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A Reliable Way to Deal with the Coupled Fractional Korteweg-De Vries Equations within the Caputo Operator

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Abstract: The development of numeric-analytic solutions and the construction of fractional order mathematical models for practical issues are of the highest concern in a variety of physics, applied mathematics, and engineering applications. The nonlinear Kersten–Krasil'shchik-coupled Korteweg–de Vries-modified Korteweg–de Vries (KdV-mKdV) system is treated analytically in this paper using a unique method, known as the Laplace residual power series (LRPS) approach to find some approximate solutions. The RPS methodology and the Laplace transform operator are combined in the LRPS method. We provide a detailed introduction to the proposed method for dealing with fractional Kersten–Krasil'shchik-linked KdV-mKdV models. When compared to exact solutions, the approach provides analytical solutions with good accuracy. We demonstrate the effectiveness of the current strategy compared to alternative methods for solving nonlinear equations using an illustrative example. The LRPS technique's results show and highlight that the method may be used for a variety of time-fractional models of physical processes with simplicity and computing effectiveness.

Keywords: Laplace transform; residual power series; Caputo operator; Korteweg–de Vries nonlinear system

MSC: 26A33; 60H15; 35R11; 34A25

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1. Introduction

Fractional calculus (FC) has attracted increased interest from scientists and academics due to its applications in a variety of scientific disciplines, including engineering, chemistry, and social science [1–3]. Iterative methodology [4], Laplace transforms [5], and the operational tool [6] are some of the methods used to explore fractional equations. Different approaches were employed by various researchers to examine various fractional nonlinear equations found in the field of nanotechnology, including the shock wave equation, the KdV Burgers–Kuramoto equation, and the differential-difference equation [7–9]. Singh et al. studied the Tricomi equation in relation to the local fractional derivative of the fractal transonic flow, which led to the discovery of the system's non-differentiable solution [10]. Using the integral transform method, Choudhary et al. [11] investigated a fractional system of heat flux in the semi-infinite solid. Liouville–Caputo operators were employed by Gomez Aguilar et al. in their study of a fractional derivative for an electrical RLC circuit [12]. In [13], a fractional-order model of the exponential kernel in the Euler–Lagrange and Hamilton equations was investigated. As a result, symmetry analysis is an excellent tool for understanding partial differential equations, particularly when looking at equations derived from accounting-related mathematical ideas [14,15]. The majority of natural observations lack symmetry, despite the fact that symmetry is the cornerstones of nature. Unexpected symmetry-breaking occurrences are a sophisticated method of hiding symmetry. Finite and infinitesimal symmetry are the two types. Finite symmetries can either be discrete or continuous. Parity and temporal inversion are discrete natural symmetries, whereas space is a continuous transformation. Patterns have always captivated mathematicians. In the seventeenth century, the classification of spatial and planar patterns really got going. Regrettably, precise solutions to fractional nonlinear differential equations have shown to be exceedingly challenging.

In recent years, people have been able to simulate numerous processes with more freedom by employing fractional differential equations (FDE). This concept has been realized in numerous disciplines, including engineering, economics, control theory, and finance, with impressive achievements. People are becoming more and more interested in using FDE to simulate challenging and real-world issues; however, integer characteristics do not make differential equations more realistic (IDE) [16–18]. FDE is non-local, has a storage effect, and performs better when compared to local IDE. For the investigation of various situations, the model's future circumstances will vary depending on both current events and historical data. These features make it possible for FDE to successfully model non-Markov events in addition to non-Gaussian phenomena realistically. Additionally, standard IDEs are unable to explain FDE's assistance or offer information between two different integer values. Numerous non-integer order derivative operators have been put out in various studies occurring to get over the restriction of only differentiating integer values. There are several disciplines where fractional-order differential operators are used [19–24].

The degree of freedom of its differential operator in modern calculus (including classical calculus) in a specific circumstance is more significant than that of the local ordinary differential operator. The main application of calculation can be found in [25–27]. As a result, the study of non-integer order differentiation and integrations is highly valued by scholars. From a geometric standpoint, the entire function's accumulation or the full global integration range is explained by the arbitrary order derivatives, which are mostly definite integrals. The optimization of differential equations, and numerical and qualitative research has benefited tremendously from the work of researchers. It is important to remember that recent derivative operators were developed using definite integral techniques. It is a well-known reality that there is no fundamental formula for solving this. As a result, different definitions are involved with both types of kernels. The Atangana, Baleanu (ABC) fractional-order derivative [28] is the fundamental formulation that is the most attractive. Nonlinear equations are typically difficult to solve analytically or exactly. As a result, numerous numerical methods have been developed to evaluate the aforementioned equations. In order to examine FDE under ABC derivatives, many academics have recently looked into numerical approaches [29–33].

There are many published papers about the interpretation of nonlinear phenomena that can be generated and propagated in different plasma models. Most researchers have focused on reducing the basic equations of different plasma models to evolution equations in the form of partial differential equations such as the Korteweg–de Vries (KdV) equation, modified KdV (mKdV) equation, extended KdV equation, Kawahara-type equation, and so on [34–36]. A few studies reduced the fluid equations of different plasma models to fractional differential equations (FDEs) to gain a lot of information about how these waves propagate. Fractional coupled systems are frequently employed to study the complicated behavior of many nonlinear structures in different plasma models. Many experts have made an effort to assess this behavior. Recent research on the KdV equation and mKdV equation was conducted by Paul Kersten and Joseph Krasil'shchik. They proposed absolute complexity between coupled KdV-mKdV nonlinear systems for the analysis of nonlinear system behavior [37–39]. This Kersten–Krasil'shchik-coupled KdV-mKdV nonlinear system has been the subject of various variations, as noted in [40–43]. The nonlinear fractional Kersten–Krasil'shchik-coupled KdV-mKdV system is one of these variations and offers a mathematical explanation for the behavior of multi-component plasma for waves traveling down the positive κ axis:

$$\begin{aligned} &D^{\rho}_{\Im}u + u_{\kappa\kappa\kappa} - 6uu_{\kappa} + 3vv_{\kappa\kappa\kappa} + 3v_{\kappa}v_{\kappa\kappa\kappa} - 3u_{\kappa}v^2 + 6uvv_{\kappa} = 0,\ \ \Im > 0,\ \ \kappa \in R,\ \ 0 < \rho \leq 1, \\ &D^{\rho}_{\Im}v + v_{\kappa\kappa\kappa} - 3v^2v_{\kappa} - 3uv_{\kappa} + 3u_{\kappa}v = 0,\ \ \Im > 0,\ \ \kappa \in R,\ \ 0 < \rho \leq 1, \end{aligned} \tag{1}$$

where κ denotes a spatial coordinate and $\Im$ denotes a time coordinate. Factor ρ denotes the order of the fractional operator. This operator is studied using the Caputo form. The fractional coupled system turns into a classical system when $\rho = 1$, as shown below:

$$\begin{aligned} &u_{\Im} + u_{\kappa\kappa\kappa} - 6uu_{\kappa} + 3vv_{\kappa\kappa\kappa} + 3v_{\kappa}v_{2\kappa} - 3u_{\kappa}v^2 + 6uvv_{\kappa} = 0,\ \ \Im > 0,\ \ \kappa \in R, \\ &v_{\Im} + v_{\kappa\kappa\kappa} - 3v^2v_{\kappa} - 3uv_{\kappa} + 3u_{\kappa}v = 0,\ \ \Im > 0,\ \ \kappa \in R. \end{aligned} \tag{2}$$

The well-known KdV system is created from the Kersten–Krasil'shchik-linked KdV-mKdV system as follows if we put $v = 0$.

$$u_{\Im} + u_{\kappa\kappa\kappa} - 6uu_{\kappa} = 0,\ \ \Im > 0,\ \ \kappa \in R. \tag{3}$$

The well-known modified KdV system is created as follows when the Kersten–Krasil' shchik-connected KdV-mKdV system at $u = 0$.

$$v_{\Im} + v_{\kappa\kappa\kappa} - 3v^2v_{\kappa} = 0,\ \ \Im > 0,\ \ \kappa \in R. \tag{4}$$

The Kersten–Krasil'shchik-linked KdV-mKdV system can, therefore, be thought of as a combination of the KdV system and the mKdV system, which are described by (2) to (4). In this work, we also look at the third-order KdV system with two components that are fractionally nonlinear and homogeneously coupled in time:

$$\begin{aligned} &D^{\rho}_{\Im}u - u_{\kappa\kappa\kappa} - uu_{\kappa} - vv_{\kappa} = 0,\ \Im > 0,\ \kappa \in R,\ 0 < \rho \leq 1, \\ &D^{\rho}_{\Im}v + 2v_{\kappa\kappa\kappa} - uv_{\kappa} = 0,\ \Im > 0,\ \kappa \in R,\ 0 < \rho \leq 1, \end{aligned} \tag{5}$$

where ρ is the order factor of the fractional operator, $\Im$ is the temporal coordinate amd κ is the spatial coordinate. This operator is studied using Caputo form. The fractional coupled system turns into a classical system when $\rho = 1$, as shown below:

$$\begin{aligned} &u_{\Im} - u_{\kappa\kappa\kappa} - uu_{\kappa} - vv_{\kappa} = 0,\ \ \Im > 0,\ \ \kappa \in R, \\ &v_{\Im} + 2v_{\kappa\kappa\kappa} - uv_{\kappa} = 0,\ \ \Im > 0,\ \kappa \in R. \end{aligned} \tag{6}$$

Finding the exact solution to nonlinear partial differential equations is still a major problem in physics and applied mathematics, necessitating the use of various techniques to obtain innovative approximate or exact solutions. Many approximation and numerical techniques have been used to solve fractional differential equations [44–46]. In this work, we propose a new easy and effective semi-analytical method to solve fractional PDE systems with variable coefficients. Our recommended method, LRPSM, which combines RPSM and the Laplace transform, was put into practice. In comparison to the RPS technique, which requires fractional differentiation in each phase, the main advantage of the present technique is that it can identify the unknown components of the suggested solutions by utilizing limits in the Laplace space, which, in turn, reduces the number of calculations needed and saves time [47–49]. For a variety of FDEs and time-fractional PDEs, the LRPS approach has been effectively used to develop approximative series solutions in closed forms [50–52].

The remaining parts of the ongoing work are structured as follows: The Laplace transform, Laplace fractional expansion, and certain fundamental definitions and theorems relating to fractional calculus are reviewed in Section 2. The concept of the suggested method for constructing the approximate solution of the fractional model under consideration (5) is provided in Section 3. To demonstrate the applicability and effectiveness of examining the solutions of time-PDEs of fractional order, the LRPS technique is used in Section 4 to solve fractional Kersten–Krasil'shchik-coupled KdV-mKdV systems. Finally, Section 5 presents possible interpretations of our findings.

2. Preliminaries

Here we presented some essential definitions related to our present work.

Definition 1. *The Caputo fractional derivative is stated as [53–55]*

$$^{C}D^{\rho}_{\Im}u(\kappa,\Im) = J^{m-\rho}_{\Im}u^{m}(\kappa,\Im), \;\; m-1<\rho\leq m, \;\; \Im>0. \tag{7}$$

where $m \in N$ and $J^{\rho}_{\Im}$ is the fractional Riemann-Liouville (RL) integral stated as

$$J^{\rho}_{\Im}u(\kappa,\Im) = \frac{1}{\Gamma(\rho)}\int_0^{\Im}(\Im-t)^{\rho-1}u(\kappa,t)dt \tag{8}$$

considering that the given integral exists.

Definition 2. *For a function $u(\kappa,\Im)$, the Laplace transform is stated as [53]*

$$u(\kappa,\varpi) = \mathcal{L}_{\Im}[u(\kappa,\Im)] = \int_0^{\infty} e^{-\varpi\Im}u(\kappa,\Im)d\Im, \;\; \varpi>\rho, \tag{9}$$

where the inverse LT is stated as

$$u(\kappa,\Im) = \mathcal{L}^{-1}_{\Im}[u(\kappa,\varpi)] = \int_{l-i\infty}^{l+i\infty} e^{\varpi\Im}u(\kappa,\varpi)d\varpi, \;\; l=Re(\varpi)>l_0, \tag{10}$$

where l_0 is in the right half-plane of the Laplace integral's absolute convergence.

Lemma 1. *Let us consider that $u(\kappa,\Im)$ is a continuous piecewise function having exponential order ζ and $U(\kappa,\varpi) = \mathcal{L}_{\Im}[u(\kappa,\Im)]$, we get*

1. $\mathcal{L}_{\Im}[J^{\rho}_{\Im}u(\kappa,\Im)] = \frac{U(\kappa,\varpi)}{\varpi^{\rho}}, \;\; \rho>0.$
2. $\mathcal{L}_{\Im}[D^{\rho}_{\Im}u(\kappa,\Im)] = \varpi^{\rho}U(\kappa,\varpi) - \sum_{k=0}^{m-1}\varpi^{\rho-k-1}u^{k}(\kappa,0), \;\; m-1<\rho\leq m.$
3. $\mathcal{L}_{\Im}[D^{n\rho}_{\Im}u(\kappa,\Im)] = \varpi^{n\rho}U(\kappa,\varpi) - \sum_{k=0}^{n-1}\varpi^{(n-k)\rho-1}D^{k\rho}_{\Im}u(\kappa,0), \;\; 0<\rho\leq 1.$

Proof. Check Ref. [56]. □

Theorem 1. *Let $u(\kappa,\Im)$ be a piecewise continuous function on $I\times[0,\infty)$ with exponential order ζ. Assume the $U(\kappa,\varpi) = \mathcal{L}_{\Im}[u(\kappa,\Im)]$ function has the below fractional expansion:*

$$U(\kappa,\varpi) = \sum_{n=0}^{\infty}\frac{f_n(\kappa)}{\varpi^{1+n\rho}}, 0<\rho\leq 1, \kappa\in I, \varpi>\zeta. \tag{11}$$

Thus, $f_n(\kappa) = D^{n\rho}_{\Im}u(\kappa,0)$.

Proof. Check Ref. [53]. □

Remark 1. *The inverse LT of Equation (11) is stated as [53]:*

$$u(\kappa,\Im) = \sum_{i=0}^{\infty}\frac{D^{\rho}_{\Im}u(\kappa,0)}{\Gamma(1+i\rho)}\Im^{i(\zeta)}, \;\; 0<\zeta\leq 1, \;\; \Im\geq 0. \tag{12}$$

which is equal to the fractional Taylor's formula as given in [57].

The FPS convergence of Theorem (1) is determined in the below theorem.

Theorem 2. *Consider $u(\kappa,\Im)$ is piecewise continuous on $\mathbf{I}\times[0,\infty)$ having order ξ and is as given in Theorem (1), $u(\kappa,\varpi)=\mathcal{L}_{\Im}[u(\kappa,\Im)]$ can be stated to be in the form of fractional Taylor's formula. If $\left|\varpi\mathcal{L}_{\Im}[D_{\Im}^{i\rho+1}u(\kappa,\Im)]\right|\leq M(\kappa)$, on $\mathbf{I}\times(\xi,\gamma]$ where $0<\rho\leq 1$, then $R_i(\kappa,\varpi)$ is the remainder of the form of fractional Taylor's formula in Theorem (1), proving the below inequality.*

$$|R_i(\kappa,\varpi)|\leq\frac{M(\kappa)}{S^{1+(i+1)\rho}},\ \ \kappa\in\mathbf{I},\ \ \xi<\varpi\leq\gamma. \tag{13}$$

Proof. Check Ref. [53]. □

3. Idea of LRPS

In this part, we will discuss the general methodology of LRPS to solve the system of fractional partial differential equations.

On taking LT of Equation (5), we have

$$\begin{aligned}
&U(\kappa,\varpi)-\frac{f_0(\kappa,\xi)}{\varpi}+\frac{1}{\varpi^{\rho}}\mathcal{L}_{\Im}\left[\mathcal{L}_{\Im}^{-1}[U_{\kappa\kappa\kappa}]+\mathcal{L}_{\Im}^{-1}[U]\mathcal{L}_{\Im}^{-1}[U_{\kappa}]+\mathcal{L}_{\Im}^{-1}[V]\mathcal{L}_{\Im}^{-1}[V_{\kappa}]\right]=0,\\
&V(\kappa,\varpi)-\frac{g_0(\kappa,\xi)}{\varpi}+\frac{1}{\varpi^{\rho}}\mathcal{L}_{\Im}\left[2\mathcal{L}_{\Im}^{-1}[V_{\kappa\kappa\kappa}]-\mathcal{L}_{\Im}^{-1}[U]\mathcal{L}_{\Im}^{-1}[V_{\kappa}]\right]=0.
\end{aligned} \tag{14}$$

Considering that Equation (14)'s solution has the below expansion

$$U(\kappa,\varpi)=\sum_{n=0}^{\infty}\frac{f_n(\kappa,\varpi)}{\varpi^{n\rho+1}},\ \ V(\kappa,\varpi)=\sum_{n=0}^{\infty}\frac{g_n(\kappa,\varpi)}{\varpi^{n\rho+1}}. \tag{15}$$

The series for the k^{th}-truncated term is

$$U(\kappa,\varpi)=\frac{f_0(\kappa,\varpi)}{\varpi}+\sum_{n=1}^{k}\frac{f_n(\kappa,\varpi)}{\varpi^{n\rho+1}},\ \ V(\kappa,\varpi)=\frac{g_0(\kappa,\varpi)}{\varpi}+\sum_{n=1}^{k}\frac{g_n(\kappa,\varpi)}{\varpi^{n\rho+1}}.\ \ k=1,2,3,4\cdots \tag{16}$$

The Laplace residual functions (LRFs) [58] are

$$\begin{aligned}
\mathcal{L}_{\Im}Res_u(\kappa,\varpi)=&U(\kappa,\varpi)-\frac{f_0(\kappa,\varpi)}{\varpi}+\frac{1}{\varpi^{\rho}}\mathcal{L}_{\Im}\left[\mathcal{L}_{\Im}^{-1}[U_{\kappa\kappa\kappa}]+\mathcal{L}_{\Im}^{-1}[U]\mathcal{L}_{\Im}^{-1}[U_{\kappa}]+\mathcal{L}_{\Im}^{-1}[V]\mathcal{L}_{\Im}^{-1}[V_{\kappa}]\right],\\
\mathcal{L}_{\Im}Res_v(\kappa,\varpi)=&V(\kappa,\varpi)-\frac{g_0(\kappa,\varpi)}{\varpi}+\frac{1}{\varpi^{\rho}}\mathcal{L}_{\Im}\left[2\mathcal{L}_{\Im}^{-1}[V_{\kappa\kappa\kappa}]-\mathcal{L}_{\Im}^{-1}[U]\mathcal{L}_{\Im}^{-1}[V_{\kappa}]\right].
\end{aligned} \tag{17}$$

The k^{th}-LRFs as:

$$\begin{aligned}
\mathcal{L}_{\Im}Res_{u,k}(\kappa,\varpi)=&U_k(\kappa,\varpi)-\frac{f_0(\kappa,\varpi)}{\varpi}+\frac{1}{\varpi^{\rho}}\mathcal{L}_{\Im}\left[\mathcal{L}_{\Im}^{-1}[U_{\kappa\kappa\kappa,k}]+\mathcal{L}_{\Im}^{-1}[U_k]\mathcal{L}_{\Im}^{-1}[U_{\kappa,k}]+\mathcal{L}_{\Im}^{-1}[V_k]\mathcal{L}_{\Im}^{-1}[V_{\kappa,k}]\right],\\
\mathcal{L}_{\Im}Res_{v,k}(\kappa,\varpi)=&V_k(\kappa,\varpi)-\frac{g_0(\kappa,\varpi)}{\varpi}+\frac{1}{\varpi^{\rho}}\mathcal{L}_{\Im}\left[2\mathcal{L}_{\Im}^{-1}[V_{\kappa\kappa\kappa,k}]-\mathcal{L}_{\Im}^{-1}[U_k]\mathcal{L}_{\Im}^{-1}[V_{\kappa,k}]\right].
\end{aligned} \tag{18}$$

There are some properties that arise in the LRPSM [58] to point out some facts:

- $\mathcal{L}_{\Im}Res(\kappa,\varpi)=0$ and $\lim_{j\to\infty}\mathcal{L}_{\Im}Res_{u,k}(\kappa,\varpi)=\mathcal{L}_{\Im}Res_u(\kappa,\varpi)$ for each $\varpi>0$.
- $\lim_{\varpi\to\infty}\varpi\mathcal{L}_{\Im}Res_u(\kappa,\varpi)=0\Rightarrow\lim_{\varpi\to\infty}\varpi\mathcal{L}_{\Im}Res_{u,k}(\kappa,\varpi)=0$.
- $\lim_{\varpi\to\infty}\varpi^{k\rho+1}\mathcal{L}_{\Im}Res_{u,k}(\kappa,\varpi)=\lim_{\varpi\to\infty}\varpi^{k\rho+1}\mathcal{L}_{\Im}Res_{u,k}(\kappa,\varpi)=0,\ \ 0<\rho\leq 1,$ $k=1,2,3,\cdots$.

Now to get coefficients $f_n(\kappa,\varpi)$ and $g_n(\kappa,\varpi)$, we recursively solve the below system

$$\begin{aligned}\lim_{\varpi\to\infty}\varpi^{k\rho+1}\mathcal{L}_{\Im}Res_{u,k}(\kappa,\varpi)=0,\quad k=1,2,\cdots,\\ \lim_{\varpi\to\infty}\varpi^{k\rho+1}\mathcal{L}_{\Im}Res_{v,k}(\kappa,\varpi)=0,\quad k=1,2,\cdots,\end{aligned}\tag{19}$$

Lastly, we employ inverse LT to Equation (16) to obtain the k^{th} approximate solutions of $u_k(\kappa,\Im)$ and $v_k(\kappa,\Im)$.

4. Numerical Problem

In this part, we implemented the suggested scheme for solving fractional Kersten–Krasil'shchik-linked KdV-mKdV systems.

4.1. Problem

Assume a time-fractional homogeneous two component-coupled KdV system of third-order as:

$$\begin{aligned}D^{\rho}_{\Im}u-u_{\kappa\kappa\kappa}-uu_{\kappa}-vv_{\kappa}=0,\\ D^{\rho}_{\Im}v+2v_{\kappa\kappa\kappa}-uv_{\kappa}=0.\end{aligned}\tag{20}$$

subjected to initial conditions:

$$\begin{aligned}u(\kappa,0)&=3-6\tanh^2\left(\frac{\kappa}{2}\right),\\ v(\kappa,0)&=-3c\sqrt{2}\tanh\left(\frac{\kappa}{2}\right).\end{aligned}\tag{21}$$

On taking LT from Equation (20) and by means of Equation (21), we obtain

$$\begin{aligned}U(\kappa,\varpi)-\frac{3-6\tanh^2\left(\frac{\kappa}{2}\right)}{\varpi}-\frac{1}{\varpi^{\rho}}\mathcal{L}_{\Im}\left[\mathcal{L}^{-1}_{\Im}[U_{\kappa\kappa\kappa}]+\mathcal{L}^{-1}_{\Im}[U]\mathcal{L}^{-1}_{\Im}[U_{\kappa}]+\mathcal{L}^{-1}_{\Im}[V]\mathcal{L}^{-1}_{\Im}[V_{\kappa}]\right]=0,\\ V(\kappa,\varpi)-\frac{-3c\sqrt{2}\tanh\left(\frac{\kappa}{2}\right)}{\varpi}-\frac{1}{\varpi^{\rho}}\mathcal{L}_{\Im}\left[-2\mathcal{L}^{-1}_{\Im}[V_{\kappa\kappa\kappa}]+\mathcal{L}^{-1}_{\Im}[U]\mathcal{L}^{-1}_{\Im}[V_{\kappa}]\right]=0,\end{aligned}\tag{22}$$

The series for k^{th}-truncated term is

$$U(\kappa,\varpi)=\frac{3-6\tanh^2\left(\frac{\kappa}{2}\right)}{\varpi}+\sum_{n=1}^{k}\frac{f_n(\kappa,\varpi)}{\varpi^{n\rho+1}},\quad V(\kappa,\varpi)=\frac{-3c\sqrt{2}\tanh\left(\frac{\kappa}{2}\right)}{\varpi}+\sum_{n=1}^{k}\frac{g_n(\kappa,\varpi)}{\varpi^{n\rho+1}},\quad k=1,2,3,4\cdots\tag{23}$$

and the k^{th}-LRFs as:

$$\begin{aligned}\mathcal{L}_{\Im}Res_{u,k}(\kappa,\varpi)=&U_k(\kappa,\varpi)-\frac{3-6\tanh^2\left(\frac{\kappa}{2}\right)}{\varpi}-\frac{1}{\varpi^{\rho}}\mathcal{L}_{\Im}\left[\mathcal{L}^{-1}_{\Im}[U_{\kappa\kappa\kappa,k}]+\mathcal{L}^{-1}_{\Im}[U_k]\mathcal{L}^{-1}_{\Im}[U_{\kappa,k}]+\mathcal{L}^{-1}_{\Im}[V_k]\mathcal{L}^{-1}_{\Im}[V_{\kappa,k}]\right],\\ \mathcal{L}_{\Im}Res_{v,k}(\kappa,\varpi)=&V_k(\kappa,\varpi)-\frac{-3c\sqrt{2}\tanh\left(\frac{\kappa}{2}\right)}{\varpi}-\frac{1}{\varpi^{\rho}}\mathcal{L}_{\Im}\left[-2\mathcal{L}^{-1}_{\Im}[V_{\kappa\kappa\kappa,k}]+\mathcal{L}^{-1}_{\Im}[U_k]\mathcal{L}^{-1}_{\Im}[V_{\kappa,k}]\right].\end{aligned}\tag{24}$$

Now, to find $f_k(\kappa,\varpi)$ and $g_k(\kappa,\varpi)$, $k=1,2,3,\cdots$, we put the k^{th}-truncated series Equation (23) into the k^{th}-Laplace residual function Equation (24), multiply the resulting equation by $\varpi^{k\rho+1}$, and now solve the relation recursively $\lim_{\varpi\to\infty}(\varpi^{k\rho+1}\mathcal{L}_{\Im}Res_{u,k}(\kappa,\varpi))=0$, and $\lim_{\varpi\to\infty}(\varpi^{k\rho+1}\mathcal{L}_{\Im}Res_{v,k}(\kappa,\varpi))=0$, $k=1,2,3,\cdots$.

We get some components as:

$$\begin{aligned}
f_1(\kappa,\varpi) =&3-6\tanh^2\left(\frac{\kappa}{2}\right),\\
g_1(\kappa,\varpi) =&-3c\sqrt{2}\tanh\left(\frac{\kappa}{2}\right),\\
f_2(\kappa,\varpi) =&6\,\mathrm{sech}^2\left(\frac{\kappa}{2}\right)\tanh\left(\frac{\kappa}{2}\right),\\
g_2(\kappa,\varpi) =&3c\sqrt{2}\,\mathrm{sech}^2\left(\frac{\kappa}{2}\right)\tanh\left(\frac{\kappa}{2}\right),\\
f_3(\kappa,\varpi) =&3[2+7\,\mathrm{sech}^2\left(\frac{\kappa}{2}\right)-15\,\mathrm{sech}^4\left(\frac{\kappa}{2}\right)]\,\mathrm{sech}^2\left(\frac{\kappa}{2}\right)\\
g_3(\kappa,\varpi) =&\frac{3c\sqrt{2}}{2}\left[2+21\,\mathrm{sech}^2\left(\frac{\kappa}{2}\right)-24\,\mathrm{sech}^4\left(\frac{\kappa}{2}\right)\right]\mathrm{sech}^2\left(\frac{\kappa}{2}\right).
\end{aligned}\tag{25}$$

and so on.

Now, by substituting $f_k(\kappa,\varpi)$ and $g_k(\kappa,\varpi)$, $k=1,2,3,\cdots$, in Equation (23), we obtain

$$\begin{aligned}
&U(\kappa,\varpi)=\frac{3-6\tanh^2(\frac{\kappa}{2})}{\varpi}+\frac{6\,\mathrm{sech}^2(\frac{\kappa}{2})\tanh(\frac{\kappa}{2})}{\varpi^{\rho+1}}+\\
&\frac{3[2+7\,\mathrm{sech}^2(\frac{\kappa}{2})-15\,\mathrm{sech}^4(\frac{\kappa}{2})]\,\mathrm{sech}^2(\frac{\kappa}{2})}{\varpi^{2\rho+1}}+\cdots,\\
&V(\kappa,\varpi)=\frac{-3c\sqrt{2}\tanh(\frac{\kappa}{2})}{\varpi}+\frac{3c\sqrt{2}\,\mathrm{sech}^2(\frac{\kappa}{2})\tanh(\frac{\kappa}{2})}{\varpi^{\rho+1}}+\\
&\frac{\frac{3c\sqrt{2}}{2}\left[2+21\,\mathrm{sech}^2(\frac{\kappa}{2})-24\,\mathrm{sech}^4(\frac{\kappa}{2})\right]\mathrm{sech}^2(\frac{\kappa}{2})}{\varpi^{2\rho+1}}+\cdots.
\end{aligned}\tag{26}$$

On taking the inverse LT, we have

$$\begin{aligned}
u(\kappa,\Im)=&\Bigg(3-6\tanh^2\left(\frac{\kappa}{2}\right)+6\,\mathrm{sech}^2\left(\frac{\kappa}{2}\right)\tanh\left(\frac{\kappa}{2}\right)\frac{\Im^\rho}{\Gamma(\rho+1)}+\\
&3[2+7\,\mathrm{sech}^2\left(\frac{\kappa}{2}\right)-15\,\mathrm{sech}^4\left(\frac{\kappa}{2}\right)]\,\mathrm{sech}^2\left(\frac{\kappa}{2}\right)\frac{\Im^{2\rho}}{\Gamma(2\rho+1)}+\cdots\Bigg),\\
v(\kappa,\Im)=&\Bigg(-3c\sqrt{2}\tanh\left(\frac{\kappa}{2}\right)+3c\sqrt{2}\,\mathrm{sech}^2\left(\frac{\kappa}{2}\right)\tanh\left(\frac{\kappa}{2}\right)\frac{\Im^\rho}{\Gamma(\rho+1)}+\\
&\frac{3c\sqrt{2}}{2}\left[2+21\,\mathrm{sech}^2\left(\frac{\kappa}{2}\right)-24\,\mathrm{sech}^4\left(\frac{\kappa}{2}\right)\right]\mathrm{sech}^2\left(\frac{\kappa}{2}\right)\frac{\Im^{2\rho}}{\Gamma(2\rho+1)}+\cdots\Bigg).
\end{aligned}$$

we obtain exact solutions by taking $\rho=1$ as

$$\begin{aligned}
u(\kappa,\Im)&=3-6\tanh^2\left(\frac{\kappa+\Im}{2}\right),\\
v(\kappa,\Im)&=-3c\sqrt{2}\tanh\left(\frac{\kappa+\Im}{2}\right).
\end{aligned}\tag{27}$$

4.2. Problem

Assume the time-fractional Kersten–Krasil'shchik-coupled KdV-mKdV system as:

$$\begin{aligned}
&D^\rho_\Im u-u_{\kappa\kappa\kappa}-6uu_\kappa+3vv_{\kappa\kappa\kappa}+3v_\kappa v_{\kappa\kappa}-3u_\kappa v^2+6uvv_\kappa=0,\\
&D^\rho_\Im v+v_{\kappa\kappa\kappa}-3v^2v_\kappa-3uv_\kappa+3u_\kappa v=0.
\end{aligned}\tag{28}$$

subjected to initial conditions:

$$\begin{aligned} u(\kappa,0) &= c - 2c\,\mathrm{sech}^2(\sqrt{c}\kappa), \\ v(\kappa,0) &= 2\sqrt{c}\,\mathrm{sech}(\sqrt{c}\kappa). \end{aligned} \tag{29}$$

On taking the LT of Equation (28) and by means of Equation (29), we obtain

$$\begin{aligned} &U(\kappa,\varpi) - \frac{c - 2c\,\mathrm{sech}^2(\sqrt{c}\kappa)}{\varpi} - \frac{1}{\varpi^\rho}\mathcal{L}_\Im\Bigg[\mathcal{L}_\Im^{-1}[U_{\kappa\kappa\kappa}] + 6\mathcal{L}_\Im^{-1}[U]\mathcal{L}_\Im^{-1}[U_\kappa] - 3\mathcal{L}_\Im^{-1}[V]\mathcal{L}_\Im^{-1}[V_{\kappa\kappa\kappa}] - \\ &3\mathcal{L}_\Im^{-1}[V_\kappa]\mathcal{L}_\Im^{-1}[V_{\kappa\kappa}] + 3\mathcal{L}_\Im^{-1}[U_\kappa]\mathcal{L}_\Im^{-1}[V^2] - 6\mathcal{L}_\Im^{-1}[U]\mathcal{L}_\Im^{-1}[V]\mathcal{L}_\Im^{-1}[V_\kappa]\Bigg] = 0, \\ &V(\kappa,\varpi) - \frac{2\sqrt{c}\,\mathrm{sech}(\sqrt{c}\kappa)}{\varpi} - \frac{1}{\varpi^\rho}\mathcal{L}_\Im\Bigg[-\mathcal{L}_\Im^{-1}[V_{\kappa\kappa\kappa}] + 3\mathcal{L}_\Im^{-1}[V^2]\mathcal{L}_\Im^{-1}[V_\kappa] + 3\mathcal{L}_\Im^{-1}[U]\mathcal{L}_\Im^{-1}[V_\kappa] - \\ &3\mathcal{L}_\Im^{-1}[U_\kappa]\mathcal{L}_\Im^{-1}[V]\Bigg] = 0, \end{aligned} \tag{30}$$

The series for the k^{th}-truncated term is

$$U(\kappa,\varpi) = \frac{c - 2c\,\mathrm{sech}^2(\sqrt{c}\kappa)}{\varpi} + \sum_{n=1}^{k}\frac{f_n(\kappa,\varpi)}{\varpi^{n\rho+1}}, \quad V(\kappa,\varpi) = \frac{2\sqrt{c}\,\mathrm{sech}(\sqrt{c}\kappa)}{\varpi} + \sum_{n=1}^{k}\frac{g_n(\kappa,\varpi)}{\varpi^{n\rho+1}}, \quad k = 1,2,3,4\cdots \tag{31}$$

and the k^{th}-LRFs as:

$$\begin{aligned} \mathcal{L}_\Im Res_{u,k}(\kappa,\varpi) =& U_k(\kappa,\varpi) - \frac{c - 2c\,\mathrm{sech}^2(\sqrt{c}\kappa)}{\varpi} + \frac{1}{\varpi^\rho}\mathcal{L}_\Im\Bigg[\mathcal{L}_\Im^{-1}[U_{\kappa\kappa\kappa,k}] + 6\mathcal{L}_\Im^{-1}[U_k]\mathcal{L}_\Im^{-1}[U_{\kappa,k}] - 3\mathcal{L}_\Im^{-1}[V_k]\mathcal{L}_\Im^{-1}[V_{\kappa\kappa\kappa,k}] - \\ &3\mathcal{L}_\Im^{-1}[V_{\kappa,k}]\mathcal{L}_\Im^{-1}[V_{\kappa\kappa,k}] + 3\mathcal{L}_\Im^{-1}[U_{\kappa,k}]\mathcal{L}_\Im^{-1}[V_k^2] - 6\mathcal{L}_\Im^{-1}[U_k]\mathcal{L}_\Im^{-1}[V_k]\mathcal{L}_\Im^{-1}[V_{\kappa,k}]\Bigg], \\ \mathcal{L}_\Im Res_{v,k}(\kappa,\varpi) =& V_k(\kappa,\varpi) - \frac{2\sqrt{c}\,\mathrm{sech}(\sqrt{c}\kappa)}{\varpi} + \frac{1}{\varpi^\rho}\mathcal{L}_\Im\Bigg[-\mathcal{L}_\Im^{-1}[V_{\kappa\kappa\kappa,k}] + 3\mathcal{L}_\Im^{-1}[V_k^2]\mathcal{L}_\Im^{-1}[V_{\kappa,k}] + 3\mathcal{L}_\Im^{-1}[U_k]\mathcal{L}_\Im^{-1}[V_{\kappa,k}] - \\ &3\mathcal{L}_\Im^{-1}[U_{\kappa,k}]\mathcal{L}_\Im^{-1}[V_k]\Bigg]. \end{aligned} \tag{32}$$

Now, to find $f_k(\kappa,\varpi)$ and $g_k(\kappa,\varpi)$, $k = 1,2,3,\cdots$, we put the k^{th}-truncated series Equation (31) into the k^{th}-Laplace residual function Equation (32), multiply the resulting equation by $s^{k\rho+1}$, and now solve the relation recursively $\lim_{\varpi\to\infty}(\varpi^{k\rho+1}\mathcal{L}_\Im Res_{u,k}(\kappa,\varpi)) = 0$, and $\lim_{\varpi\to\infty}(\varpi^{k\rho+1}\mathcal{L}_\Im Res_{v,k}(\kappa,\varpi)) = 0$, $k = 1,2,3,\cdots$.

We get some components as:

$$\begin{aligned} f_1(\kappa,\varpi) =& c - 2c\,\mathrm{sech}^2(\sqrt{c}\kappa), \\ g_1(\kappa,\varpi) =& 2\sqrt{c}\,\mathrm{sech}(\sqrt{c}\kappa), \\ f_2(\kappa,\varpi) =& 8c^{\frac{5}{2}}\sinh(\sqrt{c}\kappa)\,\mathrm{sech}^3(\sqrt{c}\kappa), \\ g_2(\kappa,\varpi) =& -4c^2\sinh(\sqrt{c}\kappa)\,\mathrm{sech}^2(\sqrt{c}\kappa), \\ f_3(\kappa,\varpi) =& -16c^4[2\cosh^2(\sqrt{c}\kappa) - 3]\,\mathrm{sech}^4(\sqrt{c}\kappa) \\ g_3(\kappa,\varpi) =& 8c^{\frac{7}{2}}[\cosh^2(\sqrt{c}\kappa) - 2]\,\mathrm{sech}^3(\sqrt{c}\kappa). \end{aligned} \tag{33}$$

and so on.

Now, by substituting $f_k(\kappa,\varpi)$ and $g_k(\kappa,\varpi)$, $k=1,2,3,\cdots$, into Equation (31), we obtain

$$\begin{aligned} U(\kappa,\varpi) &= \frac{c-2c\,\mathrm{sech}^2(\sqrt{c}\kappa)}{\varpi} + \frac{8c^{\frac{5}{2}}\sinh(\sqrt{c}\kappa)\,\mathrm{sech}^3(\sqrt{c}\kappa)}{\varpi^{\rho+1}} + \\ &\frac{-16c^4[2\cosh^2(\sqrt{c}\kappa)-3]\,\mathrm{sech}^4(\sqrt{c}\kappa)}{\varpi^{2\rho+1}} + \cdots, \\ V(\kappa,\varpi) &= \frac{2\sqrt{c}\,\mathrm{sech}(\sqrt{c}\kappa)}{\varpi} + \frac{-4c^2\sinh(\sqrt{c}\kappa)\,\mathrm{sech}^2(\sqrt{c}\kappa)}{\varpi^{\rho+1}} + \\ &\frac{8c^{\frac{7}{2}}[\cosh^2(\sqrt{c}\kappa)-2]\,\mathrm{sech}^3(\sqrt{c}\kappa)}{\varpi^{2\rho+1}} + \cdots. \end{aligned} \tag{34}$$

On taking inverse LT, we have

$$\begin{aligned} u(\kappa,\Im) &= \Bigg(c-2c\,\mathrm{sech}^2(\sqrt{c}\kappa) + 8c^{\frac{5}{2}}\sinh(\sqrt{c}\kappa)\,\mathrm{sech}^3(\sqrt{c}\kappa)\frac{\Im^\rho}{\Gamma(\rho+1)} - \\ &16c^4[2\cosh^2(\sqrt{c}\kappa)-3]\,\mathrm{sech}^4(\sqrt{c}\kappa)\frac{\Im^{2\rho}}{\Gamma(2\rho+1)} + \cdots\Bigg), \\ v(\kappa,\Im) &= \Bigg(2\sqrt{c}\,\mathrm{sech}(\sqrt{c}\kappa) - 4c^2\sinh(\sqrt{c}\kappa)\,\mathrm{sech}^2(\sqrt{c}\kappa)\frac{\Im^\rho}{\Gamma(\rho+1)} + \\ &8c^{\frac{7}{2}}[\cosh^2(\sqrt{c}\kappa)-2]\,\mathrm{sech}^3(\sqrt{c}\kappa)\frac{\Im^{2\rho}}{\Gamma(2\rho+1)} + \cdots\Bigg). \end{aligned}$$

we obtain exact solutions by taking $\rho = 1$ as

$$\begin{aligned} u(\kappa,\Im) &= c-2c\,\mathrm{sech}^2(\sqrt{c}(\kappa+2c\Im)), \\ v(\kappa,\Im) &= 2\sqrt{c}\,\mathrm{sech}(\sqrt{c}(\kappa+2c\Im)). \end{aligned} \tag{35}$$

5. Results and Discussion

The solutions to $u(\kappa,\Im)$ using the exact and suggested approach are calculated in Figure 1 with $\rho = 1$. The graphical representations of $u(\kappa,\Im)$ for $\rho = 0.8$ and 0.6 are shown in Figure 2. Figure 3 illustrates the 2D and 3D behavior of $u(\kappa,\Im)$ for various fractional orders. The solutions to $v(\kappa,\Im)$ using the actual and suggested approach are calculated in Figure 4 with $\rho = 1$. The graphical representations of $v(\kappa,\Im)$ for $\rho = 0.8$ and 0.6 are shown in Figure 5. Figure 6 illustrates the 2D and 3D behavior of $v(\kappa,\Im)$ for various fractional orders. The graphical representation of System 1 demonstrates great agreement between our solutions and the correct answer. Similar to Figure 6, Figure 7 presents the actual and suggested methods solutions for $u(\kappa,\Im)$ for $\rho = 1$. Figure 8 exhibits the graphical representations of $u(\kappa,\Im)$ for $\rho = 0.8$ and 0.6, respectively, while Figure 9 depicts the 2D and 3D behaviors of $u(\kappa,\Im)$ for various fractional orders. Additionally, Figure 10 shows both the actual and proposed technique's solutions for $v(\kappa,\Im)$ for $\rho = 1$. Figure 11 presents the graphical representations of $v(\kappa,\Im)$ for $\rho = 0.8$ and 0.6, respectively, while Figure 12 depicts the 2D and 3D behaviors of $v(\kappa,\Im)$ for various fractional orders. In a similar manner, System 2's graphical appearance demonstrates the fact that our solutions are in close agreement with the accurate solutions. Each and every figure was generated at $c = 0.1$ and $\Im \in [0, 0.1]$ within the range $-3 \leq \kappa \geq 3$. Additionally, Tables 1 and 2 display the approximate solution to System 1 for various values of κ and $\Im$, while the approximate solution of System 2 for various values of κ and $\Im$ is shown in Tables 3 and 4, respectively. The figures and tables show that the exact solution and our approach solution are quite similar to one another and have a greater level of precision. Further, from the figures and tables, it is observed that the proposed method solution gets closer to the exact solution as the value of ρ tends from the fractional order toward the integer order. Moreover, from the comparison, it is clear that the LRPS technique solution and the exact solutions are very close. Thus, the LRPS technique is a dependable new study that requires less computation, is adaptable, and is simple to use.

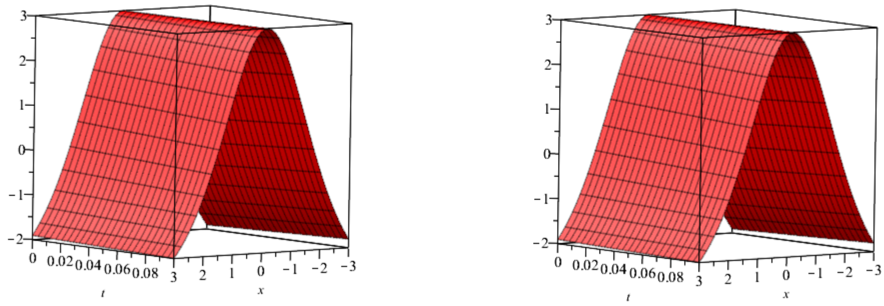

Figure 1. The accurate and suggested approach solution for $u(\kappa, \Im)$ for System 1 at $\rho = 1$.

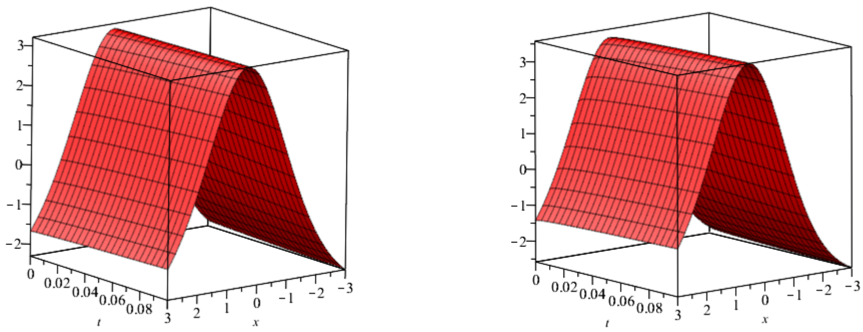

Figure 2. The suggested approach solution for System 1 at $\rho = 0.8, 0.6$ of $u(\kappa, \Im)$.

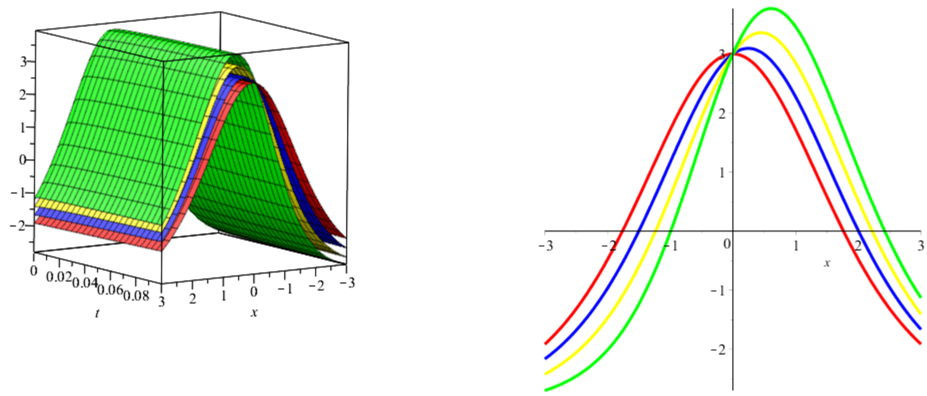

Figure 3. The solution of the suggested approach for $u(\kappa, \Im)$ of System 1 at various ρ values.

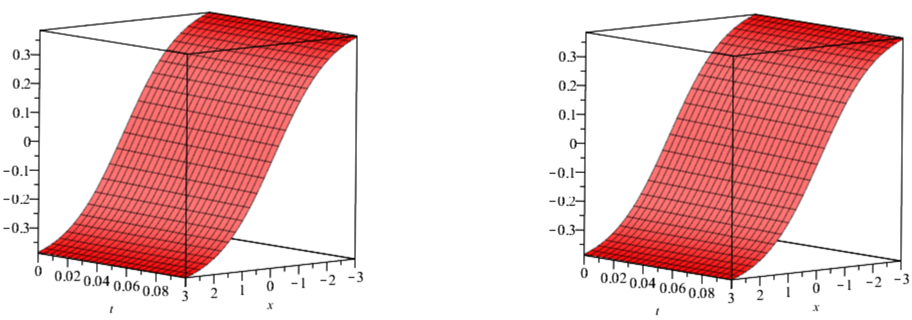

Figure 4. The accurate and suggested approach solution for $v(\kappa, \Im)$ for System 1 at $\rho = 1$.

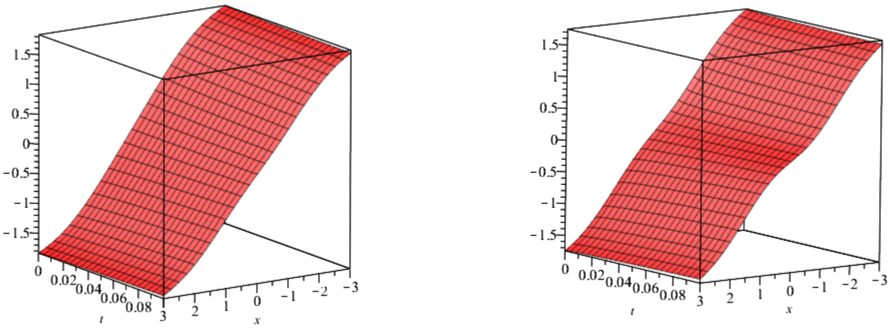

Figure 5. The suggested approach solution for System 1 at $\rho = 0.8, 0.6$ of $v(\kappa, \Im)$.

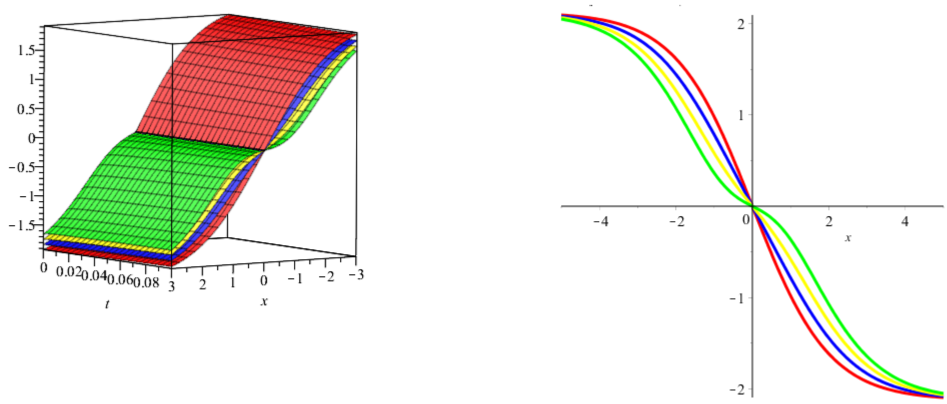

Figure 6. The solution of the suggested approach for $v(\kappa, \Im)$ of System 1 at various ρ values.

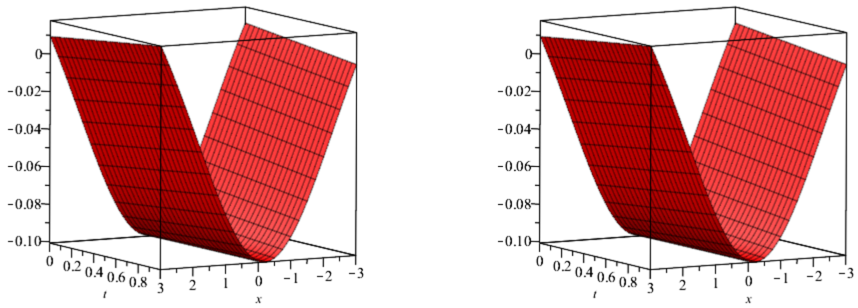

Figure 7. The accurate and suggested approach solution for $u(\kappa, \Im)$ for System 2 at $\rho = 1$.

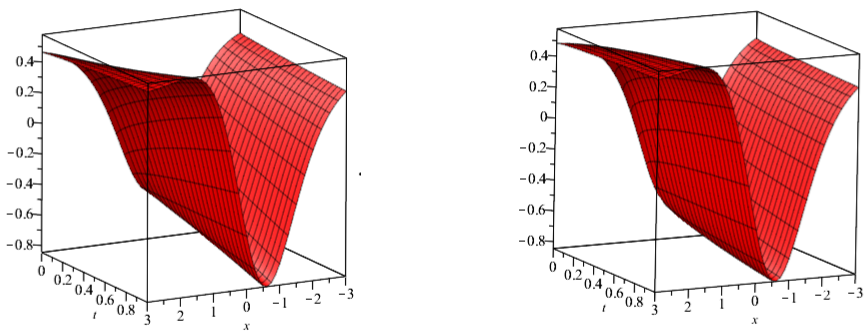

Figure 8. The suggested approach solution for System 2 at $\rho = 0.8, 0.6$ of $u(\kappa, \Im)$.

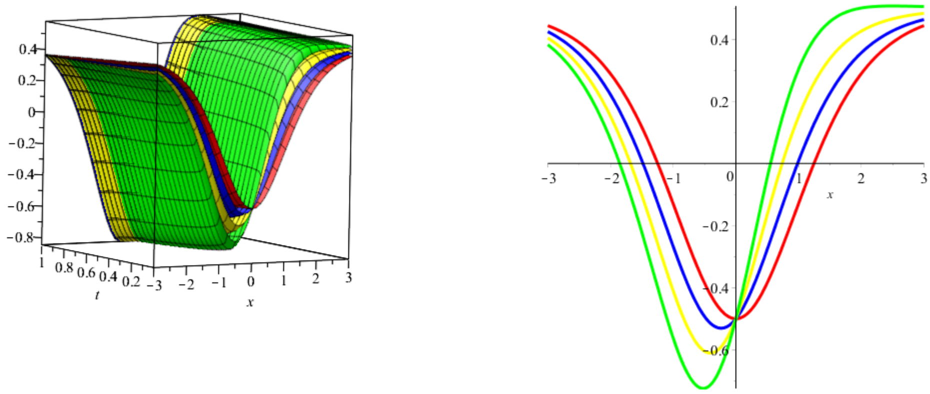

Figure 9. The solution of the suggested approach for $u(\kappa, \Im)$ of System 2 at various ρ values.

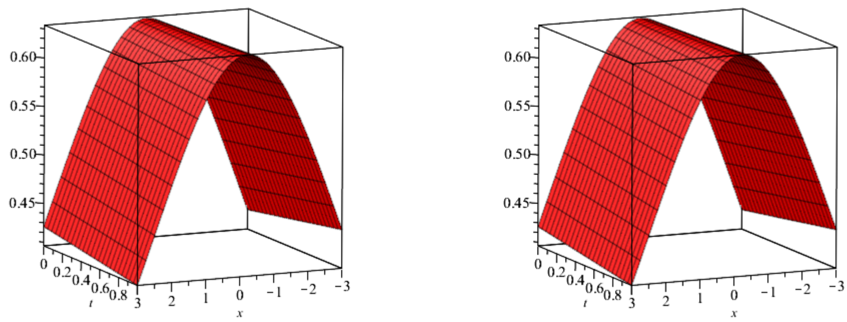

Figure 10. The accurate and suggested approach solution for $v(\kappa, \Im)$ for System 2 at $\rho = 1$.

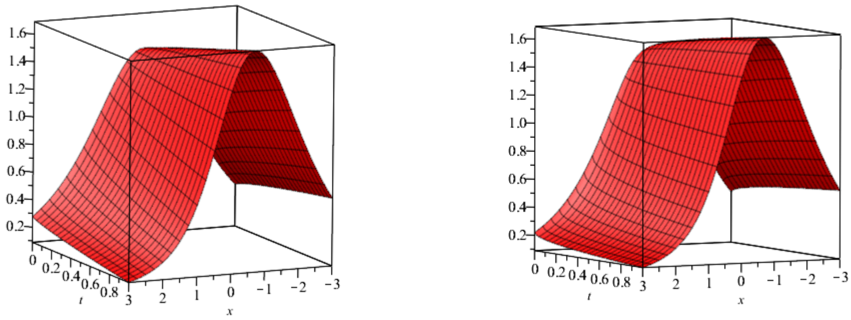

Figure 11. The suggested approach solution for System 2 at $\rho = 0.8, 0.6$ of $v(\kappa, \Im)$.

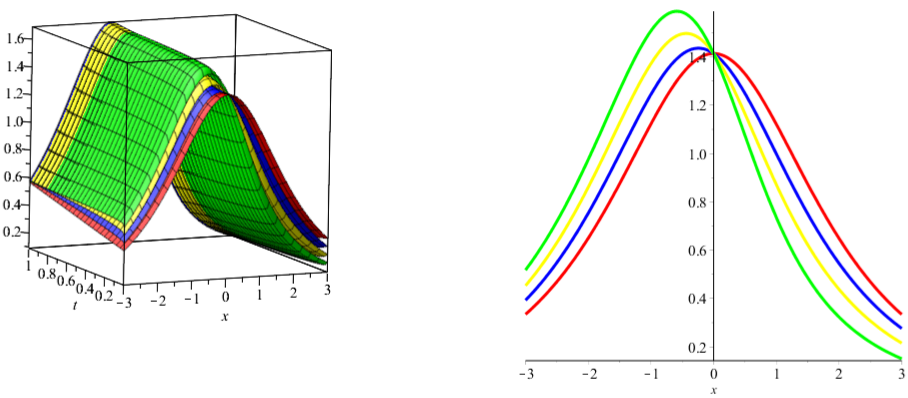

Figure 12. The solution of the suggested approach for $v(\kappa, \Im)$ of System 2 at various ρ values.

Table 1. Solution to $u(\kappa, \Im)$ at various fractional orders for System 1.

$(\kappa, \Im)$	$u(\kappa, \Im)$ at $\rho = 0.5$	$u(\kappa, \Im)$ at $\rho = 0.75$	$(LRPSM)$ at $\rho = 1$	Exact Result
(0.2, 0.01)	2.9522951	2.9463031	2.9403977	2.9403977
(0.4, 0.01)	2.7892015	2.7772263	2.7662578	2.7662578
(0.6, 0.01)	2.5227923	2.5068021	2.4908217	2.4908217
(0.2, 0.02)	2.9523242	2.9463522	2.9403977	2.9403977
(0.4, 0.02)	2.7892041	2.7762190	2.7662578	2.7662578
(0.6, 0.02)	2.5227863	2.5068015	2.4908217	2.4908217
(0.2, 0.03)	2.9523245	2.9463621	2.9403977	2.9403977
(0.4, 0.03)	2.7891987	2.7762124	2.7662578	2.7662578
(0.6, 0.03)	2.5227921	2.5068105	2.4908217	2.4908217
(0.2, 0.04)	2.9523183	2.9463561	2.9403977	2.9403977
(0.4, 0.04)	2.7892124	2.7762321	2.7662578	2.7662578
(0.6, 0.04)	2.5227833	2.5068103	2.4908217	2.4908217
(0.2, 0.05)	2.9523133	2.9463562	2.9403977	2.9403977
(0.4, 0.05)	2.7892189	2.7762488	2.7662578	2.7662578
(0.6, 0.05)	2.5228021	2.5068111	2.4908216	2.4908216

Table 2. Solution to $v(\kappa, \Im)$ at various fractional orders for System 1.

$(\kappa, \Im)$	$v(\kappa, \Im)$ at $\rho = 0.5$	$v(\kappa, \Im)$ at $\rho = 0.75$	$(LRPSM)$ at $\rho = 1$	Exact Result
(0.2, 0.01)	−0.0004074	−0.0004127	−0.0004228	−0.0004228
(0.4, 0.01)	−0.0008051	−0.0008202	−0.0008373	−0.0008373
(0.6, 0.01)	−0.0011923	−0.0012126	−0.0012359	−0.0012359
(0.2, 0.02)	−0.0004024	−0.0004087	−0.0004228	−0.0004228
(0.4, 0.02)	−0.0008051	−0.0008212	−0.0008373	−0.0008373
(0.6, 0.02)	−0.0012003	−0.0012217	−0.0012359	−0.0012359
(0.2, 0.03)	−0.0004034	−0.0004126	−0.0004228	−0.0004228
(0.4, 0.03)	−0.0008111	−0.0008202	−0.0008373	−0.0008373
(0.6, 0.03)	−0.0012004	−0.0012127	−0.0012359	−0.0012359
(0.2, 0.04)	−0.0004024	−0.0004106	−0.0004228	−0.0004228
(0.4, 0.04)	−0.0008061	−0.0008203	−0.0008374	−0.0008374
(0.6, 0.04)	−0.0012005	−0.0012127	−0.0012359	−0.0012359
(0.2, 0.05)	−0.0004033	−0.0004126	−0.0004228	−0.0004228
(0.4, 0.05)	−0.0008072	−0.0008213	−0.0008374	−0.0008374
(0.6, 0.05)	−0.0012016	−0.0012137	−0.0012359	−0.0012359

Table 3. Solution to $u(\kappa, \Im)$ at various fractional orders for System 2.

$(\kappa, \Im)$	$u(\kappa, \Im)$ at $\rho = 0.5$	$u(\kappa, \Im)$ at $\rho = 0.75$	$(LRPSM)$ at $\rho = 1$	Exact Result
(0.2, 0.01)	−0.0991522	−0.0991843	−0.0992005	−0.0992005
(0.4, 0.02)	−0.0967362	−0.0967994	−0.0968306	−0.0968306
(0.6, 0.03)	−0.0928263	−0.0929185	−0.0929647	−0.0929647
(0.2, 0.01)	−0.0991503	−0.0991826	−0.0991989	−0.0991989
(0.4, 0.02)	−0.0967322	−0.0967953	−0.0968275	−0.0968275
(0.6, 0.03)	−0.0928212	−0.0929134	−0.0929601	−0.0929601
(0.2, 0.01)	−0.0991480	−0.0991812	−0.0991973	−0.0991973
(0.4, 0.02)	−0.0967281	−0.0967922	−0.0968244	−0.0968244
(0.6, 0.03)	−0.0928152	−0.0929094	−0.0929556	−0.0929556
(0.2, 0.01)	−0.0991473	−0.0991795	−0.0991957	−0.0991957
(0.4, 0.02)	−0.0967250	−0.0967891	−0.0968212	−0.0968212
(0.6, 0.03)	−0.0928102	−0.0929044	−0.0929510	−0.0929510
(0.2, 0.01)	−0.0991454	−0.0991776	−0.0991941	−0.0991941
(0.4, 0.02)	−0.0967214	−0.0967866	−0.0968181	−0.0968181
(0.6, 0.03)	−0.0928051	−0.0928992	−0.0929464	−0.0929464

Table 4. Solution to $v(\kappa, \Im)$ at various fractional orders for System 2.

$(\kappa, \Im)$	$v(\kappa, \Im)$ at $\rho = 0.5$	$v(\kappa, \Im)$ at $\rho = 0.75$	$(LRPSM)$ at $\rho = 1$	Exact Result
(0.2, 0.01)	0.6311140	0.6311641	0.6311902	0.6311902
(0.4, 0.02)	0.6272731	0.6273743	0.6274244	0.6274244
(0.6, 0.03)	0.6210090	0.6211581	0.6212322	0.6212322
(0.2, 0.01)	0.6311112	0.6311624	0.6311876	0.6311876
(0.4, 0.02)	0.6272672	0.6273683	0.6274194	0.6274194
(0.6, 0.03)	0.6210013	0.6211506	0.6212248	0.6212248
(0.2, 0.01)	0.6311083	0.6311596	0.6311851	0.6311851
(0.4, 0.02)	0.6272620	0.6273632	0.6274144	0.6274144
(0.6, 0.03)	0.6209921	0.6211423	0.6212175	0.6212175
(0.2, 0.01)	0.6311052	0.6311564	0.6311826	0.6311826
(0.4, 0.02)	0.6272561	0.6273583	0.6274094	0.6274094
(0.6, 0.03)	0.6209844	0.6211358	0.6212101	0.6212101
(0.2, 0.01)	0.6311023	0.6311457	0.6311800	0.6311800
(0.4, 0.02)	0.6272500	0.6273532	0.6274044	0.6274044
(0.6, 0.03)	0.6209764	0.6211275	0.6212027	0.6212027

6. Conclusions

The fractional Kersten–Krasil'shchik-linked KdV-mKdV analytical approximate solution, together with the necessary initial data, was successfully developed in this work using the LRPS technique. The major goal of the suggested method is to use the limit concept to find the unknown LFSE coefficients for the new equation in Laplace space. Without perturbation, discretization, or physical hypotheses, the analytical approximations for the solved fractional Kersten–Krasil'shchik-coupled KdV-mKdV systems' starting value equations are obtained in rapidly convergent MFPS formulas. Two illustrated examples were used to study the LRPS technique's performance and reliability. As a result, the LRPS technique is a straightforward, simple, and useful tool for treating a variety of nonlinear time-fractional PDEs that occur in engineering and science problems.

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