symmetry

Article

Ulam Stability of a General Linear Functional Equation in Modular Spaces

Issam Aboutaib [1], Chaimaa Benzarouala [1], Janusz Brzdęk [2,*], Zbigniew Leśniak [3] and Lahbib Oubbi [4]

[1] Team GrAAF, Laboratory LMSA, Center CeReMAR, Department of Mathematics, Faculty of Sciences, Mohammed V University in Rabat, 4 Avenue Ibn Batouta, P.O. Box 1014 RP, Rabat 10000, Morocco
[2] Faculty of Applied Mathematics, AGH University of Science and Technology, Mickiewicza 30, 30-059 Kraków, Poland
[3] Department of Mathematics, Pedagogical University, Podchorążych 2, 30-084 Kraków, Poland
[4] Team GrAAF, Laboratory LMSA, Center CeReMAR, Ecole Normale Supérieure Takaddoum, Department of Mathematics, Mohammed V University in Rabat, Avenue Mohammed Belhassane Elouazzane, Rabat 10105, Morocco
* Correspondence: brzdek@agh.edu.pl

Abstract: Using the direct method, we prove the Ulam stability results for the general linear functional equation of the form $\sum_{i=1}^{m} A_i(f(\varphi_i(\bar{x}))) = D(\bar{x})$ for all $\bar{x} \in X^n$, where f is the unknown mapping from a linear space X over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ into a linear space Y over field $\mathbb{K}$; n and m are positive integers; $\varphi_1, \ldots, \varphi_m$ are linear mappings from X^n to X; $A_1, \ldots, A_m$ are continuous endomorphisms of Y; and $D : X^n \to Y$ is fixed . In this paper, the stability inequality is considered with regard to a convex modular on Y, which is lower semicontinuous and satisfies an additional condition (the Δ_2-condition). Our main result generalizes many similar stability outcomes published so far for modular space. It also shows that there is some kind of symmetry between the stability results for equations in modular spaces and those in classical normed spaces .

Keywords: Ulam stability; direct method; general linear functional equation; modular space

MSC: 39B62; 39B82; 46A80; 47J20

check for updates

Citation: Aboutaib, I.; Benzarouala, C.; Brzdęk, J.; Leśniak, Z.; Oubbi, L. Ulam Stability of a General Linear Functional Equation in Modular Spaces. *Symmetry* **2022**, *14*, 2468. https://doi.org/10.3390/sym14112468

Academic Editor: Ioan Rasa

Received: 28 October 2022
Accepted: 17 November 2022
Published: 21 November 2022

1. Introduction

In 1940, S. M. Ulam (cf. [1]) posed the stability problem for the functional equation of group homomorphisms. Quite soon, Hyers [2] provided the affirmative answer to this problem in real Banach spaces by using the approach that has subsequently been called the direct method. After that, the problem of the stability of various types of equations (not only functional ones) was extensively studied by many authors (see [3–10] for various types of information; examples; and further references).

For instance, in 2015, Bahyrycz and Olko [11] (see also [12]) published stability results for the following general functional equation:

$$\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij} x_j\right) + B = 0, \quad (1)$$

where f is the unknown mapping from a linear space X over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ into a Banach space Y over $\mathbb{K}$, B_i and b_{ij} are scalars, and B is a vector from Y. They used the fixed-point approach suggested in [13]. It should be mentioned that the application of fixed-point methods in Ulam stability was initiated in [14,15]. The result reads as follows.

Theorem 1. *Let $B \in Y$ be fixed and assume that either $\left(\sum_{i=1}^{m} B_i\right)B \neq 0$ or $B = 0$. Let $g\colon X \to Y$ and $\theta\colon X^n \to [0,\infty)$ satisfy the inequality*

$$\left\| \sum_{i=1}^{m} B_i g\left(\sum_{j=1}^{n} b_{ij} x_j\right) + B \right\| \leq \theta(x_1,\dots,x_n), \qquad x_1,\dots,x_n \in X. \tag{2}$$

Further, assume that $\emptyset \neq I \subset \{1,\dots,m\}$, $I_0 := \{1,\dots,m\} \setminus I \neq \emptyset$, $c_1,\dots,c_n \in \mathbb{K}$ and $\omega_1,\dots,\omega_n \in [0,\infty)$ exist such that

(i) $\sum_{j=1}^{n} a_{ij} c_j = 1$ *for* $i \in I$;

(ii) $\sum_{i\in I_0} |B_i|\omega_i < |\sum_{i\in I} B_i|$;

(iii) $\theta\left(\left(\sum_{j=1}^{n} b_{ij} c_j\right)x_1,\dots,\left(\sum_{j=1}^{n} b_{ij} c_j\right)x_n\right) \leq \omega_i \theta(x_1,\dots,x_n)$ *for* $i \in I_0$ *and* $x_1,\dots,x_n \in X$.

Then, there is a unique solution $G\colon X \to Y$ to functional Equation (1) with

$$\|g(x) - G(x)\| \leq \frac{\theta(c_1 x,\dots,c_n x)}{|\sum_{i\in I} B_i| - \sum_{i\in I_0} |B_i|\omega_i}, \qquad x \in X.$$

Moreover, G is the unique solution to (1) such that

$$\|g(x) - G(x)\| \leq \beta\theta(c_1 x,\dots,c_n x), \qquad x \in X,$$

with some constant $\beta \in (0,\infty)$.

The stability of the homogeneous version of Equation (1) (i.e., with $B = 0$) was first investigated by Forti [16]. The equation generalizes numerous functional equations that are well known. In particular, the special cases of it are the equations of Cauchy

$$f(x+y) = f(x) + f(y),$$

Jensen

$$f\left(\frac{1}{2}(x+y)\right) = \frac{1}{2}(f(x) + f(y)),$$

Jordan–von Neumann

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

but also the equations of Drygas, Fréchet, and Popoviciu; the monomial and polynomial functions (see [17]); the p-Wright affine function; and various others (e.g., cubic, quartic, quintic etc.). For examples of stability results for the mentioned equations, we refer to [18–29]. Further stability outcomes concerning (1) can be found, e.g., in [30–34] (see also [35,36] for analogous investigations concerning some particular cases of (1)). For information on the solutions to some of these functional equations, we refer to [17,37].

In [38], the authors introduced the following linear functional equation:

$$\sum_{i=1}^{m} A_i(f(\varphi_i(\bar{x}))) + b = 0, \quad \bar{x} := (x_1,\dots,x_n) \in X^n, \tag{3}$$

where again m and n are positive integers; $f : X \to Y$ is a mapping from a linear space X into a Banach space Y; and, for every $i \in \mathbb{N}_m := \{1,\dots,m\}$, φ_i is a linear mapping from X^n into X and A_i is a continuous endomorphism of Y and $b \in Y$. Using the classical Banach contraction theorem, they proved the stability of (3) in Banach spaces. Notice also that Equation (3) is a generalization of (1). The stability of another very general equation that could be considered a generalization of (1) was studied in [39].

Roughly speaking, the issue of Ulam stability can be formulated as follows: how much the approximate solutions of an equation differ from the exact solutions of this equation. The next definition explains more precisely how this notion could be understood in metric spaces ($\mathbb{R}_+$ denotes the set of non-negative reals and A^B means the family of all mappings from a set $B \neq \emptyset$ into a set $A \neq \emptyset$).

Definition 1. *Let $S \neq \emptyset$ and $U \neq \emptyset$ be nonempty sets, (W, d) and (V, ρ) be metric spaces, and $\mathcal{D} \subset W^U$ and $\mathcal{C} \subset \mathbb{R}_+^S$ be nonempty. Let $\mathcal{F}, \mathcal{E} : \mathcal{D} \to V^S$ and $\mathcal{G} : \mathcal{C} \to \mathbb{R}_+^U$ be given. If for every $\psi \in \mathcal{D}$ and $\delta \in \mathcal{C}$ with*

$$\rho\big((\mathcal{F}\psi)(s), (\mathcal{E}\psi)(s)\big) \leq \delta(s), \qquad s \in S,$$

there is $\phi \in \mathcal{D}$ satisfying the equation

$$\mathcal{F}\psi = \mathcal{E}\psi \tag{4}$$

and such that

$$d\big(\phi(t), \psi(t)\big) \leq (\mathcal{G}\delta)(t), \qquad t \in U,$$

then we say that Equation (4) is $\mathcal{G}$-stable in the Ulam sense.

However, the notion of an approximate solution and difference between two functions can be defined in different ways (see, e.g., [26,40–44]), depending on the tools that we use to measure distances. One such tool is a modular.

The notion of a modular space was introduced by Nakano [45] and next redefined and generalized by Musielak and Orlicz [46,47]. In the last decade, several authors studied the Ulam stability of functional equations in modular spaces (see [44,48–50]). For instance, using a fixed-point method due to Khamsi [51], Sadeghi established, in [49], a stability result for a generalized Jensen functional equation in a convex modular space. Additionally, using the same technique, Wongkum et al. [52] proved a stability result for a quartic functional equation.

In the present paper, we use the direct method (analogous to [2]) to investigate the stability of the functional equation

$$\sum_{i=1}^{m} A_i(f(\varphi_i(\bar{x}))) = D(\bar{x}), \quad \bar{x} := (x_1, \ldots, x_n) \in X^n, \tag{5}$$

for mappings f from a linear space X into a complete modular space Y_ρ, where n and m are positive integers, $\varphi_1, \ldots, \varphi_m$ are linear mappings from X^n to X; $A_1, \ldots, A_m$ are continuous endomorphisms of Y; and the function $D : X^n \to Y_\rho$ is non-constant.

In particular, our results generalize some earlier stability outcomes for the modular spaces in [44,48–50,52].

2. Preliminaries

We first recall some basic notions and properties in modular spaces, as in [6,7,44,46,47].

Definition 2. *A functional $\rho : Y \to [0, +\infty]$ is called a modular if, for every $x, y \in Y$,*

M1. $\rho(x) = 0$ if and only if $x = 0$;
M2. $\rho(\alpha x) = \rho(x)$ for every $\alpha \in \mathbb{K}$ with $|\alpha| = 1$;
M3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \in \mathbb{R}_+$ with $\alpha + \beta = 1$.

If we replace condition M3 with the following one:

M4. $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ *for every* $\alpha, \beta \in \mathbb{R}_+$ with $\alpha + \beta = 1$,

then the modular ρ is called a convex modular.

If ρ is a modular in Y, then the set

$$Y_\rho := \{x \in Y : \lim_{\lambda \to 0} \rho(\lambda x) = 0\}$$

is called a modular space. Let us note that Y_ρ is a linear subspace of Y.

Definition 3. *A modular ρ on Y is said to satisfy the Δ_2-condition if there is $k > 0$ such that $\rho(2y) \leq k\rho(y)$ for every $y \in Y_\rho$.*

It is easily seen that every norm is a convex modular that fulfills the Δ_2-condition. If ρ is a norm in Y, then clearly $Y_\rho = Y$, which means that our considerations also include the case where Y is a classical normed space.

Remark 1. (a) If ρ is a modular on Y and $y \in Y$, then the function $\mathbb{R}_+ \ni t \to \rho(ty)$ is non-decreasing, i.e., $\rho(ay) \leq \rho(by)$ for every $a, b \in \mathbb{R}_+$ with $a < b$ (it is enough to take $y = 0$ in M3).
(b) For a convex modular ρ on Y, we have $\rho(\alpha y) \leq |\alpha|\rho(y)$ for all $y \in Y$ and $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$ and, moreover,

$$\rho\left(\sum_{j=1}^{n} \alpha_j y_j\right) \leq \sum_{j=1}^{n} \alpha_j \rho(y_j) \tag{6}$$

for all $y_1, \ldots, y_n \in Y$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_+$ with $\sum_{j=1}^{n} \alpha_j \leq 1$.

Definition 4. *Let ρ be a modular on Y and $(y_n)_n$ be a sequence in Y. Then,*

(i) *$(y_n)_n$ is ρ-convergent to a point $y \in Y$ (which we denote by $y = \rho - \lim_n y_n$), if $\rho(y_n - y) \to 0$ as $n \to +\infty$;*
(ii) *$(y_n)_n$ is ρ-Cauchy if for any $\epsilon > 0$, we have $\rho(y_n - y_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$;*
(iii) *Y_ρ is said to be ρ-complete if every ρ-Cauchy sequence in Y_ρ is ρ-convergent.*
(iv) *A subset $C \subset Y_\rho$ is called ρ-closed if C contains every $x \in Y_\rho$ such that there is a sequence $(x_n)_n$ in C which is ρ-convergent to x.*

Notice that if $(x_n)_n$ is ρ-convergent to x, then $(\alpha x_n)_n$ is ρ-convergent to αx, for $\alpha \in \mathbb{R}_+$, $\alpha \leq 1$. This does not need to hold if $|\alpha| > 1$, unless ρ satisfies Δ_2.

Definition 5. *A modular ρ on Y is said to be lower semi-continuous if every sequence $(x_n)_n$ in Y_ρ that is ρ-convergent to some $x \in Y_\rho$, satisfies the inequality*

$$\rho(x) \leq \liminf_{n \to +\infty} \rho(x_n).$$

3. Stability of Equation (5)

In the sequel, X and Y are linear spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and ρ denotes a convex, lower semi-continuous modular on Y that satisfies the Δ_2-condition, with a constant $k > 0$. Moreover, we always assume that Y_ρ is ρ-complete.

Let $m > 1$ and n be positive integers, $\varphi_i : X^n \to X$ for $i \in \mathbb{N}_m := \{1, 2, \ldots, m\}$, and $A_1, \ldots, A_m$ be endomorphisms of Y_ρ that commute (i.e., $A_i \circ A_j = A_j \circ A_i$ for $i, j \in \mathbb{N}_m$). Moreover, we assume that each A_i is continuous with respect to the topology of the modular space Y_ρ (as in [53]).

An arbitrary element $(x_1, \ldots, x_n)$ of X^n will be denoted by $\bar{x}$, and, for every non empty $I \subset \mathbb{N}_m$, we define $A_I : Y_\rho \to Y_\rho$ by $A_I(x) := \sum_{i \in I} A_i(x)$ for $x \in Y_\rho$. If $I = \{1, 2, \ldots, m\}$, then we simply write A instead of $A_{\{1,2,\ldots,m\}}$. Next, given $I \subset \mathbb{N}_m$, by $i \notin I$ we mean that $i \in \mathbb{N}_m \setminus I$.

Our main result concerns the stability of (5) in modular spaces.

Theorem 2. *Let $D : X^n \to Y_\rho$ and $\psi : X \to X^n$ be such that, for every $\bar{x} \in X^n$,*

$$\sum_{i=1}^{m} A_i(D(\psi \circ \varphi_i(\bar{x}))) = \sum_{i=1}^{m} A_i(D(\varphi_i \circ \psi(x_1), \ldots, \varphi_i \circ \psi(x_n))). \tag{7}$$

Suppose the following exist: $\theta : X^n \to \mathbb{R}_+$, a proper subset I of $\mathbb{N}_m$ and positive real numbers ω_i and α_i, for $i \notin I$, such that A_I possesses an eigenvalue $M \neq 0$ with eigenspace Y_ρ^M and $f(X) \cup D(X^n) \subset Y_\rho^M$. Assume that

$$\varphi_j \circ \psi(x) = x, \tag{8}$$
$$\theta(\varphi_i \circ \psi(x_1), \dots, \varphi_i \circ \psi(x_n)) \le \omega_i \theta(\bar{x}), \tag{9}$$
$$\varphi_p(\varphi_i \circ \psi(x_1), \dots, \varphi_i \circ \psi(x_n)) = \varphi_i \circ \psi \circ \varphi_p(\bar{x}), \tag{10}$$
$$\theta \circ \psi \circ \varphi_i \circ \psi(x) \le \omega_i \theta \circ \psi(x), \tag{11}$$
$$\gamma := \frac{1}{|M|} \sum_{i \notin I} \alpha_i \omega_i < \min\left(1, \frac{2}{k}\right), \tag{12}$$
$$A_i(y) = \alpha_i y, \qquad \text{and} \qquad \sum_{i \notin I} \alpha_i \le |M| \tag{13}$$

for all $j \in I$, $i \notin I$, $p \in \mathbb{N}_m$, $\bar{x} \in X^n$, $x \in X$ and $y \in Y_\rho$.

If a function $f : X \to Y_\rho$ satisfies

$$\rho\left(\sum_{i=1}^{m} A_i(f(\varphi_i(\bar{x}))) - D(\bar{x})\right) \le \theta(\bar{x}), \quad \bar{x} \in X^n, \tag{14}$$

then there is a unique solution $G : X \to Y_\rho^M$ of (5) such that

$$\rho(MG(x) - Mf(x)) \le \frac{k\theta(\psi(x))}{2 - k\gamma}, \quad x \in X. \tag{15}$$

Proof. Taking $\bar{x} := \psi(x)$ in (14), for each $x \in X$ we obtain

$$\rho\left(\sum_{i \in I} A_i(f(\varphi_i(\psi(x)))) + \sum_{i \notin I} A_i(f(\varphi_i(\psi(x)))) - D(\psi(x))\right) \le \theta(\psi(x)),$$

whence, by (8), we come to

$$\rho\left(M\left[f(x) - \frac{-1}{M}\left(\sum_{i \notin I} A_i(f(\varphi_i(\psi(x)))) - D(\psi(x))\right)\right]\right) \le \theta(\psi(x)). \tag{16}$$

Let $\mathcal{M}$ denote the family of all $g : X \to Y_\rho^M$. The family $\mathcal{M}$ is nonempty since $f \in \mathcal{M}$. Now, for an arbitrary $g \in \mathcal{M}$, define the mapping $Tg : X \to Y_\rho$ by

$$Tg(t) := \frac{-1}{M}\left(\sum_{i \notin I} A_i(g(\varphi_i(\psi(t)))) - D(\psi(t))\right), \quad t \in X.$$

Then,

$$\begin{aligned} A_I(Tg(t)) &= A_I\left(\frac{-1}{M}\left[\sum_{i \notin I} A_i(g(\varphi_i \circ \psi(t))) - D(\psi(t))\right]\right) \\ &= \frac{-1}{M}\left(\sum_{i \notin I} A_i \circ A_I(g(\varphi_i \circ \psi(t))) + A_I(D(\psi(t)))\right) \\ &= \frac{-1}{M}\left(\sum_{i \notin I} A_i(Mg(\varphi_i \circ \psi(t))) - MD(\psi(t)\right) \\ &= MTg(t), \qquad t \in X, g \in \mathcal{M}. \end{aligned}$$

This shows that, for every $t \in X$ and $g \in \mathcal{M}$, $Tg(t) \in Y_\rho^M$ and consequently $Tg \in \mathcal{M}$.

Now, we show that for every $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,

$$\rho\Big(M\Big(T^{n+1}f - T^n f\Big)(t)\Big) \leq \gamma^n \theta \circ \psi(t), \quad t \in X. \tag{17}$$

The case $n = 0$ coincides with (16). So assume that (17) holds for a non-negative integer n. Then, by (6), (11), the definitions of T and γ (see (12)), and (13),

$$\begin{aligned}
&\rho\big(M\big(T^{n+2}f(t) - T^{n+1}f(t)\big)\big) \\
&\quad = \rho\Big(\sum_{i \notin I} A_i\Big(T^{n+1}f(\varphi_i \circ \psi(t)) - T^n f(\varphi_i \circ \psi(t))\Big)\Big) \\
&\quad \leq \frac{1}{|M|}\sum_{i \notin I} \alpha_i \rho\Big(M\Big(T^{n+1}f(\varphi_i \circ \psi(t)) - T^n f(\varphi_i \circ \psi(t))\Big)\Big) \\
&\quad \leq \frac{1}{|M|}\sum_{i \notin I} \alpha_i \gamma^n \theta \circ \psi \circ \varphi_i \circ \psi(t) \\
&\quad \leq \frac{1}{|M|}\sum_{i \notin I} \alpha_i \omega_i \gamma^n \theta \circ \psi(t) \\
&\quad \leq \gamma^{n+1}\theta \circ \psi(t).
\end{aligned}$$

Thus, we have shown that (17) holds for every $n \in \mathbb{N}_0$.

Next, by (6), (12), and the Δ_2 property, for all fixed $m, n \in \mathbb{N}_0$, and $t \in X$, one can obtain

$$\begin{aligned}
\rho\big(M\big(T^n f - T^{n+m} f\big)(t)\big) &= \rho\left(M\sum_{i=0}^{m-1}\left(\frac{1}{2}\right)^{i+1} 2^{i+1}\Big(T^{n+i}f - T^{n+i+1}f\Big)(t)\right) \\
&\leq \sum_{i=0}^{m-1}\left(\frac{k}{2}\right)^{i+1}\rho\Big(M\Big(T^{n+i}f - T^{n+i+1}f\Big)(t)\Big) \\
&\leq \frac{k\gamma^n}{2}\sum_{i=0}^{m-1}\left(\frac{k\gamma}{2}\right)^{i}\theta(\psi(t)) \\
&\leq k\gamma^n \frac{\theta(\psi(t))}{2 - k\gamma}.
\end{aligned}$$

Since $\gamma < 1$, we conclude that $(MT^n f(t))_n$ is a ρ-Cauchy sequence in Y_ρ for every $t \in X$. Since Y_ρ is ρ-complete and Y_ρ^M is ρ-closed, so $(MT^n f(t))_n$ is ρ-convergent in Y_ρ^M. This allows us to define a function $G : X \to Y_\rho^M$ by

$$G(t) := \frac{1}{M}\,\rho - \lim_{n \to +\infty} MT^n f(t), \quad t \in X.$$

Since ρ is lower semi-continuous, one has

$$\begin{aligned}
\rho(MG(t) - Mf(t)) &\leq \liminf_{n \to +\infty} \rho(MT^m f(t) - Mf(t)) \\
&\leq \frac{k\theta(\psi(t))}{2 - k\gamma}, \quad t \in X,
\end{aligned}$$

whereby we have (15).

Now, we prove that G satisfies Equation (5). First, we show by induction that, for every $k \in \mathbb{N}_0$,

$$\rho\left(\sum_{p=1}^{m} A_p \circ (T^k f) \circ \varphi_p(\bar{x}) - D(\bar{x})\right) \leq \gamma^k \theta(\bar{x}), \quad \bar{x} \in X^n. \tag{18}$$

The case $k = 0$ is just (14). Next, assume that (18) holds for $k \in \mathbb{N}_0$. Putting

$$\overline{\varphi_i \circ \psi(\bar{x})} := (\varphi_i \circ \psi(x_1), \ldots, \varphi_i \circ \psi(x_n))),$$

by the assumption (7), for every $i \notin I$ and every $\bar{x} \in X^n$, we obtain

$$\begin{aligned}
\sum_{p=1}^{m} A_p \circ (T^{k+1} f) \circ \varphi_p(\bar{x}) &= \sum_{p=1}^{m} A_p \circ (T(T^k f)) \circ \varphi_p(\bar{x}) \\
&= \frac{-1}{M} \sum_{i \notin I} A_i \left(\sum_{p=1}^{m} A_p \circ (T^k f) \circ \varphi_p \left(\overline{\varphi_i \circ \psi(\bar{x})} \right) \right) \\
&\quad + \frac{1}{M} \sum_{p=1}^{m} A_p (D \circ \psi \circ \varphi_p(\bar{x}))) \\
&= \frac{-1}{M} \sum_{i \notin I} A_i \left(\sum_{p=1}^{m} A_p \left((T^k f)(\varphi_p(\overline{\varphi_i \circ \psi(\bar{x})}) \right) \right. \\
&\quad + \frac{1}{M} \sum_{p=1}^{m} A_p (D(\varphi_p \circ \psi(x_1), \ldots, \varphi_p \circ \psi(x_n))) \\
&= \frac{-1}{M} \sum_{i \notin I} A_i \left(\sum_{p=1}^{m} A_p \left((T^k f)(\varphi_p(\overline{\varphi_i \circ \psi(\bar{x})}) \right) \right. \\
&\quad + \frac{1}{M} \left[\sum_{i \notin I} A_i \left(D(\overline{\varphi_i \circ \psi(\bar{x})}) \right) + \sum_{i \in I} A_i (D(\bar{x})) \right] \\
&= \frac{-1}{M} \sum_{i \notin I} A_i \left(\sum_{p=1}^{m} A_p \left((T^k f)(\varphi_p(\overline{\varphi_i \circ \psi(\bar{x})}) \right) - D(\overline{\varphi_i \circ \psi(\bar{x})}) \right) + D(\bar{x}),
\end{aligned}$$

whence

$$\begin{aligned}
&\rho\left(\sum_{p=1}^{m} A_p \circ (T^{k+1} f) \circ \varphi_p(\bar{x}) - D(\bar{x}) \right) \\
&= \rho\left(\frac{-1}{M} \sum_{i \notin I} A_i \circ \left[\sum_{p=1}^{m} A_p((T^k f)(\varphi_p(\overline{\varphi_i \circ \psi(\bar{x})}) - D(\overline{\varphi_i \circ \psi(\bar{x})} \right] \right) \\
&\leq \rho\left(\frac{-1}{M} \sum_{i \notin I} \alpha_i \left[\sum_{p=1}^{m} A_p((T^k f)(\varphi_p(\overline{\varphi_i \circ \psi(\bar{x})}) - D(\overline{\varphi_i \circ \psi(\bar{x})} \right] \right) \\
&\leq \frac{1}{|M|} \sum_{i \notin I} \alpha_i \rho\left(\sum_{p=1}^{m} A_p((T^k f)(\varphi_p(\overline{\varphi_i \circ \psi(\bar{x})}) - D(\overline{\varphi_i \circ \psi(\bar{x})} \right) \\
&\leq \gamma^k \frac{1}{|M|} \sum_{i \notin I} \alpha_i \theta\left(\overline{\varphi_i \circ \psi(\bar{x})} \right) \\
&\leq \gamma^k \frac{1}{|M|} \sum_{i \notin I} \alpha_i \omega_i \theta(\bar{x}) \\
&\leq \gamma^{k+1} \theta(\bar{x}).
\end{aligned}$$

This means that (18) holds for every $k \in \mathbb{N}_0$. Now, since the topology of the modular space Y_ρ is a linear topology, for each $\bar{x} \in X^n$, we obtain

$$\sum_{i=1}^{m} A_i \circ G \circ \varphi_i(\bar{x}) - D(\bar{x}) = \rho - \lim_{k \to +\infty} \left(\sum_{i=1}^{m} A_i \left((T^k f)(\varphi_i(\bar{x})) \right) - D(\bar{x}) \right).$$

and consequently

$$\rho\left(\sum_{i=1}^{m} A_i \circ G \circ \varphi_i(\bar{x})) - D(\bar{x})\right) \leq \liminf_{k\to+\infty} \rho\left(\sum_{i=1}^{m} A_i\Big((T^k f)(\varphi_i(\bar{x}))\Big) - D(\bar{x})\right)$$
$$\leq \liminf_{k\to+\infty} \gamma^k \theta(\bar{x}),$$

because ρ is lower semi-continuous. As $\gamma < 1$, this implies that

$$\sum_{p=1}^{m} A_p \circ G \circ \varphi_p(\bar{x}) = D(\bar{x}), \quad \bar{x} \in X^n.$$

Finally, to show the uniqueness of G, assume that $G_1 : X \to Y_\rho^M$ also is a solution of (5) that satisfies (15). First, we prove that G and G_1 are both fixed points of T. Since G is a solution of (5), we obtain

$$\sum_{i\in I} A_i \circ G \circ \varphi_i \circ \psi(t) + \sum_{i\notin I} A_i \circ G \circ \varphi_i \circ \psi(t) = D \circ \psi(t), \quad t \in X.$$

Using (8), we obtain

$$\sum_{i\in I} A_i \circ G(t) + \sum_{i\notin I} A_i \circ G \circ \varphi_i(\psi(t)) = D(\psi(t)), \quad t \in X.$$

Moreover, $A_I \circ G(t) = MG(t)$ for every $t \in X$ and therefore $TG = G$. Using the same argument, we obtain $TG_1 = G_1$.

Now, we prove by induction that, for every $n \in \mathbb{N}_0$,

$$\rho(MT^nG(t) - MT^nG_1(t)) \leq \frac{\gamma^n k^2 \theta(\psi(t))}{2 - k\gamma}, \quad t \in X. \tag{19}$$

We have

$$\rho(MG(t) - MG_1(t)) \leq \frac{1}{2}\rho(2M(G(t) - f(t))) + \frac{1}{2}\rho(2M(G_1(t) - f(t)))$$
$$\leq \frac{k}{2}\rho(MG(t) - Mf(t)) + \frac{k}{2}\rho(MG_1(t) - Mf(t))$$
$$\leq \frac{k^2\theta(\psi(t))}{2 - k\gamma}.$$

Then, (19) holds for $n = 0$. Next, if (19) holds for $n \in \mathbb{N}_0$, then

$$\rho\Big(MT^{n+1}G(t) - MT^{n+1}G_1(t)\Big) \leq \frac{1}{|M|}\sum_{i\notin I} \alpha_i \rho\big((MT^nG \circ \varphi_i - MT^nG_1 \circ \varphi_i)(\psi(t))\big)$$
$$\leq \frac{1}{|M|}\sum_{i\notin I} \alpha_i \gamma^n \frac{k^2\theta(\psi \circ \varphi_i \circ \psi(t))}{2 - k\gamma}$$
$$\leq \gamma^{n+1}\frac{k^2\theta(\psi(t))}{2 - k\gamma}.$$

Thus, by induction, we have shown (19). Therefore, for every $n \in \mathbb{N}_0$ and every $t \in X$, we have

$$\rho(MG(t) - MG_1(t)) = \rho(MT^nG(t) - MT^nG_1(t))$$
$$\leq \gamma^n \frac{k^2\theta(\psi(t))}{2 - k\gamma}.$$

Letting n tend to $+\infty$, we obtain $G(t) = G_1(t)$ for every $t \in X$. This finishes the proof. □

Using Theorem 2, we can show the stability of various linear functional equations. For instance, we can prove the stability of the following Cauchy inhomogeneous functional equation:

$$f(x_1 + x_2) - f(x_1) - f(x_2) = D(x_1, x_2). \tag{20}$$

Corollary 1. *Assume that* $\|\ \|$ *is a norm on* X, $L \in \mathbb{R}_+$, *and* $p, q \in [-\infty, s]$ *with* $s = \min\left(1, 2 - \frac{\ln(k)}{\ln(2)}\right)$. *Assume also that* $D : X^2 \to Y_\rho$ *is a given symmetric and biadditive mapping, and* $f : X \to Y_\rho$ *satisfies*

$$\rho(f(x_1 + x_2) - f(x_1) - f(x_2) - D(x_1, x_2)) \leq L(\|x_1\|^p + \|x_2\|^q),$$

for every $x_1, x_2 \in X$. *Then, there is a unique solution* $G : X \to Y_\rho$ *to the Cauchy inhomogeneous Equation* (20) *such that*

$$\rho(2f(x) - 2G(x)) \leq \frac{kL(\|x\|^p + \|x\|^q)}{2 - 2^{r-1}k}, \quad x \in X, \tag{21}$$

with $r = \max(p, q)$.

Proof. Here, we have $m = 3$, $n = 2$. Define $\psi : X \to X^2$ by $\psi(x) = (x, x)$, $x \in X$, and for every $i = 1, 2, 3$, the linear mappings $\varphi_i : X^2 \to X$ and $A_i : Y_\rho \to Y_\rho$ by $\varphi_1(x_1, x_2) := x_1 + x_2$, $\varphi_2(x_1, x_2) := x_1$, $\varphi_3(x_1, x_2) := x_2$ and $A_1(y) = -A_2(y) = -A_3(y) := y$. Then, (see the next remark) condition (7) holds. Putting $I = \{2, 3\}$, $\omega_1 = 2^r$ with $r = \max(p, q)$, $\alpha_1 = 1$, $M = -2$, and $\theta(x_1, x_2) = L(\|x_1\|^p + \|x_2\|^q)$ for $x_1, x_2 \in X$; from Theorem 2, we obtain that there is a unique solution $G : X \to Y_\rho$ to Equation (20), which satisfies (21), as desired. □

In a simplified situation when $X = \mathbb{R}$ and the modular is a norm, Corollary 1 has the following form.

Corollary 2. *Let* $\|\ \|$ *be a complete norm on* Y, $\widehat{y} \in Y$, $L \in \mathbb{R}_+$ *and* $p, q \in [-\infty, 1]$. *Moreover, let* $f : \mathbb{R} \to Y$ *be continuous at some point* $x_0 \in \mathbb{R}$ *and satisfy*

$$\|f(x_1 + x_2) - f(x_1) - f(x_2) - x_1 x_2 \widehat{y}\| \leq L(|x_1|^p + |x_2|^q),$$

for every $x_1, x_2 \in \mathbb{R}$. *Then, there is a unique vector* $y_0 \in Y$ *with*

$$\left\|f(x) - xy_0 - \frac{x^2}{2}\widehat{y}\right\| \leq \frac{L(|x|^p + |x|^q)}{2 - 2^r}, \quad x \in \mathbb{R}, \tag{22}$$

where $r = \max(p, q)$.

Proof. As we have already noticed just after Definition 3, every norm is a convex modular that satisfies the Δ_2-condition with $k = 2$. So, we can apply Corollary 1 (with $X = \mathbb{R}$ and $D(x_1, x_2) = x_1 x_2 \widehat{y}$ for $x_1, x_2 \in \mathbb{R}$). According to it there is a unique solution $G : \mathbb{R} \to Y$ to the Cauchy inhomogeneous Equation (20) such that

$$\|2f(x) - 2G(x)\| \leq \frac{kL(|x|^p + |x|^q)}{2 - 2^{r-1}k}, \quad x \in \mathbb{R}. \tag{23}$$

Note that in this case G fulfils the equation

$$G(x_1 + x_2) - G(x_1) - G(x_2) = x_1 x_2 \widehat{y}, \quad x_1, x_2 \in \mathbb{R},$$

whence

$$G(x_1+x_2)-\frac{(x_1+x_2)^2}{2}\,\widehat{y}-\left(G(x_1)-\frac{x_1^2}{2}\,\widehat{y}\right)-\left(G(x_2)-\frac{x_2^2}{2}\,\widehat{y}\right)=0,\quad x_1,x_2\in\mathbb{R}.$$

Hence, the function $G_0:\mathbb{R}\to Y$, given by

$$G_0(x)=G(x)-\frac{x^2}{2}\,\widehat{y},\quad x\in\mathbb{R},$$

is additive. Next, (23) implies that G is bounded on a neigbourhood of x_0 and so is G_0, which means (see, e.g., [37]) that there is $y_0\in Y$ such that

$$G_0(x)=xy_0,\quad x\in\mathbb{R}.$$

Now, it is easily seen that (23) yields (22). The uniqueness of G implies the uniqueness of y_0. □

Example 1. *If Y_ρ is a commutative algebra, then the function $D:X^2\to Y_\rho$ given by $D(x,y)=\varphi_1(x)\varphi_1(y)$ for all $x,y\in X$, where $\varphi_1:X\to Y_\rho$ is a linear mapping, is symmetric and biadditive.*

The next remark provides some comments on condition (7).

Remark 2. (1) Every constant function $D:X^n\to Y$ satisfies condition (7).
(2) If $D_1,D_2:X^n\to Y$ satisfy (7), then so does the function $\alpha_1D_1+\alpha_2D_2$ for any fixed scalars α_1,α_2.
(3) Consider the situation in Corollary 1 (i.e., when Equation (5) has the form (20)). Then, condition (7) has the form

$$D(x_1+x_2,x_1+x_2)-D(x_1,x_1)-D(x_2,x_2)=D(2x_1,2x_2)-2D(x_1,x_2).\tag{24}$$

It is easy to check that, for every $h:X\to Y$, the function $D:X^2\to Y$, given by

$$D(x_1,x_2)=h(x_1+x_2)-h(x_1)-h(x_2),\quad x_1,x_2\in X,\tag{25}$$

is a solution to Equation (24). In particular, if D is symmetric and biadditive, then (25) holds with $h(x)=\frac{1}{2}D(x,x)$ for $x\in X$. Thus, Equation (24) holds for every symmetric and biadditive function $D:X^2\to Y$.

4. Conclusions

We continue the investigation of the stability in the sense of Ulam of the non-homogeneous version of the very general linear functional Equation (5), which was introduced in [38] and generalizes numerous linear functional equations. Here, using the direct method, we show that this equation is stable in the context of complete modular spaces, whenever the modular is assumed to be convex and satisfies the Δ_2-condition. The outcome of this study covers most of the known results in the same context.

Author Contributions: Conceptualization, I.A., C.B., and L.O.; methodology, I.A., C.B., J.B., Z.L., and L.O.; software, I.A., C.B., J.B., and L.O.; validation, I.A., C.B., J.B., Z.L., and L.O.; formal analysis, I.A., C.B., J.B., Z.L., and L.O.; investigation, I.A., C.B., J.B., Z.L., and L.O.; data curation, I.A., C.B., J.B., Z.L., and L.O.; writing—original draft preparation, I.A., C.B., and L.O.; writing—review and editing, I.A., C.B., J.B., Z.L., and L.O.; visualization, I.A., C.B., J.B., Z.L., and L.O.; supervision, L.O.; project administration, I.A., C.B., and L.O.; and funding acquisition, I.A., C.B., J.B., Z.L., and L.O. All authors have read and agreed to the published version of the manuscript.

Funding: This work has received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience Publishers: New York, NY, USA, 1960.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef]
3. Brzdęk, J.; Popa, D.; Raşa, I.; Xu, B. *Ulam Stability of Operators*; Academic Press: Oxford, UK, 2018.
4. Hyers, D.H.; Isac, G.; Rassias, T.M. *Stability of Functional Equations in Several Variables*; Birkhäuser: Boston, MA, USA, 1998.
5. Jung, S.-M. *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*; Springer: New York, NY, USA, 2011.
6. Khodaei, H. On the hyperstability of (m,n)-derivations. *Fixed Point Theory* **2017**, *18*, 641–650. [CrossRef]
7. Kim, H.-M.; Shin, W.-Y. Approximate Lie $\star$-derivations on ρ-complete convex modular algebras. *J. Appl. Anal. Comput.* **2019**, *9*, 765–776. [CrossRef]
8. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [CrossRef]
9. Shah, S.O.; Tunç, C.; Rizwan, R.; Zada, A.; Khan, Q.U.; Ullah, I.; Ullah, I. Bielecki–Ulam's Types Stability Analysis of Hammerstein and Mixed Integro–Dynamic Systems of Non–Linear Form with Instantaneous Impulses on Time Scales. *Qual. Theory Dyn. Syst.* **2022**, *21*, 107. [CrossRef]
10. Tunç, O.; Tunç, C. Solution estimates to Caputo proportional fractional derivative delay integro-differential equations. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **2023**, *117*, 12. [CrossRef]
11. Bahyrycz, A.; Olko, J. On stability of the general linear equation. *Aequat. Math.* **2015**, *89*, 1461–1474. [CrossRef]
12. Bahyrycz, A.; Olko, J. Hyperstability of general linear functional equation. *Aequat. Math.* **2016**, *90*, 527–540. [CrossRef]
13. Brzdęk, J. Hyperstability of the Cauchy equation on restricted domains. *Acta Math. Hungarica* **2013**, *141*, 58–67. [CrossRef]
14. Cădariu, L.; Radu, V. Fixed points and the stability of Jensen's functional equation. *J. Inequal. Pure Appl. Math.* **2003**, *4*, 4.
15. Radu, V. The fixed point alternative and the stability of functional equations. *Fixed Point Theory* **2003**, *4*, 91–96.
16. Forti, G. L. Elementary remarks on Ulam-Hyers stability of linear functional equations. *J. Math. Anal. Appl.* **2007**, *328*, 109–118. [CrossRef]
17. Kuczma, M. *An Introduction to the Theory of Functional Equations and Inequalities: Cauchy's Equation and Jensen's Inequality*, 2nd ed.; Birkhäuser: Basel, Switzerland, 2009.
18. Jung, S.-M.; Rassias, M.Th. A linear functional equation of third order associated to the Fibonacci numbers. *Abstr. Appl. Anal.* **2014**, *2014*, 137468. [CrossRef]
19. Jung, S.-M.; Popa, D.; Rassias, M.T. On the stability of the linear functional equation in a single variable on complete metric groups. *J. Glob. Optim.* **2014**, *59*, 165–171. [CrossRef]
20. Lee, Y.W. On the stability on a quadratic Jensen type functional equation. *J. Math. Anal. Appl.* **2002**, *270*, 590–601. [CrossRef]
21. Lee, Y.W. Stability of a generalized quadratic functional equation with Jensen type. *Bull. Korean Math. Soc.* **2005**, *42*, 57–73. [CrossRef]
22. Mortici, C.; Rassias, M.T.; Jung, S.-M. On the stability of a functional equation associated with the Fibonacci numbers. *Abstr. Appl. Anal.* **2014**, *2014*, 546046. [CrossRef]
23. Pales, Z. Generalized stability of the Cauchy functional equation. *Aequat. Math.* **1998**, *56*, 222–232. [CrossRef]
24. Park, C. Generalized Hyers-Ulam Stability of Quadratic Functional Equations: A fixed-point approach. *Fixed Point Theory Appl.* **2008**, *2008*, 493751. [CrossRef]
25. Patel, B.M.; Patel, A.B. Stability of Quartic functional equations in 2-Banach space. *Int. J. Math. Anal.* **2013**, *7*, 1097–1107. [CrossRef]
26. Pinelas, S.; Govindan, V.; Tamilvanan, K. Stability of Cubic Functional Equation in Random Normed Space. *J. Adv. Math.* **2018**, *14*. [CrossRef]
27. Piszczek, M.; Szczawińska, J. Stability of the Drygas functional equation on restricted domain. *Results Math.* **2015**, *68*, 11–24. [CrossRef]
28. Smajdor, W. Note on a Jensen type functional equation. *Publ. Math. Debrecen* **2003**, *63*, 703–714.
29. Trif, T. Hyers-Ulam-Rassias stability of a Jensen type functional equation. *J. Math. Anal. Appl.* **2000**, *250*, 579–588. [CrossRef]
30. Benzarouala, C.; Oubbi, L. A purely fixed-point approach to the Ulam-Hyers stability and hyperstability of a general functional equation. In *Ulam Type Stability*; Brzdęk, J., Popa, D., Rassias, T.M., Eds.; Springer Nature: Cham, Switzerland, 2019; pp. 47–56.
31. Phochai, T.; Saejung, S. The hyperstability of general linear equation via that of Cauchy equation. *Aequat. Math.* **2019**, *93*, 781–789. [CrossRef]
32. Phochai, T.; Saejung, S. Hyperstability of generalized linear functional equations in several variables. *Bull. Austral. Math. Soc.* **2020**, *102*, 293–302. [CrossRef]
33. Zhang, D. On Hyperstability of generalised linear functional equations in several variables. *Bull. Aust. Math. Soc.* **2015**, *92*, 259–267. [CrossRef]
34. Zhang, D. On Hyers-Ulam stability of generalized linear functional equation and its induced Hyers-Ulam programming problem. *Aequat. Math.* **2016**, *90*, 559–568. [CrossRef]

35. Brzdęk, J.; Leśniak, Z.; Malejki, R. On the generalized Fréchet functional equation with constant coefficients and its stability. *Aeq. Math.* **2018**, *92*, 355–373. [CrossRef]
36. Brzdęk, J.; Leśniak, Z.; Malejki, R. On the stability of a generalized Fréchet functional equation with respect to hyperplanes in the parameter space. *Symmetry* **2021**, *13*, 384. [CrossRef]
37. Aczél, J.; Dhombres, J. *Functional Equations in Several Variables*; Cambridge University Press: Cambridge, UK, 1989.
38. Benzarouala, C.; Oubbi, L. Ulam-stability of a generalized linear functional equation, a fixed-point approach. *Aequat. Math.* **2020**, *94*, 989–1000. [CrossRef]
39. Bahyrycz, A.; Brzdęk, J.; Jabłońska, E.; Malejki, R. Ulam's stability of a generalization of the Fréchet functional equation *J. Math. Anal. Appl.* **2016**, *442*, 537–553. [CrossRef]
40. Bahyrycz, A.; Brzdęk, J.; El-hady, E.-S.; Leśniak, Z. On Ulam stability of functional equations in 2-normed spaces—A survey. *Symmetry* **2021**, *13*, 2200. [CrossRef]
41. Brzdęk, J.; El-hady, E.-S. On Ulam stability of functional equations in 2-normed spaces—A survey II. *Symmetry* **2022**, *14*, 1365.
42. Brzdęk, J.; Karapınar, E.; Petruşel, A. A fixed point theorem and the Ulam stability in generalized dq-metric spaces. *J. Math. Anal. Appl.* **2018**, *467*, 501–520. [CrossRef]
43. Brzdęk, J.; Popa, D.; Raşa, I. Hyers-Ulam stability with respect to gauges. *J. Math. Anal. Appl.* **2017**, *453*, 620–628. [CrossRef]
44. Karthikeyan, S.; Park, C.; Palani, P.; Kumar, T.R.K. Stability of an additive-quartic functional equation in modular spaces. *J. Math. Comput. SCI-JM* **2022**, *26*, 22–40. [CrossRef]
45. Nakano, H. *Modulared Semi-Ordered Linear Spaces*; Maruzen: Tokyo, Japan, 1950.
46. Musielak, J.; Orlicz, W. On modular spaces. *Stud. Math.* **1959**, *18*, 49–65. [CrossRef]
47. Musielak, J.; Orlicz, W. Some remarks on modular spaces. *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.* **1959**, *7*, 661–668.
48. Ramdoss, M.; Pachaiyappan, D.; Hwang, I.; Park, C. Stability of an n-variable mixed type functional equation in probabilistic modular spaces. *AIMS Math.* **2020**, *5*, 5903–5915. [CrossRef]
49. Sadeghi, G. A fixed-point approach to stability of functional equations in modular spaces. *Bull. Malays. Math. Sci. Soc.* **2014**, *37*, 333–344.
50. Wongkum, K.; Chaipunya, P.; Kumam, P. On the Generalized Ulam-Hyers-Rassias Stability of Quadratic Mappings in Modular Spaces without Δ-Conditions. *J. Funct. Spaces* **2015**, *2015*, 461719.
51. Khamsi, M.A. Quasicontraction mappings in modular spaces without δ_2-condition. *Fixed Point Theory Appl.* **2008**, 2008, 916187. [CrossRef]
52. Wongkum, K.; Kumam, P.; Cho, Y.J.; Thounthong, P.; Chaipunya, P. On the generalized Ulam-Hyers-Rassias stability for quartic functional equation in modular spaces. *J. Nonlinear Sci. Appl.* **2017**, *10*, 1399–1406. [CrossRef]
53. Hajji, A. Modular spaces topology. *Appl. Math.* **2013**, *4*, 1296–1300. [CrossRef]