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Survivability of AIDS Patients via Fractional Differential Equations with Fuzzy Rectangular and Fuzzy *b*-Rectangular Metric like Spaces

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Abstract: As it is not always true that the distance between the points in fuzzy rectangular metric spaces is one, so we introduce the notions of rectangular and b-rectangular metric-like spaces in fuzzy set theory that generalize many existing results, which can be regarded as the main advantage of this paper. In these spaces, the symmetry property is preserved, but the self distance may not be equal to one. We discuss topological properties and demonstrate that neither of these spaces is Hausdorff. Using $\alpha - \psi$-contraction and Geraghty contractions, respectively, in our main results, we establish fixed point results in these spaces. We present examples that justify our definitions and results. We use our main results to demonstrate that the solution of a nonlinear fractional differential equation for HIV is unique.

Keywords: fuzzy metric-like space; $\alpha - \psi$ contraction; Geraghty contraction; fractional differential equation

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1. Introduction

Fixed point theory has been widely used due to its applications in many fields of science. Banach fixed point theorem concerns self-mappings on a complete metric space and gives the iterative process to find the fixed point. Researchers have generalized the Banach contraction in many different ways and proved Banach fixed point theorem. For example, in 1974, Ćirić [1] generalized the Banach contraction principle by introducing Ćirić-type contraction. In 1993, Czerwik [2] generalized the Banach contraction by introducing an increasing function φ. In 2012, Wardowski [3] established F-contraction, where F is increasing and satisfies certain properties; it is also a generalization of the Banach contraction. There are many other contractions that generalize the Banach contraction but all of them need to be continuous mappings. To overcome this deficiency, Suzuki [4] introduced the Suzuki-type contraction that generalizes the Banach contraction that need not be a continuous mapping. In 2008, Berinde et al. [5] introduced the concept of almost contraction which is continuous at its fixed points. In 2017, the authors in [6] introduced generalized Suzuki-type F-contraction fuzzy mappings and to prove the existence of fixed fuzzy points for such mappings in the setup of complete ordered metric spaces. Saleem et al. [7] utilized the concepts of Suzuki and Berinde to establish Suzuki-type generalized multi-valued almost contraction mappings that generalize the Banach contraction in a natural way. In [8], the authors introduced Suzuki-type $(\alpha, \beta, \gamma_g)$-generalized and modified proximal contractive mappings and found some interesting results. The authors in [9] introduced some new generalizations of F-contraction, F-Suzuki contraction and F-expanding mappings and proved the existence and uniqueness of the fixed points for these mappings. They also investigated the existence of a unique solution of an integral boundary value problem for scalar nonlinear Caputo fractional differential equations. Fatemah et al. [10] proved fixed points results for multivalued mappings and applied their results to linear systems. On the other hand, the fuzzy set theory, which was introduced by Zadeh [11], also has significant importance as it gives more efficient results compared to the crisp set theory. It extends the ordinary set theory as it assigns the grade of membership to each element of the set. Due to their greater accuracy and efficiency, fuzzy sets have been widely used in engineering, decision making, game theory and other natural sciences. Jakhar et al. [12] adopted the fixed point method and direct method to find the solution and intuitionistic fuzzy stability of the three-dimensional cubic functional equation. Taha [13] utilized the concept of a fuzzy set and introduced the notion of (r,s)-generalized fuzzy semi-closed sets with some properties. Prasertpong et al. [14] gave the approximation approaches for rough hypersoft sets based on hesitant bipolar-valued fuzzy hypersoft relations on semigroups. Zhou et al. [15] introduced a new family of fuzzy contractions based on Proinov-type contractions and established some new results concerning the existence and uniqueness of fixed points.

Using the concept of Zadeh, Kramosil and Michálek [16] gave the notion of a fuzzy metric space and compared it to the statistical metric space and found that both concepts are the same in some sense. They discussed only left continuity and did not discuss the topological aspects of the fuzzy metric space they introduced. In 1983, Grabiec [17] introduced the convergence Cauchyness of a sequence and established the fuzzy versions of Banach and Edelstein contraction principles in fuzzy metric spaces. He also proved that the fuzzy metric space is non-decreasing with respect to the third argument. In [18], George and Veeramani discussed the topological properties of the fuzzy metric space and modified the definition of [16]. They modify the definition of Cauchy sequence discussed in [17]. They defined open ball and closed ball and proved the Hausdorffness of fuzzy metric space. They discussed the compactness of a set and proved that if it is compact then it is F-bounded. They also proved Baire's theorem in fuzzy metric space. These concepts are further utilized by many authors, see [19–21].

In 2000, Branciari [22] introduced the definition of a rectangular metric space that generalizes a metric space, while George et al. [23] introduced the concept of b-Branciari metric space that generalized the notion of Branciari metric space in a natural way. They introduced the convergence of a sequence and Cauchyness of a sequence in b-Branciari metric space. They proved the Banach and Kannan-type contraction theorems in b-Branciari metric space. They showed with an example that the b-Branciari metric space is not Hausdorff. Ding et al. [24] discussed, improved and generalized some fixed point results for mappings in b-metric, rectangular metric and b-rectangular metric spaces. Ege [25] introduced complex valued rectangular b-metric spaces and proved fixed point results. He applied fixed point results to the uniqueness of the solution of a system of n-linear equations in n-unknowns. Kadelburg et al. [26], utilized the Pata-type contraction and obtained (common) fixed point results in b-metric and b-rectangular metric spaces. Nădăban [27] gave the notions of b-metric, quasi b-metric and quasi-pseudo b-metric space using fuzzy set theory in the sense of [16]. He also defined the convergence and Cauchyness of a sequence in a fuzzy b-metric space. In [28], the author extended the concept of metric-like by giving the notion of rectangular metric-like space. He proved some convergence and fixed point results. In [29], the authors gave the fuzzy version of [23] and proved some contraction principles that also generalized some results in fuzzy metric spaces. In 2021, using controlled functions, the notions of double and triple controlled metric spaces in a fuzzy environment were introduced by [30] and [31], respectively, which generalized many metric spaces in fuzzy set theory. By discussing the topological properties, they proved that neither of these spaces is Hausdorff.

Since it is not always true that the distance between the points is zero, Hitzler et al. [32] introduced the idea of d-metric spaces. They introduced the convergence as well as Cauchyness of a sequence and proved that in d-metric space the limit of a sequence is always unique. They discussed the neighborhoods and continuity in such spaces. Alghamdi [33] introduced the concept of b-metric-like space to generalize the idea of a metric-like, partial metric and b-metric space. They used the non-expensive mappings in order to find the fixed point. Recently, Prakasam et al. [34] presented the concepts of O-generalized F-contraction of type-(1) and type-(2) and proved several fixed point theorems for a self mapping in b-metric-like space. They proved and generalized some of the well known results in the literature. The concept of metric-like spaces in fuzzy set theory was introduced by Shukla et al. [21] in the sense of [18]. They defined the convergence and Cauchyness of a sequence in fuzzy metric-like space. They used fuzzy contractive mapping to find the fixed point.

Due to the contribution of fractional calculus in many branches of mathematics and engineering, including a variety of dynamical problem analyses, scientists have paid more attention to fractional order modeling. The application of various mathematical methods to the management of these models is evident. It generalizes the integer order differentiation and integration to the variable order. After centuries of small advancements, it is now growing from an application point of view. The reason for this is that modeling using the fractional order technique gives more accuracy and hereditary properties to the system as compared with ordinary calculus models. In [35], the authors introduced an efficient meshless approach for approximating the nonlinear fractional fourth-order diffusion model described in the Riemann–Liouville sense. The spread of diseases among humans is caused by viruses, bacteria, blood, spit and many other factors. AIDS is a transmittable disease that spreads within humans by an immunodeficiency virus that weakens the human body with respect to fighting against the disease. Moreover, it leaves the body open for other diseases to attack. Nazir et al. [36] investigated the HIV model by employing the Caputo–Fabrizio fractional order derivative. They used the classical technique of fixed point to prove the existence and uniqueness of the solution. Sweilam et al. [37] used three controlled variables and investigated the fractional co-infection optimal model of HIV versus malaria in fractional order.

In fuzzy rectangular metric space, the possibility that the distance between the points might not be equal to one was not discussed earlier. This motivates us to write this paper. We define rectangular and b-rectangular metric-like spaces in a fuzzy environment and discuss some topological aspects of these spaces. These concepts are new and generalize the concepts in [21,38]. We replace the triangle inequality with a rectangular inequality, but the symmetry property remains the same. As for topological aspects, we prove neither of these newly defined spaces is Hausdorff. We find the fixed point using different techniques based on the properties of contractions and the considered metric, such as the rectangular inequality and the symmetry. The paper is organized as follows. In Section 2, some fundamental definitions are given. In Section 3, we define fuzzy rectangular and fuzzy b-rectangular metric-like spaces, we prove the Banach theorem by using $\alpha\psi$-contraction and Geraghty contraction, respectively, in these spaces. Each definition and result is supported by examples. In Section 4, we use the fixed point technique to show the uniqueness of the solution of a fractional model for HIV.

2. Preliminaries

The following section comprises some fundamental definitions and outcomes connected to our main results.

Definition 1 ([39])**.** *Let* $\Upsilon \neq \emptyset$*, then* (Υ, d_l) *is known as metric-like space* MLS*, if* $d_l : \Upsilon \times \Upsilon \longrightarrow \mathbb{R}^+ \cup \{0\}$ *satisfies:*

(L1) $d_l(\wp_1, \wp_2) = 0 \Rightarrow \wp_1 = \wp_2$;
(L2) $d_l(\wp_1, \wp_2) = d(\wp_2, \wp_1)$;
(L3) $d_l(\wp_1, \wp_3) \leq d_l(\wp_1, \wp_2) + d_l(\wp_2, \wp_3)$*, for all* $\wp_1, \wp_2, \wp_3 \in \Upsilon$.

Definition 2 ([33])**.** *Let* $\Upsilon \neq \emptyset$ *and* $b \geq 1$*, then* (Υ, d_{bl}) *is called b-metric-like space* (bMLS)*, if the function* $d_{bl} : \Upsilon \times \Upsilon \longrightarrow \mathbb{R}^+ \cup \{0\}$ *satisfies:*

(bL1) $d_{bl}(\wp_1, \wp_2) = 0 \Rightarrow \wp_1 = \wp_2$;

(bL2) $d_{bl}(\wp_1, \wp_2) = d_{bl}(\wp_2, \wp_1)$;
(bL3) $d_{bl}(\wp_1, \wp_3) \leq b[d_{bl}(\wp_1, \wp_2) + d_{bl}(\wp_2, \wp_3)]$, *for all* $\wp_1, \wp_2, \wp_3 \in \Upsilon$.

Example 1 ([33]). *Let* $\Upsilon = [0, \infty)$; *define* $d_{bl} : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ *as* $d_{bl}(\wp_1, \wp_2) = (\wp_1 + \wp_2)^2$. *Then* (Υ, d_{bl}) *is* (bMLS) *with* $b = 2$.

Definition 3 ([28]). *Let* $\Upsilon \neq \emptyset$, *then* (Υ, d_{rl}) *is called a rectangular metric-like space* (RMLS), *if the function,* $d_{rl} : \Upsilon \times \Upsilon \longrightarrow [0, \infty)$ *satisfies:*

(rL1) $d_{rl}(\wp_1, \wp_2) = 0 \Rightarrow \wp_1 = \wp_2$;
(rL2) $d_{rl}(\wp_1, \wp_2) = d_{rl}(\wp_2, \wp_1)$;
(rL3) $d_{rl}(\wp_1, \wp_4) \leq d_{rl}(\wp_1, \wp_2) + d_{rl}(\wp_2, \wp_3) + d_{rl}(\wp_3, \wp_4)$, *for all distinct* $\wp_2, \wp_3 \in \Upsilon \setminus \{\wp_1, \wp_4\}$.

Definition 4 ([40]). *Let* $I = [0, 1]$, $* : I \times I \to I$ *be a binary operation. Then* $*$ *is known as continuous triangular norm* (CTN), *if* $*$ *satisfies:*

(C1) $*(\wp_1, \wp_2) = *(\wp_2, \wp_1)$;
(C2) $*(\wp_1, *(\wp_2, \wp_3)) = *(*(\wp_1, \wp_2), \wp_3)$;
(C3) $*$ *is continuous;*
(C4) $*(\wp_1, 1) = \wp_1$ *for every* $\wp_1 \in I$;
(C5) $*(\wp_1, \wp_2) \leq *(\wp_3, \wp_4)$ *whenever* $\wp_1 \leq \wp_3$, $\wp_2 \leq \wp_4$ *for all* $\wp_1, \wp_2, \wp_3, \wp_4 \in I$.

Definition 5 ([18]). *Let* $\Upsilon \neq \emptyset$, *then the tuple* $(\Upsilon, M_s^f, *)$ *is known as fuzzy metric space with* $*$ *as a* (CTN), *if for all* $\wp_1, \wp_2, \wp_3 \in \Upsilon$, *the fuzzy set* $M_s^f : \Upsilon \times \Upsilon \times (0, \infty) \longrightarrow [0, 1]$ *satisfies:*

(F1) $M_s^f(\wp_1, \wp_2, t_1) > 0$;
(F2) $M_s^f(\wp_1, \wp_2, t_1) = 1$ *if and only if* $\wp_1 = \wp_2$;
(F3) $M_s^f(\wp_1, \wp_2, t_1) = M_s^f(\wp_2, \wp_1, t_1)$;
(F4) $M_s^f(\wp_1, \wp_3, (t_1 + t_2) \geq M_s^f(\wp_1, \wp_2, t_1) * M_s^f(\wp_2, \wp_3, t_2)$;
(F5) $M_s^f(\wp_1, \wp_2, 0) : (0, \infty) \longrightarrow [0, 1]$ *is continuous for all* $\wp_1, \wp_2, \wp_3 \in \Upsilon$ *and* $t_2, t_1 > 0$.

Definition 6 ([27]). *Let* $\Upsilon \neq \emptyset$ *and* $b \geq 1$. *Then the quadruple* $(\Upsilon, M_b^f, b, *)$ *is called a fuzzy b-metric space* (FbMS) *with* $*$ *as* (CTN), *if for all* $\wp_1, \wp_2, \wp_3 \in \Upsilon$, *the fuzzy set* $M_b^f : \Upsilon \times \Upsilon \times [0, \infty) \longrightarrow [0, 1]$ *satisfies:*

(Fb1) $M_b^f(\wp_1, \wp_2, 0) = 0$;
(Fb2) $M_b^f(\wp_1, \wp_2, t_1) = 1$ *if and only if* $\wp_1 = \wp_2$;
(Fb3) $M_b^f(\wp_1, \wp_2, t_1) = M_b^f(\wp_2, \wp_1, t_1)$;
(Fb4) $M_b^f(\wp_1, \wp_3, b(t_1 + t_2)) \geq M_b^f(\wp_1, \wp_2, t_1) * M_b^f(\wp_2, \wp_3, t_2)$;
(Fb5) $M_b^f(\wp_1, \wp_2, 0) : (0, \infty) \longrightarrow [0, 1]$ *is left continuous for all* $\wp_1, \wp_2, \wp_3 \in \Upsilon$ *and* $t_1, t_2 > 0$.

Definition 7 ([29]). *Let* $\Upsilon \neq \emptyset$ *and* $b \geq 1$. *Then the quadruple* $(\Upsilon, M_b^f, , b, *)$ *is known as fuzzy b-rectangular metric space, if for all* $\wp_1, \wp_4 - \Upsilon \cup \{\wp_2 \wp_3\}$, *the fuzzy set* $M_b^f : \Upsilon \times \Upsilon \times [0, \infty) \longrightarrow [0, 1]$ *satisfies:*

(Fbr1) $M_r^f(\wp_1, \wp_2, 0) = 0$;
(Fbr2) $M_r^f(\wp_1, \wp_2, t_1) = 1$ *if and only if* $\wp_1 = \wp_2$;
(Fbr3) $M_r^f(\wp_1, \wp_2, t_1) = M_r^f(\wp_2, \wp_1, t_1)$;
(Fbr4) $M_r^f(\wp_1, \wp_4, b(t_1 + t_2 + t_3)) \geq M_r^f(\wp_1, \wp_2, t_1) * M_r^f(\wp_2, \wp_3, t_1) * M_r^f(\wp_3, \wp_4, t_1)$;
(Fbr5) $M_r^f(\wp_1, \wp_2, 0) : (0, \infty) \longrightarrow [0, 1]$ *is left continuous for all* $\wp_1, \wp_2, \wp_3, \wp_4 \in \Upsilon$ *and* $t_1, t_2, t_3 > 0$.

Definition 8 ([41]). *Let $(\Upsilon, M, *)$ be a fuzzy metric space where $\alpha : \Upsilon \times \Upsilon \times (0,\infty) \longrightarrow (0,\infty)$ is a function. The mapping $T : \Upsilon \longrightarrow \Upsilon$ is called α-admissible if,*

$$\alpha(\wp_1, \wp_2, \mathfrak{t}_1) \geq 1 \Rightarrow \alpha(\mathcal{H}\wp_1, \mathcal{H}\wp_2, \mathfrak{t}_1) \geq 1,\ for\ all\ \mathfrak{t}_1 > 0, \wp_1, \wp_2 \in \Upsilon.$$

In 1973, Geraghty [42] generalized the Banach contraction principle by introducing Geraghty contractions that have been used extensively by many authors. We follow the concept of [43] in our main results.

Definition 9 ([43]). *Let $b > 1$ be a real number; denote F_b as the class of all $\beta : [0,\infty) \longrightarrow [0, \frac{1}{b})$ with the condition*

$$\beta(t_n) \longrightarrow \frac{1}{b}\ as\ n \longrightarrow \infty\ implies\ t_n \longrightarrow 0\ as\ n \longrightarrow \infty.$$

Example 2. *Consider the function $\beta : [0,\infty) \longrightarrow [0, \frac{1}{b})$ defined by $\beta(t) = \frac{e^{-t}}{b}$ for some $b > 1$. Then $\beta \in F_b$.*

Definition 10 ([44]). *Let $\Upsilon = [0,\infty)$, then $\psi : \Upsilon \longrightarrow \Upsilon$ is called a ψ-function, if*

1. *ψ is non-decreasing;*
2. *$\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all t, where ψ^n is the n-th iteration of ψ.*

We will denote the set Ψ such that $\psi \in \Psi$.

Example 3. *Consider the function defined by*

$$\psi(t) = \begin{cases} t - \frac{t^2}{2},\ if\ t \in [0,1], \\ \frac{1}{2},\ for\ t > 1, \end{cases}$$

clearly $\psi \in \Psi$.

3. Main Results

This section deals with the notions of our newly defined rectangular and b-recangular metric-like spaces in the context of fuzzy sets that generalize numerous results existing in the literature. In our main results, first we will use $\alpha - \psi$-contraction to prove the fixed point theorem in fuzzy rectangular metric-like space. Later, we will use Geraghty contraction in fuzzy b-rectangular metric-like space. Some examples are presented that support our results. We will also show, with examples, that neither of these spaces is Hausdorff.
Following the concept of George and Veeramani [18], we have the following definition.

Definition 11. *Let $\Upsilon \neq \emptyset$, $*$ is* (CTN). *Then $(\Upsilon, M_{rl}, *)$ is known as fuzzy rectangular metric-like space* (FRMLS) *if for all distinct $\wp_3, \wp_4 \in \Upsilon \setminus \{\wp_1, \wp_2\}$, the fuzzy set $M_{rl} : \Upsilon \times \Upsilon \times (0,\infty) \longrightarrow [0,1]$ satisfies:*

(FL1) $M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1) > 0$;
(FL2) if $M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1) = 1$ *for all* $\mathfrak{t}_1 > 0$ *then* $\wp_1 = \wp_2$;
(FL3) $M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1) = M_{rl}(\wp_2, \wp_1, \mathfrak{t}_1)$;
(FL4) $M_{rl}(\wp_1, \wp_4, \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3) \geq M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1) * M_{rl}(\wp_2, \wp_3, \mathfrak{t}_2) * M_{rl}(\wp_3, \wp_4, \mathfrak{t}_3)$, *for all* $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3 > 0$;
(FL5) $M_{rl}(\wp_1, \wp_2, \cdot) : (0,\infty) \rightarrow [0,1]$ *is continuous.*

Remark 1. *In (FL4), if $M_{rl}(\wp_3, \wp_4, \mathfrak{t}_3) = 1$, then by taking $\mathfrak{t}_2 + \mathfrak{t}_3 = \mathfrak{t}_1'$ every* (FRMLS) *reduces to fuzzy metric-like space [21].*

Example 4. *Consider* $\mathrm{Y} = [0, \infty)$ *and let* $d_{rl} : \mathrm{Y} \times \mathrm{Y} \longrightarrow [0, \infty)$ *be an* RMLS, *then*

$$M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1) = \frac{\mathfrak{t}_1}{\mathfrak{t}_1 + d_{rl}(\wp_1, \wp_2)},$$

is an (FRMLS) *with minimum* $\mathfrak{t}_1$*-norm. Conditions (FL1)–(FL3) and (FL5) are easy to prove; we only prove (FL4).*

$$M_{rl}(\wp_1, \wp_4, \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3) = \frac{\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3}{\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3 + d_{rl}(\wp_1, \wp_4)}.$$

Now assume

$$M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1) \leq M_{rl}(\wp_2, \wp_3, \mathfrak{t}_2)$$

and

$$M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1) \leq M_{rl}(\wp_3, \wp_4, \mathfrak{t}_3)$$

so

$$\frac{\mathfrak{t}_1}{\mathfrak{t}_1 + d_{rl}(\wp_1, \wp_2)} \leq \frac{\mathfrak{t}_2}{\mathfrak{t}_2 + d_{rl}(\wp_2, \wp_3)}$$

and

$$\frac{\mathfrak{t}_1}{\mathfrak{t}_1 + d_{rl}(\wp_1, \wp_2)} \leq \frac{\mathfrak{t}_3}{\mathfrak{t}_3 + d_{rl}(\wp_3, \wp_4)},$$

Thus we have

$$\mathfrak{t}_1 d_{rl}(\wp_2, \wp_3) \leq \mathfrak{t}_2 d_{rl}(\wp_1, \wp_2) \ \text{ and } \ \mathfrak{t}_1 d_{rl}(\wp_3, \wp_4) \leq \mathfrak{t}_3 d_{rl}(\wp_1, \wp_2)$$

that is

$$\mathfrak{t}_1(d_{rl}(\wp_2, \wp_3) + d_{rl}(\wp_3, \wp_4) \leq (\mathfrak{t}_2 + \mathfrak{t}_3) d_{rl}(\wp_1, \wp_2). \tag{1}$$

Note also that

$$M_{rl}(\wp_1, \wp_4, \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3) \geq M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1),$$

so

$$\frac{\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3}{\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3 + d_{rl}(\wp_1, \wp_4)} \geq \frac{\mathfrak{t}_1}{\mathfrak{t}_1 + d_{rl}(\wp_1, \wp_2)},$$

Hence

$$\frac{\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3}{\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3 + d_{rl}(\wp_1, \wp_2) + d_{rl}(\wp_2, \wp_3) + d_{rl}(\wp_3, \wp_4)} \\ \geq \frac{\mathfrak{t}_1}{\mathfrak{t}_1 + d_{rl}(\wp_1, \wp_2)},$$

After simplification, we have

$$\mathfrak{t}_1(d_{rl}(\wp_2, \wp_3) + d_{rl}(\wp_3, \wp_4) \leq (\mathfrak{t}_2 + \mathfrak{t}_3) d_{rl}(\wp_1, \wp_2). \tag{2}$$

Equations 1 and 2 are identical, so

$$M_{rl}(\wp_1, \wp_4, t_1 + t_2 + t_3) \geq M_{rl}(\wp_1, \wp_2, t_1) * M_{rl}(\wp_2, \wp_3, t_2) * M_{rl}(\wp_3, \wp_4, t_3),$$

for all $t_1, t_2, t_3 > 0$ *and hence* $(Y, M_{rl}, *)$ *is an* (FRMLS).

Example 5. *Let* $Y = \{0, 1, 2, 3\}$*; define*

$$M_{rl}(\wp_1, \wp_2, t_1) = \frac{t_1}{t_1 + d_{rl}(\wp_1, \wp_2)}, \tag{3}$$

where $d_{rl} = \max\{\wp_1, \wp_2\}$ *is the* (RMLS)*. Then* $(Y, M_{rl}, *)$ *is an* (FRMLS) *with product* t_1*-norm. We will only prove (FL4); to do this, consider the following cases:*
Case-1 *Let* $\wp_1 = 0$ *and* $\wp_4 = 3$*, then either* $\wp_2 = 1$ *and* $\wp_3 = 2$ *or* $\wp_2 = 2$ *and* $\wp_3 = 1$*. Suppose* $\wp_2 = 1$ *and* $\wp_3 = 2$*, then*

$$M_{rl}(0, 3, t_1 + t_2 + t_3) = \frac{t_1 + t_2 + t_3}{t_1 + t_2 + t_3 + 3}.$$

Now

$$M_{rl}(0, 1, t_1) = \frac{t_1}{t_1 + 1},\ M_{rl}(1, 2, t_2) = \frac{t_2}{t_2 + 2},\ M_{rl}(2, 3, t_3) = \frac{t_3}{t_3 + 3}.$$

Clearly

$$M_{rl}(0, 3, t_1 + t_2 + t_3) \geq M_{rl}(0, 1, t_1) * M_{rl}(1, 2, t_2) * M_{rl}(2, 3, t_3).$$

Case-2 *Let* $\wp_1 = 1$ *and* $\wp_4 = 3$*, then either* $\wp_2 = 0$ *and* $\wp_3 = 2$ *or* $\wp_2 = 2$ *and* $\wp_3 = 0$*. Suppose* $\wp_2 = 2$ *and* $\wp_3 = 0$*, then*

$$M_{rl}(1, 3, t_1 + t_2 + t_3) = \frac{t_1 + t_2 + t_3}{t_1 + t_2 + t_3 + 3}.$$

Now

$$M_{rl}(1, 2, t_1) = \frac{t_1}{t_1 + 2},\ M_{rl}(2, 0, t_2) = \frac{t_2}{t_2 + 2},\ M_{rl}(0, 3, t_3) = \frac{t_3}{t_3 + 3}.$$

Clearly

$$M_{rl}(1, 3, t_1 + t_2 + t_3) \geq M_{rl}(1, 2, t_1) * M_{rl}(2, 0, t_2) * M_{rl}(0, 3, t_3).$$

Case-3 *Let* $\wp_1 = 2$ *and* $\wp_4 = 3$*, then either* $\wp_2 = 0$ *and* $\wp_3 = 1$ *or* $\wp_2 = 1$ *and* $\wp_3 = 0$*. Suppose* $\wp_2 = 0$ *and* $\wp_3 = 1$*, then*

$$M_{rl}(2, 3, t_1 + t_2 + t_3) = \frac{t_1 + t_2 + t_3}{t_1 + t_2 + t_3 + 3}.$$

Now

$$M_{rl}(2, 0, t_1) == \frac{t_1}{t_1 + 2},\ M_{rl}(0, 1, t_2) = \frac{t_2}{t_2 + 1},\ M_{rl}(1, 3, t_3) = \frac{t_3}{t_3 + 3}.$$

Clearly

$$M_{rl}(2, 3, t_1 + t_2 + t_3) \geq M_{rl}(2, 0, t_1) * M_{rl}(0, 1, t_2) * M_{rl}(1, 3, t_3).$$

Along similar lines, one can prove remaining cases. Thus

$$M_{rl}(\wp_1, \wp_4, \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3) \geq M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1) * M_{rl}(\wp_2, \wp_3, \mathfrak{t}_2) * M_{rl}(\wp 3, \wp_4, \mathfrak{t}_3).$$

Hence $(\Upsilon, M_{rl}, *)$ *is an* (FRMLS).

Definition 12. *A sequence* $\{\wp_n\}$ *in* (FRMLS) $(\Upsilon, M_{rl}, *)$ *is called:*

1. *a convergent sequence, if for every* $\mathfrak{t}_1 > 0$, *there exists* $\wp$ *in* Υ *satisfying:*

$$\lim_{n\to\infty} M_{rl}(\wp_n, \wp, \mathfrak{t}_1) = M_{rl}(\wp, \wp, \mathfrak{t}_1),$$

2. *a Cauchy sequence, if for all* $\mathfrak{t}_1 > 0$ *and for* $p \geq 1$,

$$\lim_{n\to\infty} M_{rl}(\wp_{n+p}, \wp_n, \mathfrak{t}_1) \text{ exists and is finite}.$$

An (FRMLS) is *complete*, if every Cauchy sequence converges in Υ. Now we define the open ball in an (FRMLS).

Definition 13. *An open ball* $B(\wp_1, r, \mathfrak{t}_1)$, *in an* (FRMLS) $(\Upsilon, M_{rl}, *)$ *with center* $\wp_1$ *and radius* r, *is given by*

$$B(\wp_1, r, \mathfrak{t}_1) = \{\wp_2 \in \Upsilon : M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1) > 1 - r\},$$

and

$$\tau_{M_{rl}} = \{C \subset \Upsilon : B(\wp_1, r, \mathfrak{t}_1) \subset C\}$$

is the corresponding topology.

The following example shows an *(FRMLS)* is not Hausdorff.

Example 6. *Consider the Example 5 and define the open ball* $B(\wp_1, r_1, \mathfrak{t}_1)$ *with center* $\wp_1 = 0$, *radius* $r_1 = 0.3$ *and* $\mathfrak{t}_1 = 3$ *as*

$$B(0, 0.3, 3) = \{\wp \in \{0, 1, 2, 3\} : M_{rl}(0, \wp, 3) > 0.7\}.$$

Let $0 \in \Upsilon$, *then* $M_{rl}(0, 0, 3) = \frac{3}{3+d_{rl}(0,0)} = 1$, *so* $0 \in B(0, 0.3, 3)$.
Let $1 \in \Upsilon$, *then* $M_{rl}(0, 1, 3) = \frac{3}{3+d_{rl}(0,1)} = 0.75$, *so* $1 \in B(0, 0.3, 3)$.
Let $2 \in \Upsilon$, *then* $M_{rl}(0, 2, 3) = \frac{3}{3+d_{rl}(0,2)} = 0.6$, *so* $2 \notin B(0, 0.3, 3)$.
Let $3 \in \Upsilon$, *then* $M_{rl}(0, 3, 3) = \frac{3}{3+d_{rl}(0,3)} = 0.5$, *so* $3 \notin B(0, 0.3, 3)$.
Hence,

$$B(0, 0.3, 3) = \{0, 1\}$$

Now consider the open ball $B(\wp_2, r_2, \mathfrak{t}_2)$ *with center* $\wp_2 = 3$, *radius* $r_2 = 0.6$ *and* $\mathfrak{t}_2 = 7$ *as*

$$B(2, 0.6, 7) = \{\wp \in \{0, 1, 2, 3\} : M_{rl}(3, \wp, 7) > 0.4\}.$$

Let $0 \in \Upsilon$, *then* $M_{rl}(2, 0, 7) = \frac{7}{7+d_{rl}(2,0)} = 0.7777$, *so* $0 \in B(2, 0.6, 7)$.
Let $1 \in \Upsilon$, *then* $M_{rl}(2, 1, 7) = \frac{7}{7+d_{rl}(2,1)} = 0.7777$, *so* $1 \in B(2, 0.6, 7)$.
Let $2 \in \Upsilon$, *then* $M_{rl}(2, 2, 7) = \frac{7}{7+d_{rl}(2,2)} = 0.7777$, *so* $2 \in B(2, 0.6, 7)$.
Let $7 \in \Upsilon$, *then* $M_{rl}(2, 7, 7) = \frac{7}{7+d_{rl}(2,3)} = 0.7$, *so* $3 \in B(2, 0.6, 7)$.
Hence,

$$B(2, 0.6, 7) = \{0, 1, 2, 3\}$$

Clearly $B(0, 0.3, 3) \cap B(2, 0.6, 7) = \{0, 1\} \neq \emptyset$; *hence, an* (FbRMLS) *is not Hausdorff.*

Definition 14. *Let* $(\Upsilon, M_{rl}, *)$ *be an* (FRMLS) *and* $\alpha : \Upsilon \times \Upsilon \times (0,\infty) \longrightarrow (0,\infty)$ *and* $\psi : [0,\infty) \longrightarrow [0,\infty)$ *be two functions. A mapping* $\mathcal{H} : \Upsilon \longrightarrow \Upsilon$ *is called an* $\alpha - \psi$*-contractive mapping, if*

$$\alpha(\wp_1, \wp_2, \mathfrak{t}_1)\Big(\frac{1}{M_{rl}(\mathcal{H}\wp_1, \mathcal{H}\wp_2, \mathfrak{t}_1)} - 1\Big) \leq \psi\Big(\frac{1}{M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1)} - 1\Big),\ for\ all\ \mathfrak{t}_1 > 0,\ \wp_1,\ \wp_2 \in \Upsilon. \tag{4}$$

Utilizing $\alpha - \psi$-contraction, we now demonstrate the Banach contraction principle in the settings of *(FRMLS)*.

Theorem 1. *Let* $(\Upsilon, M_{rl}, *)$ *be a complete* (FRMLS) *and* $\mathcal{H} : \Upsilon \longrightarrow \Upsilon$ *be an* $\alpha - \psi$*-contractive mapping that satisfies the following:*

1. $\mathcal{H}$ *is* α*-admissible;*
2. *For all* $\mathfrak{t}_1$*, there exists* $\wp_0 \in \Upsilon$ *satisfying* $\alpha(\wp_0, \mathcal{H}\wp_0, \mathfrak{t}_1) \geq 1$*;*
3. *For a sequence* $\{\wp_n\}$ *in* Υ *with* $\alpha(\wp_n, \wp_{n+1}, \mathfrak{t}_1) \geq 1$ *for all* $\mathfrak{t}_1 \geq 0, n \geq 1$ *and* $\wp_n \longrightarrow \wp$ *as* $n \longrightarrow \infty$*, implies* $\alpha(\wp_n, \wp, \mathfrak{t}_1) \geq 1$ *for all* $\mathfrak{t}_1 \geq 0, n \geq 1$.

Then $\mathcal{H}$ *has a fixed point.*

Proof. For any arbitrary $\wp_0 \in \Upsilon$, consider the iterative sequence $\wp_n = \mathcal{H}\wp_{n-1} = \mathcal{H}^n\wp_0$ with $\wp_n \neq \wp_{n+1}$. As $\mathcal{H}$ is α-admissible, for all $\mathfrak{t}_1 > 0$, we have

$$\alpha(\wp_0, \mathcal{H}\wp_0, \mathfrak{t}_1) = \alpha(\wp_0, \wp_1, \mathfrak{t}_1) \geq 1 \Rightarrow \alpha(\mathcal{H}\wp_0, \mathcal{H}\wp_1, \mathfrak{t}_1) = \alpha(\wp_1, \wp_2, \mathfrak{t}_1) \geq 1$$

which implies

$$\alpha(\wp_1, \mathcal{H}\wp_1, \mathfrak{t}_1) = \alpha(\wp_1, \wp_2, \mathfrak{t}_1) \geq 1 \Rightarrow \alpha(\mathcal{H}\wp_1, \mathcal{H}\wp_2, \mathfrak{t}_1) = \alpha(\wp_2, \wp_3, \mathfrak{t}_1) \geq 1$$

Continuing in this way, we have

$$\alpha(\mathcal{H}\wp_{n-1}, \mathcal{H}\wp_n, \mathfrak{t}_1) = \alpha(\wp_n, \wp_{n+1}, \mathfrak{t}_1) \geq 1.$$

Now

$$\begin{aligned}
(\frac{1}{M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1)} - 1) &= (\frac{1}{M_{rl}(\mathcal{H}\wp_0, \mathcal{H}\wp_1, \mathfrak{t}_1)} - 1) \\
&\leq \alpha(\wp_0, \wp_1, \mathfrak{t}_1)(\frac{1}{M_{rl}(\mathcal{H}\wp_0, \mathcal{H}\wp_1, \mathfrak{t}_1)} - 1) \quad since \quad \alpha(\wp_0, \wp_1, \mathfrak{t}_1) \geq 1 \\
&\leq \psi(\frac{1}{M_{rl}(\wp_0, \wp_1, \mathfrak{t}_1)} - 1),
\end{aligned}$$

So, we have

$$\Big(\frac{1}{M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1)} - 1\Big) \leq \psi\Big(\frac{1}{M_{rl}(\wp_0, \wp_1, \mathfrak{t}_1)} - 1\Big). \tag{5}$$

Now

$$\begin{aligned}
(\frac{1}{M_{rl}(\wp_2, \wp_3, \mathfrak{t}_1)} - 1) &= (\frac{1}{M_{rl}(\mathcal{H}\wp_1, \mathcal{H}\wp_2, \mathfrak{t}_1)} - 1) \\
&\leq \alpha(\wp_1, \wp_2, \mathfrak{t}_1)\Big(\frac{1}{M_{rl}(\mathcal{H}\wp_1, \mathcal{H}\wp_2, \mathfrak{t}_1)} - 1\Big) \\
&\leq \psi(\frac{1}{M_{rl}(\wp_1, \wp_2, \mathfrak{t}_1)} - 1).
\end{aligned}$$

from (5), we have

$$(\frac{1}{M_{rl}(\wp_2, \wp_3, \mathfrak{t}_1)} - 1) \leq \psi(\psi(\frac{1}{M_{rl}(\wp_0, \wp_1, \mathfrak{t}_1)} - 1)) = \psi^2(\frac{1}{M_{rl}(\wp_0, \wp_1, \mathfrak{t}_1)} - 1)$$

Similarly,

$$(\frac{1}{M_{rl}(\wp_3, \wp_4, \mathfrak{t}_1)} - 1) \leq \psi^3(\frac{1}{M_{rl}(\wp_0, \wp_1, \mathfrak{t}_1)} - 1)$$

Continuing in this way, we have

$$(\frac{1}{M_{rl}(\wp_n, \wp_{n+1}, \mathfrak{t}_1)} - 1) \leq \psi^n(\frac{1}{M_{rl}(\wp_0, \wp_1, \mathfrak{t}_1)} - 1).$$

That is,

$$\lim_{n \longrightarrow \infty}(\frac{1}{M_{rl}(\wp_n, \wp_{n+1}, \mathfrak{t}_1)} - 1) \leq \lim_{n \longrightarrow \infty} \psi^n(\frac{1}{M_{rl}(\wp_0, \wp_1, \mathfrak{t}_1)} - 1) \longrightarrow 0;$$

$$\Rightarrow \lim_{n \to \infty} M_{rl}(\wp_n, \wp_{n+1}, \mathfrak{t}_1) = 1, \ for \ all \ \mathfrak{t}_1 > 0. \tag{6}$$

Similarly, we can prove

$$\lim_{n \to \infty} M_{rl}(\wp_{n-2}, \wp_n, \mathfrak{t}_1) = 1, \ for \ all \ \mathfrak{t}_1 > 0. \tag{7}$$

Now consider the sequence $\{\wp_n\}$ in Y and the cases below:
Case-1. If $p = 2q + 1$, then

$$\begin{aligned} M_{rl}(\wp_n, \wp_{n+2q+1}, \mathfrak{t}_1) &\geq M_{rl}\left(\wp_n, \wp_{n+1}, \frac{\mathfrak{t}_1}{3}\right) * M_{rl}\left(\wp_{n+1}, \wp_{n+2}, \frac{\mathfrak{t}_1}{3}\right) * M_{rl}\left(\wp_{n+2}, \wp_{n+2q+1}, \frac{\mathfrak{t}_1}{3}\right) \\ &\geq M_{rl}\left(\wp_n, \wp_{n+1}, \frac{\mathfrak{t}_1}{3}\right) * M_{rl}\left(\wp_{n+1}, \wp_{n+2}, \frac{\mathfrak{t}_1}{3}\right) * M_{rl}\left(\wp_{n+2}, \wp_{n+3}, \frac{\mathfrak{t}_1}{3^2}\right) \\ &* M_{rl}\left(\wp_{n+3}, \wp_{n+4}, \frac{\mathfrak{t}_1}{3^2}\right) * M_{rl}\left(\wp_{n+4}, \wp_{n+2q+1}, \frac{\mathfrak{t}_1}{3^2}\right) \\ &\geq M_{rl}\left(\wp_n, \wp_{n+1}, \frac{\mathfrak{t}_1}{3}\right) * M_{rl}\left(\wp_{n+1}, \wp_{n+2}, \frac{\mathfrak{t}_1}{3}\right) * M_{rl}\left(\wp_{n+2}, \wp_{n+3}, \frac{\mathfrak{t}_1}{3^2}\right) \\ &* M_{rl}\left(\wp_{n+3}, \wp_{n+4}, \frac{\mathfrak{t}_1}{3^2}\right) * M_{rl}\left(\wp_{n+4}, \wp_{n+5}, \frac{\mathfrak{t}_1}{3^3}\right) * M_{rl}\left(\wp_{n+5}, \wp_{n+6}, \frac{\mathfrak{t}_1}{3^3}\right) \\ &\vdots \\ &* M_{rl}\left(\wp_{n+2q}, \wp_{n+2q+1}, \frac{\mathfrak{t}_1}{3^q}\right). \end{aligned}$$

Taking limit $n \longrightarrow \infty$ and using (6), we have

$$\begin{aligned}
\lim_{n\longrightarrow\infty} M_{rl}(\wp_n, \wp_{n+2q+1}, \mathfrak{t}_1) \geq &\lim_{n\longrightarrow\infty} M_{rl}\left(\wp_n, \wp_{n+1}, \frac{\mathfrak{t}_1}{3}\right) * \lim_{n\longrightarrow\infty} M_{rl}\left(\wp_{n+1}, \wp_{n+2}, \frac{\mathfrak{t}_1}{3}\right)\\
&* \lim_{n\longrightarrow\infty} M_{rl}\left(\wp_{n+2}, \wp_{n+2q+1}, \frac{\mathfrak{t}_1}{3}\right)\\
&\geq \lim_{n\longrightarrow\infty} M_{rl}\left(\wp_n, \wp_{n+1}, \frac{\mathfrak{t}_1}{3}\right) * \lim_{n\longrightarrow\infty} M_{rl}\left(\wp_{n+1}, \wp_{n+2}, \frac{\mathfrak{t}_1}{3}\right)\\
&* \lim_{n\longrightarrow\infty} M_{rl}\left(\wp_{n+2}, \wp_{n+3}, \frac{\mathfrak{t}_1}{3^2}\right) * \lim_{n\longrightarrow\infty} M_{rl}\left(\wp_{n+3}, \wp_{n+4}, \frac{\mathfrak{t}_1}{3^2}\right)\\
&* \lim_{n\longrightarrow\infty} M_{rl}\left(\wp_{n+4}, \wp_{n+5}, \frac{\mathfrak{t}_1}{3^3}\right) * \lim_{n\longrightarrow\infty} M_{rl}\left(\wp_{n+5}, \wp_{n+6}, \frac{\mathfrak{t}_1}{3^3}\right)\\
&* \lim_{n\longrightarrow\infty} M_{rl}\left(\wp_{n+6}, \wp_{n+7}, \frac{\mathfrak{t}_1}{3^4}\right) * \lim_{n\longrightarrow\infty} M\left(\wp_{n+7}, \wp_{n+8}, \frac{\mathfrak{t}_1}{3^4}\right)\\
&\vdots\\
&* \lim_{n\longrightarrow\infty} M_{rl}\left(\wp_{n+2q}, \wp_{n+2q+1}, \frac{\mathfrak{t}_1}{3^q}\right) = 1
\end{aligned}$$

Case-2. If $p = 2q$, then

$$\begin{aligned}
M_{rl}(\wp_n, \wp_{n+2q}, \mathfrak{t}_1) \geq &M_{rl}\left(\wp_n, \wp_{n+1}, \frac{\mathfrak{t}_1}{3}\right) * M_{rl}\left(\wp_{n+1}, \wp_{n+2}, \frac{\mathfrak{t}_1}{3}\right) * M_{rl}\left(\wp_{n+2}, \wp_{n+3}, \frac{\mathfrak{t}_1}{3^2}\right)\\
&* M_{rl}\left(\wp_{n+3}, \wp_{n+4}, \frac{\mathfrak{t}_1}{3^2}\right) * M_{rl}\left(\wp_{n+4}, \wp_{n+5}, \frac{\mathfrak{t}_1}{3^3}\right) * M_{rl}\left(\wp_{n+5}, \wp_{n+6}, \frac{\mathfrak{t}_1}{3^3}\right)\\
&* M_{rl}\left(\wp_{n+6}, \wp_{n+7}, \frac{\mathfrak{t}_1}{3^4}\right) * M_{rl}\left(\wp_{n+7}, \wp_{n+8}, \frac{\mathfrak{t}_1}{3^4}\right)\\
&\vdots\\
&* M_{rl}\left(\wp_{n+2q-2}, \wp_{n+2q}, \frac{\mathfrak{t}_1}{3^{q-1}}\right).
\end{aligned}$$

Taking limit $n \longrightarrow \infty$ and using (6) and (7), we have

$$\lim_{n\longrightarrow\infty} M_{rl}(\wp_n, \wp_{n+2q}, \mathfrak{t}_1) \geq 1 * 1 * 1 * \cdots * 1 = 1.$$

Thus, in both cases, we have $\lim_{n\to\infty} M_{rl}(\wp_n, \wp_{n+p}, \mathfrak{t}_1) = 1$, showing $\{\wp_n\}$ is Cauchy in Y. Since Y is complete, so $\wp_n \longrightarrow \wp \in$ Y, i.e., $\lim_{n\to\infty} M_{rl}(\wp_n, \wp, \mathfrak{t}_1) = 1$. To show $\wp$ is a fixed point of $\mathcal{H}$, consider

$$\begin{aligned}
\frac{1}{M_{rl}(\wp_{n+1}, \mathcal{H}\wp, \mathfrak{t}_1)} - 1 &= \frac{1}{M_{rl}(\mathcal{H}\wp_n, \mathcal{H}\wp, \mathfrak{t}_1)} - 1\\
&\leq \alpha(\wp_n, \wp, \mathfrak{t}_1)\left(\frac{1}{M_{rl}(\mathcal{H}\wp_n, \mathcal{H}\wp, \mathfrak{t}_1)} - 1\right)\\
&\leq \psi\left(\frac{1}{M_{rl}(\wp_n, \wp, \mathfrak{t}_1)} - 1\right).
\end{aligned}$$

Taking limit $n \longrightarrow \infty$, we have

$$\begin{aligned}
\frac{1}{\lim_{n\longrightarrow\infty} M_{rl}(\wp_{n+1}, \mathcal{H}\wp, \mathfrak{t}_1)} - 1 &\leq \psi\left(\frac{1}{\lim_{n\longrightarrow\infty} M_{rl}(\wp_n, \wp, \mathfrak{t}_1)} - 1\right)\\
&\leq \psi\left(\frac{1}{1} - 1\right) \leq 0,
\end{aligned}$$

So, we have $\frac{1}{\lim_{n\longrightarrow\infty} M_{rl}(\wp_{n+1}, \mathcal{H}\wp, \mathfrak{t}_1)} - 1 = 0$; that is, $\lim_{n\longrightarrow\infty} M_{rl}(\wp_{n+1}, \mathcal{H}\wp, \mathfrak{t}_1) = 1$. Since M_{rl} is continuous and $\wp_n \longrightarrow \wp$, we have $M_{rl}(\wp, \mathcal{H}\wp, \mathfrak{t}_1) = 1$, showing $\wp$ is a fixed point of $\mathcal{H}$. $\square$

The following is an example elaborated from Theorem 1.

Example 7. *Let* $Y = [0,1]$ *and* $M_{rl} : Y \times Y \times (0,\infty) \to [0,1]$*. Define a complete* (FRMLS) *as* $M_{rl}(\wp_1, \wp_2, t_1) = \exp^{-\frac{|\wp_1+\wp_2|^2}{t_1}}$ *for all* $t_1 > 0$*. Let* $\mathcal{H} : Y \to Y$ *be given by* $\mathcal{H}(\wp) = \frac{\wp}{2}$ *and* $\alpha(\wp_1, \wp_2, t_1) = 1$ *if* $\wp_1, \wp_2 \in [0,1]$ *and* 0 *otherwise, then*

$$\alpha(\mathcal{H}\wp_1, \mathcal{H}\wp_2, t_1) = \alpha(\frac{\wp_1}{2}, \frac{\wp_2}{2}, t_1) = 1 \; for \; all \; \wp_1, \wp_2 \in [0,1].$$

Now,

$$\begin{aligned}\frac{1}{M_{rl}(\frac{\wp_1}{2}, \frac{\wp_2}{2}, t_1)} - 1 &= \frac{1}{\exp - \frac{|\frac{\wp_1}{2} + \frac{\wp_2}{2}|^2}{t_1}} - 1 \\ &= \exp \frac{|\wp_1 + \wp_2|^2}{4t_1} - 1 \\ &\leq \psi(\exp \frac{|\wp_1 + \wp_2|^2}{t_1} - 1) \\ &= \psi(\frac{1}{M_{rl}(\wp_1, \wp_2, t_1)} - 1).\end{aligned}$$

Hence, $\wp = 0$ *is a fixed point.*

Now we define b-rectangular metric-like space in fuzzy set theory.

Definition 15. *Let* $Y \neq \emptyset, *$ *be a* (CTN) *and* $b \geq 1$*. Then* $(Y, M_b^r, b, *)$ *is said to be a fuzzy b-rectangular metric-like space* (FbRMLS)*, if for all distinct* $\wp_3, \wp_4 \in Y \setminus \{\wp_1, \wp_2\}$ *the fuzzy set* $M_b^r : Y \times Y \times (0,\infty) \longrightarrow [0,1]$ *satisfies:*

(FbL1) $M_b^r(\wp_1, \wp_2, t_1) > 0$*;*
(FbL2) if $M_b^r(\wp_1, \wp_2, t_1) = 1$ *for all* $t_1 > 0$ *then* $\wp_1 = \wp_2$*;*
(FbL3) $M_b^r(\wp_1, \wp_2, t_1) = M_b^r(\wp_2, \wp_1, t_1)$*;*
(FbL4) $M_b^r(\wp_1, \wp_4, b(t_1 + t_2 + t_3)) \geq M_b^r(\wp_1, \wp_2, t_1) * M_b^r(\wp_2, \wp_3, t_2) * M_b^r(\wp_3, \wp_4, t_3)$*, for all* $t_1, t_2, t_3 > 0$*;*
(FbL5) $M_b^r(\wp_1, \wp_2, \cdot) : (0,\infty) \to [0,1]$ *is continuous.*

Remark 2. ***(i)*** *By taking* $b = 1$*, an* (FbRMLS) *reduces to an* (FRMLS)*.*
(ii) *In (FbL4), if* $M_b^r(\wp_3, \wp_4, t_3) = 1$*, then by taking* $t_2 + t_3 = t_1{}'$ *every* (FbRMLS) *reduces to* (FbMLS) *[38].*
(iii) *In (FbL4), if* $M_b^r(\wp_3, \wp_4, t_3) = 1$*, then by taking* $t_2 + t_3 = t_1{}'$ *and* $b = 1$ *every* (FbRMLS) *reduces to fuzzy metric-like space [21].*

The authors in [21,38] did not discuss the topologies of the spaces they defined. If we restrict ourselves and take $t_2 + t_3 = t_1{}'$, then our results generalize the results in [38]. In the same way, if we take $t_2 + t_3 = t_1{}'$ and $b = 1$, then the results of [21] become the special cases of (FbRMLS).

The following example elaborates on Definition (15).

Example 8. *Let* $Y = \mathbb{N} \cup \{0\}$ *and* $t_1 * t_2 = \min\{t_1, t_2\}$*; define* $M_b^r : Y \times Y \times [0,\infty) \longrightarrow [0,1]$ *as:*

$$M_b^r(\wp_1, \wp_2, t_1) = \exp - \frac{d(\wp_1, \wp_2)}{t_1},$$

where $d(\wp_1, \wp_2) = (\wp_1 + \wp_2)^2$ *is a b-rectangular metric-like space. Then* M_b^r *is not an* FRMLS; *however,* $(\mathrm{Y}, M_b^r, b, *)$ *is an* (FbRMLS) *with* $b = 2$. *Here we only prove (FbL4). Now*

$$\begin{aligned}
(\wp_1 + \wp_4)^2 &\leq (\wp_1 + \wp_2 + \wp_2 + \wp_3 + \wp_3 + \wp_4)^2 \\
&= (\wp_1 + \wp_2)^2 + (\wp_2 + \wp_3)^2 + (\wp_3 + \wp_4)^2 \\
&+ 2\Big((\wp_1 + \wp_2)(\wp_2 + \wp_3) + (\wp_2 + \wp_3)(\wp_2 + \wp_4) \\
&+ (\wp_3 + \wp_4)(\wp_1 + \wp_2)\Big) \\
&\leq 3\Big((\wp_1 + \wp_2)^2 + (\wp_2 + \wp_3)^2 + (\wp_3 + \wp_4)^2\Big).
\end{aligned}$$

Now,

$$\begin{aligned}
M_b^r(\wp_1, \wp_4, \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3) &= \exp \frac{-(\wp_1 + \wp_4)^2}{\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3} \\
&\geq \exp \frac{-3(\wp_1 + \wp_2)^2 - 3(\wp_2 + \wp_3)^2 - 3(\wp_3 + \wp_4)^2}{\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3} \\
&= \exp \frac{-3(\wp_1 + \wp_2)^2}{\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3} . \exp \frac{-3(\wp_2 + \wp_3)^2}{\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3} . \exp \frac{-3(\wp_3 + \wp_4)^2}{\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3} \\
&\geq \exp \frac{-(\wp_1 + \wp_2)^2}{\frac{\mathfrak{t}_1}{3}} . \exp \frac{-(\wp_2 + \wp_3)^2}{\frac{\mathfrak{t}_2}{3}} . \exp \frac{-(\wp_3 + \wp_4)^2}{\frac{\mathfrak{t}_3}{3}} \\
&= M_b^r(\wp_1, \wp_2, \frac{\mathfrak{t}_1}{3}) * M_b^r(\wp_2, \wp_3, \frac{\mathfrak{t}_2}{3}) * M_b^r(\wp_3, \wp_4, \frac{\mathfrak{t}_3}{3}).
\end{aligned}$$

Hence, $(\mathrm{Y}, M_b^r, b, *)$ *is an* (FbRMLS).

Definition 16. *The sequence* $\{\wp_n\}$ *in an* (FbRMLS) $(\mathrm{Y}, M_b^r, b, *)$ *is convergent, if*

$$\lim_{n \longrightarrow \infty} M_b^r(\wp_n, \wp, \mathfrak{t}_1) = M_b^r(\wp, \wp, \mathfrak{t}_1).$$

Definition 17. *The sequence* $\{\wp_n\}$ *in* (FbRMLS) $(\mathrm{Y}, M_b^r, b, *)$ *is Cauchy, if*

$$\lim_{n \longrightarrow \infty} M_b^r(\wp_n, \wp_{n+p}, \mathfrak{t}_1) \text{ exists and is finite},$$

where $\mathfrak{t}_1 > 0$ *and* $p \geq 1$.

Definition 18. *An* (FbRMLS) $(\mathrm{Y}, M_b^r, b, *)$ *is complete if every Cauchy sequence converges in* Y.

Definition 19. *Let* $(\mathrm{Y}, M_b^r, b, *)$ *be an* (FbRMLS). *Then the open ball* $B(\wp, r, \mathfrak{t}_1)$ *with center* $\wp$ *and radius r is defined as*

$$B(\wp, r, \mathfrak{t}_1) = \{\wp_2 \in \mathrm{Y} : M_b^r(\wp, \wp_2, \mathfrak{t}_1) > 1 - r\},$$

and

$$\tau_{M_b^r} = \{C \subset \mathrm{Y} : B(\wp, r, \mathfrak{t}_1) \subset C\}.$$

is the corresponding topology.

We now give an example that shows an (FbRMLS) is not a Hausdorff.

Example 9. *Consider the* (FbRMLS) *as in Example 8. Here, we choose a subset* $H = \{0,1,2,3\}$ *of* Y. *Consider the open ball* $B_1(\wp_1, r_1, \mathfrak{t}_1)$ *with center* $\wp_1 = 1$, *radius* $r_1 = 0.6$ *and* $\mathfrak{t}_1 = 5$ *as*

$$B_1(1, 0.6, 5) = \{\wp_2 \in H : M_b^r(1, \wp_2, 5) > 0.4\}.$$

Let $0 \in H$, *then* $M_b^r(1,0,5) = \exp(-\frac{(1+0)^2}{5}) = 0.8187$, *so* $0 \in B(\wp_1, r_1, \mathfrak{t}_1)$.
$1 \in H$, *then* $M_b^r(1,1,5) = \exp(-\frac{(1+1)^2}{5}) = 0.4493$, *so* $1 \in B(\wp_1, r_1, \mathfrak{t}_1)$.
$2 \in H$, *then* $M_b^r(1,2,5) = \exp(-\frac{(1+2)^2}{5}) = 0.1653$, *so* $2 \notin B(\wp_1, r_1, \mathfrak{t}_1)$.
$3 \in H$, *then* $M_b^r(1,3,5) = \exp(-\frac{(1+3)^2}{5}) = 0.0407$, *so* $3 \notin B(\wp_1, r_1, \mathfrak{t}_1)$.
Hence,

$$B_1(\wp_1, r_1, \mathfrak{t}_1) = \{0,1\}$$

Now consider the open ball $B_2(\wp_2, r_2, \mathfrak{t}_1)$ *with center* $\wp_2 = 0$, *radius* $r_2 = 0.6$ *and* $\mathfrak{t}_1 = 5$ *as*

$$B_2(0, 0.6, 5) = \{\wp_2 \in H : M_b^r(1, \wp_2, 5) > 0.4\}.$$

Let $0 \in H$, *then* $M_b^r(0,0,5) = \exp(-\frac{(1+0)^2}{5}) = 1$, *so* $0 \in B(\wp_2, r_2, \mathfrak{t}_1)$.
$1 \in H$, *then* $M_b^r(0,1,5) = \exp(-\frac{(0+1)^2}{5}) = 0.8187$, *so* $1 \in B(\wp_2, r_2, \mathfrak{t}_1)$.
$2 \in H$, *then* $M_b^r(0,2,5) = \exp(-\frac{(0+2)^2}{5}) = 0.4493$, *so* $2 \in B(\wp_2, r_2, \mathfrak{t}_1)$.
$3 \in H$, *then* $M_b^r(0,3,5) = \exp(-\frac{(0+3)^2}{5}) = 0.1652$, *so* $3 \notin B(\wp_2, r_2, \mathfrak{t}_1)$.
Hence,

$$B_2(\wp_2, r_2, \mathfrak{t}_1) = \{0,1,2\}$$

Clearly $B_1(\wp_1, r_1, \mathfrak{t}_1) \cap B_2(\wp_2, r_2, \mathfrak{t}_1) = \{0,1\} \neq \emptyset$, *showing an* (FbRMLS) *is not Hausdorff.*

We now prove Banach contraction theorem in the settings of *(FbRMLS)* by using Geraghty contraction. We will use this result in the application section of this article.

Theorem 2. *Let* $(\mathrm{Y}, M_b^r, b, *)$ *be a complete* (FbRMLS) *and* $\mathcal{H} : \mathrm{Y} \longrightarrow \mathrm{Y}$ *be a mapping which satisfies:*

$$M_b^r(\mathcal{H}\wp_1, \mathcal{H}\wp_2, \beta(M_b^r(\wp_1, \wp_2, \mathfrak{t}_1))) \geq M_b^r(\wp_1, \wp_2, \mathfrak{t}_1), \tag{8}$$

for all $\wp_1, \wp_2 \in \mathrm{Y}$ *and* $\beta \in F_b$. *Then* $\mathcal{H}$ *has a unique fixed point.*

Proof. Let $\wp_0$ be an arbitrary point and consider the iterative sequence $\mathcal{H}\wp_n = \mathcal{H}^n x_0 = \wp_n$. Using (8), we have

$$\begin{aligned} M_b^r(\wp_n, \wp_{n+1}, \mathfrak{t}_1) &= M_b^r(\mathcal{H}\wp_n, \mathcal{H}\wp_n, \mathfrak{t}_1) \\ &\geq M_b^r(\wp_n, \wp_n, \frac{\mathfrak{t}_1}{\beta(M_b^r(\wp_n, \wp_n, \mathfrak{t}_1))}) \\ &\geq M_b^r(\wp_2, \wp_n, \frac{\mathfrak{t}_1}{\beta(M_b^r(\wp_n, \wp_n, \mathfrak{t}_1))\beta(M_b^r(\wp_2, \wp_n, \mathfrak{t}_1))}) \\ &\vdots \\ &\geq M_b^r(\wp_0, \wp_1, \frac{\mathfrak{t}_1}{\beta(M_b^r(\wp_n, \wp_n, \mathfrak{t}_1))\ldots\beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}). \end{aligned}$$

Hence, we have

$$M_b^r(\wp_n, \wp_{n+1}, \mathfrak{t}_1) \geq M_b^r(\wp_0, \wp_1, \frac{\mathfrak{t}_1}{\beta(M_b^r(\wp_n, \wp_n, \mathfrak{t}_1))\ldots\beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}). \tag{9}$$

Let $\{\wp_n\}$ be a sequence in Y and consider the cases.
Case-1 If $p = 2q+1$, then using $(FbL4)$ repeatedly, we have

$$
\begin{aligned}
M_b^r(\wp_n, \wp_{n+2q+1}, \mathfrak{t}_1) \geq\ & M_b^r\Big(\wp_n, \wp_{n+1}, \frac{\mathfrak{t}_1}{3b}\Big) * M_b^r\Big(\wp_{n+1}, \wp_{n+2}, \frac{\mathfrak{t}_1}{3b}\Big) \\
& * M_b^r\Big(\wp_{n+2}, \wp_{n+3}, \frac{\mathfrak{t}_1}{(3b)^2}\Big) * M_b^r\Big(\wp_{n+3}, \wp_{n+4}, \frac{\mathfrak{t}_1}{(3b)^2}\Big) \\
& * M_b^r\Big(\wp_{n+4}, \wp_{n+5}, \frac{\mathfrak{t}_1}{(3b)^3}\Big) * M_b^r\Big(\wp_{n+5}, \wp_{n+6}, \frac{\mathfrak{t}_1}{(3b)^3}\Big) \\
& \vdots \\
& * M_b^r\Big(\wp_{n+2q}, \wp_{n+2q+1}, \frac{\mathfrak{t}_1}{(3b)^q}\Big).
\end{aligned}
$$

Using (9), we have

$$
\begin{aligned}
& M_b^r(\wp_n, \wp_{n+2q+1}, \mathfrak{t}_1) \\
& \geq M_b^r\Big(\wp_0, \wp_1, \frac{\mathfrak{t}_1}{3b\beta(M_b^r(\wp_n, \wp_n, \mathfrak{t}_1))\beta(M_b^r(\wp_2, \wp_n, \mathfrak{t}_1))\ldots\beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}\Big) \\
& * M_b^r\Big(\wp_0, \wp_1, \frac{\mathfrak{t}_1}{3b\beta(M_b^r(\wp_n, \wp_{n+1}, \mathfrak{t}_1))\beta(M_b^r(\wp_n, \wp_n, \mathfrak{t}_1))\ldots\beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}\Big) \\
& * M_b^r\Big(\wp_0, \wp_1, \frac{\mathfrak{t}_1}{(3b)^2\beta(M_b^r(\wp_{n+1}, \wp_{n+2}, \mathfrak{t}_1))\beta(M_b^r(\wp_n, \wp_{n+1}, \mathfrak{t}_1))\ldots\beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}\Big) \\
& * M_b^r\Big(\wp_0, \wp_1, \frac{\mathfrak{t}_1}{(3b)^2\beta(M_b^r(\wp_{n+2}, \wp_{n+3}, \mathfrak{t}_1))\beta(M_b^r(\wp_{n+1}, \wp_{n+2}, \mathfrak{t}_1))\ldots\beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}\Big) \\
& * M_b^r\Big(\wp_0, \wp_1, \frac{\mathfrak{t}_1}{(3b)^3\beta(M_b^r(\wp_{n+3}, \wp_{n+4}, \mathfrak{t}_1))\beta(M_b^r(\wp_{n+2}, \wp_{n+3}, \mathfrak{t}_1))\ldots\beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}\Big) \\
& \vdots \\
& * M_b^r\Big(\wp_0, \wp_1, \frac{\frac{\mathfrak{t}_1}{(3b)^q}}{\beta(M_b^r(\wp_{n+2q-1}, \wp_{n+2q}, \mathfrak{t}_1))\beta(M_b^r(\wp_{n+2q-2}, \wp_{n+2q-1}, \mathfrak{t}_1))\ldots\beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}\Big).
\end{aligned}
$$

So, we have

$$
\begin{aligned}
M_b^r(\wp_n, \wp_{n+2q+1}, \mathfrak{t}_1) \geq\ & M_b^r\Big(\wp_0, \wp_1, \frac{b^{n-1}\mathfrak{t}_1}{3}\Big) * M_b^r\Big(\wp_0, \wp_1, \frac{b^n\mathfrak{t}_1}{3}\Big) * M_b^r\Big(\wp_0, \wp_1, \frac{b^n\mathfrak{t}_1}{3^2}\Big) \\
& * M_b^r\Big(\wp_0, \wp_1, \frac{b^{n+1}\mathfrak{t}_1}{3^2}\Big) * \ldots M_b^r\Big(\wp_0, \wp_1, \frac{b^{n+q}\mathfrak{t}_1}{3^q}\Big).
\end{aligned}
$$

Applying a limit, we have

$$\lim_{n\longrightarrow\infty} M_b^r(\wp_n, \wp_{n+2q+1}, \mathfrak{t}_1) = 1.$$

Case-2 When $p = 2q$, then using $(FbL4)$ repeatedly, we have

$$\begin{aligned}M_b^r(\wp_n, \wp_{n+2q}, \mathfrak{t}_1) \geq{}& M_b^r\Big(\wp_n, \wp_{n+1}, \frac{\mathfrak{t}_1}{3b}\Big) * M_b^r\Big(\wp_{n+1}, \wp_{n+2}, \frac{\mathfrak{t}_1}{3b}\Big) \\ & * M_b^r\Big(\wp_{n+2}, \wp_{n+3}, \frac{\mathfrak{t}_1}{(3b)^2}\Big) * M_b^r\Big(\wp_{n+3}, \wp_{n+4}, \frac{\mathfrak{t}_1}{(3b)^2}\Big) \\ & * M_b^r\Big(\wp_{n+4}, \wp_{n+5}, \frac{\mathfrak{t}_1}{(3b)^3}\Big) * M_b^r\Big(\wp_{n+5}, \wp_{n+6}, \frac{\mathfrak{t}_1}{(3b)^3}\Big) \\ & \vdots \\ & * M_b^r\Big(\wp_{n+2q-2}, \wp_{n+2q}, \frac{\mathfrak{t}_1}{(3b)^{q-1}}\Big).\end{aligned}$$

Using (9), we have

$$\begin{aligned}& M_b^r(\wp_n, \wp_{n+2q}, \mathfrak{t}_1) \\ & \geq M_b^r\Big(\wp_0, \wp_1, \frac{\mathfrak{t}_1}{3b\beta(M_b^r(\wp_n, \wp_n, \mathfrak{t}_1))\beta(M_b^r(\wp_2, \wp_n, \mathfrak{t}_1)) \dots \beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}\Big) \\ & * M_b^r\Big(\wp_0, \wp_1, \frac{\mathfrak{t}_1}{3b\beta(M_b^r(\wp_n, \wp_{n+1}, \mathfrak{t}_1))\beta(M_b^r(\wp_n, \wp_n, \mathfrak{t}_1)) \dots \beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}\Big) \\ & * M_b^r\Big(\wp_0, \wp_1, \frac{\mathfrak{t}_1}{(3b)^2\beta(M_b^r(\wp_{n+1}, \wp_{n+2}, \mathfrak{t}_1))\beta(M_b^r(\wp_n, \wp_{n+1}, \mathfrak{t}_1)) \dots \beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}\Big) \\ & * M_b^r\Big(\wp_0, \wp_1, \frac{\mathfrak{t}_1}{(3b)^2\beta(M_b^r(\wp_{n+2}, \wp_{n+3}, \mathfrak{t}_1))\beta(M_b^r(\wp_{n+1}, \wp_{n+2}, \mathfrak{t}_1)) \dots \beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}\Big) \\ & \vdots \\ & * M_b^r\Big(\wp_0, \wp_1, \frac{\frac{\mathfrak{t}_1}{(3b)^{q-1}}}{\beta(M_b^r(\wp_{n+2q-1}, \wp_{n+2q-1}, \mathfrak{t}_1))\beta(M_b^r(\wp_{n+2q-2}, \wp_{n+2q-1}, \mathfrak{t}_1)) \dots \beta(M_b^r(\wp_0, \wp_1, \mathfrak{t}_1))}\Big).\end{aligned}$$

So, we have

$$\begin{aligned}M_b^r(\wp_n, \wp_{n+2q}, \mathfrak{t}_1) \geq{}& M_b^r\Big(\wp_0, \wp_1, \frac{b^{n-1}\mathfrak{t}_1}{3}\Big) * M_b^r\Big(\wp_0, \wp_1, \frac{b^n\mathfrak{t}_1}{3}\Big) * M_b^r\Big(\wp_0, \wp_1, \frac{b^n\mathfrak{t}_1}{3^2}\Big) \\ & * M_b^r\Big(\wp_0, \wp_1, \frac{b^{n+1}\mathfrak{t}_1}{3^2}\Big) * \dots M_b^r\Big(\wp_0, \wp_1, \frac{b^{n+q-1}\mathfrak{t}_1}{3^q}\Big)\end{aligned}$$

Applying a limit, we have $\lim_{n\longrightarrow\infty} M_b^r(\wp_n, \wp_{n+2q+1}, \mathfrak{t}_1) = 1$. Thus in both cases, we have $\lim_{n\to\infty} M_b^r(\wp_n, \wp_{n+p}, \mathfrak{t}_1) = 1$, showing $\{\wp_n\}$ is Cauchy in Y. Now we prove $\wp$ is the fixed point of $\mathcal{H}$; consider,

$$\begin{aligned}M_b^r(\mathcal{H}\wp, \wp, \mathfrak{t}_1) &\geq M_b^r(\mathcal{H}\wp, \mathcal{H}\wp_n, \frac{\mathfrak{t}_1}{3b}) * M_b^r(\mathcal{H}\wp_n, \mathcal{H}\wp_n, \frac{\mathfrak{t}_1}{3b}) * M_b^r(\mathcal{H}\wp_n, \wp, \frac{\mathfrak{t}_1}{3b}) \\ &\geq M_b^r(\wp, \wp_n, \frac{\mathfrak{t}_1}{3b\beta(M_b^r(\wp, \wp_n, \mathfrak{t}_1))}) * M_b^r(\wp_n, \wp_n, \frac{\mathfrak{t}_1}{3b\beta(M_b^r(\wp_n, \wp_n, \mathfrak{t}_1))}) \\ & * M_b^r(\wp_n, \wp, \frac{\mathfrak{t}_1}{3b}) \\ & \longrightarrow 1 * 1 * 1 = 1.\end{aligned}$$

That shows $\wp$ is a fixed point of $\mathcal{H}$.
Uniqueness: Let $\wp' \in \mathrm{Y}$ with $\mathcal{H}\wp' = \wp'$. Now

$$\begin{aligned}
M_b^r(\wp, \wp', \mathrm{t}_1) &= M_b^r(\mathcal{H}\wp, \mathcal{H}\wp', \mathrm{t}_1) \\
&\geq M_b^r(\wp, \wp', \frac{\mathrm{t}_1}{\beta(M_b^r(\wp, \wp', \mathrm{t}_1))}) \\
&= M_b^r(\mathcal{H}\wp, \mathcal{H}\wp', \frac{\mathrm{t}_1}{\beta(M_b^r(\wp, \wp', \mathrm{t}_1))}) \\
&\geq M_b^r(\wp, \wp', \frac{\mathrm{t}_1}{\beta(M_b^r(\wp, \wp', \mathrm{t}_1))^2}) \\
&\geq \dots \\
&\geq M_b^r(\wp, \wp', \frac{\mathrm{t}_1}{\beta(M_b^r(\wp, \wp', \mathrm{t}_1))^n}) \\
&= M_b^r(\wp, \wp', b^n t) \\
&\longrightarrow 1 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Hence $\wp = \wp'$. $\square$

Example 10. *Let* $\mathrm{Y} = [0, \infty)$ *and* $M_b^r : \mathrm{Y} \times \mathrm{Y} \times (0, \infty) \longrightarrow [0, 1]$ *be defined by*

$$M_b^r(\wp_1, \wp_2, \mathrm{t}_1) = \frac{\mathrm{t}_1}{\mathrm{t}_1 + (\wp_1 + \wp_2)^2}.$$

Then $(\mathrm{Y}, M_b^r, b, *)$ *is a complete* (FbRMLS) *with* $b = 2$ *and* $\mathrm{t}_1 * \mathrm{t}_2 = \min\{\mathrm{t}_1, \mathrm{t}_2\}$. *Now define* $\mathcal{H} : \mathrm{Y} \longrightarrow \mathrm{Y}$ *by* $\mathcal{H}\wp_1 = \dfrac{\wp_1}{2(1 - \wp_1)}$ *and* $\beta : \mathrm{Y} \longrightarrow [0, \frac{1}{2})$ *by* $\beta(\mathrm{t}_1) = \frac{1}{4}$. *Let* $\wp_1, \wp_2 \in \mathrm{Y}$, $\mathrm{t}_1 > 0$ *and consider*

$$\begin{aligned}
M_b^r(\mathcal{H}\wp_1, \mathcal{H}\wp_2, \beta(M_b^r(\wp_1, \wp_2, \mathrm{t}_1)\mathrm{t}_1)) &= \frac{\beta(M_b^r(\wp_1, \wp_2, \mathrm{t}_1)\mathrm{t}_1}{\beta(M_b^r(\wp_1, \wp_2, \mathrm{t}_1))\mathrm{t}_1 + (\mathcal{H}\wp_1 + \mathcal{H}\wp_2)^2} \\
&= \frac{\frac{1}{4}\mathrm{t}_1}{\frac{1}{4}\mathrm{t}_1 + \left(\frac{\wp_1}{2(1-\wp_1)} + \frac{\wp_2}{2(1-\wp_2)}\right)^2} \\
&= \frac{\mathrm{t}_1}{\mathrm{t}_1 + \left(\frac{\wp_1}{1-\wp_1} + \frac{\wp_2}{1-\wp_2}\right)^2} \\
&= \frac{\mathrm{t}_1}{\mathrm{t}_1 + \left(\frac{\wp_1 - \wp_1\wp_2 + \wp_2 - \wp_1\wp_2}{(1-\wp_1)(1-\wp_2)}\right)^2} \\
&= \frac{\mathrm{t}_1}{\mathrm{t}_1 + \frac{(\wp_1 + \wp_2 - 2\wp_1\wp_2)^2}{(1-\wp_1)^2(1-\wp_2)^2}}.
\end{aligned}$$

Here,

$$\begin{aligned}
\frac{(\wp_1+\wp_2-2\wp_1\wp_2)^2}{(1-\wp_1)^2(1-\wp_2)^2} &\leq (\wp_1+\wp_2)^2 \\
\mathfrak{t}_1 + \frac{(\wp_1+\wp_2-2\wp_1\wp_2)^2}{(1-\wp_1)^2(1-\wp_2)^2} &\leq \mathfrak{t}_1 + (\wp_1+\wp_2)^2 \\
\frac{1}{\mathfrak{t}_1 + \frac{(\wp_1+\wp_2-2\wp_1\wp_2)^2}{(1-\wp_1)^2(1-\wp_2)^2}} &\geq \frac{1}{\mathfrak{t}_1 + (\wp_1+\wp_2)^2} \\
\frac{\mathfrak{t}_1}{\mathfrak{t}_1 + \frac{(\wp_1+\wp_2-2\wp_1\wp_2)^2}{(1-\wp_1)^2(1-\wp_2)^2}} &\geq \frac{\mathfrak{t}_1}{\mathfrak{t}_1 + (\wp_1+\wp_2)^2}.
\end{aligned}$$

Hence,

$$M_b^r(\mathcal{H}\wp_1, \mathcal{H}\wp_2, \beta(M_b^r(\wp_1, \wp_2, \mathfrak{t}_1)\mathfrak{t}_1)) \geq M_b^r(\wp_1, \wp_2, \mathfrak{t}_1).$$

Hence, $\mathcal{H}$ has a unique fixed point $\wp_1 = 0$.

From Theorem 2, we have the following remark.

Remark 3. *Taking $\beta(\mathfrak{t}_1) = k \in (0,1)$, then Theorem 1 reduces to the Banach contraction theorem for* (FbRMLS) *as follows.*

Theorem 3. *Let $(\Upsilon, M_b^r, b, *)$ be a complete* (FbRMLS) *and $k \in [0, \frac{1}{b})$ $(b \geq 1)$ with*

$$\lim_{n\longrightarrow\infty} M_b^r(\wp_1, \wp_2, \mathfrak{t}_1) = 1, \ for\ all\ \wp_1, \wp_2 \in \Upsilon.$$

Further let $\mathcal{H}$ be a self mapping on Υ that satisfies:

$$M_b^r(\wp_1, \wp_2, k\mathfrak{t}_1) \geq M_b^r(\wp_1, \wp_2, \mathfrak{t}_1).$$

Then $\mathcal{H}$ has unique fixed point in Υ.

Example 11. *Let $v = [0,1]$, with product t-norm; define a complete* (FbRMLS) *$(\Upsilon, M_b^r, b, *)$ as*

$$M_b^r(\wp_1, \wp_2, \mathfrak{t}_1) = \frac{\mathfrak{t}_1}{\mathfrak{t}_1 + \max(\wp_1+\wp_2)^2}, \ for\ all\ \wp_1, \wp_2 \in \Upsilon,\ \mathfrak{t}_1 > 0.$$

Now define a self-mapping $\mathcal{H}$ on Υ as $\mathcal{H}\wp = \frac{1-2^{-\wp}}{3}$. Let $\wp_1, \wp_2 \in \Upsilon$, then

$$\begin{aligned}
M_b^r(\wp_1, \wp_2, k\mathfrak{t}_1) &= M_b^r\Big(\frac{1-2^{-\wp_1}}{3}, \frac{1-2^{-\wp_2}}{3}, k\mathfrak{t}_1\Big) \\
&= \frac{k\mathfrak{t}_1}{k\mathfrak{t}_1 + \left(\frac{1-2^{-\wp_1}}{3} + \frac{1-2^{-\wp_1}}{3}\right)^2} \\
&= \frac{9k\mathfrak{t}_1}{9k\mathfrak{t}_1 + \left(2 - (2^{-\wp_1} + 2^{-\wp_2})\right)^2} \\
&\geq \frac{9k\mathfrak{t}_1}{9k\mathfrak{t}_1 + (2^{-\wp_1} + 2^{-\wp_2})^2} \\
&\geq \frac{\mathfrak{t}_1}{\mathfrak{t}_1 + \max(\wp_1+\wp_2)^2} \\
&= M_b^r(\wp_1, \wp_2, \mathfrak{t}_1)
\end{aligned}$$

for all $\wp_1, \wp_2 \in \Upsilon$ *and* $k \in (\frac{11}{100}, 1)$. *By the application of Theorem 2,* $\mathcal{H}$ *has a fixed point* 0.

Remark 4. *Taking* $b = 1$, *then Theorem 2 reduces to Banach contraction theorem by using Geraghty contraction in* RMLS.

4. Application to Fractional Differential Equations

Fixed point theory plays a vital role in proving the uniqueness of the solution of certain problems in almost every branch of mathematics. On the other hand, fractional calculus has applications in diverse and widespread fields of engineering, medicine and other scientific disciplines such as signal processing, visco-elasticity, fluid mechanics, biological population models, etc. In this section, we apply our main result for the uniqueness of the solution of a nonlinear fractional differential equation. In epidemiology, mathematical modeling has developed into a useful method for comprehending the dynamics of diseases. Ross [45] developed the first epidemiological model to study malaria transmission at the beginning of 1900.

The study of fractional calculus has a long history; however, scientists study applications these days. Scientists focus on the study of HIV modeling in fractional calculus. In this direction, Ding et al. [46] introduced the HIV model in fractional order derivative in which the $\mathcal{H}$ cell gets infected. Tabassum et al. [47] established the nonlinear mathematical model of HIV using necessary requirements for well posedness and boundedness. An HIV/AIDS model with weak CD4+ $\mathcal{H}$ cells was presented by Dutta and Gupta [48] and infection-free equilibrium conditions were examined.

Let g denote the model for the survivability of AIDS patients, then we have the following fractional differential equation,

$$D^{\hbar}_{0+}\mathfrak{y}(s) = g(s, \mathfrak{y}(s)),\ s > 0, \tag{10}$$

where $\mathfrak{y}(0) + \mathfrak{y}'(0) = 0, \mathfrak{y}(1) + \mathfrak{y}'(1) = 0$, $D^{\hbar}_{0+}$ is the Caputo fractional derivative, $1 < \hbar \leq 2$ is a real number and g is a continuous function from $[0,1] \times [0,\infty)$ to $[0,\infty)$. Now define a complete *(FbRMLS)* $(\Upsilon, M^r_b, b, *)$ as

$$M^r_b(\wp_1, \wp_2, \mathfrak{t}_1) = \exp -\frac{|\wp_1 + \wp_2|^2}{\mathfrak{t}_1},\ for\ all\ \wp_1, \wp_2 \in \Upsilon,\ \mathfrak{t}_1 > 0,\ b = 2,$$

where $\mathfrak{t}_1 * \mathfrak{t}_2 = \mathfrak{t}_1\mathfrak{t}_2$. Denote the space of all continuous functions defined on $I = [0,1]$ by $\Upsilon = C([0,1], \mathbb{R})$. Observe that $\mathfrak{y} \in \Upsilon$ is the solution of (10) if and only if $\mathfrak{y}$ solves the following integral equation,

$$\begin{aligned}\mathfrak{y}(s) = &\frac{1}{\Gamma(\hbar)}\int_0^1 (1-\jmath)^{\hbar-1}(1-s)g(\jmath, \mathfrak{y}(\jmath))d\jmath + \frac{1}{\Gamma(\hbar-1)}\int_0^1 (1-\jmath)^{\hbar-2}(1-s)g(\jmath, \mathfrak{y}(\jmath))d\jmath \\ &+ \frac{1}{\Gamma(\hbar)}\int_0^s (s-\jmath)^{\hbar-1}g(\jmath, \mathfrak{y}(\jmath))d\jmath.\end{aligned} \tag{11}$$

Theorem 4. *Consider the integral operator* $\mathcal{H} : \Upsilon \longrightarrow \Upsilon$ *defined by*

$$\begin{aligned}\mathcal{H}\mathfrak{y}(s) = &\frac{1}{\Gamma(\hbar)}\int_0^1 (1-\jmath)^{\hbar-1}(1-s)g(\jmath, \mathfrak{y}(\jmath))d\jmath + \frac{1}{\Gamma(\hbar-1)}\int_0^1 (1-\jmath)^{\hbar-2}(1-s)g(\jmath, \mathfrak{y}(\jmath))d\jmath \\ &+ \frac{1}{\Gamma(\hbar)}\int_0^s (s-\jmath)^{\hbar-1}g(\jmath, \mathfrak{y}(\jmath))d\jmath,\end{aligned}$$

and assume the conditions:
(i) *for all* $\mathfrak{y}, \mathfrak{v} \in \Upsilon,\ \beta \in F_b$ *and* $g : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$, *satisfies*

$$|g(s, \mathfrak{y}(s)) + g(s, \mathfrak{v}(s))| \leq \frac{1}{4}\sqrt{\beta(M^r_b(\mathfrak{y}, \mathfrak{v}, \mathfrak{t}_1))}|\mathfrak{y}(s) + \mathfrak{v}(s)|,$$

(ii)

$$\sup_{s\in(0,1)} \frac{1}{4}\Big|\frac{1-s}{\Gamma(\hbar+1)} + \frac{1-s}{\Gamma(\hbar)} + \frac{s^{\hbar}}{\Gamma(\hbar+1)}\Big|^2 = \eta < 1,$$

holds. Then the nonlinear fractional differential Equation (10) has a unique solution.

Proof. Let $\mathfrak{y}, \mathfrak{v} \in \Upsilon$ and consider

$$\begin{aligned}
\Big|\mathcal{H}\mathfrak{y}(s) + \mathcal{H}\mathfrak{v}(s)\Big|^2 &= \Big|\frac{1}{\Gamma(\hbar)}\int_0^1 (1-\jmath)^{\hbar-1}(1-s)(g(\jmath,\mathfrak{y}(\jmath) + g(\jmath,\mathfrak{v}(\jmath))))d\jmath \\
&+ \frac{1}{\Gamma(\hbar-1)}\int_0^1 (1-\jmath)^{\hbar-2}(1-s)(g(\jmath,\mathfrak{y}(\jmath) + g(\jmath,\mathfrak{v}(\jmath))))d\jmath \\
&+ \frac{1}{\Gamma(\hbar)}\int_0^s (s-\jmath)^{\hbar-1}(g(\jmath,\mathfrak{y}(\jmath)) + g(\jmath,\mathfrak{v}(\jmath)))d\jmath\Big|^2 \\
&\leq \Big(\frac{1}{\Gamma(\hbar)}\int_0^1 (1-\jmath)^{\hbar-1}(1-s)\Big|g(\jmath,\mathfrak{y}(\jmath) + g(\jmath,\mathfrak{v}(\jmath)))\Big|d\jmath \\
&+ \frac{1}{\Gamma(\hbar-1)}\int_0^1 (1-\jmath)^{\hbar-2}(1-s)\Big|g(\jmath,\mathfrak{y}(\jmath) + g(\jmath,\mathfrak{v}(\jmath)))\Big|d\jmath \\
&+ \frac{1}{\Gamma(\hbar)}\int_0^s (s-\jmath)^{\hbar-1}\Big|g(\jmath,\mathfrak{y}(\jmath)) + g(\jmath,\mathfrak{v}(\jmath))\Big|d\jmath\Big)^2 \\
&\leq \Big(\frac{1}{\Gamma(\hbar)}\int_0^1 (1-\jmath)^{\hbar-1}(1-s)\frac{1}{4}\sqrt{\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))}|\mathfrak{y}(\jmath) \\
&+ \mathfrak{v}(\jmath)|d\jmath \\
&+ \frac{1}{\Gamma(\hbar-1)}\int_0^1 (1-\jmath)^{\hbar-2}(1-s)\frac{1}{4}\sqrt{\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))}|\mathfrak{y}(\jmath) \\
&+ \mathfrak{v}(\jmath)|d\jmath + \frac{1}{\Gamma(\hbar)}\int_0^s (s-\jmath)^{\hbar-1}\frac{1}{4}\sqrt{\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))}|\mathfrak{y}(\jmath) \\
&+ \mathfrak{v}(\jmath)|d\jmath\Big)^2 \\
&= \frac{1}{4}\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))|\mathfrak{y}(s) + \mathfrak{v}(s)|^2\Big(\frac{1}{\Gamma(\hbar)}\int_0^1 (1-\jmath)^{\hbar-1}(1-s)d\jmath \\
&+ \frac{1}{\Gamma(\hbar-1)}\int_0^1 (1-\jmath)^{\hbar-2}(1-s)d\jmath + \frac{1}{\Gamma(\hbar)}\int_0^s (s-\jmath)^{\hbar-1}d\jmath\Big)^2 \\
&= \frac{1}{4}\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))|\mathfrak{y}(s) + \mathfrak{v}(s)|^2\Big(\frac{1}{\Gamma(\hbar)}\int_0^1 (1-\jmath)^{\hbar-1}(1-s)d\jmath \\
&+ \frac{1}{\Gamma(\hbar-1)}\int_0^1 (1-\jmath)^{\hbar-2}(1-s)d\jmath + \frac{1}{\Gamma(\hbar)}\int_0^s (s-\jmath)^{\hbar-1}d\jmath\Big)^2 \\
&= \frac{1}{4}\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))|\mathfrak{y}(s) + \mathfrak{v}(s)|^2\Big(\frac{1-s}{\Gamma(\hbar)}\frac{(1-\jmath)^{\hbar}}{-\hbar}\Big|_0^1 \\
&+ \frac{1-s}{\Gamma(\hbar-1)}\frac{(1-\jmath)^{\hbar-1}}{-(\hbar-1)}\Big|_0^1 + \frac{1}{\Gamma(\hbar)}\frac{(s-\jmath)^{\hbar}}{-\hbar}\Big|_0^s\Big)^2 \\
&= \frac{1}{4}\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))|\mathfrak{y}(s) + \mathfrak{v}(s)|^2\Big(\frac{1-s}{\Gamma(\hbar+1)} + \frac{1-s}{\Gamma(\hbar)} \\
&+ \frac{s^{\hbar}}{\Gamma(\hbar+1)}\Big)^2
\end{aligned}$$

$$\leq \frac{1}{4}\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))|\mathfrak{y}(s)+\mathfrak{v}(s)|^2 \sup_{s\in(0,1)}\Big(\frac{1-s}{\Gamma(\hbar+1)}+\frac{1-s}{\Gamma(\hbar)}$$
$$+\frac{s^{\hbar}}{\Gamma(\hbar+1)}\Big)^2$$
$$=\eta.\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))|\mathfrak{y}(s)+\mathfrak{v}(s)|^2$$
$$\leq \beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))|\mathfrak{y}(s)+\mathfrak{v}(s)|^2,$$

So, we have

$$\frac{\Big|\mathcal{H}\mathfrak{y}(s)+\mathcal{H}\mathfrak{v}(s)\Big|^2}{\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))}\leq|\mathfrak{y}(s)+\mathfrak{v}(s)|^2$$

That is,

$$-\frac{\Big|\mathcal{H}\mathfrak{y}(s)+\mathcal{H}\mathfrak{v}(s)\Big|^2}{\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))\mathfrak{t}_1}\geq-\frac{|\mathfrak{y}(s)+\mathfrak{v}(s)|^2}{\mathfrak{t}_1}$$

Taking an exponential on both sides, we have

$$\exp\Big(-\frac{\Big|\mathcal{H}\mathfrak{y}(s)+\mathcal{H}\mathfrak{v}(s)\Big|^2}{\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))\mathfrak{t}_1}\Big)\geq\exp\Big(-\frac{|\mathfrak{y}(s)+\mathfrak{v}(s)|^2}{\mathfrak{t}_1}\Big)$$

Thus, we have

$$M_b^r(\mathcal{H}\mathfrak{y}(s),\mathcal{H}\mathfrak{v}(s),\beta(M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1))\mathfrak{t}_1)\geq M_b^r(\mathfrak{y},\mathfrak{v},\mathfrak{t}_1),$$

Thus, from the application of Theorem 2, the nonlinear fractional differential Equation (10) has a unique solution.
Taking $\hbar=1.1,1.2,\ldots,2,\ s\in[0,1]$, we plot $\mathfrak{y}(s)$ in Figure 1 using Matlab 2018a as follows:

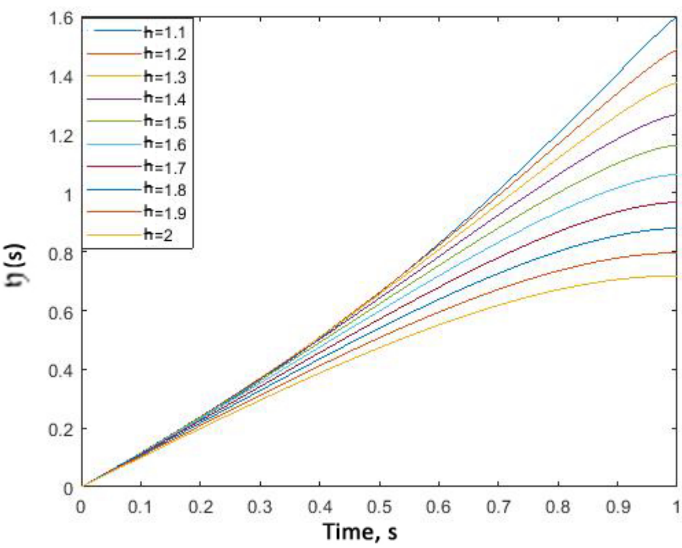

Figure 1. Shows the values of $\mathfrak{y}(s)$ for different values of $\hbar$.

□

The following numerical example illustrates Theorem 4.

Example 12. *Consider the fractional order differential equation*

$$D^{2}_{0+}\mathfrak{y}(s) = g(s, \mathfrak{y}(s)),\ s > 0. \tag{12}$$

with $\beta(M^{r}_{b}(\mathfrak{y}, \mathfrak{v}, \mathfrak{t}_1)) = \frac{1}{4}$ *and* $g(s, \mathfrak{y}(s)) = \frac{\mathfrak{y}(s)}{8} - e^{s}$. *Let* $\mathcal{H}$ *be the integral operator as defined in Theorem 4. Note that*
(i)

$$\begin{aligned}
\left|g(s, \mathfrak{y}(s)) + g(s, \mathfrak{v}(s))\right| &= \left|\frac{\mathfrak{y}(s)}{8} - e^{s} + \frac{\mathfrak{v}(s)}{8} - e^{s}\right| \\
&= \frac{1}{8}\left|\mathfrak{y}(s) + \mathfrak{v}(s) - 2e^{s}\right| \\
&\leq \frac{1}{4}\sqrt{\frac{1}{4}}\left|(\mathfrak{y}(s) + \mathfrak{v}(s))\right| \\
&= \frac{1}{4}\sqrt{\beta}\left|(\mathfrak{y}(s) + \mathfrak{v}(s))\right|,\ \beta = \frac{1}{4}.
\end{aligned}$$

*and **(ii)***

$$\begin{aligned}
\eta &= \frac{1}{4}\sup_{s\in(0,1)}\left|\frac{1-s}{\Gamma(\hbar+1)} + \frac{1-s}{\Gamma(\hbar)} + \frac{s^{\hbar}}{\Gamma(\hbar+1)}\right|^{2} \\
&= \frac{1}{4}\sup_{s\in(0,1)}\left|\frac{1-s}{\Gamma(3)} + \frac{1-s}{\Gamma(2)} + \frac{s^{2}}{\Gamma(3)}\right|^{2},\ \textit{here } \hbar = 2 \\
&= \frac{1}{4}\sup_{s\in(0,1)}\left|\frac{1-s}{2} + \frac{1-s}{1} + \frac{s^{2}}{2}\right|^{2} \\
&= \frac{1}{4}\sup_{s\in(0,1)}\left|\frac{s^{2} - 3s + 2}{2}\right|^{2} \\
&< 1.
\end{aligned}$$

Since conditions (i) and (ii) of Theorem 4 are fulfilled, fractional Equation (12) has a unique solution in Y.

5. Conclusions

We defined rectangular and b-rectangular metric-like spaces using fuzzy set theory, which are generalizations of numerous fuzzy metric spaces previously described in the literature. We proved with an example that neither of these spaces is Hausdorff. In these spaces, we demonstrated our main results through the use of Geraghty and $\alpha - \psi$ contractions. Our definitions and results are supported by examples. Some remarks have also been given that show the generalization of our results and definitions as compared to some other existing results in the literature. In the end, we provided an application for the survivability of AIDS patients via a fractional differential equation. Our newly defined result and application can be employed in the existing literature. In summary, our results are original, meaningful and valuable in the context of the existing literature. We hope that our new results can be applied to fields such as nonlinear analysis, fractional calculus models and other related fields in the future.

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