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Article

Intuitionistic Fuzzy Stability of an Euler–Lagrange Symmetry Additive Functional Equation via Direct and Fixed Point Technique (FPT)

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Abstract: In this article, a new class of real-valued Euler–Lagrange symmetry additive functional equations is introduced. The solution of the equation is provided, assuming the unknown function to be continuous and without any regularity conditions. The objective of this research is to derive the Hyers–Ulam–Rassias stability (HURS) in intuitionistic fuzzy normed spaces (IFNS) by applying the classical direct method and fixed point techniques (FPT). Furthermore, it is proven that the Euler–Lagrange symmetry additive functional equation and the control function, which is the IFNS of the sums and products of powers of norms, is stable. In addition, a few examples where the solution of this equation can be applied in Fourier series and Fourier transforms are demonstrated.

Keywords: Euler–Lagrange symmetry additive functional equations; generalised Hyers–Ulam–Rassias; stability intuitionistic fuzzy normed spaces; fixed point technique

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1. Introduction

The study of functional equations is one of the most important aspects of modern research and it is becoming an increasingly popular topic among academics all over the world. As functional equations may be used in a diverse array of contexts, an increasing number of empirical researchers and mathematicians are directing their attentions on studying them. The study of functional equations in a variety of domains, including differential equations, differential geometry, queueing theory, probability theory, abstract algebra, and number theory has led to an increase in the significance of functional equations [1–3].

The stability of equations is essential because it gives a helpful approach to estimating the error that is introduced when exact solutions are substituted for functions that fulfil some equations approximately. Modern mathematicians state that an equation is stable within a specific type of function if every function in that category that significantly fulfils the equation is close to the optimal solution of the equation. Mathematicians have investigated quite a number of stability problems with diverse functional equations (radical, reciprocal, logarithmic, and algebraic) in recent times [4–8].

Ulam [9] posed a significant problem regarding the stability of group homomorphisms in 1940. Hyers [10] found a solution to the problem for the Cauchy additive functional equation in the subsequent year. Rassias [11] generalised Hyers' result after more than two decades, and Gavruta [12] then expanded Rassias' result by introducing unbounded control functions. Today, the term "generalised HURS" of functional equations is commonly used to refer to the stability notion developed by Rassias and Gavruta. The Ulam stability analysis can be studied using a number of different approaches, but one of the most well-known is the FPT, which invokes a central result from fixed point theory [13–17].

The symmetry properties of functions used to define an equation or an inequality can be studied in order to determine the solutions with particular properties. As far as inequalities are concerned, the study of special functions such as hypergeometric functions and special polynomials based on their symmetry properties may provide some interesting outcomes. The symmetry properties for different types of operators associated with the concept of quantum functional calculus may also be investigated.

The functional equation

$$\mathfrak{Y}(\theta_1+\theta_2)+\mathfrak{Y}(\theta_1-\theta_2)=2\mathfrak{Y}(\theta_1)+2\mathfrak{Y}(\theta_2) \tag{1}$$

is related to a symmetric bi-additive mapping [18] . It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic Equation (1) is said to be a quadratic mapping. It is well-known that a mapping $\mathfrak{Y}$ between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive mapping B such that $\mathfrak{Y}(\theta)=B(\theta_1,\theta_2)$ for all $\mathfrak{Y}$. The bi-additive mapping B is given by

$$B(\theta_1,\theta_2)=\frac{1}{4}[\mathfrak{Y}(\theta_1+\theta_2)-\mathfrak{Y}(\theta_1-\theta_2)].$$

Rassias introduced a variety of Euler–Lagrange functional equations [19–23] and established the HURS.

The HURS of the Cauchy functional equation (CFE), defined as

$$\mathfrak{Y}(\theta_1+\theta_2)=\mathfrak{Y}(\theta_1)+\mathfrak{Y}(\theta_2)$$

in random normed spaces was studied by Mihet et al. [24].

Kim et al. [25] proposed a generalised version of CFE, as follows:

$$\mathfrak{Y}\left(\frac{\theta_1-\theta_2}{n}+\theta_3\right)+\mathfrak{Y}\left(\frac{\theta_2-\theta_3}{n}+\theta_1\right)+\mathfrak{Y}\left(\frac{\theta_3-\theta_1}{n}+\theta_2\right)=\mathfrak{Y}(\theta_1+\theta_2+\theta_3).$$

In addition, Kim et al. determined the Ulam stability in fuzzy normed spaces for any non-zero fixed integer n. Firstly, it is self-evident that a function $\mathfrak{Y}$ satisfies the given equation if and only if it is additive. Consequently, the equation is known as the CFE. Baktash et al. [26] addressed the stability of cubic and quartic mappings in random normed spaces, and Ghaffari et al. [27] demonstrated the stability of cubic mappings in fuzzy normed spaces.

In the year 2020, Saha et al. [28] applied fixed point techniques to stability problems in intuitionistic fuzzy Banach spaces. In the same year, Alanazi et al. [29] proved the fuzzy stability results of a finite variable additive functional equation using direct and fixed point methods. Madadi et al. [30] published ground-breaking work on the stability of unbounded differential equations in Menger k-normed spaces using a fixed point technique. Liu et al. [31] presented a fixed-point approach to the Hyers–Ulam stability (HUS) of Caputo–Fabrizio fractional differential equations.

In the year 2021, Badora et al. [32] studied the applications of the Banach limit in Ulam stability. Subsequently, Alzabut et al. [33] derived the existence, uniqueness, and stability analysis of the discrete fractional three-point boundary value problem for the elastic beam equation. Bahyrycz et al. [34] presented a survey on the Ulam stability of functional equations in two-normed spaces. Tamilvanan et al. [35] investigated the Ulam stabilities and instabilities of the Euler–Lagrange–Rassias quadratic functional equation in non-Archimedean IFNS.

In 2022, Govindan et al. [36] derived the stability of an additive functional equation originating from the characteristic polynomial of degree three. Lupas [37] presented a summary of symmetry in functional equations and analytic inequalities. El-Hady et al. [38] studied the stability of the equation of q-wright affine functions in non-Archimedean (n,β)-Banach Spaces. Uthirasamy et al. [39] derived the Ulam stability and nonstability of additive functional equations in IFNS and two-Banach spaces using different methods.

Quite recently, Agilan et al. [40] investigated the generalised HUS of an additive functional equation

$$\mathfrak{Y}\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right)+\mathfrak{Y}\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right)+\eta^{\ell+\wp}\mathfrak{Y}(\gamma-\kappa)+\eta^{\hbar}\mathfrak{Y}(\kappa-\mu)$$
$$=2\left(\eta^{\ell+\wp}\mathfrak{Y}(\gamma)+\eta^{\hbar}\mathfrak{Y}(\kappa)\right)$$

with $\eta^{\ell+\wp},\eta^{\hbar}\neq 0$ in Banach spaces and quasi β-normed spaces using direct and FPT. The counter example for non-stable cases is also demonstrated. A few more results on applying the fixed point techniques on fuzzy and dominated mappings are presented in [41–47].

Extending the above research, a new class of real-valued Euler–Lagrange symmetry additive functional equations is introduced in this article, and the generalised HURS of various general control functions of this equation are established. The unique approach of analysis of the HURS for this class of equations using two different techniques has not been performed before. Hence, the results presented in the upcoming sections are novel and significant in the study of functional equations.

The Euler–Lagrange symmetry additive functional equation introduced in this article is defined as follows:

$$\begin{aligned}(\varsigma+\wp+\hbar)\mathfrak{Z}(\varsigma\mathfrak{p}+\wp\mathfrak{q}+\hbar\mathfrak{r})+\varsigma\mathfrak{Z}(\wp\hbar(\mathfrak{p}-\mathfrak{q}))\\+\wp\mathfrak{Z}(\varsigma\hbar(\mathfrak{q}-\mathfrak{r}))+\hbar\mathfrak{Z}(\varsigma\wp(\mathfrak{r}-\mathfrak{p}))\\=(\varsigma+\wp+\hbar)(\varsigma\mathfrak{Z}(p)+\wp\mathfrak{Z}(\mathfrak{q})+\hbar\mathfrak{Z}(r))\end{aligned} \quad (2)$$

where $\varsigma,\wp,\hbar\in\mathbb{R}$, with $\varsigma,\wp,\hbar\neq 0$ and $\varsigma+\wp+\hbar\neq 0$ in IFNS using direct and FPT.

In the following section, the general solution of the above equation is derived and discussed. In Section 3, some basics concepts and notations related to IFNS are explained briefly. In Sections 4 and 5, the stability results are derived using the direct and fixed point techniques, respectively. In Section 6, the applications of the proposed equation are analysed using Fourier series and Fourier transform over various intervals.

2. General Solution

In this section, the solution of the Euler–Lagrange symmetry additive functional equation is discussed.

Lemma 1. *Let $\mathbb{A},\mathbb{B}$ be non-empty sets, and let the mapping $\mathfrak{Z}:\mathbb{A}\to\mathbb{B}$ satisfy the additive Cauchy equation with $\mathfrak{Z}(0)=0$*

$$\mathfrak{Z}(\theta_1+\theta_2)=\mathfrak{Z}(\theta_1)+\mathfrak{Z}(\theta_2). \quad (3)$$

If $\theta_1,\theta_2\in\mathbb{A}$ and if $\mathfrak{Z}:\mathbb{A}\to\mathbb{B}$ satisfies the functional equation, then

$$\begin{aligned}(\varsigma+\wp+\hbar)\mathfrak{Z}(\varsigma\mathfrak{p}+\wp\mathfrak{q}+\hbar\mathfrak{r})+\varsigma\mathfrak{Z}(\wp\hbar(\mathfrak{p}-\mathfrak{q}))\\+\wp\mathfrak{Z}(\varsigma\hbar(\mathfrak{q}-\mathfrak{r}))+\hbar\mathfrak{Z}(\varsigma\wp(\mathfrak{r}-\mathfrak{p}))\\=(\varsigma+\wp+\hbar)(\varsigma\mathfrak{Z}(\mathfrak{p})+\wp\mathfrak{Z}(\mathfrak{q})+\hbar\mathfrak{Z}(\mathfrak{r}))\end{aligned} \quad (4)$$

for all $\mathfrak{p},\mathfrak{q},\mathfrak{r}\in\mathbb{A}$.

Proof. Let $\mathbb{A},\mathbb{B}$ be non-empty sets and the mapping $\mathfrak{Z}:\mathbb{A}\to\mathbb{B}$. From (3), the following conditions are derived:

(1) $\theta_1=\theta_2=0$ in (3) $\Longrightarrow\mathfrak{Z}(0)=0$;
(2) $\theta_1=-\theta_2$ in (3) $\Longrightarrow\mathfrak{Z}(-\theta_2)=-\mathfrak{Z}(\theta_2)$ then $\mathfrak{Z}$ is odd;
(3) $\theta_2=\theta_1,2\theta_1$ in (3) $\Longrightarrow\mathfrak{Z}(2\theta_1)=2\mathfrak{Z}(\theta_1),\mathfrak{Z}(3\theta_1)=3\mathfrak{Z}(\theta_1)$;
(4) In general, $\mathfrak{Z}(a\theta_1)=a\mathfrak{Z}(\theta_1)$, for any a.

From (3)

$$\mathfrak{Z}(\theta_1 + \theta_2 + \theta_3) = \mathfrak{Z}(\theta_1) + \mathfrak{Z}(\theta_2) + \mathfrak{Z}(\theta_3). \tag{5}$$

Replacing $(\theta_1, \theta_2, \theta_3)$ by $(\varsigma\mathfrak{p}, \wp\mathfrak{q}, \hbar\mathfrak{r})$ in (3) and using condition (4),

$$\mathfrak{Z}(\varsigma\mathfrak{p} + \wp\mathfrak{q} + \hbar\mathfrak{r}) = \varsigma\mathfrak{Z}(\mathfrak{p}) + \wp\mathfrak{Z}(\mathfrak{q}) + \hbar\mathfrak{Z}(\mathfrak{r}). \tag{6}$$

Multiplying (6) by $(\varsigma + \wp + \hbar)$,

$$(\varsigma + \wp + \hbar)\mathfrak{Z}(\varsigma\mathfrak{p} + \wp\mathfrak{q} + \hbar\mathfrak{r}) = (\varsigma + \wp + \hbar)[\varsigma\mathfrak{Z}(p) + \wp\mathfrak{Z}(\mathfrak{q}) + \hbar\mathfrak{Z}(\mathfrak{r})]. \tag{7}$$

Replacing (θ_1, θ_2) by $(\wp\hbar\mathfrak{p}, -\wp\hbar\mathfrak{q})$ in (3) and by (4)

$$\mathfrak{Z}(\wp\hbar(\mathfrak{p} - \mathfrak{q})) = \wp\hbar\mathfrak{Z}(\mathfrak{p}) + \wp\hbar\mathfrak{Z}(-\mathfrak{q}). \tag{8}$$

Multiplying by ς in (8), and by (2), the following result is obtained.

$$\varsigma\mathfrak{Z}(\wp\hbar(\mathfrak{p} - \mathfrak{q})) = \varsigma\wp\hbar\mathfrak{Z}(\mathfrak{p}) - \varsigma\wp\hbar\mathfrak{Z}(\mathfrak{q}). \tag{9}$$

Replacing (θ_1, θ_2) by $(\varsigma\hbar\mathfrak{q}, -\varsigma\hbar\mathfrak{r})$ in (3), and by (4),

$$\mathfrak{Z}(\varsigma\hbar(\mathfrak{q} - \mathfrak{r})) = \varsigma\hbar\mathfrak{Z}(\mathfrak{q}) + \varsigma\hbar\mathfrak{Z}(-\mathfrak{r}). \tag{10}$$

Multiplying by $\wp$ in (10) and applying condition (2),

$$\wp\mathfrak{Z}(\varsigma\hbar(\mathfrak{q} - \mathfrak{r})) = \wp\varsigma\hbar\mathfrak{Z}(\mathfrak{q}) - \wp\varsigma\hbar\mathfrak{Z}(\mathfrak{r}). \tag{11}$$

Replacing (θ_1, θ_2) by $(\varsigma\wp\mathfrak{r} - \varsigma\wp\mathfrak{p})$ in (3), and using condition (4),

$$\mathfrak{Z}(\varsigma\wp(\mathfrak{r} - \mathfrak{p})) = \varsigma\wp\mathfrak{Z}(\mathfrak{r}) + \varsigma\wp\mathfrak{Z}(-\mathfrak{p}). \tag{12}$$

Multiplying by $\hbar$ on both sides of (10) and by condition (2), the following result is obtained.

$$\hbar\mathfrak{Z}(\varsigma\wp(\mathfrak{r} - \mathfrak{p})) = \hbar\varsigma\wp\mathfrak{Z}(\mathfrak{r}) - \hbar\varsigma\wp\mathfrak{Z}(p). \tag{13}$$

Finally, adding these derived Equations (7), (9), (11), and (13), Equation (4) can be obtained. □

3. Fundamentals of Intuitionistic Fuzzy Normed Spaces

The basic definitions and notations in the context of IFNS are provided in [48–56].

Definition 1. *The five-tuple $(\mathbb{A}, \mu_a, \nu_a, *, \diamond)$ is said to be an IFNS if $\mathbb{A}$ is a vector space, $*$ is a continuous κ-norm, $\diamond$ is a continuous $\kappa-$conorm, and μ, ν are fuzzy sets on $\mathbb{A} \times (0, \infty)$ satisfying the following conditions. For every $\mathfrak{p}, \mathfrak{q} \in \mathbb{A}$ and $s, \kappa > 0$,*
(A1) $\mu_a(\mathfrak{p}, \kappa) + \nu_a(\mathfrak{p}, \kappa) \leq 1$,
(A2) $\mu_a(\mathfrak{p}, \kappa) > 0$,
(A3) $\mu_a(\mathfrak{p}, \kappa) = 1$, *if and only if* $\mathfrak{p} = 0$,
(A4) $\mu_a(\alpha\mathfrak{p}, \kappa) = \mu_a\left(\mathfrak{p}, \frac{\kappa}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,
(A5) $\mu_a(\mathfrak{p}, \kappa) * \mu_a(\mathfrak{q}, \mathfrak{s}) \leq \mu_a(\mathfrak{p} + \mathfrak{q}, \kappa + \mathfrak{s})$,
(A6) $\mu_a(\mathfrak{p}, \cdot) : (0, \infty) \to [0, 1]$ *is continuous,*
(A7) $\lim_{\kappa \to \infty} \mu_a(\mathfrak{p}, \kappa) = 1$ *and* $\lim_{\kappa \to 0} \mu_a(p, \kappa) = 0$,
(A8) $\nu_a(\mathfrak{p}, \kappa) < 1$,
(A9) $\nu_a(\mathfrak{p}, \kappa) = 0$, *if and only if* $\mathfrak{p} = 0$,
(A10) $\nu_a(\alpha\mathfrak{p}, \kappa) = \nu\left(\mathfrak{p}, \frac{\kappa}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,
(A11)$\nu_a(\mathfrak{p}, \kappa) \diamond \nu_a(\mathfrak{q}, s) \geq \nu_a(\mathfrak{p} + \mathfrak{q}, \kappa + \mathfrak{s})$,

(A12) $\nu_a(\mathfrak{p}, \cdot) : (0, \infty) \to [0, 1]$ *is continuous,*
(A13) $\lim_{\kappa \to \infty} \nu_a(\mathfrak{p}, \kappa) = 0$ *and* $\lim_{\kappa \to 0} \nu_a(\mathfrak{p}, \kappa) = 1$.

Example 1. *Let* $(\mathbb{A}, \|\cdot\|)$ *be a normed space. Let* $a * b = ab$ *and* $a \diamond b = \min\{a + b, 1\}$ *for all* $a, b \in [0, 1]$. *For all* $\mathfrak{p} \in \mathbb{A}$ *and every* $\kappa > 0$, *consider*

$$\mu(\mathfrak{p}, \kappa) = \begin{cases} \frac{\kappa}{\kappa + \|\mathfrak{p}\|} & if \quad \kappa > 0; \\ 0 & if \quad \kappa \le 0; \end{cases} \qquad and$$

$$\nu(\mathfrak{p}, \kappa) = \begin{cases} \frac{\|\mathfrak{p}\|}{\kappa + \|\mathfrak{p}\|} & if \quad \kappa > 0; \\ 0 & if \quad \kappa \le 0. \end{cases}$$

Then, $(\mathbb{A}, \mu_a, \nu_a, *, \diamond)$ *is an IFNS.*

4. Stability Results: Direct Method

Let $\mathbb{A}$ be a linear space and $(\mathbb{B}, \mu'_a, \nu'_a)$ be an IFNS. Then,

$$\begin{aligned} H\mathfrak{Z}^{\mathfrak{pqr}}_{\varsigma\wp\hbar}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) &= (\varsigma + \wp + \hbar)\mathfrak{Z}(\varsigma\mathfrak{p} + \wp\mathfrak{q} + \hbar\mathfrak{r}) + \varsigma\mathfrak{Z}(\wp\hbar(\mathfrak{p} - \mathfrak{q})) \\ &\quad + \wp\mathfrak{Z}(\varsigma\hbar(\mathfrak{q} - \mathfrak{r})) + \hbar\mathfrak{Z}(\varsigma\wp(\mathfrak{r} - \mathfrak{p})) \\ &\quad - (\varsigma + \wp + \hbar)(\varsigma\mathfrak{Z}(\mathfrak{p}) + \wp\mathfrak{Z}(\mathfrak{q}) + \hbar\mathfrak{Z}(\mathfrak{r})) \end{aligned}$$

where $\varsigma, \wp, \hbar \in \mathbb{R}$ with $\varsigma, \wp, \hbar \neq 0$ for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \mathbb{A}$.

Theorem 1. *Let* $\mathbb{A}, \mathbb{B}$ *be non-empty sets, and* $\Psi : \mathbb{A} \times \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{B}$ *such that* $0 < \left(\frac{\mathfrak{p}}{\varsigma + \wp + \hbar}\right)^{\eta} < 1$. *Then,*

$$\left.\begin{aligned} \mu'_a\big(\Psi\big((\varsigma + \wp + \hbar)^{n\eta}\mathfrak{p}, (\varsigma + \wp + \hbar)^{n\eta}\mathfrak{q}, (\varsigma + \wp + \hbar)^{n\eta}\mathfrak{r}\big), \kappa\big) \ge \mu'_a(\mathfrak{p}^{n\eta}\Psi(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa) \\ \nu'_a\big(\Psi\big((\varsigma + \wp + \hbar)^{n\eta}\mathfrak{p}, (\varsigma + \wp + \hbar)^{n\eta}\mathfrak{q}, (\varsigma + \wp + \hbar)^{n\eta}\mathfrak{r}\big), \kappa\big) \le \nu'_a(\mathfrak{p}^{n\eta}\Psi(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa) \end{aligned}\right\} \tag{14}$$

and

$$\left.\begin{aligned} \lim_{n \to \infty} \mu'_a\big(\Psi\big((\varsigma + \wp + \hbar)^{\eta n}\mathfrak{p}, (\varsigma + \wp + \hbar)^{\eta n}\mathfrak{q}, (\varsigma + \wp + \hbar)^{\eta n}\mathfrak{r}\big), a^{\eta n}\kappa\big) = 1 \\ \lim_{n \to \infty} \nu'_a\big(\Psi\big((\varsigma + \wp + \hbar)^{\eta n}\mathfrak{p}, (\varsigma + \wp + \hbar)^{\eta n}\mathfrak{q}, (\varsigma + \wp + \hbar)^{\eta n}\mathfrak{r}\big), a^{\eta n}\kappa\big) = 0. \end{aligned}\right\} \tag{15}$$

Let the odd function $\mathfrak{Z} : \mathbb{A} \to \mathbb{B}$ *satisfy the inequality*

$$\left.\begin{aligned} \mu_a\Big(H\mathfrak{Z}^{\mathfrak{pqr}}_{\varsigma\wp\hbar}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa\Big) \ge \mu'_a(\Psi(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa) \\ \nu_a\Big(H\mathfrak{Z}^{\mathfrak{pqr}}_{\varsigma\wp\hbar}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa\Big) \le \nu'_a(\Psi(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa) \end{aligned}\right\} \tag{16}$$

$\exists$ *unique additive mapping* $\mathfrak{L} : \mathbb{A} \longrightarrow \mathbb{B}$

$$\left.\begin{aligned} \mu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) \ge \mu'_a(\Psi(\mathfrak{p}, \mathfrak{p}, \mathfrak{p}), (\varsigma + \wp + \hbar)|(\varsigma + \wp + \hbar) - \mathfrak{p}|\kappa) \\ \nu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) \le \nu'_a(\Psi(\mathfrak{p}, \mathfrak{p}, \mathfrak{p}), (\varsigma + \wp + \hbar)|(\varsigma + \wp + \hbar) - \mathfrak{p}|\kappa) \end{aligned}\right\} \tag{17}$$

where $\eta \in \{1, -1\}$.

Proof. Replacing the variables $(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})$ by $(\mathfrak{p}, \mathfrak{p}, \mathfrak{p})$ in (16),

$$\left.\begin{array}{l} \mu_a\Big((\varsigma+\wp+\hbar)\mathfrak{Z}((\varsigma+\wp+\hbar)\mathfrak{p})-(\varsigma+\wp+\hbar)^2\mathfrak{Z}(\mathfrak{p}),\kappa\Big) \geq \mu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa) \\ \nu_a\Big((\varsigma+\wp+\hbar)\mathfrak{Z}((\varsigma+\wp+\hbar)\mathfrak{p})-(\varsigma+\wp+\hbar)^2\mathfrak{Z}(\mathfrak{p}),\kappa\Big) \leq \nu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa). \end{array}\right\} \quad (18)$$

Using intuitionistic fuzzy conditions (A4) and (A10), the following result is arrived.

$$\left.\begin{array}{l} \mu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)\mathfrak{p})}{(\varsigma+\wp+\hbar)}-\mathfrak{Z}(\mathfrak{p}),\frac{\kappa}{(\varsigma+\wp+\hbar)^2}\Big) \geq \mu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa) \\ \nu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)\mathfrak{p})}{(\varsigma+\wp+\hbar)}-\mathfrak{Z}(\mathfrak{p}),\frac{\kappa}{(\varsigma+\wp+\hbar)^2}\Big) \leq \nu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa). \end{array}\right\} \quad (19)$$

Replacing $\mathfrak{p}$ by $(\varsigma+\wp+\hbar)^n\mathfrak{p}$ in (19),

$$\left.\begin{array}{l} \mu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+1}\mathfrak{p})}{(\varsigma+\wp+\hbar)}-\mathfrak{Z}((\varsigma+\wp+\hbar)^n\mathfrak{p}),\frac{\kappa}{(\varsigma+\wp+\hbar)^2}\Big) \\ \qquad \geq \mu'_a(\Psi((\varsigma+\wp+\hbar)^n\mathfrak{p},(\varsigma+\wp+\hbar)^n\mathfrak{p},(\varsigma+\wp+\hbar)^n\mathfrak{p}),\kappa) \\ \nu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+1}\mathfrak{p})}{(\varsigma+\wp+\hbar)}-\mathfrak{Z}((\varsigma+\wp+\hbar)^n\mathfrak{p}),\frac{\kappa}{(\varsigma+\wp+\hbar)^2}\Big) \\ \qquad \leq \nu'_a(\Psi((\varsigma+\wp+\hbar)^n\mathfrak{p},(\varsigma+\wp+\hbar)^n\mathfrak{p},(\varsigma+\wp+\hbar)^n\mathfrak{p}),\kappa). \end{array}\right\} \quad (20)$$

It is understood from (20) and the intuitionistic fuzzy conditions (A4) and (A10) that

$$\left\{\begin{array}{l} \mu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+1}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(n+1)}}-\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{n}},\frac{\kappa}{(\varsigma+\wp+\hbar)^{n+2}}\Big) \geq \mu'_a\Big(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\frac{\kappa}{\mathfrak{p}^n}\Big) \\ \nu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+1}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(n+1)}}-\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{n}},\frac{\kappa}{(\varsigma+\wp+\hbar)^{n+2}}\Big) \leq \nu'_a\Big(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\frac{\kappa}{\mathfrak{p}^n}\Big). \end{array}\right\} \quad (21)$$

Taking κ into $\mathfrak{p}^n\kappa$ in (21),

$$\left\{\begin{array}{l} \mu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+1}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(n+1)}}-\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{n}},\frac{\kappa\cdot\mathfrak{p}^n}{(\varsigma+\wp+\hbar)^{n+2}}\Big) \geq \mu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa) \\ \nu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+1}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(n+1)}}-\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{n}},\frac{\kappa\cdot p^n}{(\varsigma+\wp+\hbar)^{n+2}}\Big) \leq \nu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa), \end{array}\right\} \quad (22)$$

the following result is obtained,

$$\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{n}}-\mathfrak{Z}(\mathfrak{p})=\sum_{i=0}^{n-1}\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{i+1}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(i+1)}}-\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{i}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{i}}. \quad (23)$$

From the above two equations,

$$\left.\begin{aligned}
&\mu_a\Big(\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^n\mathfrak{p})}{(\varsigma+\wp+\hbar)^n}-\mathfrak{z}(\mathfrak{p}),\sum_{i=0}^{n-1}\frac{\mathfrak{p}^i\,\kappa}{(\varsigma+\wp+\hbar)^{i+2}}\Big)\\
&=\mu_a\Big(\sum_{i=0}^{n-1}\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^{i+1}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(i+1)}}-\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^{i}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{i}},\sum_{i=0}^{n-1}\frac{\mathfrak{p}^i\,\kappa}{(\varsigma+\wp+\hbar)^{i+2}}\Big)\\
&\nu_a\Big(\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^n\mathfrak{p})}{(\varsigma+\wp+\hbar)^n}-\mathfrak{z}(\mathfrak{p}),\sum_{i=0}^{n-1}\frac{\mathfrak{p}^i\,\kappa}{(\varsigma+\wp+\hbar)^{i+2}}\Big)\\
&=\nu_a\Big(\sum_{i=0}^{n-1}\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^{i+1}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(i+1)}}-\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^{i}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{i}},\sum_{i=0}^{n-1}\frac{\mathfrak{p}^i\,\kappa}{(\varsigma+\wp+\hbar)^{i+2}}\Big).
\end{aligned}\right\}\tag{24}$$

From the above equations,

$$\left.\begin{aligned}
&\mu_a\Big(\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^n\mathfrak{p})}{(\varsigma+\wp+\hbar)^n}-\mathfrak{z}(\mathfrak{p}),\sum_{i=0}^{n-1}\frac{\mathfrak{p}^i\kappa}{(\varsigma+\wp+\hbar)^{i+2}}\Big)\\
&\qquad\geq\prod_{i=0}^{n-1}\mu_a\Big(\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^{i+1}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(i+1)}}-\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^{i}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{i}},\frac{\mathfrak{p}^i\,\kappa\mathfrak{r}}{(\varsigma+\wp+\hbar)^{i+2}}\Big)\\
&\nu_a\Big(\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^n\mathfrak{p})}{(\varsigma+\wp+\hbar)^n}-\mathfrak{z}(\mathfrak{p}),\sum_{i=0}^{n-1}\frac{\mathfrak{p}^i\kappa}{(\varsigma+\wp+\hbar)^{i+2}}\Big)\\
&\qquad\leq\prod_{i=0}^{n-1}\nu_a\Big(\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^{i+1}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(i+1)}}-\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^{i}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{i}},\frac{\mathfrak{p}^i\,\kappa}{(\varsigma+\wp+\hbar)^{i+2}}\Big)
\end{aligned}\diamond\right\}\tag{25}$$

where

$$\prod_{i=0}^{n-1}c_j=c_1*c_2*\cdots*c_n\quad\textit{and}\quad{}_{i=0}^{n-1}d_j=d_1\diamond d_2\diamond\cdots\diamond d_n.$$

Finally,

$$\left\{\begin{aligned}
&\mu_a\Big(\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^n\mathfrak{p})}{(\varsigma+\wp+\hbar)^n}-\mathfrak{z}(\mathfrak{p}),\sum_{i=0}^{n-1}\frac{\mathfrak{p}^i\,\kappa}{(\varsigma+\wp+\hbar)^{i+2}}\Big)\geq\prod_{i=0}^{n-1}\mu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa)=\mu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa)\\
&\nu_a\Big(\frac{\mathfrak{z}((\varsigma+\wp+\hbar)^n\mathfrak{p})}{(\varsigma+\wp+\hbar)^n}-\mathfrak{z}(\mathfrak{p}),\sum_{i=0}^{n-1}\frac{\mathfrak{p}^i\,\kappa}{(\varsigma+\wp+\hbar)^{i+2}}\Big)\leq\prod_{i=0}^{n-1}\nu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa)=\nu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa).
\end{aligned}\right\}\tag{26}$$

Replacing $\mathfrak{p}$ by $(\varsigma+\wp+\hbar)^m\mathfrak{p}$ in (26) and using (15), (A4), (A10),

$$\left\{\begin{aligned}
&\mu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(n+m)}}-\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{m}},\sum_{i=0}^{n-1}\frac{\mathfrak{p}^i\,\kappa}{(\varsigma+\wp+\hbar)^{(i+m+2)}}\Big)\\
&\quad\geq\mu'_a(\Psi((\varsigma+\wp+\hbar)^m\mathfrak{p},(\varsigma+\wp+\hbar)^m\mathfrak{p},(\varsigma+\wp+\hbar)^m\mathfrak{p}),\kappa)\\
&\quad=\mu'_a\left(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\frac{\kappa}{\mathfrak{p}^m}\right)\\
&\nu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(n+m)}}-\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{m}},\sum_{i=0}^{n-1}\frac{\mathfrak{p}^i\,\kappa}{(\varsigma+\wp+\hbar)^{(i+m+2)}}\Big)\\
&\quad\leq\nu'_a(\Psi((\varsigma+\wp+\hbar)^m\mathfrak{p},(\varsigma+\wp+\hbar)^m\mathfrak{p},(\varsigma+\wp+\hbar)^m\mathfrak{p}),\kappa)\\
&\quad=\nu'_a\left(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\frac{\kappa}{\mathfrak{p}^m}\right).
\end{aligned}\right\} \tag{27}$$

Replacing κ by $\mathfrak{p}^m\kappa$ in (27),

$$\left.\begin{aligned}
&\mu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(n+m)}}-\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{m}},\sum_{i=0}^{n-1}\frac{p^{i+m}\,\kappa}{(\varsigma+\wp+\hbar)^{(i+m+2)}}\Big)\geq\mu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa)\\
&\nu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(n+m)}}-\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{m}},\sum_{i=0}^{n-1}\frac{\mathfrak{p}^{i+m}\,\kappa}{(\varsigma+\wp+\hbar)^{(i+m+2)}}\Big)\leq\nu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\kappa)
\end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned}
&\mu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(n+m)}}-\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{m}},\kappa\Big)\geq\mu'_a\left(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\frac{\kappa}{\sum_{i=m}^{n-1}\frac{\mathfrak{p}^i}{(\varsigma+\wp+\hbar)^{i+2}}}\right)\\
&\nu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n+m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{(n+m)}}-\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{m}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{m}},\kappa\Big)\leq\nu'_a\left(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\frac{\kappa}{\sum_{i=m}^{n-1}\frac{\mathfrak{p}^i}{(\varsigma+\wp+\hbar)^{i+2}}}\right).
\end{aligned}\right\} \tag{29}$$

Since $0<\mathfrak{p}<1$ and $\sum_{i=0}^{n}\left(\frac{\mathfrak{p}}{1}\right)^i<\infty$. The sequence $\left\{\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{n}}\right\}$ is Cauchy in $(\mathbb{B},\mu_a,\nu_a)$. Since $(\mathbb{B},\mu_a,\nu_a)$ is a complete IFNS, this sequence converges to some point $\mathfrak{L}(\mathfrak{p})\in\mathbb{B}$. Defining $\mathfrak{L}:\mathbb{A}\longrightarrow\mathbb{B}$ by

$$\lim_{n\to\infty}\mu_a\left(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{n}}-\mathfrak{L}(\mathfrak{p}),\kappa\right)=1,$$

$$\lim_{n\to\infty}\nu_a\left(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{n}}-\mathfrak{L}(\mathfrak{p}),\kappa\right)=0.$$

Then,

$$\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^{n}\mathfrak{p})}{(\varsigma+\wp+\hbar)^{n}}\xrightarrow{IF}\mathfrak{L}(\mathfrak{p}),\quad as\quad n\to\infty.$$

Substituting $m = 0$ in (29),

$$\left.\begin{array}{l} \mu_a\left(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^n\mathfrak{p})}{(\varsigma+\wp+\hbar)^n} - \mathfrak{Z}(\mathfrak{p}), \kappa\right) \geq \mu'_a\left(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}), \frac{\kappa}{\sum_{i=0}^{n-1}\frac{p^i}{(\varsigma+\wp+\hbar)^{i+2}}}\right) \\ \nu_a\left(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)^n\mathfrak{p})}{(\varsigma+\wp+\hbar)^n} - \mathfrak{Z}(\mathfrak{p}), \kappa\right) \leq \nu'_a\left(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}), \frac{\kappa}{\sum_{i=0}^{n-1}\frac{p^i}{(\varsigma+\wp+\hbar)^{i+2}}}\right). \end{array}\right\} \quad (30)$$

As $n \longrightarrow \infty$ in (30),

$$\left.\begin{array}{l} \mu_a\Big(\mathfrak{L}(\mathfrak{p}) - \mathfrak{Z}(\mathfrak{p}), \kappa\Big) \geq \mu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}), (\varsigma+\wp+\hbar)\,\kappa((\varsigma+\wp+\hbar)-\mathfrak{p})) \\ \nu_a\Big(\mathfrak{L}(\mathfrak{p}) - \mathfrak{Z}(\mathfrak{p}), \kappa\Big) \leq \nu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}), (\varsigma+\wp+\hbar)\,\kappa((\varsigma+\wp+\hbar)-\mathfrak{p})). \end{array}\right\} \quad (31)$$

Finally, $\mathfrak{L}$ satisfies (2). Replacing $(\mathfrak{p},\mathfrak{q},\mathfrak{r})$ by

$$((\varsigma+\wp+\hbar)^{n\mathfrak{p}}, (\varsigma+\wp+\hbar)^{n\mathfrak{q}}, (\varsigma+\wp+\hbar)^{n\mathfrak{r}})$$

in (16) ,

$$\left.\begin{array}{l} \mu_a\left(\frac{1}{(\varsigma+\wp+\hbar)^n} H\mathfrak{Z}_{\varsigma\wp\hbar}^{\mathfrak{p}\mathfrak{q}\mathfrak{r}}((\varsigma+\wp+\hbar)^{n\mathfrak{p}}, (\varsigma+\wp+\hbar)^{n\mathfrak{q}}, (\varsigma+\wp+\hbar)^{n\mathfrak{r}}), \kappa\right) \\ \quad \geq \mu'_a(\Psi((\varsigma+\wp+\hbar)^{n\mathfrak{p}}, (\varsigma+\wp+\hbar)^{n\mathfrak{q}}, (\varsigma+\wp+\hbar)^{n\mathfrak{r}}), (\varsigma+\wp+\hbar)^n\kappa) \\ \nu_a\left(\frac{1}{(\varsigma+\wp+\hbar)^n} H\mathfrak{Z}_{\varsigma\wp\hbar}^{\mathfrak{p}\mathfrak{q}\mathfrak{r}}((\varsigma+\wp+\hbar)^{n\mathfrak{p}}, (\varsigma+\wp+\hbar)^{n\mathfrak{q}}, (\varsigma+\wp+\hbar)^{n\mathfrak{r}}), \kappa\right) \\ \quad \leq \nu'_a(\Psi((\varsigma+\wp+\hbar)^{n\mathfrak{p}}, (\varsigma+\wp+\hbar)^{n\mathfrak{q}}, (\varsigma+\wp+\hbar)^{n\mathfrak{r}}), (\varsigma+\wp+\hbar)^n\kappa). \end{array}\right\} \quad (32)$$

Here,

$$\begin{aligned} &\mu_a\Big(\varsigma\mathfrak{L}(\wp\hbar(\mathfrak{p}-\mathfrak{q})) + \wp\mathfrak{L}(\varsigma\hbar(\mathfrak{q}-\mathfrak{r})) + \\ &\qquad \hbar\mathfrak{L}(\varsigma\wp(\mathfrak{r}-\mathfrak{p})) + (\varsigma+\wp+\hbar)\mathfrak{L}(\varsigma\mathfrak{p}+\wp\mathfrak{q}+\hbar) \\ &\qquad\qquad - (\varsigma+\wp+\hbar)(\varsigma\mathfrak{L}(\mathfrak{p}) + \wp\mathfrak{L}(\mathfrak{q}) + \hbar\mathfrak{L}(\mathfrak{r}))\Big) \\ &\geq \mu_a\left(\varsigma\mathfrak{L}(\wp\hbar(\mathfrak{p}-\mathfrak{q})) - \frac{\varsigma}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\wp\hbar(\mathfrak{p}-\mathfrak{q})), \frac{\kappa}{6}\right) \\ &* \mu_a\left(\wp\mathfrak{L}(\varsigma\hbar(\mathfrak{q}-\mathfrak{r})) - \frac{\wp}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\varsigma\hbar(\mathfrak{q}-\mathfrak{r})), \frac{\kappa}{6}\right) \\ &* \mu_a\left((\varsigma+\wp+\hbar)\mathfrak{L}(\varsigma\mathfrak{p}+\wp\mathfrak{q}+\hbar) + \frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\varsigma\mathfrak{p}+\wp\mathfrak{q}+\hbar), \frac{\kappa}{6}\right) \\ &* \mu_a\Bigg(-(\varsigma+\wp+\hbar)(\varsigma\mathfrak{L}(\mathfrak{p}) + \wp\mathfrak{L}(\mathfrak{q}) + \hbar\mathfrak{L}(r)) + \frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^n} \\ &(\varsigma\mathfrak{Z}(\mathfrak{p}) + \wp\mathfrak{Z}(\mathfrak{q}) + \hbar\mathfrak{Z}(\mathfrak{r})), \frac{\kappa}{6}\Bigg) * \mu_a\Bigg(\frac{\varsigma}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\wp\hbar(\mathfrak{p}-\mathfrak{q})) \\ &+ \frac{\wp}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\varsigma\hbar(\mathfrak{q}-\mathfrak{r})) + \frac{\hbar}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\varsigma\wp(\mathfrak{r}-\mathfrak{p})) \\ &+ \frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\varsigma\mathfrak{p}+\wp\mathfrak{q}+\hbar) - \frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^n}(\varsigma\mathfrak{Z}(p) + \wp\mathfrak{Z}(\mathfrak{q}) + \hbar\mathfrak{Z}(\mathfrak{r})), \frac{\kappa}{6}\Bigg) \end{aligned} \quad (33)$$

and

$$\begin{aligned}
&\nu_a\Big(\varsigma\mathfrak{L}(\wp\hbar(\mathfrak{p}-\mathfrak{q}))+\wp\mathfrak{L}(\varsigma\hbar(\mathfrak{q}-\mathfrak{r}))+\\
&\qquad\hbar\mathfrak{L}(\varsigma\wp(\mathfrak{r}-\mathfrak{p}))+(\varsigma+\wp+\hbar)\mathfrak{L}(\varsigma\mathfrak{p}+\wp\mathfrak{q}+\hbar)\\
&\qquad\qquad-(\varsigma+\wp+\hbar)(\varsigma\mathfrak{L}(p)+\wp\mathfrak{L}(\mathfrak{q})+\hbar\mathfrak{L}(\mathfrak{r}))\Big)\\
&\geq\nu_a\Big(\varsigma\mathfrak{L}(\wp\hbar(\mathfrak{p}-\mathfrak{q}))-\frac{\varsigma}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\wp\hbar(\mathfrak{p}-\mathfrak{q})),\frac{\kappa}{6}\Big)\\
&\diamond\,\nu_a\Big(\wp\mathfrak{L}(\varsigma\hbar(\mathfrak{q}-\mathfrak{r}))-\frac{\wp}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\varsigma\hbar(\mathfrak{q}-\mathfrak{r})),\frac{\kappa}{6}\Big)\\
&\diamond\,\nu_a\Big((\varsigma+\wp+\hbar)\mathfrak{L}(\varsigma\mathfrak{p}+\wp\mathfrak{q}+\hbar)+\frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\varsigma\mathfrak{p}+\wp\mathfrak{q}+\hbar),\frac{\kappa}{6}\Big)\\
&\diamond\,\nu_a\Big(-(\varsigma+\wp+\hbar)(\varsigma\mathfrak{L}(p)+\wp\mathfrak{L}(\mathfrak{q})+\hbar\mathfrak{L}(\mathfrak{r}))+\frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^n}\\
&(\varsigma\mathfrak{Z}(p)+\wp\mathfrak{Z}(\mathfrak{q})+\hbar\mathfrak{Z}(\mathfrak{r})),\frac{\kappa}{6}\Big)\diamond\,\nu_a\Big(\frac{\varsigma}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\wp\hbar(\mathfrak{p}-\mathfrak{q}))\\
&+\frac{\wp}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\varsigma\hbar(\mathfrak{q}-\mathfrak{r}))+\frac{\hbar}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\varsigma\wp(\mathfrak{r}-\mathfrak{p}))\\
&+\frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^n}\mathfrak{Z}(\varsigma\mathfrak{p}+\wp\mathfrak{q}+\hbar)-\frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^n}(\varsigma\mathfrak{Z}(p)+\wp\mathfrak{Z}(\mathfrak{q})+\hbar\mathfrak{Z}(\mathfrak{r})),\frac{\kappa}{6}\Big).
\end{aligned}\tag{34}$$

Also,

$$\left.\begin{aligned}
&\lim_{n\to\infty}\mu_a\Big(\tfrac{1}{(\varsigma+\wp+\hbar)^n}H\mathfrak{Z}^{\mathfrak{pqr}}_{\varsigma\wp\hbar}((\varsigma+\wp+\hbar)^{n\mathfrak{p}},(\varsigma+\wp+\hbar)^{n\mathfrak{q}},(\varsigma+\wp+\hbar)^{n\mathfrak{r}}),\tfrac{\kappa}{6}\Big)=1,\\
&\lim_{n\to\infty}\nu_a\Big(\tfrac{1}{(\varsigma+\wp+\hbar)^n}H\mathfrak{Z}^{\mathfrak{pqr}}_{\varsigma\wp\hbar}((\varsigma+\wp+\hbar)^{n\mathfrak{p}},(\varsigma+\wp+\hbar)^{n\mathfrak{q}},(\varsigma+\wp+\hbar)^{n\mathfrak{r}}),\tfrac{\kappa}{6}\Big)=0.
\end{aligned}\right\}\tag{35}$$

As $n\to\infty$ in (33), (34), and applying (35), $\mathfrak{L}$ fulfills (2). Hence, $\mathfrak{L}$ is an additive function. Here, it is proven that $\mathfrak{L}(\mathfrak{p})$ is unique. Let $\mathfrak{L}'(\mathfrak{p})$ be another additive map (2) and (17). Therefore,

$$\begin{aligned}
&\mu_a(\mathfrak{L}(\mathfrak{p})-\mathfrak{L}'(\mathfrak{p}),\kappa)\\
&\geq\mu_a\Big(\mathfrak{L}((\varsigma+\wp+\hbar)^n\mathfrak{p})-\mathfrak{Z}((\varsigma+\wp+\hbar)^n\mathfrak{p}),\frac{\kappa.(\varsigma+\wp+\hbar)^n}{2}\Big)\\
&*\,\mu_a\Big(\mathfrak{Z}((\varsigma+\wp+\hbar)^n\mathfrak{p})-\mathfrak{L}'((\varsigma+\wp+\hbar)^n\mathfrak{p}),\frac{\kappa.(\varsigma+\wp+\hbar)^n}{2}\Big)\\
&\geq\mu'_a(\Psi((\varsigma+\wp+\hbar)^{n\mathfrak{p}},(\varsigma+\wp+\hbar)^{n\mathfrak{p}},(\varsigma+\wp+\hbar)^n\mathfrak{p}),\\
&\qquad\qquad\frac{\kappa(\varsigma+\wp+\hbar)^{n+1}}{2}\,|(\varsigma+\wp+\hbar)-\mathfrak{p}|\Big)\\
&\geq\mu'_a\Big(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),\frac{\kappa\,(\varsigma+\wp+\hbar)^{n+1}|(\varsigma+\wp+\hbar)-\mathfrak{p}|}{2\cdot\mathfrak{p}^n}\Big),
\end{aligned}$$

$$
\begin{aligned}
&\nu_a(\mathfrak{L}(\mathfrak{p}) - \mathfrak{L}'(\mathfrak{p}), \kappa) \\
&\leq \nu_a\left(\mathfrak{L}((\varsigma + \wp + \hbar)^n \mathfrak{p}) - \mathfrak{Z}((\varsigma + \wp + \hbar)^n \mathfrak{p}), \frac{\kappa.(\varsigma + \wp + \hbar)^n}{2}\right) \\
&\diamond \nu_a\left(\mathfrak{Z}((\varsigma + \wp + \hbar)^n \mathfrak{p}) - \mathfrak{L}'((\varsigma + \wp + \hbar)^n \mathfrak{p}), \frac{\kappa.(\varsigma + \wp + \hbar)^n}{2}\right) \\
&\leq \nu'_a(\Psi((\varsigma + \wp + \hbar)^{n\mathfrak{p}}, (\varsigma + \wp + \hbar)^{n\mathfrak{p}}, (\varsigma + \wp + \hbar)^n \mathfrak{p}), \\
&\qquad \frac{\kappa(\varsigma + \wp + \hbar)^{n+1}}{2} |(\varsigma + \wp + \hbar) - \mathfrak{p}|\Bigg) \\
&\leq \nu'_a\left(\Psi(\mathfrak{p}, \mathfrak{p}, \mathfrak{p}), \frac{\kappa\ (\varsigma + \wp + \hbar)^{n+1}|(\varsigma + \wp + \hbar) - \mathfrak{p}|}{2 \cdot \mathfrak{p}^n}\right).
\end{aligned}
$$

Since $\lim_{n\to\infty} \dfrac{\kappa\ (\varsigma + \wp + \hbar)^{n+1}|(\varsigma + \wp + \hbar) - \mathfrak{p}|}{2\ \mathfrak{p}^n} = \infty$,

$$
\left.
\begin{aligned}
\lim_{n\to\infty} \mu'_a\left(\Psi(\mathfrak{p}, \mathfrak{p}, \mathfrak{p}), \tfrac{\kappa\ (\varsigma+\wp+\hbar)^{n+1}|(\varsigma+\wp+\hbar)-\mathfrak{p}|}{2\cdot\mathfrak{p}^n}\right) = 1, \\
\lim_{n\to\infty} \nu'_a\left(\Psi(\mathfrak{p}, \mathfrak{p}, \mathfrak{p}), \tfrac{\kappa\ (\varsigma+\wp+\hbar)^{n+1}|(\varsigma+\wp+\hbar)-\mathfrak{p}|}{2\cdot\mathfrak{p}^n}\right) = 0.
\end{aligned}
\right\}
$$

Hence,

$$
\left.
\begin{aligned}
\mu_a(\mathfrak{L}(\mathfrak{p}) - \mathfrak{L}'(\mathfrak{p}), \kappa) = 1, \\
\nu_a(\mathfrak{L}(\mathfrak{p}) - \mathfrak{L}'(\mathfrak{p}), \kappa) = 0.
\end{aligned}
\right\}
$$

Thus, $\mathfrak{L}(\mathfrak{p}) = \mathfrak{L}'(\mathfrak{p})$. Hence, $\mathfrak{L}(\mathfrak{p})$ is unique.

Category 1. *Assume $\eta = -1$. Substituting p by $\frac{\mathfrak{p}}{(\varsigma+\wp+\hbar)}$ in (18),*

$$
\left.
\begin{aligned}
\mu_a\left((\varsigma + \wp + \hbar)\mathfrak{Z}(\mathfrak{p}) - (\varsigma + \wp + \hbar)^2 \mathfrak{Y}\left(\tfrac{\mathfrak{p}}{(\varsigma+\wp+\hbar)}\right), \kappa\right) \geq \mu'_a\left(\Psi\left(\tfrac{\mathfrak{p}}{2}, \tfrac{\mathfrak{p}}{2}, \tfrac{\mathfrak{p}}{2}\right), \kappa\right) \\
\nu_a\left((\varsigma + \wp + \hbar)\mathfrak{Z}(\mathfrak{p}) - (\varsigma + \wp + \hbar)^2 \mathfrak{Y}\left(\tfrac{\mathfrak{p}}{(\varsigma+\wp+\hbar)}\right), \kappa\right) \leq \nu'_a\left(\Psi\left(\tfrac{\mathfrak{p}}{2}, \tfrac{\mathfrak{p}}{2}, \tfrac{\mathfrak{p}}{2}\right), \kappa\right)
\end{aligned}
\right\} \quad (36)
$$

□

Corollary 1. *Let $\mathfrak{Z}$ be an approximately additive mapping that satisfies the inequality*

$$
\begin{aligned}
&\mu_a\left(H\mathfrak{Z}^{pqr}_{\varsigma\wp\hbar}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa\right) \\
&\geq \begin{cases}
\mu'_a(\mathcal{Q}, \kappa), \\
\mu'_a\left(\mathcal{Q}\left(||\mathfrak{p}||^a + ||\mathfrak{q}||^b + ||\mathfrak{r}||^c\right), \kappa\right), a, b, c \neq 1 \\
\mu'_a\left(\mathcal{Q}||\mathfrak{p}||^a||\mathfrak{q}||^b||\mathfrak{r}||^c, \kappa\right), a + b + c \neq 1 \\
\mu'_a\left(\mathcal{Q}\left\{||\mathfrak{p}||^a\ ||\mathfrak{q}||^b\ ||\mathfrak{r}||^c + \left(||\mathfrak{p}||^{a+b+c} + ||\mathfrak{q}||^{a+b+c} + ||\mathfrak{r}||^{a+b+c}\right)\right\}, \kappa\right), \\
\qquad a + b + c \neq 1
\end{cases} \\
&\nu_a\left(H\mathfrak{Z}^{pqr}_{\varsigma\wp\hbar}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa\right) \\
&\leq \begin{cases}
\nu'_a(\mathcal{Q}, \kappa), \\
\nu'_a\left(\mathcal{Q}\left(||\mathfrak{p}||^a + ||\mathfrak{q}||^b + ||\mathfrak{r}||^c\right), \kappa\right), a, b, c \neq 1 \\
\nu'_a\left(\mathcal{Q}||\mathfrak{p}||^a||\mathfrak{p}||^b||\mathfrak{q}||^c, \kappa\right), a + b + c \neq 1 \\
\nu'_a\left(\mathcal{Q}\left\{||\mathfrak{p}||^a\ ||\mathfrak{q}||^b\ ||\mathfrak{r}||^c + \left(||\mathfrak{p}||^{a+b+c} + ||\mathfrak{q}||^{a+b+c} + ||\mathfrak{r}||^{a+b+c}\right)\right\}, \kappa\right), \\
\qquad a + b + c \neq 1
\end{cases}
\end{aligned} \quad (37)
$$

such that

$$\mu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) \geq \begin{cases} \mu'_a(\mathcal{Q}, (\varsigma + \wp + \hbar)\,\kappa|(\varsigma + \wp + \hbar) - 1|), \\ \mu'_a\Big(\Big[\mathcal{Q}||\mathfrak{p}||^a|\varsigma + \wp + \hbar|^a + \mathcal{Q}||\mathfrak{p}||^b|\varsigma + \wp + \hbar|^b + \mathcal{Q}||\mathfrak{p}||^c|\varsigma + \wp + \hbar|^c\Big], \\ \quad (\varsigma + \wp + \hbar)\,\kappa\big[|(\varsigma + \wp + \hbar) - (\varsigma + \wp + \hbar)^a| \\ \qquad +|(\varsigma + \wp + \hbar) - (\varsigma + \wp + \hbar)^b| + |(\varsigma + \wp + \hbar) - (\varsigma + \wp + \hbar)^c|\big]\Big), \\ \mu'_a\Big(\mathcal{Q}||\mathfrak{p}||^{a+b+c}|\varsigma + \wp + \hbar|^{a+b+c}, \\ \quad (\varsigma + \wp + \hbar)\,\kappa|(\varsigma + \wp + \hbar) - (\varsigma + \wp + \hbar)^{a+b+c}|\Big), \\ \mu'_a\Big(\mathcal{Q}||\mathfrak{p}||^{a+b+c}|\varsigma + \wp + \hbar|^{a+b+c} \\ \quad +\Big[\mathcal{Q}||\mathfrak{p}||^a|\varsigma + \wp + \hbar|^a + \mathcal{Q}||\mathfrak{p}||^b|\varsigma + \wp + \hbar|^b + \mathcal{Q}||\mathfrak{p}||^c|\varsigma + \wp + \hbar|^c\Big], \\ \quad (\varsigma + \wp + \hbar)\,\kappa|(\varsigma + \wp + \hbar) - (\varsigma + \wp + \hbar)^{a+b+c}|\Big), \end{cases} \tag{38}$$

$$\nu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) \leq \begin{cases} \nu'_a(\mathcal{Q}, (\varsigma + \wp + \hbar)\,\kappa|(\varsigma + \wp + \hbar) - 1|), \\ \nu'_a\Big(\Big[\mathcal{Q}||\mathfrak{p}||^a|\varsigma + \wp + \hbar|^a + \mathcal{Q}||\mathfrak{p}||^b|\varsigma + \wp + \hbar|^b + \mathcal{Q}||\mathfrak{p}||^c|\varsigma + \wp + \hbar|^c\Big], \\ \quad (\varsigma + \wp + \hbar)\,\kappa\big[|(\varsigma + \wp + \hbar) - (\varsigma + \wp + \hbar)^a| \\ \qquad +|(\varsigma + \wp + \hbar) - (\varsigma + \wp + \hbar)^b| + |(\varsigma + \wp + \hbar) - (\varsigma + \wp + \hbar)^c|\big]\Big), \\ \nu'_a\Big(\mathcal{Q}||\mathfrak{p}||^{a+b+c}|\varsigma + \wp + \hbar|^{a+b+c}, \\ \quad (\varsigma + \wp + \hbar)\,\kappa|(\varsigma + \wp + \hbar) - (\varsigma + \wp + \hbar)^{a+b+c}|\Big), \\ \nu'_a\Big(\mathcal{Q}||\mathfrak{p}||^{a+b+c}|\varsigma + \wp + \hbar|^{a+b+c} \\ \quad +\Big[\mathcal{Q}||\mathfrak{p}||^a|\varsigma + \wp + \hbar|^a + \mathcal{Q}||\mathfrak{p}||^b|\varsigma + \wp + \hbar|^b + \mathcal{Q}||\mathfrak{p}||^c|\varsigma + \wp + \hbar|^c\Big], \\ \quad (\varsigma + \wp + \hbar)\,\kappa|(\varsigma + \wp + \hbar) - (\varsigma + \wp + \hbar)^{a+b+c}|\Big). \end{cases} \tag{39}$$

5. Stability Results: Fixed Point Method

Using the FPT [57], the generalised HURS of the functional Equation (2) is derived.

Theorem 2. *Let* $\Psi : \mathbb{A} \times \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{B}$ *such that*

$$\left.\begin{aligned} \lim_{n\to\infty} \mu'_a\big(\Psi\big(\mathcal{T}_i^n\mathfrak{p}, \mathcal{T}_i^n\mathfrak{q}, \mathcal{T}_i^n\mathfrak{r}\big), \mathcal{T}^n\kappa\big) = 1, \\ \lim_{n\to\infty} \nu'_a\big(\Psi\big(\mathcal{T}_i^n\mathfrak{p}, \mathcal{T}_i^n\mathfrak{q}, \mathcal{T}_i^n\mathfrak{r}\big), \mathcal{T}^n\kappa\big) = 0, \end{aligned}\right\} \tag{40}$$

where

$$\mathcal{T}_i = \begin{cases} \varsigma + \wp + \hbar & if \quad i = 0 \\ \frac{1}{\varsigma + \wp + \hbar} & if \quad i = 1, \end{cases} \tag{41}$$

then

$$\left.\begin{aligned} \mu_a\Big(H\mathfrak{Z}^{\mathfrak{pqr}}_{\varsigma\wp\hbar}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa\Big) \geq \mu'_a(\Psi(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa) \\ \nu_a\Big(H\mathfrak{Z}^{\mathfrak{pqr}}_{\varsigma\wp\hbar}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa\Big) \leq \nu'_a(\Psi(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa). \end{aligned}\right\} \tag{42}$$

If $L = L(i)$,

$$\mathfrak{Z}(\mathfrak{p}) = \frac{1}{\varsigma + \wp + \hbar}\Psi\left(\frac{\mathfrak{p}}{\varsigma + \wp + \hbar}, \frac{\mathfrak{p}}{\varsigma + \wp + \hbar}, \frac{\mathfrak{p}}{\varsigma + \wp + \hbar}\right), \tag{43}$$

and

$$\left.\begin{array}{l}\mu'_a\left(L\frac{\mathfrak{Z}(\mathcal{T}_i\mathfrak{p})}{\mathcal{T}_i},\kappa\right)=\mu'_a(\mathfrak{Z}(\mathfrak{p}),\kappa)\\ \nu'_a\left(L\frac{\mathfrak{Z}(\mathcal{T}_i\mathfrak{p})}{\mathcal{T}_i},\kappa\right)=\nu'_a(\mathfrak{Z}(\mathfrak{p}),\kappa)\end{array}\right\}\quad(44)$$

then

$$\left.\begin{array}{l}\mu_a(\mathfrak{Z}(\mathfrak{p})-\mathfrak{L}(\mathfrak{p}),\kappa)\geq\mu'_a\left(\mathfrak{Z}(\mathfrak{p}),\frac{L^{1-i}}{1-L}\kappa\right)\\ \nu_a(\mathfrak{Z}(\mathfrak{p})-\mathfrak{L}(\mathfrak{p}),\kappa)\leq\nu'_a\left(\mathfrak{Z}(\mathfrak{p}),\frac{L^{1-i}}{1-L}\kappa\right).\end{array}\right\}\quad(45)$$

Proof. Let

$$\mathcal{Q}=\{h|\,h:\mathfrak{p}\longrightarrow Y,\ \mathfrak{Z}(0)=0\}$$

$$d(h,f)=\inf\{L\in(0,\infty):\left\{\begin{array}{l}\mu_a(\mathfrak{Z}(\mathfrak{p})-\mathfrak{Z}(\mathfrak{p}),\kappa)\geq\mu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa),\kappa>0\\ \nu_a(\mathfrak{Z}(\mathfrak{p})-\mathfrak{Z}(\mathfrak{p}),\kappa)\leq\nu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa),\kappa>0\end{array}\right\}\}\quad(46)$$

Now, from (46)

$$\inf\left\{L\in(0,\infty):\begin{array}{l}\left\{\begin{array}{l}\mu_a(\mathfrak{Z}(\mathfrak{p})-\mathfrak{Z}(\mathfrak{p}),\kappa)\geq\mu'_a(\mathfrak{Z}(\mathfrak{p}),\kappa)\}\\ \mu_a(\frac{1}{\mathcal{T}_i}\mathfrak{Z}(\mathcal{T}_i\mathfrak{p})-\frac{1}{\mathcal{T}_i}\mathfrak{Z}(\mathcal{T}_i\mathfrak{p}),\kappa)\geq\mu'_a(\mathfrak{Z}(\mathcal{T}_i\mathfrak{p}),\mathcal{T}_i\kappa)\}\\ \mu_a(\frac{1}{\mathcal{T}_i}\mathfrak{Z}(\mathcal{T}_i\mathfrak{p})-\frac{1}{\mathcal{T}_i}\mathfrak{Z}(\mathcal{T}_i\mathfrak{p}),\kappa)\geq\mu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa)\}\\ \mu_a(J\mathfrak{Z}(\mathfrak{p})-J\mathfrak{Z}(\mathfrak{p}),\kappa)\geq\mu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa)\}\end{array}\right.\\ \left\{\begin{array}{l}\nu_a(\mathfrak{Z}(\mathfrak{p})-\mathfrak{Z}(\mathfrak{p}),\kappa)\leq\nu'_a(\mathfrak{Z}(\mathfrak{p}),\kappa)\}\\ \nu_a(\frac{1}{\mathcal{T}_i}\mathfrak{Z}(\mathcal{T}_i\mathfrak{p})-\frac{1}{\mathcal{T}_i}\mathfrak{Z}(\mathcal{T}_i\mathfrak{p}),\kappa)\leq\nu'_a(\mathfrak{Z}(\mathcal{T}_i\mathfrak{p}),\mathcal{T}_i\kappa)\}\\ \nu_a(\frac{1}{\mathcal{T}_i}\mathfrak{Z}(\mathcal{T}_i\mathfrak{p})-\frac{1}{\mathcal{T}_i}\mathfrak{Z}(\mathcal{T}_i\mathfrak{p}),\kappa)\leq\nu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa)\}\\ \nu_a(J\mathfrak{Z}(\mathfrak{p})-J\mathfrak{Z}(\mathfrak{p}),\kappa)\leq\nu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa)\}\end{array}\right.\end{array}\right\}$$

and

$$\inf\{1\in(0,\infty):\left\{\begin{array}{l}\mu_a\left(\mathfrak{Z}((\varsigma+\wp+\hbar)\mathfrak{p})-(\varsigma+\wp+\hbar)\mathfrak{Z}(\mathfrak{p}),\kappa\right)\geq\mu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),(\varsigma+\wp+\hbar)\kappa)\\ \nu_a\left(\mathfrak{Z}((\varsigma+\wp+\hbar)\mathfrak{p})-(\varsigma+\wp+\hbar)\mathfrak{Z}(\mathfrak{p}),\kappa\right)\leq\nu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),(\varsigma+\wp+\hbar)\kappa)\end{array}\right\}\}\quad(47)$$

Assuming $i = 0$, and

$$\inf\Big\{L^{1-0} \in (0,\infty) : \left.\begin{array}{l} \left\{\begin{array}{l} \mu_a\Big(\mathfrak{Z}((\varsigma+\wp+\hbar)\mathfrak{p}) - (\varsigma+\wp+\hbar)\mathfrak{Z}(\mathfrak{p}),\kappa\Big) \geq \mu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),(\varsigma+\wp+\hbar)\kappa) \\ \mu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)\mathfrak{p})}{(\varsigma+\wp+\hbar)} - \mathfrak{Z}(\mathfrak{p}),\kappa\Big) \geq \mu'_a\Big(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),(\varsigma+\wp+\hbar)^2\kappa\Big) \\ \mu_a\Big(J\mathfrak{Z}(\mathfrak{p}) - \mathfrak{Z}(\mathfrak{p}),\kappa\Big) \geq \mu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa) \\ \mu_a\Big(J\mathfrak{Z}(\mathfrak{p}) - \mathfrak{Z}(\mathfrak{p}),\kappa\Big) \geq \mu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa) \\ \mu_a\Big(J\mathfrak{Z}(\mathfrak{p}) - \mathfrak{Z}(\mathfrak{p}),\kappa\Big) \geq \mu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa) \end{array}\right. \\ \left\{\begin{array}{l} \nu_a\Big(\mathfrak{Z}((\varsigma+\wp+\hbar)\mathfrak{p}) - (\varsigma+\wp+\hbar)\mathfrak{Z}(\mathfrak{p}),t\Big) \leq \nu'_a(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),(\varsigma+\wp+\hbar)\kappa) \\ \nu_a\Big(\frac{\mathfrak{Z}((\varsigma+\wp+\hbar)\mathfrak{p})}{(\varsigma+\wp+\hbar)} - \mathfrak{Z}(\mathfrak{p}),\kappa\Big) \leq \nu'_a\Big(\Psi(\mathfrak{p},\mathfrak{p},\mathfrak{p}),(\varsigma+\wp+\hbar)^2\kappa\Big) \\ \nu_a\Big(J\mathfrak{Z}(\mathfrak{p}) - \mathfrak{Z}(\mathfrak{p}),\kappa\Big) \leq \nu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa) \\ \nu_a\Big(J\mathfrak{Z}(\mathfrak{p}) - \mathfrak{Z}(\mathfrak{p}),\kappa\Big) \leq \nu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa) \\ \nu_a\Big(J\mathfrak{Z}(\mathfrak{p}) - \mathfrak{Z}(\mathfrak{p}),\kappa\Big) \leq \nu'_a(\mathfrak{Z}(\mathfrak{p}),L\kappa) \end{array}\right. \end{array}\right\} \tag{48}$$

If $i = 1$, and

$$\inf\Big\{L^{1-1} \in (0,\infty) : \left.\begin{array}{l} \left\{\begin{array}{l} \mu_a\Big(\mathfrak{Z}(\mathfrak{p}) - (\varsigma+\wp+\hbar)\mathfrak{Y}\Big(\frac{\mathfrak{p}}{(\varsigma+\wp+\hbar)}\Big),\kappa\Big) \\ \qquad\qquad \geq \mu'_a\Big(\Psi\Big(\frac{\mathfrak{p}}{(\varsigma+\wp+\hbar)},\frac{\mathfrak{p}}{(\varsigma+\wp+\hbar)}\Big),(\varsigma+\wp+\hbar)\kappa\Big) \\ \mu_a\Big(\mathfrak{Z}(\mathfrak{p}) - J\mathfrak{Z}(\mathfrak{p}),\kappa\Big) \geq \mu'_a(\mathfrak{Z}(\mathfrak{p}),\kappa) \\ \mu_a\Big(\mathfrak{Z}(\mathfrak{p}) - J\mathfrak{Z}(\mathfrak{p}),\kappa\Big) \geq \mu'_a(\mathfrak{Z}(\mathfrak{p}),\kappa) \\ \mu_a\Big(\mathfrak{Z}(\mathfrak{p}) - J\mathfrak{Z}(\mathfrak{p}),\kappa\Big) \geq \mu'_a(\mathfrak{Z}(\mathfrak{p}),\kappa) \end{array}\right. \\ \left\{\begin{array}{l} \nu_a\Big(\mathfrak{Z}(\mathfrak{p}) - (\varsigma+\wp+\hbar)\mathfrak{Y}\Big(\frac{\mathfrak{p}}{(\varsigma+\wp+\hbar)},\frac{\mathfrak{p}}{(\varsigma+\wp+\hbar)}\Big),t\Big) \\ \qquad\qquad \leq \nu'_a\Big(\Psi\Big(\frac{\mathfrak{p}}{(\varsigma+\wp+\hbar)}\Big),(\varsigma+\wp+\hbar)\kappa\Big) \\ \nu_a\Big(\mathfrak{Z}(\mathfrak{p}) - J\mathfrak{Z}(\mathfrak{p}),\kappa\Big) \leq \nu'_a(\mathfrak{Z}(\mathfrak{p}),\kappa) \\ \nu_a\Big(\mathfrak{Z}(\mathfrak{p}) - J\mathfrak{Z}(\mathfrak{p}),\kappa\Big) \leq \nu'_a(\mathfrak{Z}(\mathfrak{p}),\kappa) \\ \nu_a\Big(\mathfrak{Z}(\mathfrak{p}) - J\mathfrak{Z}(\mathfrak{p}),\kappa\Big) \leq \nu'_a(\mathfrak{Z}(\mathfrak{p}),\kappa) \end{array}\right. \end{array}\right\} \tag{49}$$

and

$$\inf\left\{L^{1-i} \in (0,\infty) : \left\{\begin{array}{l} \mu_a(\mathfrak{Z}(\mathfrak{p}) - J\mathfrak{Z}(\mathfrak{p}),\kappa) \geq \mu'_a(\mathfrak{Z}(\mathfrak{p}),L^{1-i}\kappa), \\ \nu_a(\mathfrak{Z}(\mathfrak{p}) - J\mathfrak{Z}(\mathfrak{p}),\kappa) \leq \nu'_a(\mathfrak{Z}(\mathfrak{p}),L^{1-i}\kappa), \end{array}\right\}\right\} \tag{50}$$

Hence property [57] Condition: 1 holds.

By [57] Condition: 2, there exists a fixed point $\mathfrak{L}$ of J in $\mathcal{Q}$ such that

$$\lim_{n\to\infty} \mu_a\left(\frac{\mathfrak{Z}(\mathcal{T}_i^n \mathfrak{p})}{\mathcal{T}_i^n} - \mathfrak{L}(\mathfrak{p}), \kappa\right) = 1,$$

$$\lim_{n\to\infty} \nu_a\left(\frac{\mathfrak{Z}(\mathcal{T}_i^n \mathfrak{p})}{\mathcal{T}_i^n} - \mathfrak{L}(\mathfrak{p}), \kappa\right) = 0.$$

To show $\mathfrak{Z}$ is additive, replacing $(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})$ by $\left(\mathcal{T}_i^n \mathfrak{p}, \mathcal{T}_i^n \mathfrak{q}, \mathcal{T}_i^n \mathfrak{r}\right)$ and applying the necessary condition, the functional equation is derived.

By [57] Condition: 3, $\mathfrak{L}$ is the unique fixed point of J in the set $\Delta = \{\mathfrak{L} \in \mathcal{Q} : d(\mathfrak{Z}, A) < \infty\}$.

$\mathfrak{L}$ is a unique function, such that

$$\left.\begin{aligned} \mu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) &\geq \mu_a'(\mathfrak{Z}(\mathfrak{p}), L^{1-i}\kappa), \mathfrak{p} \in \mathbb{A} \\ \nu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) &\leq \nu_a'(\mathfrak{Z}(\mathfrak{p}), L^{1-i}\kappa), \mathfrak{p} \in \mathbb{A}. \end{aligned}\right\}$$

Finally, using [57], condition: 4

$$\left.\begin{aligned} \mu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) &\geq \mu_a'\left(\mathfrak{Z}(\mathfrak{p}), \tfrac{L^{1-i}}{1-L}\kappa\right) \\ \nu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) &\leq \nu_a'\left(\mathfrak{Z}(\mathfrak{p}), \tfrac{L^{1-i}}{1-L}\kappa\right). \end{aligned}\right\}$$

Hence, proven. □

Corollary 2. *Let $\mathfrak{Z}$ be an approximately additive mapping satisfying the inequality*

$$\begin{aligned} &\mu_a\left(H\mathfrak{Z}_{\varsigma\wp\hbar}^{pqr}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa\right) \\ &\geq \begin{cases} \mu_a'(\mathcal{Q}, \kappa), \\ \mu_a'(\mathcal{Q}(||\mathfrak{p}||^a + ||\mathfrak{q}||^a + ||\mathfrak{r}||^a), \kappa), a \neq 1 \\ \mu_a'(\mathcal{Q}||\mathfrak{p}||^a||\mathfrak{q}||^a||\mathfrak{r}||^a, \kappa), \qquad 3a \neq 1 \\ \mu_a'\left(\mathcal{Q}\left\{||\mathfrak{p}||^a\ ||\mathfrak{q}||^a\ ||\mathfrak{r}||^a + \left(||\mathfrak{p}||^{3a} + ||\mathfrak{q}||^{3a} + ||\mathfrak{r}||^{3a}\right)\right\}, \kappa\right), \qquad 3a \neq 1 \end{cases} \\ &\nu_a\left(H\mathfrak{Z}_{\varsigma\wp\hbar}^{pqr}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), \kappa\right) \\ &\leq \begin{cases} \nu_a'(\mathcal{Q}, \kappa), \\ \nu_a'(\mathcal{Q}(||\mathfrak{p}||^a + ||\mathfrak{q}||^a + ||\mathfrak{r}||^a), \kappa), a \neq 1 \\ \nu_a'(\mathcal{Q}||\mathfrak{p}||^a||\mathfrak{q}||^a||\mathfrak{r}||^a, \kappa), \qquad 3a \neq 1 \\ \nu_a'\left(\mathcal{Q}\left\{||\mathfrak{p}||^a\ ||\mathfrak{q}||^a\ ||\mathfrak{r}||^a + \left(||\mathfrak{p}||^{3a} + ||\mathfrak{q}||^{3a} + ||\mathfrak{r}||^{3a}\right)\right\}, \kappa\right), \qquad 3a \neq 1 \end{cases} \end{aligned} \tag{51}$$

such that

$$\mu_a(\mathfrak{Z}(\mathfrak{p})-\mathfrak{L}(\mathfrak{p}),\kappa) \geq \begin{cases} \mu_a'\left(\frac{\mathcal{Q}}{(\varsigma+\wp+\hbar)},\frac{(\varsigma+\wp+\hbar)}{1-(\varsigma+\wp+\hbar)}\kappa\right) \\ \mu_a'\left(\frac{\mathcal{Q}||\mathfrak{p}||^a}{(\varsigma+\wp+\hbar)}\frac{3}{|\varsigma+\wp+\hbar|^a},\frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^a-(\varsigma+\wp+\hbar)}\kappa\right) \\ \mu_a'\left(\frac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\frac{1}{|\varsigma+\wp+\hbar|^{3a}},\frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^{3a}-(\varsigma+\wp+\hbar)}\kappa\right) \\ \mu_a'\left(\frac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\left(\frac{3}{|\varsigma+\wp+\hbar|^{3a}}+\frac{1}{|\varsigma+\wp+\hbar|^{3a}},\frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^{3a}-(\varsigma+\wp+\hbar)}\kappa\right)\right) \end{cases} \tag{52}$$

$$\nu_a(\mathfrak{Z}(\mathfrak{p})-\mathfrak{L}(\mathfrak{p}),\kappa) \leq \begin{cases} \nu_a'\left(\frac{\mathcal{Q}}{(\varsigma+\wp+\hbar)},\frac{(\varsigma+\wp+\hbar)}{1-(\varsigma+\wp+\hbar)}\kappa\right) \\ \nu_a'\left(\frac{\mathcal{Q}||\mathfrak{p}||^a}{(\varsigma+\wp+\hbar)}\frac{3}{|\varsigma+\wp+\hbar|^a},\frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^a-(\varsigma+\wp+\hbar)}\kappa\right) \\ \nu_a'\left(\frac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\frac{1}{|\varsigma+\wp+\hbar|^{3a}},\frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^{3a}-(\varsigma+\wp+\hbar)}\kappa\right) \\ \nu_a'\left(\frac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\left(\frac{3}{|\varsigma+\wp+\hbar|^{3a}}+\frac{1}{|\varsigma+\wp+\hbar|^{3a}},\frac{(\varsigma+\wp+\hbar)}{(\varsigma+\wp+\hbar)^{3a}-(\varsigma+\wp+\hbar)}\kappa\right)\right) \end{cases}.$$

Proof. Let

$$\mu_a'\Big(\Psi(\mathcal{T}_i^n\mathfrak{p},\mathcal{T}_i^n\mathfrak{q},\mathcal{T}_i^n\mathfrak{r}),\mathcal{T}_i^k\kappa\Big)$$

$$= \begin{cases} \mu_a'\Big(\mathcal{Q},\mathcal{T}_i^k\kappa\Big), \\ \mu_a'\Big(\mathcal{Q}(||\mathfrak{p}||^a+||\mathfrak{q}||^a+||\mathfrak{r}||^a),\mathcal{T}_i^{k-a}\kappa\Big), \\ \mu_a'\Big(\mathcal{Q}||\mathfrak{p}||^a||\mathfrak{q}||^a||\mathfrak{r}||^a,\mathcal{T}_i^{k-3a}\kappa\Big), \\ \mu_a'\Big(\mathcal{Q}\{||\mathfrak{p}||^a\,||\mathfrak{q}||^a||\mathfrak{r}||^a+\big(||\mathfrak{p}||^{3a}+||\mathfrak{q}||^{3a}+||\mathfrak{r}||^{3a}\big)\},\mathcal{T}_i^{k-3a}\kappa\Big), \end{cases}$$

$$= \begin{cases} \to 1 \text{ as } k\to\infty \\ \to 1 \text{ as } k\to\infty \\ \to 1 \text{ as } k\to\infty \\ \to 1 \text{ as } k\to\infty \end{cases}$$

$$\nu'_a\Big(\Psi(\mathcal{T}_i^n\mathfrak{p},\mathcal{T}_i^n\mathfrak{q}),\mathcal{T}_i^k\kappa\Big)$$
$$=\begin{cases}\nu'_a\Big(\mathcal{Q},\mathcal{T}_i^k\kappa\Big),\\ \nu'_a\Big(\mathcal{Q}(||\mathfrak{p}||^a+||\mathfrak{q}||^a+||\mathfrak{r}||^a),\mathcal{T}_i^{k-a}\kappa\Big),\\ \nu'_a\Big(\mathcal{Q}||\mathfrak{p}||^a||\mathfrak{q}||^a||\mathfrak{r}||^a,\mathcal{T}_i^{k-3a}\kappa\Big),\\ \nu'_a\Big(\mathcal{Q}\big\{||\mathfrak{p}||^a\,||\mathfrak{q}||^a||\mathfrak{r}||^a+\big(||\mathfrak{p}||^{3a}+||\mathfrak{q}||^{3a}+||\mathfrak{r}||^{3a}\big)\big\},\mathcal{T}_i^{k-3a}\kappa\Big),\end{cases}$$
$$=\begin{cases}\to 0 \text{ as } k\to\infty\\ \to 0 \text{ as } k\to\infty\\ \to 0 \text{ as } k\to\infty\\ \to 0 \text{ as } k\to\infty\end{cases}$$

At the end, the relation (40) holds; hence

$$\left.\begin{array}{l}\mu'_a\Big(\frac{1}{(\varsigma+\wp+\hbar)}\Psi\Big(\frac{\mathfrak{p}}{(\varsigma+\wp+\hbar)}\Big),\kappa\Big)\\ =\begin{cases}\mu'_a\Big(\dfrac{\mathcal{Q}}{(\varsigma+\wp+\hbar)},\kappa\Big)\\ \mu'_a\Big(\dfrac{\mathcal{Q}||\mathfrak{p}||^a}{(\varsigma+\wp+\hbar)}\dfrac{3}{|\varsigma+\wp+\hbar|^a},\kappa\Big)\\ \mu'_a\Big(\dfrac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\dfrac{1}{|\varsigma+\wp+\hbar|^{3a}},\kappa\Big)\\ \mu'_a\Big(\dfrac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\Big(\dfrac{3}{|\varsigma+\wp+\hbar|^{3a}}+\dfrac{1}{|\varsigma+\wp+\hbar|^{3a}}\Big),\kappa\Big)\end{cases}\\ \nu'_a\Big(\frac{1}{(\varsigma+\wp+\hbar)}\Psi\Big(\frac{\mathfrak{p}}{(\varsigma+\wp+\hbar)}\Big),\kappa\Big)\\ =\begin{cases}\nu'_a\Big(\frac{\mathcal{Q}}{(\varsigma+\wp+\hbar)},\kappa\Big)\\ \nu'_a\Big(\dfrac{\mathcal{Q}||\mathfrak{p}||^a}{(\varsigma+\wp+\hbar)}\dfrac{3}{|\varsigma+\wp+\hbar|^a},\kappa\Big)\\ \nu'_a\Big(\dfrac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\dfrac{1}{|\varsigma+\wp+\hbar|^{3a}},\kappa\Big)\\ \nu'_a\Big(\dfrac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\Big(\dfrac{3}{|\varsigma+\wp+\hbar|^{3a}}+\dfrac{1}{|\varsigma+\wp+\hbar|^{3a}}\Big),\kappa\Big).\end{cases}\end{array}\right\}$$

From (44),

$$\left.\begin{aligned}\mu'_a\left(\frac{\mathfrak{Z}(\mathcal{T}_i\mathfrak{p})}{\mathcal{T}_i},\kappa\right) &= \begin{cases}\mu'_a(\mathcal{Q},\mathcal{T}_i\kappa)\\ \mu'_a\left(\frac{\mathcal{Q}||\mathfrak{p}||^a}{(\varsigma+\wp+\hbar)}\frac{3}{|\varsigma+\wp+\hbar|^a},\mathcal{T}_i^{1-a}\kappa\right)\\ \mu'_a\left(\frac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\frac{1}{|\varsigma+\wp+\hbar|^{3a}},\mathcal{T}_i^{1-3a}\kappa\right)\\ \mu'_a\left(\frac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\left(\frac{3}{|\varsigma+\wp+\hbar|^{3a}}+\frac{1}{|\varsigma+\wp+\hbar|^{3a}}\right),\mathcal{T}_i^{1-3a}\kappa\right)\end{cases}\\ \nu'_a\left(\frac{\mathfrak{Z}(\mathcal{T}_i\mathfrak{p})}{\mathcal{T}_i},\kappa\right) &= \begin{cases}\nu'_a(\mathcal{Q},\mathcal{T}_i\kappa)\\ \nu'_a\left(\frac{\mathcal{Q}||\mathfrak{p}||^a}{(\varsigma+\wp+\hbar)}\frac{3}{|\varsigma+\wp+\hbar|^a},\mathcal{T}_i^{1-a}\kappa\right)\\ \nu'_a\left(\frac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\frac{1}{|\varsigma+\wp+\hbar|^{3a}},\mathcal{T}_i^{1-3a}\kappa\right)\\ \nu'_a\left(\frac{\mathcal{Q}||\mathfrak{p}||^{3a}}{(\varsigma+\wp+\hbar)}\left(\frac{3}{|\varsigma+\wp+\hbar|^{3a}}+\frac{1}{|\varsigma+\wp+\hbar|^{3a}}\right),\mathcal{T}_i^{1-3a}\kappa\right).\end{cases}\end{aligned}\right\}$$

Hence, (45) is true for

	L	$a, i = 0$	L	$a, i = 1$
1.	$(\varsigma+\wp+\hbar)$	0	$(\varsigma+\wp+\hbar)^{-1}$	0
2.	$(\varsigma+\wp+\hbar)^{1-a}$	$a<1$	$(\varsigma+\wp+\hbar)^{a-1}$	$a>1$
3.	$(\varsigma+\wp+\hbar)^{1-3a}$	$3a<1$	$(\varsigma+\wp+\hbar)^{3a-1}$	$3a>1$
4.	$(\varsigma+\wp+\hbar)^{3a-1}$	$3a<1$	$(\varsigma+\wp+\hbar)^{3a-1}$	$3a>1.$

Criteria 1. *For $i = 0$,*

$$\left.\begin{aligned}\mu_a(\mathfrak{Z}(\mathfrak{p})-\mathfrak{L}(\mathfrak{p}),\kappa) &\geq \mu'_a\left(\mathfrak{Z}(\mathfrak{p}),\frac{(\varsigma+\wp+\hbar)^{1-0}}{1-(\varsigma+\wp+\hbar)}\kappa\right)\\ &= \mu'_a\left(\frac{\mathcal{Q}}{(\varsigma+\wp+\hbar)},\frac{(\varsigma+\wp+\hbar)}{1-(\varsigma+\wp+\hbar)}\kappa\right)\\ \nu_a(\mathfrak{Z}(\mathfrak{p})-\mathfrak{L}(\mathfrak{p}),\kappa) &\leq \nu'_a\left(\mathfrak{Z}(\mathfrak{p}),\frac{(\varsigma+\wp+\hbar)^{1-0}}{1-(\varsigma+\wp+\hbar)}\kappa\right)\\ &= \nu'_a\left(\frac{\mathcal{Q}}{(\varsigma+\wp+\hbar)},\frac{(\varsigma+\wp+\hbar)}{1-(\varsigma+\wp+\hbar)}\kappa\right).\end{aligned}\right\}$$

Criteria 2. *For* $i = 1$,

$$\left.\begin{aligned}\mu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) &\geq \mu'_a\left(\mathfrak{Z}(\mathfrak{p}), \frac{(((\varsigma + \wp + \hbar))^{-1})^{1-1}}{1 - ((\varsigma + \wp + \hbar))^{-1}}\kappa\right)\\ &= \mu'_a\left(\frac{\mathcal{Q}}{(\varsigma + \wp + \hbar)}, \frac{(\varsigma + \wp + \hbar)}{(\varsigma + \wp + \hbar) - 1}\kappa\right)\\ \nu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) &\leq \nu'_a\left(\mathfrak{Z}(\mathfrak{p}), \frac{(((\varsigma + \wp + \hbar))^{-1})^{1-1}}{1 - ((\varsigma + \wp + \hbar))^{-1}}\kappa\right)\\ &= \nu'_a\left(\frac{\mathcal{Q}}{(\varsigma + \wp + \hbar)}, \frac{(\varsigma + \wp + \hbar)}{(\varsigma + \wp + \hbar) - 1}\kappa\right).\end{aligned}\right\}$$

Criteria 3. *For* $i = 0$,

$$\left.\begin{aligned}\mu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) &\geq \mu'_a\left(\mathfrak{Z}(\mathfrak{p}), \frac{((\varsigma + \wp + \hbar)^{1-a})^{1-0}}{1 - ((\varsigma + \wp + \hbar)^{1-a})}\kappa\right)\\ &= \mu'_a\left(\frac{\mathcal{Q}||\mathfrak{p}||^a}{(\varsigma + \wp + \hbar)}\frac{3}{|\varsigma + \wp + \hbar|^a}, \frac{(\varsigma + \wp + \hbar)}{(\varsigma + \wp + \hbar)^a - (\varsigma + \wp + \hbar)^k}\kappa\right)\\ \nu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) &\leq \nu'_a\left(\mathfrak{Z}(\mathfrak{p}), \frac{((\varsigma + \wp + \hbar)^{1-a})^{1-0}}{1 - ((\varsigma + \wp + \hbar)^{1-a})}\kappa\right)\\ &= \nu'_a\left(\frac{\mathcal{Q}||\mathfrak{p}||^a}{(\varsigma + \wp + \hbar)}\frac{3}{|\varsigma + \wp + \hbar|^a}, \frac{(\varsigma + \wp + \hbar)}{(\varsigma + \wp + \hbar)^a - (\varsigma + \wp + \hbar)}\kappa\right).\end{aligned}\right\}$$

Criteria 4. *For* $i = 1$,

$$\left.\begin{aligned}\mu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) &\geq \mu'_a\left(\mathfrak{Z}(\mathfrak{p}), \frac{((\varsigma + \wp + \hbar)^{a-1})^{1-1}}{1 - ((\varsigma + \wp + \hbar)^{a-1})}\kappa\right)\\ &= \mu'_a\left(\frac{\mathcal{Q}||\mathfrak{p}||^a}{(\varsigma + \wp + \hbar)}\frac{3}{|\varsigma + \wp + \hbar|^a}, \frac{(\varsigma + \wp + \hbar)}{(\varsigma + \wp + \hbar)^k - (\varsigma + \wp + \hbar)^a}\kappa\right)\\ \nu_a(\mathfrak{Z}(\mathfrak{p}) - \mathfrak{L}(\mathfrak{p}), \kappa) &\leq \nu'_a\left(\mathfrak{Z}(\mathfrak{p}), \frac{((\varsigma + \wp + \hbar)^{a-1})^{1-1}}{1 - ((\varsigma + \wp + \hbar)^{a-1})}\kappa\right)\\ &= \nu'_a\left(\frac{\mathcal{Q}||\mathfrak{p}||^a}{(\varsigma + \wp + \hbar)}\frac{3}{|\varsigma + \wp + \hbar|^a}, \frac{(\varsigma + \wp + \hbar)}{(\varsigma + \wp + \hbar) - (\varsigma + \wp + \hbar)^a}\kappa\right).\end{aligned}\right\}$$

□

6. Applications of the Euler–Lagrange Symmetry Additive Functional Equation

In this section, the various applications of the solutions of the Euler–Lagrange symmetry additive functional equation are analysed using Fourier series and Fourier transform for various intervals.

Example 2. *If an odd mapping* $\mathfrak{Z} : \mathbb{A} \to \mathbb{B}$ *satisfies the additive functional Equation (2) and the solution of the functional equation is* $\mathfrak{Z}(\mathfrak{p}) = \mathfrak{p}$, *then its Fourier cosine series for the interval* $(0, \mathcal{T})$ *is*

$$\mathfrak{Z}(\mathfrak{p}) = \frac{\mathcal{T}}{2} + \sum_{n=1,3,5...}^{\infty} \frac{-4}{\mathcal{T}n^2} \cos n\mathfrak{p}$$

to deduce the series $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ... = \frac{\mathcal{T}^4}{96}$ *using Parseval's theorem.*

Proof. The cosine series is $\mathfrak{z}(\mathfrak{p}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\mathcal{T}$; we have to find a_0 and a_n

$$a_0 = \frac{2}{\mathcal{T}} \int_0^{\mathcal{T}} \mathfrak{p}\, d\mathfrak{p} = \frac{2}{\mathcal{T}} \left[\frac{\mathfrak{p}^2}{2}\right]_0^{\mathcal{T}} = \frac{2}{2\mathcal{T}} \left[\mathcal{T}^2 - 0\right] = \mathcal{T}$$

$$a_n = \frac{2}{\mathcal{T}} \int_0^{\mathcal{T}} \mathfrak{p} \cos n\mathfrak{p}\ d\mathfrak{p} = \frac{2}{n^2\mathcal{T}} [(-1)^n - 1]$$

$$a_n = \begin{cases} \frac{-4}{n^2\mathcal{T}}, & \text{if } n = 1, 3, 5, ..., \\ 0, & \text{if } n = 2, 4, 6, ... \end{cases}$$

The required Fourier cosine series is

$$\mathfrak{z}(\mathfrak{p}) = \frac{\mathcal{T}}{2} + \sum_{n=1,3,5...}^{\infty} \frac{-4}{\mathcal{T}n^2} \cos n\mathfrak{p}.$$

Let the Parseval's identity for the Fourier cosine series be

$$\frac{2}{\mathcal{T}} \int_0^{\mathcal{T}} \mathfrak{p}^2 d\mathfrak{p} = \frac{\mathcal{T}^2}{2} + \sum_{n=1,3,5,...}^{\infty} \left(\frac{-4}{n^2\mathcal{T}}\right)^2$$

$$\frac{2}{\mathcal{T}} \left(\frac{\mathfrak{p}^3}{3}\right)_0^{\mathcal{T}} = \frac{\mathcal{T}^2}{2} + \sum_{n=1,3,5...}^{\infty} \frac{16}{\mathcal{T}^2 n^4}$$

$$\frac{\mathcal{T}^2}{6} \times \frac{\mathcal{T}^2}{16} = \sum_{n=1,3,5...}^{\infty} \frac{1}{n^4}$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ... = \frac{\mathcal{T}^4}{96}.$$

□

Example 3. *If an odd mapping* $\mathfrak{z} : \mathbb{A} \to \mathbb{B}$ *satisfies the additive functional Equation (2) and the solution is* $\mathfrak{z}(\mathfrak{p}) = \mathfrak{p}$*, then the Fourier series in the interval* $-l < \mathfrak{p} < l$ *is*

$$\mathfrak{z}(\mathfrak{p}) = \frac{2}{\mathcal{T}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\mathcal{T}\mathfrak{p}}{\ell}.$$

Proof. Since the given function is odd, the Fourier series is

$$\mathfrak{z}(\mathfrak{p}) = \sum_{n=1}^{\infty} b_n \sin \frac{n\mathcal{T}\mathfrak{p}}{\ell}.$$

We have to find b_n.

$$b_n = \frac{2}{l} \int_0^{l} \mathfrak{p} \sin \frac{n\mathcal{T}\mathfrak{p}}{\ell}\ d\mathfrak{p} = \frac{2}{n\mathcal{T}} \left[(-1)^{n+1}\right].$$

The required Fourier cosine series is

$$\mathfrak{Z}(\mathfrak{p}) = \frac{2}{\mathcal{T}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\mathcal{T}\mathfrak{p}}{\ell}.$$

□

Example 4. *If an odd mapping $\mathfrak{Z} : \mathbb{A} \to \mathbb{B}$ satisfies the additive functional Equation (2) and the solution is $\mathfrak{Z}(\mathfrak{p}) = \mathfrak{p}$, then the Fourier series for the interval $0 < \mathfrak{p} < 2l$ is*

$$\mathfrak{Z}(\mathfrak{p}) = l - \frac{2l}{\mathcal{T}} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\mathcal{T}\mathfrak{p}}{\ell}.$$

Proof. The General Fourier series is

$$\mathfrak{Z}(\mathfrak{p}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\mathcal{T}\mathfrak{p}}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\mathcal{T}\mathfrak{p}}{\ell}.$$

We have to find a_0, a_n, and b_n.

$$a_0 = \frac{1}{\ell} \int_0^{2\ell} \mathfrak{Z}(\mathfrak{p})\, d\mathfrak{p} = \frac{1}{\ell} \left[\int_0^{2\ell} \mathfrak{p}\, d\mathfrak{p} \right] = 2l.$$

$$a_n = \frac{1}{\ell} \int_0^{2\ell} \mathfrak{Z}(\mathfrak{p}) \cos \frac{n\mathcal{T}\mathfrak{p}}{\ell} d\mathfrak{p} = \frac{1}{\ell} \left[\int_0^{2\ell} \mathfrak{p} \cos \frac{n\mathcal{T}\mathfrak{p}}{\ell} d\mathfrak{p} \right] = \frac{1}{\ell} \left[\frac{l^2}{n^2\mathcal{T}^2} \cos \frac{n\,\mathcal{T}\mathfrak{p}}{\ell} \right]_0^{2l} = 0.$$

$$a_n = \frac{1}{\ell} \int_0^{2\ell} \mathfrak{Z}(\mathfrak{p}) \sin \frac{n\mathcal{T}\mathfrak{p}}{\ell} d\mathfrak{p} = \frac{1}{\ell} \left[\int_0^{2\ell} \mathfrak{p} \sin \frac{n\mathcal{T}\mathfrak{p}}{\ell} d\mathfrak{p} \right] = \frac{1}{\ell} \left[\frac{-\mathfrak{p}l}{n\mathcal{T}} \cos \frac{n\,\mathcal{T}\mathfrak{p}}{\ell} \right]_0^{2l} = \frac{-2l}{n\,\mathcal{T}}.$$

Then, the required Fourier series is

$$f(\mathfrak{p}) = l - \frac{2l}{\mathcal{T}} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\mathcal{T}\mathfrak{p}}{\ell}.$$

□

Example 5. *If an odd mapping $\mathfrak{Z} : \mathbb{A} \to \mathbb{B}$ satisfies the additive functional Equation (2) and the solution is $\mathfrak{Z}(\mathfrak{p}) = \mathfrak{p}$, then the Fourier Transform of the function*

$$\mathfrak{Z}(\mathfrak{p}) = \begin{cases} \mathfrak{p}, & \textit{if } -a < \mathfrak{p} < a, \\ 0, & \textit{otherwise}, \end{cases}$$

is

$$\mathfrak{Z}(s) = i\sqrt{\frac{2}{\mathcal{T}}} \left[\left(\frac{\sin sa - as \cos sa}{s^2} \right) \right].$$

Proof. Given

$$\mathfrak{Z}(\mathfrak{p}) = \begin{cases} \mathfrak{p}, & \text{if } -a < \mathfrak{p} < a, \\ 0, & \text{otherwise}. \end{cases}$$

The Fourier transform $\mathfrak{Z}(\mathfrak{p})$ is

$$F[\mathfrak{Z}(\mathfrak{p})] = \mathfrak{Z}(s) = \frac{1}{\sqrt{2\mathcal{T}}} \int_{-\infty}^{\infty} \mathfrak{Z}(\mathfrak{p}) e^{is\mathfrak{p}} d\mathfrak{p}.$$

$$\mathfrak{Z}(s) = \frac{1}{\sqrt{2\mathcal{T}}}\left[\int_{-\infty}^{-a} 0\, e^{is\mathfrak{p}}\,d\mathfrak{p} + \int_{-a}^{a} \mathfrak{p}\, e^{is\mathfrak{p}}\,d\mathfrak{p} + \int_{a}^{\infty} 0\, e^{is\mathfrak{p}}\,d\mathfrak{p}\right].$$

$$\mathfrak{Z}(s) = \frac{1}{\sqrt{2\mathcal{T}}}\int_{-a}^{a} \mathfrak{p}(\cos s\mathfrak{p} + i\sin s\mathfrak{p})\,d\mathfrak{p}.$$

$$\mathfrak{Z}(s) = i\sqrt{\frac{2}{\mathcal{T}}}\left[\left(\frac{\sin sa - as\cos sa}{s^2}\right)\right].$$

□

7. Conclusions

In this study, a new class of real-valued Euler–Lagrange symmetry additive functional equations has been introduced, and its general solution has been derived. Subsequently, the HURS of the equation has been determined by applying direct and FPT in IFNS. Also, it has been proven that if the control function is the IFNS of the products of powers of norms, then the additive functional equation is stable. The results obtained are useful as the estimates for the difference between the exact and approximate solutions of the equation of interest can be further determined. The computed results will bridge the gap existing in the literature concerning the stability results of equations of interest in IFNS. Some significant potential applications of the results have also been explored in this article. This article improves on several earlier outcomes presented in the literature. Finally, some applications in which the solution of this Euler–Lagrange symmetry additive functional equation can be applied by the Fourier series, and Fourier transforms with various intervals are also demonstrated. In addition, the stability of a given FE can also be determined in some other known spaces such as Felbin's and Menger probabilistic normed spaces, as they have not been explored yet. This is left as an open problem for future research.

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