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The Exact Solutions of Fractional Differential Systems with n Sinusoidal Terms under Physical Conditions

Laila F. Seddek [1,2,*], Essam R. El-Zahar [1,3] and Abdelhalim Ebaid [4]

[1] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdul-Aziz University, Al-Kharj 11942, Saudi Arabia
[2] Department of Engineering Mathematics and Physics, Faculty of Engineering, Zagazig University, Zagazig 44519, Egypt
[3] Department of Basic Engineering Science, Faculty of Engineering, Menofia University, Shebin El-Kom 32511, Egypt
[4] Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia
* Correspondence: l.morad@psau.edu.sa

Abstract: This paper considers the classes of the first-order fractional differential systems containing a finite number n of sinusoidal terms. The fractional derivative employs the Riemann–Liouville fractional definition. As a method of solution, the Laplace transform is an efficient tool to solve linear fractional differential equations. However, this method requires to express the initial conditions in certain fractional forms which have no physical meaning currently. This issue formulated a challenge to solve fractional systems under real/physical conditions when applying the Riemann–Liouville fractional definition. The principal incentive of this work is to overcome such difficulties via presenting a simple but effective approach. The proposed approach is successfully applied in this paper to solve linear fractional systems of an oscillatory nature. The exact solutions of the present fractional systems under physical initial conditions are derived in a straightforward manner. In addition, the obtained solutions are given in terms of the entire exponential and periodic functions with arguments of a fractional order. The symmetric/asymmetric behaviors/properties of the obtained solutions are illustrated. Moreover, the exact solutions of the classical/ordinary versions of the undertaken fractional systems are determined smoothly. In addition, the properties and the behaviors of the present solutions are discussed and interpreted.

Keywords: Riemann–Liouville fractional derivative; fractional differential equation; sinusoidal; exact solution

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1. Introduction

Unlike the classical calculus (CC) with integer derivatives, the fractional calculus (FC) implements the derivatives of an arbitrary order (non-integer) [1–3]. So, the FC is considered as a generalization of the CC. During the past decades, numerous physical, engineering, and biological problems have been investigated by means of the FC ([4–9]). There are several definitions for the derivatives of an arbitrary order, such as the Caputo fractional derivative (CFD) [10–22], the Riemann–Liouville fractional derivative (RLFD) [23–25], and the conformable derivative [26–29]. However, some difficulties arise when applying the RLFD to solve fractional models under real physical conditions. The present paper is an attempt to face such an issue by considering the following class of first-order fractional ordinary equations (FODEs):

$$\begin{aligned} {}^{RL}_{-\infty}D^{\alpha}_{t}y(t)+\omega^2 y(t) &= b_1\sin(\Omega_1 t)+b_2\sin(\Omega_2 t)+\cdots+b_n\sin(\Omega_n t), \\ &= \sum_{j=1}^{n} b_j\sin(\Omega_j t), \qquad y(0)=A, \quad \alpha\in(0,1], \end{aligned} \tag{1}$$

where α is the non-integer order of the RLFD. The constant A is real while ω, b_j, and Ω_j may be real or complex $\forall\ j = 1, 2, 3, \ldots, n$.

The applications of the class (1) may arise in oscillatory models in engineering when the FC is incorporated. This class splits to other physical classes. As examples, for complex ω, i.e., $\omega = i\mu$ (μ is real), where i is the imaginary number, the model (1) becomes

$$ {}^{RL}_{-\infty}D^{\alpha}_{t} y(t) - \mu^2 y(t) = \sum_{j=1}^{n} b_j \sin(\Omega_j t), \;\; y(0) = A, \;\; \alpha \in (0,1]. \tag{2}$$

In addition, if $\Omega_j = i\sigma_j$ and $b_j = -id_j$, the classes (1) and (2) take the form:

$$ {}^{RL}_{-\infty}D^{\alpha}_{t} y(t) + \omega^2 y(t) = \sum_{j=1}^{n} d_j \sinh(\sigma_j t), \;\; y(0) = A, \;\; \alpha \in (0,1], \tag{3}$$

and

$$ {}^{RL}_{-\infty}D^{\alpha}_{t} y(t) - \mu^2 y(t) = \sum_{j=1}^{n} d_j \sinh(\sigma_j t), \;\; y(0) = A, \;\; \alpha \in (0,1], \tag{4}$$

in terms of hyperbolic functions, respectively.

In Refs. [1–3], the RLFD of order $\alpha \in \mathbb{R}^{+}_{0}$ of function $f : [c, d] \to \mathbb{R}$ $(-\infty < c < d < \infty)$ is defined as

$$ {}^{RL}_{c}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left(\int_{c}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \right), \quad n = [\alpha] + 1, \;\; t > c, \tag{5}$$

where $[\alpha]$ is the integral part of α. If $0 < \alpha \leq 1$ and $c \to -\infty$, then

$$ {}^{RL}_{-\infty}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_{-\infty}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau \right). \tag{6}$$

It is important to refer to the initial condition (IC) $y(0) = A$ being physical, unlike the nonphysical condition $D^{\alpha-1}_{t} y(0) = A$ that has been considered by the authors [30]. In fact, the IC in the last fractional form is required when solving an FODE via the Laplace transform (LT). This is, simply, because the LT of the RLFD as $c \to 0$, i.e., ${}^{RL}_{0}D^{\alpha}_{t}$, is [1–3,23,30]

$$ \mathcal{L}\left[{}^{RL}_{0}D^{\alpha}_{t} y(t) \right] = s^{\alpha} Y(s) - D^{\alpha-1}_{t} y(0), \tag{7}$$

which is given in terms of $D^{\alpha-1}_{t} y(0)$. Really, the main difference between ${}^{RL}_{-\infty}D^{\alpha}_{t}$ and ${}^{RL}_{0}D^{\alpha}_{t}$ lies in the nature of the considered IC of the problem. In the literature, one can see that the obtained solutions of the physical models depend on both the nature of the given classical/fractional ICs along with the implemented method of solution.

In this regard, Ebaid and Al-Jeaid [30] applied the RLFDs ${}^{RL}_{-\infty}D^{\alpha}_{t}$ and ${}^{RL}_{0}D^{\alpha}_{t}$ to obtain a dual solution for a similar model under the nonphysical IC $D^{\alpha-1}_{t} y(0)$ using the LT. Although the LT was shown as an effective tool to exactly investigate several models [31–37], it may not be appropriate to deal with the class (1) under the physical IC $y(0) = A$ by means of the RLFD operator ${}^{RL}_{0}D^{\alpha}_{t}$. However, the solution is still available under this physical condition via the RLFD operator ${}^{RL}_{-\infty}D^{\alpha}_{t}$ along with avoiding the LT, as will be shown through this paper.

Therefore, the main incentive of the present work is to introduce a new approach to obtain the real solution of the current model under the physical IC $y(0) = A$ through the following properties (see Refs. [30,38]):

$$ {}^{RL}_{-\infty}D^{\alpha}_{t} e^{i\omega t} = (i\omega)^{\alpha} e^{i\omega t}, \tag{8} $$

$$ {}^{RL}_{-\infty}D^{\alpha}_{t} \cos(\omega t) = \omega^{\alpha} \cos\Big(\omega t + \frac{\alpha\pi}{2}\Big), \tag{9} $$

$$ {}^{RL}_{-\infty}D^{\alpha}_{t} \sin(\omega t) = \omega^{\alpha} \sin\Big(\omega t + \frac{\alpha\pi}{2}\Big). \tag{10} $$

By using the above properties, it will be shown that the real solution of class (1) exists at specific values of the fractional-order α. The symmetric/asymmetric behaviors/properties of the obtained solutions will be demonstrated. Furthermore, it will be declared that the solution of the class (2) is real at any arbitrary value α. In addition, the solutions of the corresponding classes with the classical/ordinary derivative, i.e., as $\alpha \to 1$, will be evaluated.

A brief description of the structure of this paper is as follows. In Section 2, an analysis of the complementary and particular solutions is presented. Section 3 is devoted to obtaining the exact solutions for the fractional classes. In Section 4, the exact solutions for the ordinary classes are obtained. The behaviors/properties of the solution are introduced in Section 5. The paper is concluded in Section 6.

2. Analysis

The complementary solution $y_c(t)$ of Equation (1) can be obtained in the form, see [30]:

$$ y_c(t) = c\, e^{i\delta t}, \quad \delta = -i\Big(-\omega^2\Big)^{1/\alpha}, \tag{11} $$

which satisfies the homogeneous equation:

$$ {}^{RL}_{-\infty}D^{\alpha}_{t} y(t) + \omega^2 y(t) = 0. \tag{12} $$

In order to evaluate the constant c, the given IC will be applied on the general solution $y(t) = y_c(t) + y_p(t)$ in a subsequent section where $y_p(t)$ is a particular solution of the non-homogeneous Equation (1). A simple method to calculate $y_p(t)$ is explained through the following theorem.

Theorem 1. *The $y_p(t)$ of the class (1) is in the form:*

$$ y_p(t) = \sum_{j=1}^{n} b_j \left(\frac{\omega^2 \sin(\Omega_j t) + \Omega_j^{\alpha} \sin(\Omega_j t - \frac{\pi\alpha}{2})}{\omega^4 + \Omega_j^{2\alpha} + 2\omega^2 \Omega_j^{\alpha} \cos(\frac{\pi\alpha}{2})} \right), \tag{13} $$

Proof. Let us assume that

$$ y_p(t) = \sum_{j=1}^{n} (\rho_{1j} \cos(\Omega_j t) + \rho_{2j} \sin(\Omega_j t)). \tag{14} $$

Using the preceding properties of the RLFD operator ${}^{RL}_{-\infty}D^{\alpha}_{t}$, we have

$$ \begin{aligned} {}^{RL}_{-\infty}D^{\alpha}_{t} y_p &= \sum_{j=1}^{n} \Big(\rho_{1j}\, {}^{RL}_{-\infty}D^{\alpha}_{t} \cos(\Omega_j t) + \rho_{2j}(\alpha)\, {}^{RL}_{-\infty}D^{\alpha}_{t} \sin(\Omega_j t) \Big), \\ &= \sum_{j=1}^{n} \Omega_j^{\alpha} \cos(\Omega_j t) \Big(\rho_{1j} \cos\Big(\frac{\pi\alpha}{2}\Big) + \rho_{2j} \sin\Big(\frac{\pi\alpha}{2}\Big) \Big) + \\ &\quad \sum_{j=1}^{n} \Omega_j^{\alpha} \sin(\Omega_j t) \Big(\rho_{2j} \cos\Big(\frac{\pi\alpha}{2}\Big) - \rho_{1j} \sin\Big(\frac{\pi\alpha}{2}\Big) \Big). \end{aligned} \tag{15} $$

Thus,

$$
\begin{aligned}
{}^{RL}_{-\infty}D_t^{\alpha} y_p + \omega^2 y_p \quad = \quad & \sum_{j=1}^{n} \Big[\Big(\Omega_j^{\alpha} \cos\Big(\frac{\pi\alpha}{2}\Big) + \omega^2 \Big) \rho_{1j} + \Omega_j^{\alpha} \sin\Big(\frac{\pi\alpha}{2}\Big) \rho_{2j} \Big] \cos(\Omega_j t) + \\
& \sum_{j=1}^{n} \Big[\Big(\Omega_j^{\alpha} \cos\Big(\frac{\pi\alpha}{2}\Big) + \omega^2 \Big) \rho_{2j} - \Omega_j^{\alpha} \sin\Big(\frac{\pi\alpha}{2}\Big) \rho_{1j} \Big] \sin(\Omega_j t). \qquad (16)
\end{aligned}
$$

Inserting the last result into Equation (1) yields

$$
\begin{aligned}
& \Big(\Omega_j^{\alpha} \cos\Big(\frac{\pi\alpha}{2}\Big) + \omega^2 \Big) \rho_{1j} + \Omega_j^{\alpha} \sin\Big(\frac{\pi\alpha}{2}\Big) \rho_{2i} = 0, \\
& \Big(\Omega_j^{\alpha} \cos\Big(\frac{\pi\alpha}{2}\Big) + \omega^2 \Big) \rho_{2j} - \Omega_j^{\alpha} \sin\Big(\frac{\pi\alpha}{2}\Big) \rho_{1j} = b_j,
\end{aligned} \qquad (17)
$$

which can be easily solved to obtain ρ_{1j} and ρ_{2j} in the forms:

$$
\rho_{1j} = -\frac{\Omega^{\alpha} b_j \sin\left(\frac{\pi\alpha}{2}\right)}{\omega^4 + \Omega_j^{2\alpha} + 2\omega^2 \Omega_j^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right)}, \quad \rho_{2j} = \frac{b_j \omega^2 + \Omega_j^{\alpha} b_j \cos\left(\frac{\pi\alpha}{2}\right)}{\omega^4 + \Omega_j^{2\alpha} + 2\omega^2 \Omega_j^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right)}. \qquad (18)
$$

Employing (18) into (14), we find

$$
y_p(t) = \sum_{j=1}^{n} b_j \left(\frac{\omega^2 \sin(\Omega_j t) + \Omega_j^{\alpha} \sin\left(\Omega_j t - \frac{\pi\alpha}{2}\right)}{\omega^4 + \Omega_j^{2\alpha} + 2\omega^2 \Omega_j^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right)} \right), \qquad (19)
$$

which completes the proof. □

3. Solution of the Fractional Models: $\alpha \in (0, 1)$

Lemma 1. *The solution of the fractional class (1) is*

$$
y(t) = \left(A + \sum_{j=1}^{n} \frac{\Omega_j^{\alpha} b_j \sin\left(\frac{\pi\alpha}{2}\right)}{\omega^4 + \Omega_j^{2\alpha} + 2\omega^2 \Omega_j^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right)} \right) e^{\left(-\omega^2\right)^{\frac{1}{\alpha}} t} + \sum_{j=1}^{n} b_j \left(\frac{\omega^2 \sin(\Omega_j t) + \Omega_j^{\alpha} \sin\left(\Omega_j t - \frac{\pi\alpha}{2}\right)}{\omega^4 + \Omega_j^{2\alpha} + 2\omega^2 \Omega_j^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right)} \right). \qquad (20)
$$

Proof. The preceding analysis reveals that the general solution of the class (1) is in the form:

$$
y(t) = c\, e^{i\delta t} + \sum_{j=1}^{n} b_j \left(\frac{\omega^2 \sin(\Omega_j t) + \Omega_j^{\alpha} \sin\left(\Omega_j t - \frac{\pi\alpha}{2}\right)}{\omega^4 + \Omega_j^{2\alpha} + 2\omega^2 \Omega_j^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right)} \right). \qquad (21)
$$

From this equation, at $t = 0$, we obtain

$$
y(0) = c - \sum_{j=1}^{n} \frac{\Omega_j^{\alpha} b_j \sin\left(\frac{\pi\alpha}{2}\right)}{\omega^4 + \Omega_j^{2\alpha} + 2\omega^2 \Omega_j^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right)}, \qquad (22)
$$

and hence the IC can be applied to give

$$
c = A + \sum_{j=1}^{n} \frac{\Omega_j^{\alpha} b_j \sin\left(\frac{\pi\alpha}{2}\right)}{\omega^4 + \Omega_j^{2\alpha} + 2\omega^2 \Omega_j^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right)}. \qquad (23)
$$

Substituting (23) into (21), the solution reads

$$
y(t) = \left(A + \sum_{j=1}^{n} \frac{\Omega_j^{\alpha} b_j \sin\left(\frac{\pi\alpha}{2}\right)}{\omega^4 + \Omega_j^{2\alpha} + 2\omega^2 \Omega_j^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right)} \right) e^{\left(-\omega^2\right)^{\frac{1}{\alpha}} t} + \sum_{j=1}^{n} b_j \left(\frac{\omega^2 \sin(\Omega_j t) + \Omega_j^{\alpha} \sin\left(\Omega_j t - \frac{\pi\alpha}{2}\right)}{\omega^4 + \Omega_j^{2\alpha} + 2\omega^2 \Omega_j^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right)} \right). \qquad (24)
$$

It can be seen that the above solution satisfies the IC. In addition, the solution (24) is real at specific values of α; this point will be discussed later. □

Lemma 2. *The solution of the fractional class (2) is*

$$y(t) = \left(A + \sum_{j=1}^{n} \frac{\Omega_j^{\alpha} b_j \sin\left(\frac{\pi\alpha}{2}\right)}{\mu^4 + \Omega_j^{2\alpha} - 2\mu^2\Omega_j^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)}\right) e^{\mu^{\frac{2}{\alpha}} t} - \sum_{j=1}^{n} b_j \left(\frac{\mu^2 \sin\left(\Omega_j t\right) - \Omega_j^{\alpha} \sin\left(\Omega_j t - \frac{\pi\alpha}{2}\right)}{\mu^4 + \Omega_j^{2\alpha} - 2\mu^2\Omega_j^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)}\right). \tag{25}$$

Proof. As mentioned in Section 1, the class (2) is a transformed version of the class (1) when $\omega = i\mu$. Hence, the solution of the class (2) can be directly obtained from the solution of the class (1), given in lemma 1, with the aide of the substitution $\omega = i\mu$, which yields

$$y(t) = \left(A + \sum_{j=1}^{n} \frac{\Omega_j^{\alpha} b_j \sin\left(\frac{\pi\alpha}{2}\right)}{\mu^4 + \Omega_j^{2\alpha} - 2\mu^2\Omega_j^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)}\right) e^{\mu^{\frac{2}{\alpha}} t} + \sum_{j=1}^{n} b_j \left(\frac{-\mu^2 \sin\left(\Omega_j t\right) + \Omega_j^{\alpha} \sin\left(\Omega_j t - \frac{\pi\alpha}{2}\right)}{\mu^4 + \Omega_j^{2\alpha} - 2\mu^2\Omega_j^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)}\right), \tag{26}$$

or

$$y(t) = \left(A + \sum_{j=1}^{n} \frac{\Omega_j^{\alpha} b_j \sin\left(\frac{\pi\alpha}{2}\right)}{\mu^4 + \Omega_j^{2\alpha} - 2\mu^2\Omega_j^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)}\right) e^{\mu^{\frac{2}{\alpha}} t} - \sum_{j=1}^{n} b_j \left(\frac{\mu^2 \sin\left(\Omega_j t\right) - \Omega_j^{\alpha} \sin\left(\Omega_j t - \frac{\pi\alpha}{2}\right)}{\mu^4 + \Omega_j^{2\alpha} - 2\mu^2\Omega_j^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)}\right), \tag{27}$$

which completes the proof. □

Remark 1. *The analytic method used to obtain the exact solutions of the fractional classes (1) and (2) is shown in this section. The other fractional classes (3) and (4) can also be obtained similarly. It can be seen from the solution (20) of the fractional class (1) that it is not always a real solution for $\alpha \in (0,1)$. This is simply because $(-\omega^2)^{1/\alpha} \notin \mathbb{R}\ \forall\ \alpha \in (0,1)$, but there are certain values of the fractional-order α at which the solution (20) is real, $y(t) \in \mathbb{R}$. Such values of α will be addressed in a subsequent section.*

However, the solution (25) of the fractional class (2) is always a real solution $\forall\ \alpha \in (0,1)$ where $\mu^{2/\alpha} \in \mathbb{R}$ for $\mu \in \mathbb{R}$. In the case of the ordinary/classical derivative, i.e., as $\alpha \to 1$, then the solutions (20) and (25) are real. The solution of the fractional classes (3) and (4) can be obtained via substituting $\Omega_j = i\sigma_j$ and $b_j = -id_j$ into the solutions (20) and (25), respectively. Although, the resulting solutions of fractional classes (3) and (4) are not real at any value of α. In fact, the solutions of classes (3) and (4) are only real when $\alpha \to 1$. The solutions of the four classes (1)–(4), as $\alpha \to 1$, are determined in the next section.

4. Solution of the Classical/Ordinary Models: $\alpha \to 1$

This section focuses on obtaining the exact solutions of the classical/ordinary versions of the classes (1)–(4) when $\alpha \to 1$,

4.1. Class (1)

As $\alpha \to 1$, the class (1) is transformed to the following class of ODEs:

$$y'(t) + \omega^2 y(t) = \sum_{j=1}^{n} b_j \sin(\Omega_j t),\ \ y(0) = A. \tag{28}$$

The solution of this class can be derived from Equation (20) by letting $\alpha \to 1$, and accordingly, we have

$$y(t) = \left(A + \sum_{j=1}^{n} \frac{\Omega_j b_j}{\omega^4 + \Omega_j^2}\right) e^{-\omega^2 t} + \sum_{j=1}^{n} b_j \left(\frac{\omega^2 \sin\left(\Omega_j t\right) + \Omega_j \sin\left(\Omega_j t - \frac{\pi}{2}\right)}{\omega^4 + \Omega_j^2}\right), \tag{29}$$

which is equivalent to

$$y(t) = \left(A + \sum_{j=1}^{n} \frac{\Omega_j b_j}{\omega^4 + \Omega_j^2}\right) e^{-\omega^2 t} + \sum_{j=1}^{n} b_j \left(\frac{\omega^2 \sin\left(\Omega_j t\right) - \Omega_j \cos\left(\Omega_j t\right)}{\omega^4 + \Omega_j^2}\right). \tag{30}$$

The validity of the solution (30) can be easily verified by direct substitution into (28). Moreover, this solution satisfies the given IC.

4.2. Class (2)

The class (2), as $\alpha \to 1$, reduces to ODEs:

$$y'(t) - \mu^2 y(t) = \sum_{j=1}^{n} b_j \sin(\Omega_j t), \;\; y(0) = A. \tag{31}$$

From Equation (24), we obtain as $\alpha \to 1$ that

$$y(t) = \left(A + \sum_{j=1}^{n} \frac{\Omega_j b_j}{\mu^4 + \Omega_j^2} \right) e^{\mu^2 t} - \sum_{j=1}^{n} b_j \left(\frac{\mu^2 \sin(\Omega_j t) - \Omega_j \sin(\Omega_j t - \frac{\pi}{2})}{\mu^4 + \Omega_j^2} \right), \tag{32}$$

or

$$y(t) = \left(A + \sum_{j=1}^{n} \frac{\Omega_j b_j}{\mu^4 + \Omega_j^2} \right) e^{\mu^2 t} - \sum_{j=1}^{n} b_j \left(\frac{\mu^2 \sin(\Omega_j t) + \Omega_j \cos(\Omega_j t)}{\mu^4 + \Omega_j^2} \right). \tag{33}$$

4.3. Class (3)

The class (3) as $\alpha \to 1$ becomes

$$y'(t) + \omega^2 y(t) = \sum_{j=1}^{n} d_j \sinh(\sigma_j t), \;\; y(0) = A. \tag{34}$$

Because this class is transformed from the class (1) when $\Omega_j = i\sigma_j$, and $b_j = -id_j$, then the solution of the current class is determined from Equation (30) as

$$y(t) = \left(A + \sum_{j=1}^{n} \frac{\sigma_j d_j}{\omega^4 - \sigma_j^2} \right) e^{-\omega^2 t} - \sum_{j=1}^{n} i d_j \left(\frac{\omega^2 \sin(i\sigma_j t) - i\sigma_j \cos(i\sigma_j t)}{\omega^4 - \sigma_j^2} \right), \tag{35}$$

i.e.,

$$y(t) = \left(A + \sum_{j=1}^{n} \frac{\sigma_j d_j}{\omega^4 - \sigma_j^2} \right) e^{-\omega^2 t} + \sum_{j=1}^{n} d_j \left(\frac{\omega^2 \sinh(\sigma_j t) - \sigma_j \cosh(\sigma_j t)}{\omega^4 - \sigma_j^2} \right). \tag{36}$$

4.4. Class (4)

If $\omega = i\mu$, $\Omega_j = i\sigma_j$, and $b_j = -id_j$, then the class (1) as $\alpha \to 1$ is equivalent to the following class of ODEs:

$$y'(t) - \mu^2 y(t) = \sum_{j=1}^{n} d_j \sinh(\sigma_j t), \;\; y(0) = A. \tag{37}$$

In this case, we have three possible ways to obtain the solution of the current class. The first way is to substitute $\omega = i\mu$, $\Omega_j = i\sigma_j$, and $b_j = -id_j$ into Equation (30). The second is to substitute $\Omega_j = i\sigma_j$ and $b_j = -id_j$ into Equation (33). The third way is the simplest one, by substituting only $\omega = i\mu$ into Equation (36). Following the third option, one can obtain the exact solution:

$$y(t) = \left(A + \sum_{j=1}^{n} \frac{\sigma_j d_j}{\mu^4 - \sigma_j^2} \right) e^{\mu^2 t} + \sum_{j=1}^{n} d_j \left(\frac{-\mu^2 \sinh(\sigma_j t) - \sigma_j \cosh(\sigma_j t)}{\mu^4 - \sigma_j^2} \right), \tag{38}$$

or

$$y(t) = \left(A + \sum_{j=1}^{n} \frac{\sigma_j d_j}{\mu^4 - \sigma_j^2} \right) e^{\mu^2 t} - \sum_{j=1}^{n} d_j \left(\frac{\mu^2 \sinh(\sigma_j t) + \sigma_j \cosh(\sigma_j t)}{\mu^4 - \sigma_j^2} \right), \tag{39}$$

for the present class of ODEs.

Remark 2. *The obtained exact solutions for the four classes of ODEs satisfy the condition $y(0) = A$. On the other hand, the validity of the obtained solutions can be easily checked through direct substitutions into the governing ODEs of these classes. We can say that the FC is of great importance and benefits. This is because the FC not only gives the solutions of fractional models but also helps in deriving the solutions of corresponding classical/ordinary models.*

5. Behavior of Solution

It is seen from the previous sections that the fractional systems (1) and (2) have the exact solutions given by Equation (20) and Equation (24), respectively. The main observation is that the solution (20) of the class (1) is real if the quantity $(-\omega^2)^{1/\alpha}$ is real. For real ω, we note that $(-\omega^2)^{1/\alpha} = \nu\omega^{2/\alpha}$ where $\nu = (-1)^{1/\alpha}$. So, the solution (20) is real when ν is real. The authors [31] were able to specify the α-values such that $\nu = (-1)^{1/\alpha}$ is real and this occurs that the α-values follow the next theorem [30].

Theorem 2. *For $n, k \in \mathbb{N}^+$, the solution (20) is real when $\alpha = \frac{2n-1}{2(k+n-1)}$ ($\nu = 1$) and $\alpha = \frac{2n-1}{2(k+n)-1}$ ($\nu = -1$).*

Based on the above theorem, the solution (20) for the fractional class (1) is plotted in Figure 1 for $\alpha = \frac{1}{2}$ at different numbers of the sinusoidal terms. Figure 2 shows the variation in the solution (20) for the fractional class (1) with two sinusoidal terms at different values of the initial condition A. In addition, Figure 3 indicates the behavior of the solution at various values of the fractional-order α when ten sinusoidal terms are incorporated in the fractional class (1). Furthermore, the solution is depicted in Figure 4 at some selected values α close to unity. This figure declares that the fractional solution becomes identical to the ordinary/classical solution as $\alpha \to 1$ which validates the present results.

For the fractional class (2), the solution (25) is displayed in Figure 5 when $\alpha = \frac{1}{2}$ at different numbers of the sinusoidal terms. The behavior of the solution of this class is similar to Figure 1 but with a slightly higher magnitude of the oscillations for the same numbers of the sinusoidal terms. Figure 6 gives us a picture of the solution profile as the fractional-order α varies regarding the fractional class (2). Moreover, Figure 7 displays the profile of the solution (25) at various values of the parameter μ. The current results reveal the oscillatory nature of the obtained solutions for the fractional systems (1) and (2). Finally, the present analysis may be extended to effectively analyze higher-order fractional systems containing a finite number of sinusoidal terms.

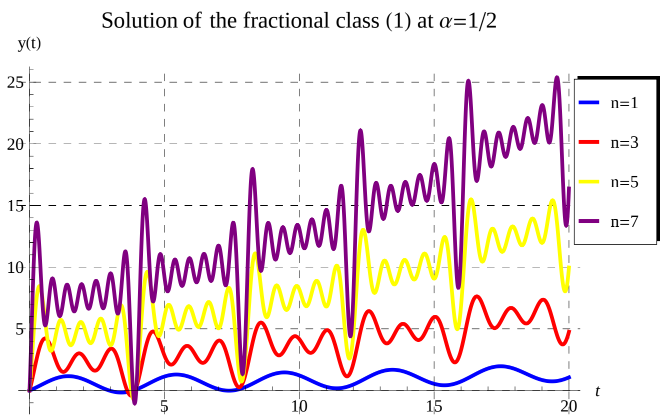

Figure 1. Plots of the solution for the fractional class (1) when $\alpha = \frac{1}{2}$, $A = 0$, $\omega = \frac{1}{2}$, $b_j = j$, and $\Omega_j = j\pi/2$ at different values of n (number of sinusoidal terms).

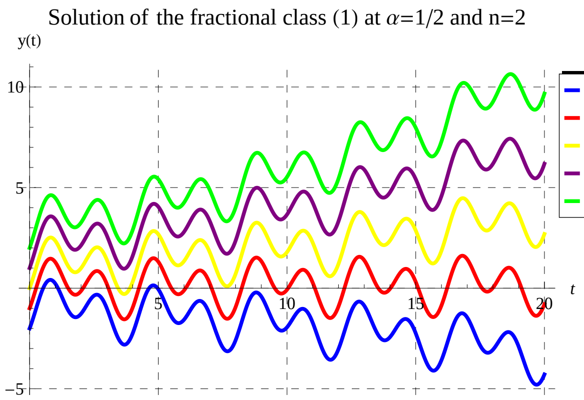

Figure 2. Plots of the solution for the fractional class (1) when $\alpha = \frac{1}{2}, \omega = \frac{1}{2}, b_j = j$, and $\Omega_j = j\pi/2$ at different values of $A = -2, -1, 0, 1, 2$ for two sinusoidal terms ($n = 2$).

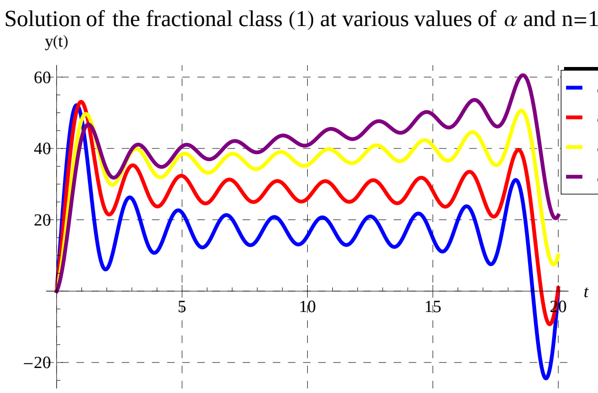

Figure 3. Plots of the solution for the fractional class (1) when $\alpha = \frac{1}{2}$, $A = 0$, $\omega = \frac{1}{5}$, $b_j = j$, and $\Omega_j = j\pi/10$ at different values of $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}$ for ten sinusoidal terms ($n = 10$).

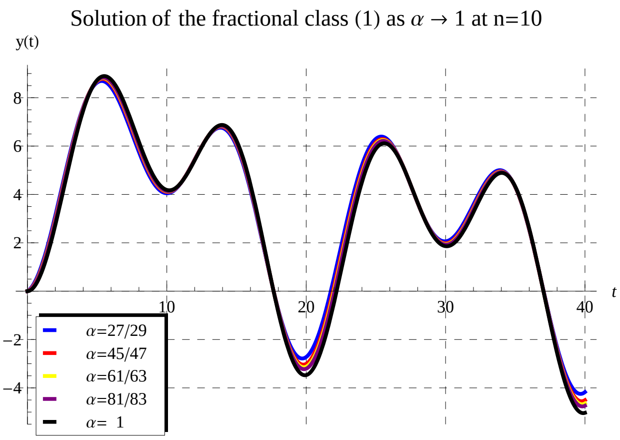

Figure 4. Plots of the solution for the fractional class (1) when $A = 0, \omega = \frac{1}{5}, b_j = j$, and $\Omega_j = j\pi/10$ at different values of $\alpha = \frac{27}{29}, \frac{45}{47}, \frac{61}{63}, \frac{81}{83}, 1$ for ten sinusoidal terms ($n = 10$).

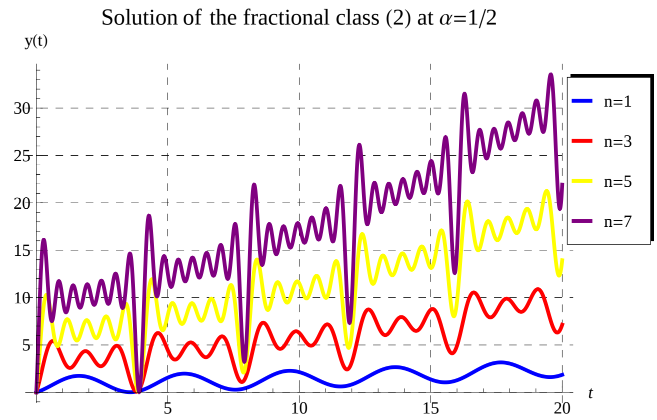

Figure 5. Plots of the solution for the fractional class (2) when $\alpha = \frac{1}{2}$, $A = 0$, $\mu = \frac{1}{2}$, $b_j = j$, and $\Omega_j = j\pi/2$ at different values of n (number of sinusoidal terms).

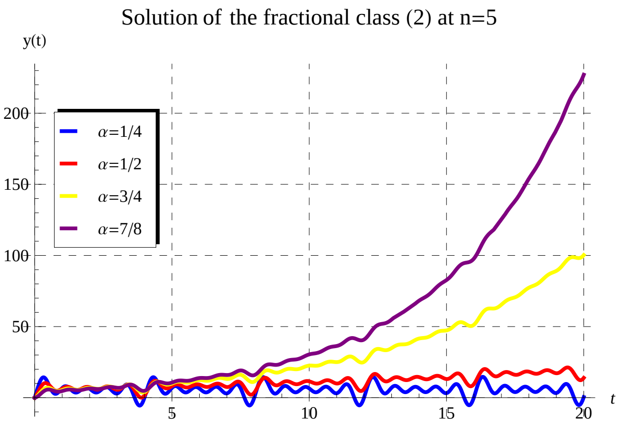

Figure 6. Plots of the solution for the fractional class (2) when $\mu = \frac{1}{2}$, $A = 0$, $b_j = j$, and $\Omega_j = j\pi/2$ at different values of α for five sinusoidal terms ($n = 5$).

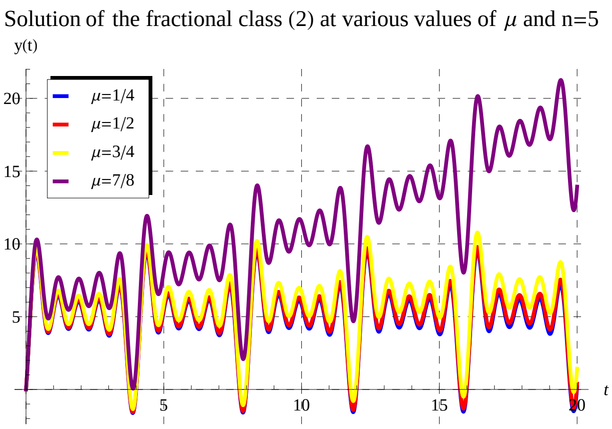

Figure 7. Plots of the solution for the fractional class (2) when $\alpha = \frac{1}{2}$, $A = 0$, $b_j = j$, and $\Omega_j = j\pi/2$ at different values of μ.

6. Conclusions

In this paper, a class of first-order fractional differential systems containing a finite number n of sinusoidal terms was analyzed by means of the Riemann–Liouville fractional definition. The difficulties in solving fractional systems under real/physical initial conditions using the Riemann–Liouville fractional definition are overcome in this paper. This task was achieved via a straightforward method. The suggested method was successfully applied to extract the exact solutions of the considered fractional systems. In addition, the corresponding exact solutions of the classical/ordinary versions were determined. The obtained results reveal the oscillatory nature of the present fractional systems. Moreover, the properties/behaviors of the obtained solutions were investigated graphically and hence interpreted. Accordingly, the current approach may deserve a further extension to include fractional systems of a higher order when the sinusoidal terms of a finite number are incorporated. Finally, the current approach may be applied to include other ideas [39–47].

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References

1. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; John Wiley & Sons: New York, NY, USA, 1993.
2. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
3. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific Publishing Company: Singapore, 2000.
4. Achar, B.N.N.; Hanneken, J.W.; Enck, T.; Clarke, T. Dynamics of the fractional oscillator. *Phys. A* **2001**, *297*, 361–367. [CrossRef]
5. Sebaa, N.; Fellah, Z.E.A.; Lauriks, W.; Depollier, C. Application of fractional calculus to ultrasonic wave propagation in human cancellous bone. *Signal Process.* **2006**, *86*, 2668–2677. [CrossRef]
6. Tarasov, V.E. Fractional Heisenberg equation. *Phys. Lett. A* **2008**, *372*, 2984–2988. [CrossRef]
7. Ding, Y.; Yea, H. A fractional-order differential equation model of HIV infection of CD4+T-cells. *Math. Comput. Model.* **2009**, *50*, 386–392. [CrossRef]
8. Wang, S.; Xu, M.; Li, X. Green's function of time fractional diffusion equation and its applications in fractional quantum mechanics. *Nonlinear Anal. Real World Appl.* **2009**, *10*, 1081–1086. [CrossRef]
9. Song, L.; Xu, S.; Yang, J. Dynamical models of happiness with fractional order. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 616–628. [CrossRef]
10. Ebaid, A. Analysis of projectile motion in view of the fractional calculus. *Appl. Math. Model.* **2011**, *35*, 1231–1239. [CrossRef]
11. Machado, J.T.; Kiryakova, V.; Mainardi, F. Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 1140–1153. [CrossRef]
12. Ebaid, A.; El-Sayed, D.M.M.; Aljoufi, M.D. Fractional calculus model for damped Mathieu equation: Approximate analytical solution. *Appl. Math. Sci.* **2012**, *6*, 4075–4080.
13. Gómez-Aguilara, J.F.; Rosales-García, J.J.; Bernal-Alvarado, J.J. Fractional mechanical oscillators. *Rev. Mex. De Física* **2012**, *58*, 348–352.
14. Garcia, J.J.R.; Calderon, M.G.; Ortiz, J.M.; Baleanu, D. Motion of a particle in a resisting medium using fractional calculus approach. *Proc. Rom. Acad. Ser. A* **2013**, *14*, 42–47.
15. Kumar, D.; Singh, J.; Baleanu, D.; Rathore, S. Analysis of a fractional model of the Ambartsumian equation. *Eur. Phys. J. Plus* **2018**, *133*, 133–259. [CrossRef]
16. Ebaid, A.; El-Zahar, E.R.; Aljohani, A.F.; Salah, B.; Krid, M.; Machado, J.T. Analysis of the two-dimensional fractional projectile motion in view of the experimental data. *Nonlinear Dyn.* **2019**, *97*, 1711–1720. [CrossRef]
17. Ebaid, A.; Cattani, C.; Juhani1, A.S.A.; El-Zahar, E.R. A novel exact solution for the fractional Ambartsumian equation. *Adv. Differ. Equations* **2021**, *2021*, 88. [CrossRef]
18. Kaur, D.; Agarwal, P.; Rakshit, M.; Chand, M. Fractional Calculus involving (p,q)-Mathieu Type Series. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 15–34. [CrossRef]

19. Agarwal, P.; Mondal, S.R.; Nisar, K.S. On fractional integration of generalized struve functions of first kind. *Thai J. Math.* **2020**, *to appear*.
20. Agarwal, P.; Singh, R. Modelling of transmission dynamics of Nipah virus (Niv): A fractional order approach. *Phys. A Stat. Mech. Its Appl.* **2020**, *547*, 124243. [CrossRef]
21. Alderremy, A.A.; Saad, K.M.; Agarwal, P.; Aly, S.; Jain, S. Certain new models of the multi space-fractional Gardner equation. *Phys. A Stat. Mech. Its Appl.* **2020**, *545*, 123806. [CrossRef]
22. Aljohani, A.F.; Ebaid, A.; Algehyne, E.A.; Mahrous, Y.M.; Cattani, C.; Al-Jeaid, H.K. The Mittag-Leffler function for re-evaluating the chlorine transport model: Comparative analysis. *Fractal Fract.* **2022**, *6*, 125. [CrossRef]
23. Ahmad, B.; Batarfi, H.; Nieto, J.J.; Oscar, O.-Z.; Shammakh, W. Projectile motion via Riemann-Liouville calculus. *Adv. Differ. Equ.* **2015**, *2015*, 63. [CrossRef]
24. Elzahar, E.R.; Gaber, A.A.; Aljohani, A.F.; Machado, J.T.; Ebaid, A. Generalized Newtonian fractional model for the vertical motion of a particle. *Appl. Math. Model* **2020**, *88*, 652–660. [CrossRef]
25. El-Zahar, E.R.; Alotaibi, A.M.; Ebaid, A.; Aljohani, A.F.; Gomez Aguilar, J.F. The Riemann-Liouville fractional derivative for Ambartsumian equation. *Results Phys.* **2020**, *19*, 103551. [CrossRef]
26. Ebaid, A.; Masaedeh, B.; El-Zahar, E. A new fractional model for the falling body problem. *Chin. Phys. Lett.* **2017**, *34*, 020201. [CrossRef]
27. Khaled, S.M.; El-Zahar, E.R.; Ebaid, A. Solution of Ambartsumian delay differential equation with conformable derivative. *Mathematics* **2019**, *7*, 425. [CrossRef]
28. Alharbi, F.M.; Baleanu, D.; Ebaid, A. Physical properties of the projectile motion using the conformable derivative. *Chin. J. Phys.* **2019**, *58*, 18–28. [CrossRef]
29. Algehyne, E.A.; El-Zahar, E.R.; Alharbi, F.M.; Ebaid, A. Development of analytical solution for a generalized Ambartsumian equation. *AIMS Math.* **2020**, *5*, 249–258. [CrossRef]
30. Ebaid, A.; Al-Jeaid, H.K. The Mittag–Leffler Functions for a Class of First-Order Fractional Initial Value Problems: Dual Solution via Riemann–Liouville Fractional Derivative. *Fractal Fract.* **2022**, *6*, 85. [CrossRef]
31. Khaled, S.M.; Ebaid, A.; Mutairi, F.A. The exact endoscopic effect on the peristaltic flow of a nanofluid. *J. Appl. Math.* **2014**, *2014*, 367526. [CrossRef]
32. Ebaid, A.; Sharif, M.A. Application of Laplace transform for the exact effect of a magnetic field on heat transfer of carbon-nanotubes suspended nanofluids. *Z. Nature. A* **2015**, *70*, 471–475. [CrossRef]
33. Saleh, H.; Alali, E.; Ebaid, A. Medical applications for the flow of carbon-nanotubes suspended nanofluids in the presence of convective condition using Laplace transform. *J. Assoc. Arab Univ. Basic Appl. Sci.* **2017**, *24*, 206–212. [CrossRef]
34. Ebaid, A.; Wazwaz, A.M.; Alali, E.; Masaedeh, B. Hypergeometric Series Solution to a Class of Second-Order Boundary Value Problems via Laplace Transform with Applications to Nanofuids. *Commun. Theor. Phys.* **2017**, *67*, 231. [CrossRef]
35. Ebaid, A.; Alali, E.; Saleh, H. The exact solution of a class of boundary value problems with polynomial coefficients and its applications on nanofluids. *J. Assoc. Arab Univ. Basi Appl. Sci.* **2017**, *24*, 156–159. [CrossRef]
36. Ali, H.S.; Alali, E.; Ebaid, A.; Alharbi, F.M. Analytic solution of a class of singular second-order boundary value problems with applications. *Mathematics* **2019**, *7*, 172. [CrossRef]
37. Ebaid, A.; Alharbi, W.; Aljoufi, M.D.; El-Zahar, E.R. The exact solution of the falling body problem in three-dimensions: Comparative study. *Mathematics* **2020**, *8*, 1726. [CrossRef]
38. El-Dib, Y.O.; Elgazery, N.S. Effect of Fractional Derivative Properties on the Periodic Solution of the Nonlinear Oscillations. *Fractals* **2020**, *28*, 2050095. [CrossRef]
39. Azam, M. Effects of Cattaneo-Christov heat flux and nonlinear thermal radiation on MHD Maxwell nanofluid with Arrhenius activation energy. *Case Stud. Therm. Eng.* **2022**, *34*, 102048. [CrossRef]
40. Azam, M. Bioconvection and nonlinear thermal extrusion in development ofchemically reactive Sutterby nano-material due to gyrotactic microorganisms. *Int. Commun. Heat Mass Transf.* **2022**, *130*, 105820. [CrossRef]
41. Azam, M.; Abbas, N.; Ganesh, K.K.; Wali, S. Transient bioconvection and activation energy impacts on Casson nanofluid with gyrotactic microorganisms and nonlinear radiation. *Waves Random Complex Media* **2022**, 1–20. [CrossRef]
42. Azam, M.; Nayak, M.K.; Khan, W.A.; Khan, M. Significance of bioconvection and variable thermal properties on dissipative Maxwell nanofluid due to gyrotactic microorganisms and partial slip. *Waves Random Complex Media* **2022**, 1–21. [CrossRef]
43. Azam, M.; Xu, T.; Nayak, M.K.; Khan, W.A.; Khan, M. Gyrotactic microorganisms and viscous dissipation features on radiative Casson nanoliquid over a moving cylinder with activation energy. *Waves Random Complex Media* **2022**, 1–23. [CrossRef]
44. Oderinu, R.A.; Owolabi, J.A.; Taiwo, M. Approximate solutions of linear time-fractional differential equations. *J. Math. Comput. Sci.* **2022**, *29*, 60–72. [CrossRef]
45. Akram, T.; Abbas, M.; Ali, A. A numerical study on time fractional Fisher equation using an extended cubic B-spline approximation. *J. Math. Comput. Sci.* **2021**, *22*, 85–96. [CrossRef]
46. AlAhmad, R.; AlAhmad, Q.; Abdelhadi, A. Solution of fractional autonomous ordinary differential Equations. *J. Math. Comput. Sci.* **2022**, *27*, 59–64. [CrossRef]
47. Nikan, O.; Avazzadeh, Z.; Machado, J.T. An efficient local meshless approach for solving nonlinear time-fractional fourth-order diffusion model. *J. King Saud Univ.-Sci.* **2021**, *33*, 101243. [CrossRef]