in the same fashion as in Theorem 4, we obtain the required results. □

We obtain the following corollary as $q \to 1^-$.

Corollary 5 ([26]). *Let $f \in CV_{\mathcal{L}}(s)$. Then, for $s < \frac{\sqrt{2}}{2}$*

$$|a_2a_3 - a_4| \le \frac{s}{6}.$$

The bound is sharp for the function

$$\Psi_3(s;\xi) = \xi + \frac{s}{6}\xi^4 + \dots .$$

4. Fekete Szegö Inequalities

Theorem 6. *Let $f \in ST_{\mathcal{L}_q}(s)$. Then, for a real number μ,*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{s^2(3q^2+3q+4-4\mu(1+q)^2)}{4q^2}, & \text{for } \mu \le \rho_1, \\ \frac{s}{q}, & \text{for } \rho_1 \le \mu \le \rho_2, \\ \frac{s^2(4\mu(1+q)^2-(3q^2+3q+4))}{4q^2}, & \text{for } \mu \ge \rho_2. \end{cases}$$

It is asserted also that

$$|a_3 - \mu a_2^2| + \left[\mu - \frac{s(3q^2+3q+4)-4q}{4s(1+q)^2}\right]|a_2|^2 \le \frac{s}{q}, \quad \rho_1 < \mu \le \frac{3q^2+3q+4}{4(1+q)^2}$$

and

$$|a_3 - \mu a_2^2| - \left[\mu - \frac{s(3q^2+3q+4)+4q}{4s(1+q)^2}\right]|a_2|^2 \le \frac{s}{q}, \quad \frac{3q^2+3q+4}{4(1+q)^2} < \mu \le \rho_2,$$

where

$$\rho_1 = \frac{(3q^2+3q+4)s-4q}{4s(1+q)^2} \quad \text{and} \quad \rho_2 = \frac{(3q^2+3q+4)s+4q}{4s(1+q)^2}. \tag{19}$$

These inequalities are sharp for the functions

$$\begin{cases} \overline{\lambda}\Phi_1(q,s;\lambda\xi), & \text{for } \mu \in (-\infty,\rho_1)\cup(\rho_2,\infty), \\ \overline{\lambda}\Phi_2(q,s;\lambda\xi), & \text{for } \rho_1 \le \mu \le \rho_2, \\ \overline{\lambda}\mathcal{P}_x(q,s;\lambda\xi), & \text{for } \mu = \rho_1, \\ \overline{\lambda}\mathcal{Q}_x(q,s;\lambda\xi), & \text{for } \mu = \rho_2, \end{cases}$$

where $|\lambda| = 1$ *and*

$$\frac{\xi D_q \mathcal{P}_x(q,s;\xi)}{\mathcal{P}_x(q,s;\xi)} = \mathbb{L}_{q,s}\left(\frac{\xi(x+\xi)}{1+x\xi}\right), \quad \frac{\xi D_q \mathcal{Q}_x(q,s;\xi)}{\mathcal{Q}_x(q,s;\xi)} = \mathbb{L}_{q,s}\left(-\frac{\xi(x+\xi)}{1+x\xi}\right), \quad 0 \le x \le 1 .$$

Proof. The proof is direct from (17) and Lemma 1. □

Setting $\mu = 1$ in Theorem 6, we have

Corollary 6. *Let* $f \in ST_{\mathcal{L}_q}(s)$*. Then, for a real number* μ*,*

$$|a_3 - a_2^2| \le \frac{s}{q}$$

The bound is sharp for the function

$$\Phi_2(q,s;\xi) = \xi + \frac{s}{q}\xi^3 + \dots .$$

Theorem 7. *Let* $f \in CV_{\mathcal{L}_q}(s)$*. Then, for a real number* μ*,*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{s^2(3q^2+3q+4-4\mu(q^2+q+1))}{4q^2(q^2+q+1)}, & \text{for } \mu \le \rho_3, \\ \frac{s}{q(q^2+q+1)}, & \text{for } \rho_3 \le \mu \le \rho_4, \\ \frac{s^2(4\mu(q^2+q+1)-(3q^2+3q+4))}{4q^2(q^2+q+1)}, & \text{for } \mu \ge \rho_4. \end{cases}$$

It is asserted also that

$$|a_3 - \mu a_2^2| + \left[\mu - \frac{s(3q^2+3q+4)-4q}{4s(q^2+q+1)}\right]|a_2|^2 \le \frac{s}{q(q^2+q+1)}, \quad \rho_3 < \mu \le \frac{3q^2+3q+4}{4(q^2+q+1)}$$

and

$$|a_3 - \mu a_2^2| - \left[\mu - \frac{s(3q^2+3q+4)+4q}{4s(q^2+q+1)}\right]|a_2|^2 \le \frac{s}{q(q^2+q+1)}, \quad \frac{3q^2+3q+4}{4(q^2+q+1)} < \mu \le \rho_4,$$

where

$$\rho_3 = \frac{(3q^2+3q+4)s-4q}{4s(q^2+q+1)} \quad \text{and} \quad \rho_4 = \frac{(3q^2+3q+4)s+4q}{4s(q^2+q+1)}. \tag{20}$$

These inequalities are sharp for the functions

$$\begin{cases} \overline{\lambda}\Psi_1(q,s;\lambda\xi), & \text{for } \mu \in (-\infty,\rho_3)\cup(\rho_4,\infty), \\ \overline{\lambda}\Psi_2(q,s;\lambda\xi), & \text{for } \rho_3 \le \mu \le \rho_4, \\ \overline{\lambda}\mathcal{P}_x(q,s;\lambda\xi), & \text{for } \mu = \rho_3, \\ \overline{\lambda}\mathcal{Q}_x(q,s;\lambda\xi), & \text{for } \mu = \rho_4, \end{cases}$$

where $|\lambda| = 1$ *and*

$$\frac{D_q(\xi D_q \mathcal{P}_x(q,s;\xi))}{D_q\mathcal{P}_x(q,s;\xi)} = \mathbb{L}_{q,s}\left(\frac{\xi(x+\xi)}{1+x\xi}\right), \quad \frac{D_q(\xi D_q \mathcal{Q}_x(q,s;\xi))}{D_q\mathcal{Q}_x(q,s;\xi)} = \mathbb{L}_{q,s}\left(-\frac{\xi(x+\xi)}{1+x\xi}\right), \quad 0 \le x \le 1\,.$$

Proof. The proof is straightforward by using (18) and Lemma 1. □

Setting $\mu = 1$ in Theorem 7, we have

Corollary 7. *Let* $f \in CV_{\mathcal{L}_q}(s)$*. Then, for a real number* μ*,*

$$|a_3 - a_2^2| \le \frac{s}{q(q^2+q+1)}\,.$$

The bound is sharp for the function

$$\Psi_2(q,s;\xi) = \xi + \frac{s}{q(q^2+q+1)}\xi^3 + \dots\,.$$

Theorem 8. *Let* $f \in ST_{\mathcal{L}_q}(s)$*. Then,*

$$|\mathcal{H}_3(1)| \le \frac{s^2((1+s)q+s)}{q^5(q^2+q+1)^2(q^2+1)}\Big[sq^8 + 2sq^7 + (2+4s)q^6 + (4+6s)q^5 + (5+9s)q^4 + (s^2+9s+4)q^3 + (2+2s^2+7s)q^2 + (2s^2+3s)q + s^2\Big]\,.$$

Proof. The proof follows easily from (8), Theorems 1, 2, 4, and Corollary 6. □

As $q \to 1^-$, we have

Corollary 8. *Let* $f \in ST_{\mathcal{L}}(s)$*. Then,*

$$|\mathcal{H}_3(1)| \le \frac{s^2(1+2s)(6s^2+41s+17)}{18}\,.$$

Theorem 9. *Let* $f \in CV_{\mathcal{L}_q}(s)$*. Then, for* $s < \min\left\{\frac{\sqrt{2}}{2}, \frac{4q}{q^2+q+1}, \frac{2(q^2-q+1)}{q^2+q+2}\right\}$*,*

$$|\mathcal{H}_3(1)| \le \frac{1}{q^5(q^2+q+1)^3(q^2+1)^2(q^4+q^3+q^2+q+1)}\Big[s^2(sq^9 + (2+s)q^8 + (4+5s)q^7 + (7+8s)q^6 + (13s+7+s^2)q^5 + (13s+7+2s^2)q^4 + (13s+3s^2+4)q^3 + (7s+2+3s^2)q^2 + (2s^2+4s)q + s^2)((1+s)q+s)\Big]\,.$$

Proof. The proof is straightforward from (8), Theorems 1, 3, 5, and Corollary 7. □

As $q \to 1^-$, we obtain

Corollary 9 ([26])**.** *Let* $f \in CV_{\mathcal{L}}(s)$*. Then, for* $s < \frac{1}{2}$*,*

$$|\mathcal{H}_3(1)| \le \frac{s^2(1+2s)(12s^2+65s+33)}{540}\,.$$

5. Coefficient Estimates

Theorem 10. *If* $f \in ST_{\mathcal{L}_q}(s)$*, then*

$$\sum_{n=1}^{\infty}\left[[n]_q^2\left(\frac{2+s(1-q)}{1+s}\right)^4 - 16\right]|a_n|^2 \le 0, \quad a_1 = 1.$$

Proof. Let $f \in ST_{\mathcal{L}_q}(s)$. Then,

$$\frac{\xi D_q f(\xi)}{f(\xi)} = p(\xi),$$

where $p(\xi) \prec \left(\frac{2(1+s\xi)}{2+s(1-q)\xi}\right)^2 := \mathbb{L}_{q,s}(\xi)$. Therefore, from the subordination property, we have

$$\frac{1}{|p(\xi)|} \geq \frac{1}{4}\left(\frac{2+s(1-q)}{1+s}\right)^2. \tag{21}$$

Using (21) and Parseval's identity, we arrive at

$$\begin{aligned} 2\pi\sum_{n=1}^{\infty}|a_n|^2 r^{2n} &= \int_0^{2\pi}|f(\xi)|\,d\theta\,, \quad \xi = re^{i\theta} \\ &= \int_0^{2\pi}|\xi D_q f(\xi)|^2\,|p(\xi)|^2\,d\theta \\ &\geq \frac{1}{16}\left(\frac{2+s(1-q)}{1+s}\right)^4\int_0^{2\pi}|\xi D_q f(\xi)|^2\,d\theta \\ &= \frac{1}{16}\left(\frac{2+s(1-q)}{1+s}\right)^4\cdot 2\pi\sum_{n=1}^{\infty}[n]_q^2|a_n|^2 r^{2n}. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}|a_n|^2 r^{2n} \geq \frac{1}{16}\left(\frac{2+s(1-q)}{1+s}\right)^4\sum_{n=1}^{\infty}[n]_q^2|a_n|^2 r^{2n},$$

which implies that

$$\sum_{n=1}^{\infty}\left[[n]_q^2\left(\frac{2+s(1-q)}{1+s}\right)^4 - 16\right]|a_n|^2 r^{2n} \leq 0, \quad a_1 = 1.$$

Hence, we have the desired result as $r \to 1^-$. □

Corollary 10. *If $f \in ST_{\mathcal{L}_q}(s)$, then*

$$\sum_{n=2}^{\infty}\left[[n]_q^2\left(\frac{2+s(1-q)}{1+s}\right)^4 - 16\right]|a_n|^2 \leq 16 - \left(\frac{2+s(1-q)}{1+s}\right)^4.$$

Corollary 11. *If $f \in CV_{\mathcal{L}_q}(s)$, then*

(i)

$$\sum_{n=1}^{\infty}[n]_q^2\left[[n]_q^2\left(\frac{2+s(1-q)}{1+s}\right)^4 - 16\right]|a_n|^2 \leq 0, \quad a_1 = 1,$$

(ii)

$$\sum_{n=2}^{\infty}[n]_q^2\left[[n]_q^2\left(\frac{2+s(1-q)}{1+s}\right)^4 - 16\right]|a_n|^2 \leq 16 - \left(\frac{2+s(1-q)}{1+s}\right)^4.$$

As $q \to 1^-$ in Theorem 10, we have the following corollary:

Corollary 12. *If $f \in ST_{\mathcal{L}}(s)$, then*

(i)

$$\sum_{n=1}^{\infty}\left[n^2-(1+s)^4\right]|a_n|^2 \leq 0, \quad a_1=1,$$

(ii)

$$\sum_{n=2}^{\infty}\left[n^2-(1+s)^4\right]|a_n|^2 \leq (1+s)^4-1\,.$$

As $q \to 1^-$ in Corollary 11, we have the following corollary:

Corollary 13. *If $f \in CV_{\mathcal{L}}(s)$, then*

(i)

$$\sum_{n=1}^{\infty}n^2\left[n^2-(1+s)^4\right]|a_n|^2 \leq 0, \quad a_1=1,$$

(ii)

$$\sum_{n=2}^{\infty}n^2\left[n^2-(1+s)^4\right]|a_n|^2 \leq (1+s)^4-1\,.$$

6. Radius Results

Theorem 11. *$P_n(A,B) \subset P(\mathbb{L}_s)$ for all ξ in the disc*

$$|\xi| < r_s(A,B) = \begin{cases} r_1 & \text{if} \quad B>0, \\ r_2 & \text{if} \quad B<0, \\ r_3 & \text{if} \quad B=0,\ A\neq 0, \\ r_4 & \text{if} \quad B=-1, \end{cases}$$

where

$$r_1 = \left(\frac{2s-s^2}{A-B+B(2s-s^2)}\right)^{\frac{1}{n}}, \quad r_2 = \left(\frac{2s-s^2}{A-B-B(2s-s^2)}\right)^{\frac{1}{n}},$$

and

$$r_3 = \left(\frac{2s-s^2}{A}\right)^{\frac{1}{n}}, \quad r_4 = \left(\frac{2s-s^2}{A+1+(2s-s^2)}\right)^{\frac{1}{n}}.$$

All the radii are sharp.

Proof. Let $p \in P_n(A,B)$. We need to find the largest radius for which the disc $|w-a|<R$ is contained in the disc $|w-1|<1-(1-s)^2$, where

$$a = \frac{1-ABr^{2n}}{1-B^2r^{2n}} \quad \text{and} \quad R = \frac{(A-B)r^n}{1-B^2r^{2n}}.$$

Now, for $B>0$, it is noticed that $a<1$. Therefore, by triangle inequality, we have

$$\begin{aligned} |p(\xi)-a|<R \Longleftrightarrow & |p(\xi)-1| \\ & < R+|1-a| \\ & = \frac{(A-B)r^n}{1-B^2r^{2n}} - \frac{1-ABr^{2n}}{1-B^2r^{2n}} + 1 \\ & = \frac{B(A-B)r^{2n}+(A-B)r^n}{1-B^2r^{2n}}. \end{aligned}$$

Therefore, $p \in P(\mathbb{L}_s)$ if

$$\frac{B(A-B)r^{2n}+(A-B)r^n}{1-B^2r^{2n}} < 2s-s^2.$$

Hence,

$$r_1 = \left(\frac{2s-s^2}{A-B+B(2s-s^2)}\right)^{\frac{1}{n}}.$$

For the sharpness, consider the function $p(\xi) + \frac{1+A\xi^n}{1+B\xi^n}$. Then,

$$|p(\xi)-1| = \left|\frac{(A-B)\xi^n}{1+B\xi^n}\right|.$$

Choosing $\xi^n = -r_1^n$, then

$$|p(\xi)-1| = 2s-s^2. \tag{22}$$

For $B < 0$, we have $a > 1$. So,

$$\begin{aligned} |p(\xi)-a| <& R+a-1 \\ =& \frac{(A-B)r^n - B(A-B)r^{2n}}{1-B^2r^{2n}}. \end{aligned}$$

Continuing in the same fashion as in the case $B > 0$, we find

$$r_2 = \left(\frac{2s-s^2}{A-B-B(2s-s^2)}\right)^{\frac{1}{n}}.$$

The sharpness is achieved by setting $\xi^n = r^n$ in (22).
For $B = 0$, we have $a = 1$. Thus, $p \in P(\mathbb{L}_s)$ if

$$Ar^n < 2s-s^2.$$

That is,

$$r_3 = \left(\frac{2s-s^2}{A}\right)^{\frac{1}{n}}.$$

In addition, following the same line of arguments as in the case of $B < 0$ for $B = -1$, we have

$$r_3 = \left(\frac{2s-s^2}{A+1+2s-s^2}\right)^{\frac{1}{n}},$$

which is sharp for the function

$$p(\xi) = \frac{1+A\xi^n}{1-\xi^n}.$$

□

Corollary 14. *The relation $S_n^*(A,B) \subset ST_{\mathcal{L}_n}(s)$ and $C_n(A,B) \subset CV_{\mathcal{L}_n}(s)$ hold, respectively, in the disc $|\xi| < r_s(A,B)$. This radius is sharp for the function $f_0(\xi) \in S_n^*(A,B)$ and $g_0(\xi) \in C_n(A,B)$ defined by*

$$f_0(\xi) = \begin{cases} \xi(1+B\xi^n)^{\frac{A-B}{nB}}, & \text{if } B \neq 0, \\ \xi \exp\left(\frac{A\xi^n}{n}\right), & \text{if } B = 0, \end{cases}$$

and

$$g_0(\xi) = \int_0^z \frac{f_0(t)}{t}\,dt.$$

7. $ST_{\mathcal{L}_n}(s)$-Radius for Ratio Functions

In this section, we study $ST_{\mathcal{L}_n}(s)$-radius for some classes of functions $f \in \mathcal{A}_n$ characterized by its ratio with a certain function $g(\xi)$.

Consider the functions

$$\begin{aligned} \mathcal{G}_1 &= \left\{ f \in \mathcal{A}_n : \frac{f}{\xi} \in P_n \right\}, \\ \mathcal{G}_2 &= \left\{ f \in \mathcal{A}_n : \frac{f(\xi)}{g(\xi)} \in P_n, \quad g \in S_n^*(\alpha) \right\}, \\ \mathcal{G}_3 &= \left\{ f \in \mathcal{A}_n : \frac{f(\xi)}{g(\xi)} \in P_n, \quad \frac{g}{\xi} \in P_n \right\}, \\ \mathcal{G}_4 &= \left\{ f \in \mathcal{A}_n : \left| \frac{f(\xi)}{g(\xi)} - 1 \right| < 1, \quad \frac{g}{\xi} \in P_n \right\}. \end{aligned}$$

Theorem 12. *The $ST_{\mathcal{L}_n}(s)$-radii for the functions in the class $\mathcal{G}_i$, $i = 1, 2, 3, 4$ are*

(i)

$$RST_{\mathcal{L}_n}(s)[\mathcal{G}_1] = \left(\frac{2s - s^2}{n + \sqrt{n^2 + (2s - s^2)^2}} \right)^{\frac{1}{n}},$$

(ii)

$$RST_{\mathcal{L}_n}(s)[\mathcal{G}_2] = \left(\frac{2s - s^2}{(n+1-\alpha) + \sqrt{(n+1-\alpha)^2 + (2s - s^2)\left(2(1-\alpha) + (2s - s^2)\right)}} \right)^{\frac{1}{n}},$$

(iii)

$$RST_{\mathcal{L}_n}(s)[\mathcal{G}_3] = \left(\frac{2s - s^2}{2n + \sqrt{4n^2 + (2s - s^2)^2}} \right)^{\frac{1}{n}},$$

(iv)

$$RST_{\mathcal{L}_n}(s)[\mathcal{G}_4] = \left(\frac{2(2s - s^2)}{3n + \sqrt{9n^2 + 4(2s - s^2)(2s - s^2 + n)}} \right)^{\frac{1}{n}}.$$

Proof.

(i) Let $f \in \mathcal{G}_1$ and assume $p(\xi) = \frac{f(\xi)}{\xi}$. Then,

$$\frac{\xi f'(\xi)}{f(\xi)} - 1 = \frac{\xi p'(\xi)}{p(\xi)}.$$

Then, by Lemma 4,

$$\left| \frac{\xi f'(\xi)}{f(\xi)} - 1 \right| \leq \frac{2nr^n}{1 - r^{2n}}.$$

Therefore, $f \in ST_{\mathcal{L}_n}(s)$ if

$$\frac{2nr^n}{1 - r^{2n}} < 2s - s^2,$$

which holds for $r \leq RST_{\mathcal{L}_n}(s)[\mathcal{G}_1]$. To see the sharpness, we consider

$$f(\xi) = \frac{\xi(1 + \xi^n)}{1 - \xi^n}.$$

Obviously, $\frac{f}{\xi} \in P_n$. Therefore, at $\xi = RST_{\mathcal{L}_n}(s)[\mathcal{G}_1]$, we have

$$\frac{\xi f'(\xi)}{f(\xi)} - 1 = \frac{2n\xi^n}{1-\xi^{2n}} = 2s - s^2.$$

This confirms the sharpness.

(ii) Let $f \in \mathcal{G}_2$ and assume $p(\xi) = \frac{f(\xi)}{g(\xi)}$. Then by logarithmic differentiation,

$$\frac{\xi f'(\xi)}{f(\xi)} - 1 = \frac{\xi g'(\xi)}{g(\xi)} - 1 + \frac{\xi p'(\xi)}{p(\xi)}.$$

In view of Lemmas 3 and 4, we have

$$\left|\frac{\xi f'(\xi)}{f(\xi)} - 1\right| \leq \frac{2(1-\alpha)r^n}{1-r^{2n}} + \frac{2(1-\alpha)r^{2n}}{1-r^{2n}} + \frac{2nr^n}{1-r^{2n}}.$$

Therefore, $f \in ST_{\mathcal{L}_n}(s)$ if

$$\frac{2(1-\alpha)r^{2n} + 2(n+1-\alpha)r^n}{1-r^{2n}} < 2s - s^2,$$

which holds for $r \leq RST_{\mathcal{L}_n}(s)[\mathcal{G}_2]$. To prove the sharpness, we consider

$$f(\xi) = \frac{\xi(1+\xi^n)}{(1-\xi^n)^{\frac{n+2(1-\alpha)}{n}}} \quad \text{and} \quad g(\xi) = \frac{\xi}{(1-\xi^n)^{\frac{2(1-\alpha)}{n}}}.$$

Then, $\frac{f(\xi)}{g(\xi)} = \frac{1+\xi^n}{1-\xi^n} \in P_n$ and $\operatorname{Re}\frac{\xi g'(\xi)}{g(\xi)} > \alpha$. Therefore, $f \in \mathcal{G}_2$. At $\xi = RST_{\mathcal{L}_n}(s)[\mathcal{G}_2]$, we have

$$\frac{\xi f'(\xi)}{f(\xi)} - 1 = \frac{2n\xi^n}{1-\xi^{2n}} = 2s - s^2.$$

(iii) Let $f \in \mathcal{G}_3$ and assume $p(\xi) = \frac{f(\xi)}{g(\xi)}$. Then, by logarithmic differentiation,

$$\frac{\xi f'(\xi)}{f(\xi)} - 1 = \frac{\xi g'(\xi)}{g(\xi)} + \frac{\xi p'(\xi)}{p(\xi)}.$$

Following the technique of the proof of (ii), we obtain the result. To establish the sharpness, we consider the function

$$f(\xi) = \xi\left(\frac{1+\xi^n}{1-\xi^n}\right)^2 \quad \text{and} \quad g(\xi) = \frac{\xi(1+\xi^n)}{1-\xi^n}.$$

Verily, $\frac{g}{\xi} \in P_n$ and $\frac{f(\xi)}{g(\xi)} \in P_n$. Therefore,

$$\frac{\xi f'(\xi)}{f(\xi)} - 1 = \frac{4n\xi^n}{1-\xi^{2n}},$$

and at $\xi = RST_{\mathcal{L}_n}(s)[\mathcal{G}_3]$, we have

$$\frac{\xi f'(\xi)}{f(\xi)} - 1 = 2s - s^2.$$

This proves the sharpness.

(iv) Let $h(\xi) = \frac{g(\xi)}{f(\xi)}$ with $f \in \mathcal{G}_4$, and assume $p(\xi) = \frac{g(\xi)}{\xi}$. Then,

$$\left|\frac{1}{h(\xi)} - 1\right| < 1 \Longrightarrow h(\xi) = \frac{1}{1 + w(\xi)},$$

where $w \in \mathcal{W}$. Therefore, $h(\xi) \prec \frac{1}{1+\xi}$, $\xi \in U$. This implies that Re $h((\xi)) > \frac{1}{2}$. We have $\xi p(\xi) = h(\xi) f(\xi)$ and by logarithmic differentiation,

$$\frac{\xi f'(\xi)}{f(\xi)} - 1 = \frac{zh'(\xi)}{h(\xi)} + \frac{\xi p'(\xi)}{p(\xi)}.$$

Using Lemmas 3 and 4, we arrive at

$$\left|\frac{\xi f'(\xi)}{f(\xi)} - 1\right| \le \frac{n(3 + r^n) r^n}{1 - r^{2n}}.$$

Hence, $f \in ST_{\mathcal{L}_n}(s)$ if

$$\left|\frac{\xi f'(\xi)}{f(\xi)} - 1\right| < 2s - s^2,$$

which is valid for $r \le RST_{\mathcal{L}_n}(s)[\mathcal{G}_4]$. To establish the sharpness, we consider the function

$$f(\xi) = \xi\left(\frac{1 + \xi^n}{1 - \xi^n}\right)^2 \quad \text{and} \quad g(\xi) = \frac{\xi(1 + \xi^n)}{1 - \xi^n}.$$

Since $\left|\frac{f(\xi)}{g(\xi)} - 1\right| = |\xi^n| < 1$ and Re $\frac{g(\xi)}{\xi} > 0$, then at $\xi = RST_{\mathcal{L}_n}(s)[\mathcal{G}_4]$, we have

$$\frac{\xi f'(\xi)}{f(\xi)} - 1 = \frac{3n\xi^n - n\xi^{2n}}{1 - \xi^{2n}} = -(2s - s^2).$$

This proves the sharpness.

□

8. Conclusions

In this work, we introduced a q-limaçon function and used it to present the classes of q-limaçon starlike and convex functions. The coefficient bounds and third Hankel determinant for these families were obtained. Furthermore, at a particular instance, we obtained sharp radii of inclusion between $ST_{\mathcal{L}_n}(s)$ and the classes of the ratio of the analytic functions. Overall, many consequences of our findings were demonstrated. In addition, to have more new hypotheses under the present assessments, new extensions and applications are being investigated with some positive and novel results in different fields of science, particularly in GFT. These new studies will be introduced in future research work being prepared by the authors of the current paper.

However, the purported trivial (p, q)-calculus extension was clearly demonstrated to be a relatively insignificant and inconsequential variation of classical q-calculus, with the extra parameter p being redundant or superfluous (for details, see [13] (p. 340) and [36] (pp. 1511–1512)). This observation by Srivastava (see [13,36]) will indeed apply to any future attempts to produce the rather straightforward (p, q)-variants of the results we have presented in this paper.

Author Contributions: Conceptualization, A.S.; Formal analysis, A.S.; Investigation, A.S., I.A.-S., N.A., S.N.M. and K.J.; Methodology, A.S., I.A.-S., N.A., S.N.M. and K.J.; Project administration, A.S., I.A.-S., N.A., S.N.M. and K.J.; Validation, A.S., I.A.-S., N.A., S.N.M. and K.J.; Writing—original draft, A.S.; Writing—review and editing, A.S., I.A.-S., N.A., S.N.M. and K.J. All authors have read and agreed to the published version of the manuscript.

Funding: The fourth author would like to thank the Deanship of scientific research at the University of Qassim for supporting this research work.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Jackson, F.H. On q-functions and a certain difference operator. *Trans. R. Soc. Edinb.* **1909**, *2*, 253–281. [CrossRef]
2. Jackson, F.H. On q-definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
3. Ismail, M.E.H.; Merkes, E.; Styer, D. A generalization of starlike function. *Complex Var. Elliptic Equ.* **1990**, *14*, 77–84. [CrossRef]
4. Ademogullari, K.; Kahramaner, Y. q-Harmonic mappings for which analytic part is q-convex function. *Nonlinear Anal. Differ. Equ.* **2016**, *4*, 283–293. [CrossRef]
5. Ahmad, B.; Darus, M.; Khan, N.; Khan, R.; Khan, M.G. On a Class of Analytic Multivalent Analytic Functions in q-Analogue Associated with Leminiscate of Bernoulli. *Bull. Math. Anal. Appl.* **2021**, *13*, 71–82.
6. Adweby, H.; Darus, M. On Fekete-Szego Problems for Certain Subclasses Defined by-Derivative. *J. Funct. Spacese* **2017**, *2017*, 7156738.
7. Noor, K.I.; Riaz, S.; Noor, M.A. On q-Bernardi integral operator. *TWMS J. Pure Appl. Math.* **2017**, *8*, 3–11.
8. Noor, K.I.; Riaz, S. Generalized q-starlike functions. *Stud. Sci. Math. Hung.* **2017**, *54*, 509–522. [CrossRef]
9. Saliu, A.; Noor, K.I.; Hussain, S.; Darus, M. On Quantum Differential Subordination Related with Certain Family of Analytic Functions. *J. Math.* **2020**, 2020, 6675732. [CrossRef]
10. Saliu, A.; Al-Shbeil, I.; Gong, J.; Malik, S.N.; Aloraini, N. Properties of q-Symmetric Starlike Functions of Janowski Type. *Symmetry* **2022**, *14*, 1907. [CrossRef]
11. Srivastava, H.M.; Bansal, D. Close-to-convexity of a certain family of q-Mittag-Leffer functions. *J. Nonlinear Var. Anal.* **2017**, *1*, 61–69.
12. Srivastava, H.M.; Tahir, B.; Khan, B.; Ahmad, Q.Z.; Khan, N. Some general classes of q-starlike functions associated with the Janowski functions. *Symmetry* **2019**, *11*, 292. [CrossRef]
13. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theoryof complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]
14. Ucar, H.E.Ö. Coefficient inequality for q-starlike functions. *Appl. Math. Comput.* **2016**, *276*, 122–126.
15. Ul-Haq, M.; Raza, M.; Arif, M.; Khan, Q.; Tang, H. q-Analogue of differential subordinations. *Mathematics* **2019**, *7*, 724. [CrossRef]
16. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis*; International Press Inc.: Boston, MA, USA, 1992; pp. 157–169.
17. Kanas, S.; Wisniowska, A. Conic regions and k-uniform convexity. *J. Comput. Appl. Math.* **1999**, *105* 327–336. [CrossRef]
18. Kanas, S.; Wisniowska, A. Conic domains and starlike functions. *Rev. Roum. Math. Pures Appl.* **2000**, *45*, 647–658.
19. Kanas, S.; Masih, V.S.; Ebadian, A. Relations of a planar domains bounded by hyperbolas with families of holomorphic functions. *J. Inequalities Appl.* **2019**, *2019*, 246. [CrossRef]
20. Kanas, S.; Masih, V.S.; Ebadian, A. Coefficients problems for families of holomorphic functions related to hyperbola. *Math. Slovaca* **2020**, *70*, 605–616. [CrossRef]
21. Kumar, S.; Ravichandran, V. A Subclass of Starlike Functions Associated With a Rational Function. *Southeast Asian Bull. Math.* **2016**, *40*, 199–212.
22. Masih, V.S.; Kanas, S. Subclasses of Starlike and Convex Functions Associated with the Limaçon Domain. *Symmetry* **2020**, *12*, 942. [CrossRef]
23. Noor, K.I.; Malik, S.N. On coefficient inequalities of functions associated with conic domains. *Comput. Math. Appl.* **2011**, *62*, 2209–2217. [CrossRef]
24. Saliu, A. On Generalized k-Uniformly Close-to-Convex Functions of Janowski Type. *Int. J. Appl. Anal.* **2019**, *17*, 958–973.
25. Saliu, A.; Noor, K.I. On Janowski Close-to-Convex Functions Associated with Conic Regions. *Int. J. Appl. Anal.* **2020**, *18*, 614–623.
26. Saliu, A.; Noor, K.I.; Hussain, S.; Darus, M. Some results for the family of univalent functions related with Limaçon domain. *AIMS Math.* **2021**, *6*, 3410–3431. [CrossRef]
27. Robertson, M.I. On the theory of univalent functions. *Ann. Math.* **1936**, *37*, 374–408. [CrossRef]
28. Janowski, W. Some extremal problems for certain families of analytic functions I. *Ann. Pol. Math.* **1973**, *28*, 297–326. [CrossRef]
29. Aral, A.; Gupta, V.; Agarwal, R.P. *Applications of q-Calculus in Operator Theory*; Springer: New York, NY, USA, 2013.
30. Noonan, J.W.; Thomas, D.K. On the Hankel determinants of areally mean p-valent functions. *Proc. Lond. Math. Soc.* **1972**, *3*, 503–524. [CrossRef]
31. Babalola, K.O. On $H_3(1)$ Hankel determinant for some classes of univalent functions. *Inequal. Theory Appl.* **2010**, *6*, 1–7.
32. Ali, R.M.; Ravichandran, V.; Seenivasagan, N. Coefficient bounds for p-valent functions. *Appl. Math. Comput.* **2007**, *187*, 35–46. [CrossRef]

33. Ravichandran, V.; Rønning, F.; Shanmugam, T.N. Radius of convexity and radius of starlikeness for some classes of analytic functions. *Complex Var. Theory Appl.* **1997**, *33*, 265–280. [CrossRef]
34. Shah, G.M. On the univalence of some analytic functions. *Pac. J. Math.* **1972**, *43*, 239–250. [CrossRef]
35. Rogosinski, W. On the coefficients of subordinate functions. *Proc. Lond. Math. Soc.* **1943**, *48*, 48–82. [CrossRef]
36. Srivastava, H.E. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.