

Article

Esakia Duality for Heyting Small Spaces

Artur Piękosz 

Department of Applied Mathematics, Cracow University of Technology, Warszawska 24, 31-155 Kraków, Poland; apiekosz@pk.edu.pl

Abstract: We continue our research plan of developing the theory of small and locally small spaces, proposing this theory as a realisation of Grothendieck's idea of tame topology on the level of general topology. In this paper, we develop the theory of Heyting small spaces and prove a new version of Esakia Duality for such spaces. To do this, we notice that spectral spaces may be seen as sober small spaces with all smops compact and introduce the method of the standard spectralification. This helps to understand open continuous definable mappings between definable spaces over o-minimal structures.

Keywords: Esakia Duality; small space; spectral space; Heyting algebra; tame topology

MSC: primary 54A05; 06D20; 18F60; secondary 18F70; 54D80; 54F05; 18F10

1. Introduction

Equivalences or dualities between categories are a form of symmetry on the category theory level. While this form of symmetry is very abstract, it is also extremely fruitful since it can connect apparently distant branches of mathematics.

Extensions of Stone Duality and their applications have been quite popular in recent years. One can mention the locally compact Hausdorff case of [1] and remove the zero-dimensionality together with the commutativity assumptions in [2]. We also have generalisations of Gelfand–Naimark–Stone Duality to completely regular spaces (see [3]) and its application to the characterisation of normal, Lindelöf and locally compact Hausdorff spaces in [4]. Some extensions of Stone Duality drop the compactness assumption completely: for example, the paper [5] extends this duality to all zero-dimensional Hausdorff spaces. From the point of view of algebraic structures, Stone Duality has been extended in [6] to (non-distributive, in general) orthomodular lattices, which correspond to spectral presheaves, while [7] extends it to some non-distributive (implicative, residuated, or co-residuated) lattices and applies to the semantics of substructural logics, and [8] extends this duality to a non-commutative case of left-handed skew Boolean algebras. Moreover, Esakia Duality has been extended to implicative semilattices in [9]. Applications of Stone Duality have been developed in [10] (canonical extensions of lattice-ordered algebras) and [11] (the semantics of non-distributive propositional logics). Some recent applications of Stone Duality appear also in the theory of C^* -algebras, see [12].

On the other hand, one of the greatest mathematicians in history, Alexander Grothendieck, suggested in his scientific programme [13] creating a new type of topology, called *tame topology*, that would eliminate pathological phenomena such as space-filling curves. O-minimality is widely recognized as a realisation of Grothendieck's programme. It is usually understood as studying o-minimal structures (in the sense of model theory). The fundamental monograph about o-minimal structures is [14]. It appears that arbitrary open sets are not so important in o-minimality, but the definable open sets play the main role.

Although Grothendieck's programme has been being realised for many decades in many special situations, the authors of [15] are disappointed that no clear definition of the notion of tame topology suggested by A. Grothendieck was given (Cruz Morales [16] gives



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some history of the evolution of Grothendieck's ideas about the notion of space to physical and philosophical questions).

It seems that not enough attention was paid to linking Stone-like dualities to the tame topology of A. Grothendieck. This paper fills this gap by clearly proposing the theory of small spaces and locally small spaces as a realisation of Grothendieck's postulate on a purely topological level (i.e., on the level of general topology) and making another step in developing this sort of topology (apparently a kind of generalised topology, but in fact the usual topology with additional structure).

Dropping some of the requirements for a topology leads to the notions of a unitary smopology and a small space (already used in [17–20]) as well as the notions of an arbitrary smopology and a locally small space (used, for example, in [19–21]). In this way, we construct a kind of algebra-friendly topology. Recall that the small spaces are the underlying topological structures of the definable spaces and the locally small spaces are the underlying topological structures of the locally definable spaces. (Both definable and locally definable spaces were used by many authors. See, for example, [14,22,23], implicitly even [24]).

The main objective of this paper is to give a version of Esakia Duality for suitably chosen small spaces. We call them Heyting small spaces, following the conventions of [25]. The first tool to analyse small spaces is the theory of spectral spaces, which is already developed enough (see the monograph [25]). The present paper is a continuation of the paper [20] about some versions of Stone Duality [26] or Priestley Duality [27] for locally small spaces. This time we focus on giving a new version of a related duality due to Leo Esakia [28] for small spaces. Basic facts about spectral spaces may be found in [25,29]. On the other hand, refs. [30,31] are good resources about Esakia Duality and its connections to modal logics. Our version of Esakia Duality helps to understand open continuous definable mappings between definable spaces over o-minimal structures.

Recognizing the role of smopologies (implicit in [32] (Definition 7.1.14) or [33] (p. 12)) should be useful in many areas of mathematics such as the generalisations of o-minimality, analytic geometry, algebraic geometry and in many other contexts since families of sets closed under only finite unions are much more natural in many branches of mathematics than usual topologies. In contrast to the usual topology, where only spectral reflections (see [25] (Chapter 11)) are available, using dualities or equivalences for suitably chosen smopologies allows transferring structural information without any losses between algebraic and topological structures. Translating between the topological and the algebraic languages made available by our version of Esakia Duality should give more understanding of locally definable spaces over structures with topologies, especially in the case of definable topologies. We want to stress that our approach considers the geometry of (first-order) definable sets but does not reduce to it.

As far as the set-theoretical axiomatics is concerned, we follow Saunders Mac Lane's standard Zermelo–Fraenkel axioms with the Axiom of Choice plus the existence of a set which is a universe, see [34] (p. 23). This allows speaking about proper classes of sets (in particular, about categories) while using methods from the usual mathematics developed in the axiomatic system ZFC (See “Axiomatic assumptions” in [35] for the full explanation of an axiomatic system using a universe).

Notation. We shall use a special notation for operations on families of sets. For example, for a family intersection

$$\mathcal{U} \cap_1 \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$$

or for other operations

$$\bigcup_1 \mathcal{P}(\mathcal{A}) = \{\bigcup \mathcal{B} : \mathcal{B} \subseteq \mathcal{A}\}, \quad \bigcap_1 \mathcal{P}(\mathcal{A}) = \{\bigcap \mathcal{B} : \mathcal{B} \subseteq \mathcal{A}\}.$$

2. Pre-Heyting and Pre-Boolean Small Spaces

Definition 1 (cf. [19] (Definition 2.21) and [20] (Definition 2)). A small space \mathcal{X} is a pair (X, \mathcal{L}_X) , where X is any set and $\mathcal{L}_X \subseteq \mathcal{P}(X)$ satisfies the following conditions:

- (S1) $\emptyset, X \in \mathcal{L}_X$,
- (S2) if $A, B \in \mathcal{L}_X$, then $A \cap B, A \cup B \in \mathcal{L}_X$.

Elements of \mathcal{L}_X are called smops (i.e., small open subsets of X , for reasons that become clear after reading [21] or [19]). Their complements will be called co-smops or inverse smops. The family \mathcal{L}_X will be called a (unitary) smopology, and the family $\mathcal{L}'_X = X \setminus \mathcal{L}_X$ of co-smops will be called a co-smopology. The Boolean combinations of smops will be called the constructible sets and the family of all constructible sets of a small space (X, \mathcal{L}_X) will be denoted by $\text{Con}(\mathcal{X})$.

Definition 2. A mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ between small spaces is:

1. continuous (or strictly continuous) if the preimage of any smop is a smop ($f^{-1}(\mathcal{L}_Y) \subseteq \mathcal{L}_X$),
2. a strict homeomorphism if f is a bijection and $f^{-1}(\mathcal{L}_Y) = \mathcal{L}_X$.

We have the category **SS** of small spaces and continuous mappings.

Example 1. (1) Each topology on X and each Boolean subalgebra of $\mathcal{P}(X)$ is a unitary smopology on X . (2) The spaces $\mathbb{R}_{om}, \mathbb{R}_{rom}, \mathbb{R}_{slom}, \mathbb{R}_{sl+om}, \mathbb{R}_{st}$ from [19] (Example 2.14) are examples of small spaces. (3) Since the category **SS** of small spaces and continuous mappings in our sense is concretely isomorphic ([36] (Remark 5.12)) to the category **SS** in the sense of [17], also [17,18] give examples of small spaces.

Definition 3. The topology $\tau(\mathcal{L}_X)$, generated by the smops, will be called the original topology. The closure, the interior and the exterior operations in the original topology will be denoted by $\bar{\cdot}, \text{int}(\cdot), \text{ext}(\cdot)$, respectively. The topology $\tau(\mathcal{L}'_X)$, generated by the co-smops, will be called the inverse topology. The closure and the interior operations in the inverse topology will be denoted by $\bar{\cdot}^{inv}, \text{int}_{inv}(\cdot)$, respectively. The topology $\tau(\text{Con}(\mathcal{X}))$, generated by the constructible sets, will be called the constructible topology. The closure and the interior operations in the constructible topology will be denoted by $\bar{\cdot}^{con}, \text{int}_{con}(\cdot)$, respectively.

Example 2. For each small space (X, \mathcal{L}_X) , we can produce the following small spaces:

$$\mathcal{X}_{inv} = (X, \mathcal{L}'_X), \mathcal{X}_{con} = (X, \text{Con}(\mathcal{X})).$$

Definition 4. A small space will be called pre-Boolean if any of the equivalent conditions:

1. $\mathcal{L}_X = \mathcal{L}'_X$,
2. $\mathcal{L}_X = \text{Con}(\mathcal{X})$,
3. $\mathcal{L}'_X = \text{Con}(\mathcal{X})$

are satisfied.

Fact 1. In a pre-Boolean small space, all the three above-mentioned topologies are equal.

Fact 2. For each \mathcal{X} , the space \mathcal{X}_{con} is pre-Boolean.

Fact 3. For each \mathcal{X} , the space \mathcal{X}_{inv} is pre-Boolean iff \mathcal{X} is pre-Boolean.

Definition 5. For any subset $Y \subseteq X$ in a small space (X, \mathcal{L}_X) , the pair $\mathcal{Y} = (Y, \mathcal{L}_X \cap_1 Y)$ is called a subspace of (X, \mathcal{L}_X) .

Remark 1. The original, the inverse and the constructible topologies in a subspace \mathcal{Y} are topological subspace topologies of, respectively, the original, the inverse and the constructible topologies of the whole small space \mathcal{X} .

Definition 6. A small space (X, \mathcal{L}_X) will be called:

1. pre-semi-Heyting if the closure in the original topology of any smop is a co-smop (i.e., $\bar{A} \in \mathcal{L}'_X$ for any $A \in \mathcal{L}_X$),
2. pre-Heyting if the closure in the original topology of any constructible set is a co-smop (i.e., $\bar{A} \in \mathcal{L}'_X$ for any $A \in \text{Con}(\mathcal{X})$).

The following proposition and theorem are inspired by Section 8.3 of [25].

Proposition 1. Assume that \mathcal{X} is a pre-semi-Heyting small space. Then:

- (1) for each $F \in \mathcal{L}'_X$, we have $\text{int}(F) \in \mathcal{L}_X$,
- (2) the following two mappings are well defined:
 - (a) the open regularisation mapping $N : \mathcal{L}_X \rightarrow \mathcal{L}_X$ given by the formula $N(A) = \text{int}\bar{A}$,
 - (b) the closed regularisation mapping $\bar{N} : \mathcal{L}'_X \rightarrow \mathcal{L}'_X$ given by the formula $\bar{N}(F) = \overline{\text{int}F}$,
- (3) for each $V \in \mathcal{L}_X$, the subspace $(V, \mathcal{L}_X \cap_1 V)$ is pre-semi-Heyting.

Proof. (1) Follows from the definition by taking complements.

(2) Obvious by the above.

(3) For $A \in \mathcal{L}_X$, we get $\overline{A \cap V}^V = V \cap (\overline{A \cap V}) \in V \cap_1 \mathcal{L}'_X$, a co-smop in V . \square

Theorem 1 (characterisation of pre-Heyting spaces). For a small space $\mathcal{X} = (X, \mathcal{L}_X)$, the following conditions are equivalent:

- (1) \mathcal{X} is pre-Heyting,
- (2) for each $A \in \text{Con}(\mathcal{X})$, we have $\text{int}(A) \in \mathcal{L}_X$,
- (3) for each $B \in \mathcal{L}_X \cap_1 \mathcal{L}'_X$, we have $\bar{B} \in \mathcal{L}'_X$,
- (4) for each $A \in \text{Con}(\mathcal{X})$, the subspace $(A, \mathcal{L}_X \cap_1 A)$ is pre-Heyting,
- (5) for each $A \in \text{Con}(\mathcal{X})$, the subspace $(A, \mathcal{L}_X \cap_1 A)$ is pre-semi-Heyting,
- (6) for each $F \in \mathcal{L}'_X$, the subspace $(F, \mathcal{L}_X \cap_1 F)$ is pre-semi-Heyting.

Proof. (1) \Leftrightarrow (2) By taking complements.

(1) \Leftrightarrow (3) By taking finite unions.

(1) \Rightarrow (4) For $A \in \text{Con}(\mathcal{X})$ and $V \in \text{Con}(\mathcal{A}) = \text{Con}(\mathcal{X}) \cap_1 A \subseteq \text{Con}(\mathcal{X})$, we have $\bar{V} \in \mathcal{L}'_X$.

Now $\bar{V}^A = \bar{V} \cap A \in \mathcal{L}'_A$ is a (relative to A) co-smop.

(4) \Rightarrow (5) Trivial.

(5) \Rightarrow (6) Trivial.

(6) \Rightarrow (3) If $A \in \mathcal{L}_X, F \in \mathcal{L}'_X$, then $A \cap F \in \mathcal{L}_X \cap_1 F \subseteq \mathcal{L}_X \cap_1 \mathcal{L}'_X$. Since $\overline{A \cap F} \cap F = \overline{A \cap F} \in \mathcal{L}'_X \cap_1 F$ by (6), we have $\overline{A \cap F} \in \mathcal{L}'_X \cap_1 \mathcal{L}'_X \subseteq \mathcal{L}'_X$. \square

3. Spectral Small Spaces

Notation. We use the following notation for some distinguished families of subsets in a small space $\mathcal{X} = (X, \mathcal{L}_X)$:

1. $WOp(\mathcal{X}) = \bigcup_1 \mathcal{P}(\mathcal{L}_X) = \mathcal{U}\mathcal{L}_X = \tau(\mathcal{L}_X)$, the weakly open sets,
2. $WCl(\mathcal{X}) = \bigcap_1 \mathcal{P}(\mathcal{L}'_X) = \Omega\mathcal{L}'_X$, the weakly closed sets,
3. $WOp(\mathcal{X}_{inv}) = \bigcup_1 \mathcal{P}(\mathcal{L}'_X) = \mathcal{U}\mathcal{L}'_X = \tau(\mathcal{L}'_X)$, the inversely weakly open sets,
4. $WCl(\mathcal{X}_{inv}) = \bigcap_1 \mathcal{P}(\mathcal{L}_X) = \Omega\mathcal{L}_X$, the inversely weakly closed sets,
5. $WOp(\mathcal{X}_{con}) = \bigcup_1 \mathcal{P}(\text{Con}(\mathcal{X})) = \mathcal{U}\text{Con}(\mathcal{X}) = \tau(\text{Con}(\mathcal{X}))$, the constructibly weakly open sets,
6. $WCl(\mathcal{X}_{con}) = \bigcap_1 \mathcal{P}(\text{Con}(\mathcal{X})) = \Omega\text{Con}(\mathcal{X})$, the constructibly weakly closed sets.

We use the symbols \mathcal{U}, Ω after [37].

Remark 2. By **Spec** we denote the category of spectral topological spaces (the name introduced by Hochster [29]) and spectral mappings (compare [20,25]). By the Stone Duality ([25] (Chapter 3)), this category is dually equivalent to the category of bounded distributive lattices and their homomorphisms (denoted in [20] by **Lat**). Recall that the category of Priestley spaces and Priestley mappings

([25] (1.5.15), denoted here by **Pri**) is concretely isomorphic to **Spec**, so these two categories may be identified.

Definition 7. A spectral small space is a small space $(X, \text{CO}(X))$ where $(X, \bigcup \text{CO}(X))$ is a spectral topological space and $\text{CO}(X)$ is the family of all compact open sets of this space. The category **SpSS** of spectral small spaces and (strictly) continuous mappings is concretely isomorphic with **Spec**, so these categories may be identified.

Example 3. For a spectral topological space (X, τ_X) , the corresponding spectral small space $(X, \text{CO}(X))$ has the property, that the constructible weakly open sets are exactly the compact open sets, so smops, i.e., $\mathcal{L}_X = \text{CO}(X)$ and $\text{Con}(X) \cap \bigcup \mathcal{L}_X = \mathcal{L}_X$ (See [25] (1.3)).

Example 4. A weakly open constructible set in a small space may not be a smop. Take the small space $(\mathbb{N}, (\{\emptyset, 2\mathbb{N}\} \cup_1 \text{Fin}(\mathbb{N})) \cup \{\mathbb{N}\})$ where $\text{Fin}(\mathbb{N})$ is the family of all finite subsets of \mathbb{N} . Then the original topology is discrete and the set $2\mathbb{N} + 1$ is a weakly open co-smop but not a smop, so $\mathcal{L}'_X \cap \bigcup \mathcal{L}_X \not\subseteq \mathcal{L}_X$. In particular, $\mathcal{L}_X \subset \text{Con}(X) \cap \bigcup \mathcal{L}_X$.

Example 5. A T_0 small space may have all smops (topologically) compact but not be sober. Let X be an infinite set. Then X is irreducible in the small space $(X, \{\emptyset\} \cup \text{Cofin}(X))$ but not the closure of a point (Cofin(X) is the family of all cofinite subsets of X .)

Example 6. A T_0 small space may be (topologically) sober but not compact. Take any non-compact Hausdorff topological space.

Example 7. A T_0 small space may be sober and compact with not all smops compact. Take the interval $[0, 1]$ with the natural topology as the smopology.

Proposition 2. If X is a T_0 sober small space with all smops compact, then X is a spectral small space.

Proof. Notice that $\mathcal{L}_X \subseteq \text{CO}(X)$, so $\text{CO}(X) = \mathcal{L}_X$. The other conditions for spectrality are obvious. \square

Proposition 3. If a small space has all smops compact, then:

- (1) the ideals in \mathcal{L}_X are in a bijective correspondence with the weakly open sets, so also with the weakly closed sets,
- (2) the prime ideals of \mathcal{L}_X are in a bijective correspondence with the non-empty, irreducible, weakly closed sets.

Proof. (1) For an ideal I in \mathcal{L}_X , the set $s(I) = \bigcup I$ is weakly open (and its complement is weakly closed). For a weakly open set $W \in \bigcup \mathcal{L}_X$, the set $i(W) = \{A \in \mathcal{L}_X : A \subseteq W\}$ is an ideal in \mathcal{L}_X . Obviously, $s(i(W)) = W$ and $I \subseteq i(s(I))$. On the other hand, if $B \in \mathcal{L}_X$ is compact and $B \subseteq \bigcup I$, then there exist a finite family of members of I covering B , so $B \in I$. This means that $I = i(s(I))$ and the mappings s and i are bijections.

(2) By Proposition 1.1.11 of [25], the ideal $I \subseteq \mathcal{L}_X$ is prime ($A \cap B \in I$ implies $A \in I$ or $B \in I$) iff $\bigcup I$ is a prime weakly open set iff $X \setminus \bigcup I$ is a non-empty, irreducible, weakly closed set. \square

Example 8. If not all smops are compact, then it may happen that $I \subset i(s(I))$, using notation from the above proof.

1. In the small space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ we can take $I = \text{Fin}(\mathbb{N}) \subset \mathcal{P}(\mathbb{N}) = i(s(I))$.
2. Consider the interval $[0, 1]$ with the natural topology as the unitary smopology. Finite unions of open subintervals in the open interval $(0, 1)$ form an ideal that covers $(0, 1)$ but not all smops that are contained in $(0, 1)$ are of this form.

4. The Specialization Relation

Definition 8. We recall (compare [25] (1.1.3)) the relation of specialization between points of a topological space (X, τ_X) :

We say that x specializes to y (and write $x \rightsquigarrow y$) if each neighbourhood of y also contains x (it is enough to check this condition only for sets from some basis of the topology). We say then that x is a generalisation of y , and y is a specialization of x . The set of all generalisations of a point y is denoted by $\text{Gen}(y)$, and the set of all specializations of a point x is denoted by $\text{Spez}(x)$. Similarly for sets, if $A \subseteq X$, then:

$$\begin{aligned}\text{Gen}(A) &= \{x \in X : x \in \text{Gen}(a) \text{ for some } a \in A\} \quad \text{and} \\ \text{Spez}(A) &= \{x \in X : x \in \text{Spez}(a) \text{ for some } a \in A\}.\end{aligned}$$

Remark 3. A specialization relation is always a preorder (called a quasi-order in [25]) on X and does not depend on the ambient topological space. More precisely: if $Y \subseteq X$ and $x, y \in Y$, then $x \rightsquigarrow y$ in X iff $x \rightsquigarrow y$ in Y .

Fact 4. For a small space (X, \mathcal{L}_X) and $x, y \in X$, the following conditions are equivalent for the original topology on X :

1. x specializes to y (denoted by: $x \rightsquigarrow y$),
2. $y \in \overline{\{x\}}$,
3. $y \notin \text{ext}\{x\}$,
4. each smop containing y also contains x ,
5. each co-smop containing x also contains y ,
6. $y \rightsquigarrow_{\text{inv}} x$ (read: y specializes to x in \mathcal{X}_{inv} .)

Fact 5. For each small space (X, \mathcal{L}_X) and $A \subseteq X$, we have:

$$\text{Spez}(A) \subseteq \overline{A} \quad \text{and} \quad \text{Gen}(A) \subseteq \overline{A}^{\text{inv}}.$$

Fact 6. In a T_0 pre-Boolean small space, $x \rightsquigarrow y$ iff $x = y$.

Fact 7. In any small space (X, \mathcal{L}_X) , we have the following:

- (1) If $A \subseteq X$ is weakly closed ($A \in \Omega\mathcal{L}'_X$), then $\text{Spez}(A) = A$.
- (2) If $A \subseteq X$ is weakly open ($A \in \cup\mathcal{L}_X$), then $\text{Gen}(A) = A$.

Definition 9 (cf. [25] (1.1.20)). A subset $Q \subseteq X$ is called saturated if it is an intersection of weakly open sets, that is $Q \in \Omega\cup\mathcal{L}_X$.

Definition 10. A small space (X, \mathcal{L}_X) is called T_0 (or Kolmogorov) if the family \mathcal{L}_X separates points ([25] (Reminder 1.1.4)), which means that for $x, y \in X$ the following condition is satisfied:

$$\text{if } x \in A \iff y \in A \text{ for each } A \in \mathcal{L}_X, \text{ then } x = y.$$

Fact 8 (cf. [25] (1.1.6)). If (X, \mathcal{L}_X) is T_0 , then the specialization relation \rightsquigarrow is a partial order.

Fact 9 (cf. [25] (1.1.20)). If (X, \mathcal{L}_X) is T_0 and $Q \subseteq X$, then the following are equivalent:

1. Q is saturated,
2. $\text{Gen}(Q) = Q$.

Fact 10 ([25] (Corollary 1.5.5)). In a spectral topological space (X, τ_X) , we have the following facts for any $A \subseteq X$:

1. $\overline{A} = \text{Spez}(\overline{A}^{\text{con}})$,
2. $\overline{A}^{\text{inv}} = \text{Gen}(\overline{A}^{\text{con}})$.

5. The Standard Spectralification

Definition 11. An embedding of small spaces $e : \mathcal{X} \rightarrow \mathcal{Y}$ is an injective mapping $e : X \rightarrow Y$ such that $e(\mathcal{L}_X) = \mathcal{L}_Y \cap_1 e(X)$.

Definition 12 ([20]). A spectralification of a small space \mathcal{X} is the pair (e, \mathcal{Y}) where: \mathcal{Y} is a spectral small space and $e : \mathcal{X} \rightarrow \mathcal{Y}$ is an embedding between small spaces with the image $e(X)$ constructibly dense in \mathcal{Y} .

Remark 4. Recall that each T_0 small space \mathcal{X} admits a spectralification (in the sense of the above definition) that is given by applying the functor $\bar{S}\bar{A}$ from the proof of Theorem 1 (see also Theorem 2) of [20] to \mathcal{X} and embedding \mathcal{X} into the resulting spectral space (the existence of such an embedding is guaranteed by the functor \bar{R} of that proof). This spectralification, treating \mathcal{X} as a subspace of \mathcal{Y} and dropping e in the notation, will be called the standard spectralification of \mathcal{X} and will be denoted by \mathcal{X}^{sp} . Hence we have:

1. $\text{CO}(\mathcal{X}^{sp}) \cap_1 X = \mathcal{L}_X$,
2. $\text{CC}(\mathcal{X}^{sp}) \cap_1 X = \mathcal{L}'_X$,
3. $\text{Con}(\mathcal{X}^{sp}) \cap_1 X = \text{Con}(X)$.

Here $\text{CC}(\mathcal{X}^{sp})$ denotes the family of all closed constructible sets in the spectral space \mathcal{X}^{sp} .

Since X is constructibly dense in \mathcal{X}^{sp} , we have the bijection

$$\text{Con}(X) \ni A \mapsto A^{sp} \in \text{Con}(\mathcal{X}^{sp})$$

where A^{sp} is the only member of $\text{Con}(\mathcal{X}^{sp})$ such that $A^{sp} \cap X = A$. One can see that $A^{sp} = \overline{A}^{con}$ taken in \mathcal{X}^{sp} . This bijection does not extend to the proconstructible sets (i.e., arbitrary intersections of constructible sets).

Remark 5. Consider the operation $cl : \text{Con}(X) \rightarrow \mathcal{P}(X)$ introduced below for a T_0 small space \mathcal{X} . For $A \in \text{Con}(X)$, let $A^{sp} \in \text{Con}(\mathcal{X}^{sp})$ be the corresponding constructible set in the standard spectralification \mathcal{X}^{sp} of our space \mathcal{X} , so $A = A^{sp} \cap X$. Define

$$cl(A) = \overline{A^{sp}} \cap X.$$

(The closure is taken in the original topology in \mathcal{X}^{sp} .) The operation cl is equal to the closure in the original topology in X since for each $B \in \text{Con}(\mathcal{X}^{sp})$ we have $\overline{B \cap X}^{con} = B$, so $\overline{B} = \overline{B \cap X}$. This means

$$cl(A) = cl(A^{sp} \cap X) = \overline{A^{sp} \cap X} \cap X = \overline{A^{sp}} \cap X^X = \overline{A}^X.$$

Proposition 4. If C is a non-empty, weakly closed set in X , then the following conditions are equivalent:

- (1) C is irreducible,
- (2) the closure \overline{C} in the standard spectralification \mathcal{X}^{sp} is irreducible.

Proof. (1) \Rightarrow (2) Assume $C \subseteq X$ is (non-empty, weakly closed) irreducible in X , the sets A, B are weakly closed in the original topology in \mathcal{X}^{sp} , $A \cup B = \overline{C}$ and $A \neq \overline{C} \neq B$. Then $(A \cap X) \cup (B \cap X) = C$, hence either $A \cap X = C$ or $B \cap X = C$. We get either $\overline{C} = \overline{A \cap X} \subseteq A \subseteq \overline{C}$ or, similarly, $B = \overline{C}$, a contradiction.

(2) \Rightarrow (1) If $C = A \cup B$ with A, B weakly closed in X , then $\overline{C} = \overline{A \cup B}$ in \mathcal{X}^{sp} . However, \overline{C} is irreducible, hence $\overline{C} = \overline{A}$ or $\overline{C} = \overline{B}$. That is why $C = \overline{C} \cap X = \overline{A} \cap X = A$ or, similarly, $C = B$. \square

6. Heyting Spectral Spaces

Definition 13 ([25] (Section 8.3)). A topological spectral space X is called Heyting if the closure of any constructible set is a constructible set, so belongs to $\text{CC}(X)$. A mapping between Heyting spectral spaces $g : X \rightarrow Y$ is called a Heyting spectral mapping if g is spectral and the condition

$f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$ for $C \in \text{Con}(Y)$ holds. We have the category **HSpec** of (topological) Heyting spectral spaces and Heyting spectral mappings.

Fact 11. Each homeomorphism between Heyting spectral spaces is an isomorphism in **HSpec**.

Example 9. The real spectrum $\text{Sper } R[X_1, \dots, X_n]$ of the ring of polynomials $R[X_1, \dots, X_n]$ over a real closed field R is an object of **HSpec** (see [32] (Theorem 7.2.3), [25] (Corollary 13.5.11)).

Remark 6. One can similarly form the category **HSpSS** of Heyting spectral small spaces and Heyting (strictly) continuous mappings. Since **HSpSS** is concretely isomorphic with **HSpec**, these two categories may be identified.

Definition 14. A T_0 pre-Heyting small space will be called a Heyting small space.

Proposition 5. If the spectralification \mathcal{X}^{sp} of a T_0 small space \mathcal{X} is Heyting, then \mathcal{X} is Heyting.

Proof. Assume $\forall A \in \text{Con}(\mathcal{X}^{sp}) \overline{A} \in \text{CC}(\mathcal{X}^{sp})$. Then by taking traces with X , we have $\forall B \in \text{Con}(\mathcal{X}) \overline{B}^X \in \mathcal{L}'_X$ (see Remark 4). \square

Proposition 6. If \mathcal{X} is a Heyting small space, then:

- (1) \mathcal{L}_X is a Heyting algebra,
- (2) \mathcal{X}^{sp} is a Heyting spectral space.

Proof. (1) The formula $U \rightarrow V = \text{int}(V \cup (X \setminus U))$ gives the intuitionistic implication for the bounded lattice \mathcal{L}_X .

(2) Since the lattices \mathcal{L}_X and $\text{CO}(\mathcal{X}^{sp})$ are isomorphic Heyting algebras, \mathcal{X}^{sp} is a Heyting spectral space by [25] (Theorem 8.3.12). \square

7. Heyting and Boolean Small Spaces

Definition 15 (Heyting mappings). A mapping between pre-Heyting small spaces $f : \mathcal{X} \rightarrow \mathcal{Y}$ will be called a Heyting (strictly) continuous mapping if it is (strictly) continuous and satisfies any of the following equivalent conditions:

1. $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$ for $C \in \text{Con}(\mathcal{Y})$,
2. $f^{-1}(\text{int}(C)) = \text{int}(f^{-1}(C))$ for $C \in \text{Con}(\mathcal{Y})$.

Definition 16. We have the following categories:

1. the category **HSS** of Heyting small spaces and Heyting continuous mappings,
2. the category **BSS** of Boolean (i.e., Hausdorff compact pre-Boolean) small spaces and continuous mappings.

Fact 12. Each strict homeomorphism between Heyting small spaces is an isomorphism of **HSS**.

Proposition 7. Boolean small spaces $\mathcal{X} = (X, \mathcal{L}_X)$ are spectral and satisfy $\mathcal{L}_X = \text{ClOp}(X)$, where $\text{ClOp}(X)$ is the family of all clopen subsets of X in the original topology of X .

Proof. Boolean spaces are Hausdorff, so sober. All their smops are clopen, so compact. They are spectral by Proposition 2. Each clopen subset is compact open, so $\text{ClOp}(X) = \text{CO}(X) = \mathcal{L}_X$. \square

Example 10. The small space $(X, \text{Fin}(X) \cup \text{Cofin}(X))$ for an infinite set X is Hausdorff pre-Boolean but $\mathcal{L}_X = \text{Con}(X) \subset \text{ClOp}(X) = \mathcal{P}(X)$.

Example 11. For a structure with a topology (\mathcal{M}, σ) , see [19] or [21], each definable (with parameters) set $D \subseteq M^n$ gives an example of an object $(D, \text{DefOp}(D))$ of **SS** where $\text{DefOp}(D)$ is the

family of definable open subsets of D in the topology τ_D induced from M^n . (Since both the topology on D and the family of all definable subsets of D are bounded sublattices of $\mathcal{P}(D)$, their intersection is also a bounded sublattice.) Similarly, when a definable set is replaced by a definable space over (\mathcal{M}, σ) . (The latter case follows from the former using finite unions by the very definition of a definable space, see [19] (Definition 3.13).)

Example 12. In an o-minimal structure (M, \leq, \dots) , see [14], any definable set $D \subseteq M^n$ gives an example of an object $(D, \text{DefOp}(D))$ of **HSS** since the closure of a definable set is always definable. Similarly, when a definable set is replaced by a definable space over this o-minimal structure ([14] (Chapter 10)).

Example 13. Not all definable continuous mappings between definable sets in o-minimal structures are Heyting continuous. For example, consider the o-minimal structure (\mathbb{R}, \leq) and the natural projection $p : \{1, 2\} \times \mathbb{R} \rightarrow \mathbb{R}$. Define f to be the restriction of p to the definable set $D = \{1\} \times (\mathbb{R} \setminus \{0\}) \cup \{2\} \times \{0\}$ and let $C = \{0\} \in \text{Con}(\mathbb{R})$. Then $\emptyset = f^{-1}(\text{int}(C)) \neq \text{int}(f^{-1}(C)) = \{(2, 0)\}$.

Example 14. All open continuous definable mappings between definable sets in o-minimal structures are Heyting continuous mappings. A similar property holds when a definable set is replaced by a definable space over the same o-minimal structure. Indeed, any continuous definable mapping is strictly continuous and the property of being Heyting is a weak form of openness, see [38] (Exercise 1.4.C) working for any mapping between topological spaces.

8. Other Useful Categories of Small Spaces

Definition 17. Let (X, \leq_X) be a partially ordered set. For any subset $A \subseteq X$, we have:

$$\uparrow A = \{x \in X : \exists a \in A \ a \leq_X x\}, \text{ the upset generated by } A,$$

$$\downarrow A = \{x \in X : \exists a \in A \ x \leq_X a\}, \text{ the downset generated by } A.$$

When $A = \{x\}$, we write $\uparrow x$ instead of $\uparrow A$.

Definition 18. A Priestley small space is a system $(X, \mathcal{L}_X, \leq_X)$ where (X, \mathcal{L}_X) is a Boolean small space and \leq_X is a partial order on X satisfying the Priestley separation axiom

$$\text{if } x \not\leq_X y, \text{ then } \exists V \in \mathcal{L}_X \text{ such that } V = \uparrow V, x \in V, y \notin V.$$

A Priestley mapping is a strictly (equivalently: weakly) continuous non-decreasing mapping between Priestley small spaces. We have the category **PSS** of Priestley small spaces and Priestley mappings.

Remark 7. The category **PSS** is concretely isomorphic to the category **Pri** of topological Priestley spaces and Priestley morphisms (see [25,31]) as well as to **Spec** and to **SpSS**, so all these categories may be identified.

Definition 19. An Esakia small space is a Priestley small space that satisfies the condition

$$\text{for all } A \in \mathcal{L}_X \text{ we have } \downarrow A \in \mathcal{L}_X.$$

An Esakia mapping is a Priestley mapping that is a p-morphism, i.e., it satisfies $f(\uparrow x) = \uparrow f(x)$. We have the category **ESS** of Esakia small spaces and Esakia mappings.

Remark 8. Following [30,31], we understand $x \leq_X y$ as saying that y is a generalisation of x , denoted also by $y \rightsquigarrow x$, while [25] uses the opposite convention.

Remark 9. The category **ESS** is concretely isomorphic to both the category **Esa** of topological Esakia spaces and Esakia morphisms (see [25,30,31]) and to **HSpec** from Definition 13, hence these

categories may be identified. Similarly, the category **ESSD** of Esakia small spaces with distinguished decent sets and Esakia mappings respecting the decent sets is concretely isomorphic to **HSpecD**.

In analogy to the category **SpecD** from [20], we have the following category.

Definition 20. An object of **HSpecD** is a system $((X, \tau_X), X_d)$ where (X, τ_X) is a Heyting (topological) spectral space and $X_d \subseteq X$ is a constructibly dense subset. (Such X_d is called a decent subset of X .) A morphism from $((X, \tau_X), X_d)$ to $((Y, \tau_Y), Y_d)$ in **HSpecD** is such a Heyting spectral mapping between Heyting spectral spaces $g : (X, \tau_X) \rightarrow (Y, \tau_Y)$ that respects the decent set, i.e., $g(X_d) \subseteq Y_d$.

In analogy to the category **Lat** from [20], we have the following category.

Definition 21. A Heyting algebra is a system $L = (L, \vee, \wedge, \rightarrow, 0, 1)$ where $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and for each $a, b \in L$ there exists the largest $c \in L$ such that $a \wedge c \leq b$. We then denote c by $a \rightarrow b$. A homomorphism of Heyting algebras is a homomorphism $h : L \rightarrow M$ of bounded lattices satisfying additionally the condition

$$h(a \rightarrow b) = h(a) \rightarrow h(b) \text{ for all } a, b \in L.$$

We have the category **HA** of Heyting algebras and their homomorphisms.

Remark 10 (The classical Esakia Duality). The category **HSpec** is dually equivalent to **HA**. (See [30] or [31] for a good exposition.) Namely, we have the covariant functors:

- (1) $Sp : \mathbf{HA}^{op} \rightarrow \mathbf{HSpec}$ given by $Sp(L) = (\mathcal{PF}(L), \tau(\tilde{L}))$ and $Sp(h^{op}) = h^\bullet$ where $h : L \rightarrow M$ is a homomorphism, $h^\bullet(\mathcal{F}) = h^{-1}(\mathcal{G})$ for $\mathcal{G} \in \mathcal{PF}(M)$ and $\mathcal{PF}(L)$ denotes the set of all prime filters in L ,
- (2) $Co : \mathbf{HSpec} \rightarrow \mathbf{HA}^{op}$ given by $Co(X) = CO(X)$ with the obvious set-theoretic operations and $Co(g) = (\mathcal{L}g)^{op}$ where $(\mathcal{L}g)(V) = g^{-1}(V)$ for $g : X \rightarrow Y$ and $V \in CO(Y)$.

In analogy to the category **LatD** from [20], we have the following category.

Definition 22. Objects of **HAD** are pairs (L, \mathbf{D}_L) where L is a Heyting algebra $(L, \vee, \wedge, \rightarrow, 0, 1)$ and $\mathbf{D}_L \subseteq \mathcal{PF}(L)$ is a constructibly dense subset (i.e., a decent set of prime filters). Morphisms of **HAD** are such homomorphisms of Heyting algebras $h : L \rightarrow M$ that $h^\bullet(\mathbf{D}_M) \subseteq \mathbf{D}_L$.

9. Esakia Duality for Heyting Small Spaces

Theorem 2. The categories **HSS**, **HSpecD** and $(\mathbf{HAD})^{op}$ are equivalent.

Proof. Since the functors we shall consider are restrictions of the functors from Stone Duality [20] (Theorem 2), we concentrate on objects and morphisms being Heyting.

Step 1: The restriction functor R .

Restrict the functor \bar{R} from [20] (Theorem 1) given by $\bar{R}((X, \tau_X), X_d) = (X_d, CO(X)_d)$, $\bar{R}(g) = g_d$, where $g_d : X_d \rightarrow Y_d$ is the restriction of $g : X \rightarrow Y$ in both the domain and the codomain, to the functor $R : \mathbf{HSpecD} \rightarrow \mathbf{HSS}$. For (X, τ_X) a Heyting spectral space, the induced small space $(X_d, CO(X)_d)$ is T_0 and pre-Heyting: for $A \in Con(X_d)$ we have $cl(A) \in CC(X)_d$, see Remark 5. For a Heyting spectral mapping $g : X \rightarrow Y$ satisfying $g(X_d) \subseteq Y_d$, by the density of X_d , we have $g_d^{-1}(cl(A \cap Y_d)) = X_d \cap g^{-1}(A) = X_d \cap g^{-1}(A) = X_d \cap g_d^{-1}(A \cap Y_d) = cl(g_d^{-1}(A \cap Y_d))$ for $A \in Con(Y)$, so g_d is a Heyting continuous map. Hence R is a well-defined functor.

Step 2: The spectrum functor S .

Restrict the functor \bar{S} from [20] (Theorem 1) defined by $\bar{S}(L, \mathbf{D}_L) = ((\mathcal{PF}(L), \tau(\tilde{L})), \mathbf{D}_L)$, $\bar{S}(h^{op}) = h^\bullet$, where h^{op} in \mathbf{HAD}^{op} is the (renamed) morphism h in **HAD**, to the functor

$S : \mathbf{HAD}^{op} \rightarrow \mathbf{HSpecD}$. That $(\mathcal{PF}(L), \tau(\tilde{L}))$ is a Heyting spectral space follows from [25] (Theorem 8.3.12) since the Heyting algebra L is isomorphic to $\tilde{L} = CO(\mathcal{PF}(L))$.

For a morphism $h : L \rightarrow M$ in \mathbf{HAD} we have $(h^\bullet)^{-1}(\tilde{L}) \subseteq \tilde{M}$, so $h^\bullet : \mathcal{PF}(M) \rightarrow \mathcal{PF}(L)$ is spectral. Moreover, h^\bullet is a Heyting spectral mapping by [25] (Theorem 8.3.20). Hence S is a well-defined functor.

Step 3: The algebraization functor A .

Restrict the functor \bar{A} from [20] (Theorem 1) defined by formulas $\bar{A}(X, \mathcal{L}_X) = (\mathcal{L}_X, \hat{X})$, $\bar{A}(f) = (\mathcal{L}f)^{op}$ to the functor $A : \mathbf{HSS} \rightarrow \mathbf{HAD}^{op}$. We have the obvious Heyting algebra operations on \mathcal{L}_X (see Proposition 6) and, for a Heyting continuous mapping $f : X \rightarrow Y$, the mapping $\mathcal{L}f : \mathcal{L}_Y \ni W \mapsto f^{-1}(W) \in \mathcal{L}_X$ is a homomorphism of Heyting algebras: $(\mathcal{L}f)(U \rightarrow V) = f^{-1}(\text{int}(V \cup (Y \setminus U))) = \text{int}(f^{-1}(V) \cup (X \setminus f^{-1}(U))) = (\mathcal{L}f)(U) \rightarrow (\mathcal{L}f)(V)$. Hence A is a well-defined functor.

Step 4: The functor RSA is naturally isomorphic to $Id_{\mathbf{HSS}}$.

Restrict the natural transformation η from [20] (Theorem 1) to one from RSA to $Id_{\mathbf{HSS}}$. We have $(f \circ \eta_X)(\hat{x}) = (\eta_Y \circ RSA(f))(\hat{x})$ and each $\eta_X : (\hat{X}, \tilde{\mathcal{L}}_X^d) \rightarrow (X, \mathcal{L}_X)$ is a strict homeomorphism, so, by Fact 12, an isomorphism of Heyting small spaces. Hence η is truly a natural isomorphism of endofunctors of \mathbf{HSS} .

Step 5: The functor SAR is naturally isomorphic to $Id_{\mathbf{HSpecD}}$.

Restrict the natural transformation θ from [20] (Theorem 1) to one from SAR to $Id_{\mathbf{HSpecD}}$. We have $g \circ \theta_X = \theta_Y \circ SAR(g)$ and each θ_X is a bijection preserving both ways the compact open sets, so an isomorphism of Heyting spectral spaces (Fact 11). Hence θ is truly a natural isomorphism of endofunctors of \mathbf{HSpecD} .

Step 6: The functor ARS is naturally isomorphic to $Id_{\mathbf{HAD}^{op}}$.

Restrict the natural transformation κ^{op} from [20] (Theorem 1) to one from ARS to $Id_{\mathbf{HAD}^{op}}$. We have $\kappa_L^{op} \circ ARS(h^{op}) = h^{op} \circ \kappa_M^{op}$. As an order isomorphism, each κ_L^{op} is an isomorphism of Heyting algebras ([39] (Exercise IX.4.3)), so κ^{op} is truly a natural isomorphism of endofunctors of $(\mathbf{HAD})^{op}$. \square

Corollary 1. *The categories \mathbf{ESSD} and \mathbf{HSS} are equivalent.*

Corollary 2. *Open continuous definable mappings between definable spaces over o-minimal structures are restrictions to decent subsets of Heyting spectral mappings between spectral spaces. The latter mappings correspond to homomorphisms of Heyting algebras.*

10. Conclusions

After introducing pre-Heyting and pre-Boolean small spaces (Section 2) and giving a characterisation of pre-Heyting small spaces (Theorem 1), we have noticed that topological spectral spaces may be seen as T_0 sober small spaces with all smops compact (Section 3). We have recalled the basic properties of the specialization relation in the context of small spaces (Section 4). We have introduced the method of the standard spectralification of a T_0 small space (Section 5) and inferred some of its properties including preserving of being Heyting both ways (Section 6). After introducing the category \mathbf{HSS} of Heyting small spaces and Heyting continuous mappings, we have given an example of such mappings: open continuous definable mappings between definable spaces over o-minimal structures (Section 7), which means that those mappings can be described using homomorphisms of Heyting algebras. Next, we have constructed the category \mathbf{ESS} of Esakia small spaces and Esakia mappings, concretely isomorphic to \mathbf{HSpec} , and other useful categories (Section 8). Finally, we have proven a version of Esakia Duality for Heyting small spaces (Section 9, Theorem 2), making a new step in developing *tame topology*.

As extending our version of Esakia Duality to locally small spaces needs another set of notions, we postpone this to another paper. Future research may also concern the use of

the (already developed) theory of (up-)spectral spaces to understand (locally) small spaces, especially exploiting the notions of normal spectral spaces and spectral root systems.

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