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Applicability of Mönch's Fixed Point Theorem on a System of (k, ψ)-Hilfer Type Fractional Differential Equations

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Abstract: In this article, we study a system of Hilfer (k, ψ)-fractional differential equations, subject to nonlocal boundary conditions involving Hilfer (k, ψ)-derivatives and (k, ψ)-integrals. The results for the mentioned system are established by using Mönch's fixed point theorem, then the Ulam–Hyers technique is used to verify the stability of the solution for the proposed system. In general, symmetry and fractional differential equations are related to each other. When a generalized Hilfer fractional derivative is modified, asymmetric results are obtained. This study concludes with an applied example illustrating the existence results obtained by Mönch's theorem.

Keywords: generalized Hilfer derivative; existence; mixed boundary conditions; Ulam–Hyers stability

MSC: 26A33; 34B15; 34B18

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1. Introduction

Fractional derivatives are generalizations of classical derivatives. The concept of fractional differentiation has coincided with ordinary differentiation until late in our present era. However, recently, researchers have begun to focus their attention on fractional differentiation, which is considered a generalization of ordinary differentiation. Fractional analysis is the branch of mathematical analysis that deals with the several different possibilities of defining real number powers or complex powers of the differentiation operator and of integration. Fractional order differential equations are generalized and noninteger differential equations that are achieved in time and space with a power-law memory kernel of nonlocal relationships [1].

In [2], the authors studied the drug concentration in a blood model via Psi-Caputo fractional derivative, where the fractional model showed more accurate results in estimating the drug concentration in the blood. In [3], based on real data, the authors showed the superiority of a fractional model of blood ethanol concentration over a classical model. Moreover, in [4], a fractional modeling of the logistic population growth again showed superiority over an ordinary one. It should be noted that most of the works on fractional differentiation in the literature deal with Riemann–Liouville and Caputo fractional derivatives; see [5–10]. It is impossible to list all of the research papers that have addressed the issue of fractional stability of differential equation solutions in various known ways. On the other hand, many studies have focused on the stability of solutions using the Ulam–Hyers technique [11–16]. The study of the Hyers type of stability contributes significantly to more practical problems and applications such as population dynamics and fluid movement. While others have reported results using other types of stability, Ulam's group designed and implemented a type of stability for ordinary, fractional differential, and difference equations; see [17–19]. In [20], the (k, ψ)-fractional derivative of Hilfer was linked to the familiar Riemann–Liouville (k, ψ)-fractional derivative and the Caputo (k, ψ)-fractional derivative so the authors concluded that each of them was a special case of that derivative, and they provided some properties on that derivative. In addition, they investigated the existence of solutions to the following

$$
\begin{aligned}
&{}^{k,H}\mathcal{D}_{a+}^{\Xi,\nu;\psi}\mathcal{W}(\varsigma) = \mathfrak{f}(\varsigma,\mathcal{W}(\varsigma)), \varsigma \in (a,T],\ 0 < \Xi < k,\ 0 \le \nu \le 1,\\
&{}^{k}\mathcal{I}^{k-\xi_k;\psi}\mathcal{W}(a) = \mathcal{W}_a \in \mathcal{R} \quad \xi_k = \Xi + \nu(k-\Xi),
\end{aligned}
$$

where ${}^{k,H}\mathcal{D}_{a+}^{\Xi,\nu;\psi}(\cdot)$ is the Hilfer (k,ψ)-fractional derivative of order Ξ and type ν, ${}^{k}\mathcal{I}^{k-\xi_k;\psi}(\cdot)$ is the Riemann–Liouville (k,ψ)-fractional integral of order $k-\xi_k$.

The authors considered the Cauchy problem for the Hilfer ψ-fractional differential equations and investigated the existence and uniqueness of solutions in the weighted space of functions for the following FDE [21].

$$
\begin{aligned}
&{}^{H}\mathcal{D}_{a+}^{\Xi,\nu;\psi}\mathcal{W}(\varsigma) = \mathfrak{f}(\varsigma,\mathcal{W}(\varsigma)), \varsigma \in \varpi_1 = (a,a+\zeta],\ 0 < \Xi <\ 0 \le \nu \le 1,\\
&\mathcal{I}^{1-\xi;\psi}\mathcal{W}(a) = \mathcal{W}_a \in \mathcal{R} \quad \xi = \Xi + \nu(k-\Xi),
\end{aligned}
$$

where ${}^{H}\mathcal{D}_{a+}^{\Xi,\nu;\psi}(\cdot)$ is the Hilfer (ψ)-derivative of order Ξ and type ν, $\mathcal{I}^{1-\xi;\psi}(\cdot)$ is the Riemann–Liouville (ψ)-fractional integral of order $1-\xi$, and $\mathfrak{f} : \varpi_1 \times \mathcal{R} \to \mathcal{R}$ is an appropriate function.

The aforementioned works inspired us to employ Mönch's fixed point theorem for investigating the following system:

$$
\begin{cases}
{}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_1,\varphi_1;\psi}\mathcal{W}(\varsigma) = \mathcal{B}_1(\varsigma,\mathcal{W}(\varsigma),\mathcal{S}(\varsigma)),\ \varsigma \in (a,T],\ k > 0, 1 < \vartheta_1 \le 2,\ \varphi_1 \in [0,1],\\
{}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_2,\varphi_2;\psi}\mathcal{S}(\varsigma) = \mathcal{B}_2(\varsigma,\mathcal{W}(\varsigma),\mathcal{S}(\varsigma)),\ \varsigma \in (a,T],\ k > 0, 1 < \vartheta_2 \le 2,\ \varphi_2 \in [0,1],\\
\mathcal{W}(a) = 0,\ \ \mathcal{W}(T) = \varpi_1{}^{k,\mathcal{H}}\mathcal{D}^{\rho_1,q_1;\psi}\mathcal{W}(\Xi_1) + \chi_1{}^{k}\mathfrak{J}^{\mu_1;\psi}\mathcal{W}(\varrho_1),\\
\mathcal{S}(a) = 0,\ \ \mathcal{S}(T) = \varpi_2{}^{k,\mathcal{H}}\mathcal{D}^{\rho_2,q_2;\psi}\mathcal{S}(\Xi_2) + \chi_2{}^{k}\mathfrak{J}^{\mu_2;\psi}\mathcal{S}(\varrho_2),
\end{cases}
\tag{1}
$$

where ${}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_1,\varphi_1;\psi}$ denotes the Hilfer (k,ψ)-fractional derivative of order $\vartheta_1, \vartheta_2, 1 \le \vartheta_1, \vartheta_2 < 2$ and parameter $\varphi_i, i = 1,2.,\ 0 \le \varphi_i \le 1, \mathcal{B}_i : [a,T] \times \mathcal{R}^2 \to \mathcal{R}$ is a continuous function, ${}^{k,\mathcal{H}}\mathcal{D}^{\rho_i,q_i;\psi}, i = 1,2$, denotes the (k,ψ)-Hilfer type fractional derivative of order $\rho_i, 1 < \rho_i < 2$ and parameter q_i, $1 < q_i < 2$, $p < \vartheta_i$, ${}^{k}\mathfrak{J}^{\mu_i;\psi}$ is the the Riemann–Liouville (ψ)-fractional integral of order $v > 0$, $\varpi_1, \chi \in \mathcal{R}$, and $0 < \Xi_i, \varrho_i < 1, i = 1,2$.

The idea and originality of this study are summed up by using the Hilfer (k, ψ)-fractional derivative, which has received little attention in the literature. In addition, the existence of solutions to the system of fractional and nonlinear equations is investigated via Mönch's fixed point theorem, which has not been paid much attention to, as the Leray–Schauder alternative. Finally, the stability of the solution to the system of nonlinear equations given in Equation (1) is verified.

The manuscript is organized as follows: In the second section, we present the most important mathematical tools in the form of definitions and theories. In the third part, the main findings are presented. In the fourth part, the stability of the solutions to the system of equations studied is examined. In the fifth part, a numerical example is presented, documenting the theoretical results obtained in the third part. Finally, we present the most important observations and special cases and summarize the main results in the conclusion section.

2. Preliminaries

Definition 1 ([22])**.** *Consider* $\mathfrak{h} \in \mathcal{L}^1([\mathfrak{a},\mathcal{T}],\mathcal{R})$ *and* $k,\vartheta \in \mathcal{R}^+$. *Then, the Riemann–Liouville k-fractional derivative of order* ϑ *of function* $\mathfrak{h}$ *is presented by*

$$
{}^{k}\mathfrak{J}_{a+}^{\vartheta}\mathfrak{h}(\varsigma) = \frac{1}{k\Gamma_k(\vartheta)}\int_{\vartheta}^{\varsigma}(\varsigma - w)^{\frac{\vartheta}{k}-1}\mathfrak{h}(w)dw, \tag{2}
$$

where Γ_k is the k-Gamma function for $z \in \mathbb{C}$ with $\Re(z) > 0$ and $k \in \mathcal{R}$, $k > 0$, which is given in [23] by

$$\Gamma_k(z) = \int_0^{+\infty} s^{z-1} e^{-\frac{s^k}{k}} ds.$$

Definition 2 ([24])**.** *Let $\mathfrak{h} \in \mathcal{L}^1([\mathfrak{a}, \mathcal{T}], \mathcal{R})$ and $k, \vartheta \in \mathcal{R}^+$. Then, the Riemann–Liouville k-fractional derivative of order ϑ of function $\mathfrak{h}$ is given by*

$$^{k,\mathcal{RL}}\mathcal{D}^{\vartheta}_{a+}\mathfrak{h}(\varsigma) = \left(k\frac{d}{d\varsigma}\right)^n {}^k\mathfrak{J}^{nk-\vartheta}_{a+}\mathfrak{h}(\varsigma),\ n = \left[\frac{\vartheta}{k}\right], \tag{3}$$

where $\left[\frac{\vartheta}{k}\right]$ is the ceiling function of $\frac{\vartheta}{k}$.

Definition 3 ([25])**.** *Let $\mathfrak{h} \in \mathcal{L}^1([\mathfrak{a}, \mathcal{T}], \mathcal{R})$ and a monotone increasing function $\psi : [\mathfrak{a}, \mathcal{T}] \to \mathcal{R}$ with $\psi'(\varsigma) \neq 0$ for all $\varsigma \in [\mathfrak{a}, \mathcal{T}]$. Then, the Riemann–Liouville ψ-fractional integral of function $\mathfrak{h}$ is given by*

$$\mathfrak{J}^{\vartheta;\psi}\mathfrak{h}(\varsigma) = \frac{1}{\Gamma_k(\vartheta)} \int_a^{\varsigma} \psi'(u)(\psi(\varsigma) - \psi(u))^{\vartheta-1}\mathfrak{h}(u)du. \tag{4}$$

Definition 4. *Let $n - 1 < \vartheta \leq n, \psi \in \mathcal{C}^n([\mathfrak{a}, \mathcal{T}], \mathcal{R})$ is a monotone increasing function with $\psi'(\varsigma) \neq 0, \varsigma \in [\mathfrak{a}, \mathcal{T}]$, and $\mathfrak{h} \in \mathcal{C}([\mathfrak{a}, \mathcal{T}], \mathcal{R})$.*

(a) The Riemann–Liouville ψ-fractional derivative of function $\mathfrak{h}$ of order ϑ is presented in [25] as

$$^{\mathcal{RL}}\mathcal{D}^{\vartheta;\psi}\mathfrak{h}(\varsigma) = \left(\frac{1}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right)^n \mathfrak{J}^{n-\vartheta;\psi}_{a+}\mathfrak{h}(\varsigma). \tag{5}$$

(b) The Caputo ψ-fractional derivative of function $\mathfrak{h}$ of order ϑ is presented in [26] as

$$^{\mathcal{C}}\mathcal{D}^{\vartheta;\psi}\mathfrak{h}(\varsigma) = \mathfrak{J}^{n-\vartheta;\psi}_{a+}\left(\frac{1}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right)^n \mathfrak{h}(\varsigma). \tag{6}$$

(b) The Hilfer ψ-fractional derivative of function $\mathfrak{h} \in \mathcal{C}([\mathfrak{a}, \mathcal{T}], \mathcal{R})$ of order $\vartheta \in (n-1, n]$ and type $\varphi \in [0, 1]$ is presented in [27] as

$$^{\mathcal{H}}\mathcal{D}^{\vartheta,\varphi;\psi}\mathfrak{h}(\varsigma) = \mathfrak{J}^{\varphi(n-\vartheta);\psi}_{a+}\left(\frac{1}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right)^n \mathfrak{J}^{(1-\varphi)(n-\vartheta);\psi}_{a+}\mathfrak{h}(\varsigma). \tag{7}$$

Definition 5 ([28])**.** *Let $\psi : [\mathfrak{a}, \mathcal{T}] \to \mathcal{R}$ be a monotone increasing function with $\psi'(\varsigma) \neq 0$ for all $\varsigma \in [\mathfrak{a}, \mathcal{T}]$. Then, the Riemann–Liouville (k, ψ)-fractional integral of order $\vartheta > 0 (\vartheta \in \mathcal{R})$ of a function $\mathfrak{h} \in \mathcal{L}^1([\mathfrak{a}, \mathcal{T}], \mathcal{R})$ is given by*

$$^k\mathfrak{J}^{\vartheta;\psi}\mathfrak{h}(\varsigma) = \frac{1}{k\Gamma_k(\vartheta)} \int_a^{\varsigma} \psi'(u)(\psi(\varsigma) - \psi(u))^{\frac{\vartheta}{k}-1}\mathfrak{h}(u)du, k > 0. \tag{8}$$

Definition 6 ([20])**.** *Let $\vartheta, k \in \mathcal{R}^+ = (0, +\infty), \varphi \in [0, 1], \psi \in \mathcal{C}^n([0, 1], \mathcal{R})$ be a monotone increasing function with $\psi'(\varsigma) \neq 0, \varsigma \in [\mathfrak{a}, \mathcal{T}]$ and $\mathfrak{h} \in \mathcal{C}^n([\mathfrak{a}, \mathcal{T}], \mathcal{R})$. Then, the Hilfer (k, ψ)-fractional derivative of function $\mathfrak{h}$ of order ϑ and type φ is defined as*

$$^{k,\mathcal{H}}\mathcal{D}^{\vartheta,\varphi;\psi}\mathfrak{h}(\varsigma) = {}^k\mathfrak{J}^{\varphi(nk-\vartheta);\psi}_{a+}\left(\frac{k}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right)^n {}^k\mathfrak{J}^{(1-\varphi)(nk-\vartheta);\psi}_{a+}\mathfrak{h}(\varsigma),\ n = \left[\frac{\vartheta}{k}\right]. \tag{9}$$

Remark 1. *The Hilfer (k,ψ)-fractional derivative can be presented using the (k,ψ)-Riemann–Liouville fractional derivative as*

$$
\begin{aligned}
{}^{k,\mathcal{H}}\mathcal{D}^{\vartheta,\varphi;\psi}\mathfrak{h}(\varsigma) &= {}^{k}\mathfrak{J}_{a+}^{\phi_k-\vartheta;\psi}\left(\frac{k}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right)^{n}{}^{k}\mathfrak{J}_{a+}^{(nk-\phi_k);\psi}\mathfrak{h}(\varsigma),\\
&= {}^{k}\mathfrak{J}_{a+}^{\phi_k-\vartheta;\psi}\left({}^{k,\mathcal{RL}}\mathcal{D}^{\vartheta,\varphi;\psi}\mathfrak{h}\right)(\varsigma),
\end{aligned}
\tag{10}
$$

where $\phi_k = \vartheta + \varphi(nk-\vartheta)$, $\varphi(nk-\vartheta) = \phi_k - \vartheta$ and $(1-\varphi)(nk-\vartheta) = nk - \phi_k$, $\varphi \in [0,1]$. Note that $n-1 < \frac{\phi_k}{k} \le n$, where $n-1 < \frac{\vartheta}{k} \le n$.

We now recall some useful lemmas.

Lemma 1 ([20]). *Let $\mathfrak{h} \in \mathcal{C}^n([\mathfrak{a},\mathcal{T}],\mathcal{R})$ and ${}^{k}\mathcal{J}^{nk-\chi,\psi}\langle \in \mathcal{C}^n([\mathfrak{a},\mathcal{T}],\mathcal{R})$ with $\chi, k \in \mathcal{R}^+ = (0,+\infty)$ and $n = \left[\frac{\chi}{k}\right]$. Then,*

$$
{}^{k,\mathcal{H}}\mathcal{D}^{\vartheta,\varphi;\psi}\mathfrak{h}(\varsigma) = \mathfrak{h}(\varsigma) - \sum_{j=1}^{n}\frac{(\psi(\varsigma)-\psi(a))^{\frac{\chi}{k}-1}}{\Gamma_k(\chi - jk + k)}\left[\left(\frac{k}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right)^{n}{}^{k}\mathfrak{J}_{a+}^{nk-\chi;\psi}\mathfrak{h}(\varsigma)\right]_{z=a},
\tag{11}
$$

Lemma 2 ([20]). *Let $\phi_k = \vartheta + \varphi(k-\vartheta)$ with $\vartheta, k \in \mathcal{R}^+ = (0,+\infty)$, $\varphi \in [0,1]$. Then,*

$$
{}^{k,\mathcal{RL}}\mathcal{D}^{\vartheta,\varphi;\psi}\mathfrak{h}(\varsigma) = {}^{k,\mathcal{H}}\mathcal{D}^{\vartheta,\varphi;\psi}\mathfrak{h}(\varsigma).\ \mathfrak{h} \in \mathcal{C}^n([\mathfrak{a},\mathcal{T}],\mathcal{R}).
$$

Lemma 3 ([20]). *Let $\zeta, k \in \mathcal{R}^+ = (0,+\infty)$ and $\Xi \in \mathcal{R}$ such that $\frac{\Xi}{k} > -1$. Then,*

(1)

$$
{}^{k}\mathcal{I}^{\zeta,\psi}(\psi(\varsigma)-\psi(a))^{\frac{\Xi}{k}} = \frac{\Gamma_k(\Xi + k)}{\Gamma_k(\Xi + k + \zeta)}(\psi(\varsigma)-\psi(a))^{\frac{\Xi+\zeta}{k}}.
$$

(2)

$$
{}^{k}\mathcal{D}^{\zeta,\psi}(\psi(\varsigma)-\psi(a))^{\frac{\Xi}{k}} = \frac{\Gamma_k(\Xi + k)}{\Gamma_k(\Xi + k - \zeta)}(\psi(\varsigma)-\psi(a))^{\frac{\Xi-\zeta}{k}}.
$$

Definition 7 ([29]). *Let $\mathcal{W}$ be a bounded set on a Banach space $\hat{\mathcal{E}}$, then the Kuratowski measure of noncompactness of $\mathcal{W}$ is defined as:*

$$
k(\mathcal{W}) := \inf\{r > 0 : \mathcal{W} = \mathcal{W}_i \ \text{and diam} \ \ (\mathcal{W}_i) \le r \ \text{for} \ 1 \le i \le m\}.
$$

Lemma 4 ([29]). *Let $\hat{\mathcal{E}}$ be a Banach space with $\mathcal{U}$ and μ_1 two bounded proper subsets of $\hat{\mathcal{E}}$, then the following properties hold true*

(1) If $\mathcal{W} \subset \mathcal{S}$, then $k(\mathcal{W}) \le k(\mathcal{S})$;

(2) $k(\mathcal{W}) = k(\bar{\mathcal{W}}) = k(\overline{conv}\mathcal{W})$;

(3) $\mathcal{W}$ is relatively compact $k(\mathcal{W}) = 0$;

(4) $k(\delta\mathcal{W}) = |\delta| k(\mathcal{W}), \delta \in$;

(5) $k(\mathcal{W} \cup \mathcal{S}) = \max\{k(\mathcal{W}), k(\mathcal{S})\}$;

(6) $k(\mathcal{W} + \mathcal{S}) = k(\mathcal{W}) + k(\mathcal{S}), \mathcal{W} + \mathcal{S} = \{x | x = \mathcal{W} + \mathcal{S}, \mathcal{W} \in \mathcal{W}, \mathcal{S} \in \mathcal{S}\}$;

(7) $k(\mathcal{W} + \mathcal{S}) = k(\mathcal{W}), \forall y \in \hat{\mathcal{E}}$.

Lemma 5 ([29]). *Given an equicontinuous and bounded set $\mathcal{G} \subset \mathcal{C}([\mathfrak{a}, \mathcal{T}], \hat{\mathcal{E}})$, then the function $\rightarrowtail k(\mathcal{G}())$ is continuous on $[\mathfrak{a}, \mathcal{T}], k_{\mathcal{C}}(\mathcal{G}) = \max_{\in[\mathfrak{a},\mathcal{T}]} k(\mathcal{G}())$, and*

$$k\left(\int_{\mathfrak{a}}^{\mathcal{T}} x(\varsigma)d\varsigma\right) \leq \left(\int_{\mathfrak{a}}^{\mathcal{T}} (x(\varsigma))d\varsigma\right), \mathcal{G}(\varsigma) = \{x(\varsigma) : x \in \mathcal{G}\}. \tag{12}$$

Definition 8 ([29]). *Given the function $\Psi : [\mathfrak{a}, \mathcal{T}] \times \hat{\mathcal{E}} \to \hat{\mathcal{E}}, \Psi$ satisfies the Carathéodory's conditions, if the following conditions apply:*

$\Psi(\varsigma, z)$ is measurable in ς for $z \in \hat{\mathcal{E}}$;

$\Psi(\varsigma, z)$ is continuous in $z \in \hat{\mathcal{E}}$ for $\in [\mathfrak{a}, \mathcal{T}]$.

Theorem 1 ([30]). *(Mönch's fixed point theorem) Let Ω be a closed, bounded and convex subset of $\hat{\mathcal{E}}$, such that $0 \in \Omega$; let also $\mathcal{M}$ be a continuous map of Ω into itself.*

If $\mathcal{G} = \overline{conv}\mathcal{M}(\mathcal{G})$ or $\mathcal{G} = \mathcal{M}(\mathcal{G}) \cup \{0\}$, then $k(\mathcal{G}) = 0$, such that $\forall\ \mathcal{G} \subset \Omega$, and $\mathcal{M}$ has a fixed point.

In the next section, we present an auxiliary result dealing with the linear variant of problem (1).

3. Main Result

In this section, through the following lemma, we find a solution to the proposed system of fractional equations given in (1), then set the necessary conditions to verify the existence of solutions to the system of equations mentioned. Accordingly, we verify the applicability of Mönch's fixed point theorem to the system.

Lemma 6. *Let $\mathfrak{g} \in \mathcal{C}(\mathfrak{a}, \mathcal{T}) \cup \mathcal{L}^1(\mathfrak{a}, \mathcal{T})$ and*

$$\hat{\Lambda}_1 = \frac{(\psi(T) - \psi(a))^{\frac{\phi_k}{k}-1}}{\Gamma_k(\phi_k)} - \varpi_1 \Gamma_k(\phi_k) \frac{(\psi(\Xi_1) - \psi(a))^{\frac{\phi_k - \rho_1}{k}-1}}{\Gamma_k(\phi_k)} - \chi_1 \Gamma_k(\phi_k) \frac{(\psi(\varrho_1) - \psi(a))^{\frac{\phi_k + \mu_1}{k}-1}}{\Gamma_k(\phi_k)} \neq 0. \tag{13}$$

and

$$\hat{\Lambda}_2 = \frac{(\psi(T) - \psi(a))^{\frac{\phi_k}{k}-1}}{\Gamma_k(\phi_k)} - \varpi_2 \Gamma_k(\phi_k) \frac{(\psi(\Xi_2) - \psi(a))^{\frac{\phi_k - \rho_2}{k}-1}}{\Gamma_k(\phi_k)} - \chi_2 \Gamma_k(\phi_k) \frac{(\psi(\varrho_2) - \psi(a))^{\frac{\phi_k + \mu_2}{k}-1}}{\Gamma_k(\phi_k)} \neq 0. \tag{14}$$

Then, $\mathcal{W}, \mathcal{S}$ are a solution for the following BVP

$$\begin{cases} {}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_1,\varphi_1;\psi}\mathcal{W}(\varsigma) = \mathfrak{g}_1(\varsigma),\ \varsigma \in (a, T],\ k > 0, 1 < \vartheta_1 \leq 2,\ \varphi_1 \in [0,1], \\ {}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_2,\varphi_2;\psi}\mathcal{S}(\varsigma) = \mathfrak{g}_2(\varsigma),\ \varsigma \in (a, T],\ k > 0, 1 < \vartheta_2 \leq 2,\ \varphi_2 \in [0,1], \\ \mathcal{W}(a) = 0,\ \ \mathcal{W}(T) = \varpi_1{}^{k,\mathcal{H}}\mathcal{D}^{\rho_1,q_1;\psi}\mathcal{W}(\Xi_1) + \chi_1{}^{k}\mathfrak{J}^{\mu_1;\psi}\mathcal{W}(\varrho_1), \\ \mathcal{S}(a) = 0,\ \ \mathcal{S}(T) = \varpi_2{}^{k,\mathcal{H}}\mathcal{D}^{\rho_2,q_2;\psi}\mathcal{S}(\Xi_2) + \chi_2{}^{k}\mathfrak{J}^{\mu_2;\psi}\mathcal{S}(\varrho_2), \end{cases} \tag{15}$$

if and only if

$$\mathcal{W}(\varsigma) = {}^{k}\mathfrak{J}^{\vartheta_1;\psi}\mathfrak{g}_1(\varsigma) + \frac{(\psi(\varsigma) - \psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_1 \Gamma_k(\phi_k)} \Big[\varpi_1{}^{k}\mathfrak{J}^{\vartheta_1-\rho_1;\psi}\mathfrak{g}_1(\Xi_1) + \chi_1{}^{k}\mathfrak{J}^{\mu_1+\vartheta_1;\psi}\mathfrak{g}_1(\varrho_1) - {}^{k}\mathfrak{J}^{\vartheta_1;\psi}\mathfrak{g}_1(T)\Big], \tag{16}$$

and

$$\mathcal{S}(\varsigma) = {}^{k}\mathfrak{J}^{\vartheta_2;\psi}\mathfrak{g}_2(\varsigma) + \frac{(\psi(\varsigma) - \psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_2 \Gamma_k(\phi_k)} \Big[\varpi_1{}^{k}\mathfrak{J}^{\vartheta_2-\rho_2;\psi}\mathfrak{g}_2(\Xi_2) + \chi_2{}^{k}\mathfrak{J}^{\mu_2+\vartheta_2;\psi}\mathfrak{g}_2(\varrho_2) - {}^{k}\mathfrak{J}^{\vartheta_2;\psi}\mathfrak{g}_2(T)\Big], \tag{17}$$

where $\phi_k = \vartheta_i + \varphi_i(2k - \vartheta_i)$, $i = 1,2$.

Proof. Applying the operator ${}^{k}\mathfrak{J}^{\vartheta_i;\psi}, i = 1,2$ and using Lemma 1 and 2 yield

$$\begin{aligned}{}^{k}\mathfrak{J}^{\vartheta_1;\psi}\left({}^{k,\mathcal{H}}\mathcal{D}^{\phi_k;\psi}\mathcal{W}\right)(\varsigma) =& {}^{k}\mathfrak{J}^{\phi_k;\psi}\left({}^{k,\mathcal{RL}}\mathcal{D}^{\phi_k;\psi}\mathcal{W}\right)(\varsigma)\\ =&\mathcal{W}(\varsigma) - \frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\Gamma_k(\phi_k)}\left[\left(\frac{k}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right){}^{k}\mathfrak{J}^{2k-\phi_k;\psi}_{a+}\mathcal{W}(\varsigma)\right]_{w=a}\\ &- \frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-2}}{\Gamma_k(\phi_k - k)}\left[{}^{k}\mathfrak{J}^{2k-\phi_k;\psi}_{a+}\mathcal{W}(\varsigma)\right]_{w=a},\end{aligned}$$

consequently

$$\mathcal{W}(\varsigma) = {}^{k}\mathfrak{J}^{\vartheta_1;\psi}\mathfrak{g}_1(\varsigma) + c_0\frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\Gamma_k(\phi_k)} + c_1\frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-2}}{\Gamma_k(\phi_k - k)}, \tag{18}$$

where

$$c_0 = \left[\left(\frac{k}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right){}^{k}\mathfrak{J}^{2k-\phi_k;\psi}_{a+}\mathcal{W}(\varsigma)\right]_{w=a},\quad c_1 = \left[{}^{k}\mathfrak{J}^{2k-\phi_k;\psi}_{a+}\mathcal{W}(\varsigma)\right]_{w=a},$$

and

$$\begin{aligned}{}^{k}\mathfrak{J}^{\vartheta_2;\psi}\left({}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_2,\varphi_2;\psi}\mathcal{S}\right)(\varsigma) =& {}^{k}\mathfrak{J}^{\phi_k;\psi}\left({}^{k,\mathcal{RL}}\mathcal{D}^{\phi_k;\psi}\mathcal{S}\right)(\varsigma)\\ =&\mathcal{S}(\varsigma) - \frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\Gamma_k(\phi_k)}\left[\left(\frac{k}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right){}^{k}\mathfrak{J}^{2k-\phi_k;\psi}_{a+}\mathcal{S}(\varsigma)\right]_{w=a}\\ &- \frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-2}}{\Gamma_k(\phi_k - k)}\left[{}^{k}\mathfrak{J}^{2k-\phi_k;\psi}_{a+}\mathcal{S}(\varsigma)\right]_{w=a}\end{aligned}$$

implies

$$\mathcal{S}(\varsigma) = {}^{k}\mathfrak{J}^{\vartheta_2;\psi}\mathfrak{g}_2(\varsigma) + d_0\frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\Gamma_k(\phi_k)} + d_2\frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-2}}{\Gamma_k(\phi_k - k)}, \tag{19}$$

where

$$d_0 = \left[\left(\frac{k}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right){}^{k}\mathfrak{J}^{2k-\phi_k;\psi}_{a+}\mathcal{S}(\varsigma)\right]_{w=a},\quad d_1 = \left[{}^{k}\mathfrak{J}^{2k-\phi_k;\psi}_{a+}\mathcal{S}(\varsigma)\right]_{w=a},$$

by the condition $\mathcal{W}(a) = 0, \mathcal{S}(a) = 0$, we find that $c_1 = 0, d_1 = 0$ as $\frac{\phi_k}{k} - 2 < 0$ by Remark 1. By using Lemma 3, we obtain

$${}^{k}\mathcal{D}^{\rho_1,\sigma_1;\psi}(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1} = \frac{\Gamma_k(\phi_k)}{\Gamma_k(\phi_k - \rho_1)}(\psi(\varsigma)-\psi(a))^{\frac{\phi_k-\rho_1}{k}-1}. \tag{20}$$

and

$${}^{k}\mathfrak{J}^{\mu_1;\psi}(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1} = \frac{\Gamma_k(\phi_k)}{\Gamma_k(\phi_k - \rho_1)}(\psi(\varsigma)-\psi(a))^{\frac{\phi_k-\mu_1}{k}-1}, \tag{21}$$

from (20) and (21) and the boundary conditions $\mathcal{W}(T) = \varpi_1\mathcal{W}(\Xi_1) + \chi_1{}^{k}\mathcal{I}^{\mu_1;\psi}\mathcal{W}(\varrho)$, $\mathcal{S}(T) = \varpi_2\mathcal{S}(\Xi_2) + \chi_2{}^{k}\mathcal{I}^{\mu_2;\psi}\mathcal{S}(\varrho)$ and we obtain

$$c_0 = \frac{1}{\hat{\Lambda}_1}\left(\varpi_1{}^{k}\mathfrak{J}^{\vartheta_1-\rho_1;\psi}\mathfrak{g}_1(\Xi_1) + \chi_1{}^{k}\mathfrak{J}^{\mu_1+\vartheta_1;\psi}\mathfrak{g}_1(\varrho_1) - {}^{k}\mathfrak{J}^{\vartheta_1;\psi}\mathfrak{g}_1(T)\right).$$

and

$$d_0 = \frac{1}{\hat{\Lambda}_2}\Big(\varpi_2{}^k\mathfrak{I}^{\vartheta_2-\rho_2;\psi}\mathfrak{g}_2(\Xi_2) + \chi_2{}^k\mathfrak{I}^{\mu_2+\vartheta_2;\psi}\mathfrak{g}_2(\varrho_2) - {}^k\mathfrak{I}^{\vartheta_2;\psi}\mathfrak{g}_2(T)\Big),$$

by substituting the values of the constants c_0, d_0, d_1, and c_1 where necessary, we obtain the solution (16) and (17). By a trivial computation, the converse of the lemma can be easily verified. □

Denote the Banach space by the set $\widehat{\mathcal{E}} = \{(\mathcal{W}(\varsigma), \mathcal{S}(\varsigma)) | (\mathcal{W}, \mathcal{S}) \in \mathcal{C}([a,T], \mathcal{R}_e) \times \mathcal{C}([a,T], \mathcal{R}_e)\}$, endowed with the norm defined as

$$||(\mathcal{W}, \mathcal{S})||_{\widehat{\mathcal{E}}} = ||\mathcal{W}||_\infty + ||\mathcal{S}||_\infty,$$

for simpler computations, we set

$$\begin{aligned}\Delta_1 =& \frac{(\psi(T)-\psi(a))^{\frac{\vartheta_1}{k}}}{\Gamma_k(\vartheta_1+k)} + \frac{(\psi(T)-\psi(a))^{\frac{\phi_k}{k}-1}}{|\hat{\Lambda}_1|\Gamma_k(\vartheta_1+k)}\Bigg[|\varpi_1|\frac{(\psi(\Xi_1)-\psi(a))^{\frac{\vartheta_1-\rho}{k}}}{\Gamma_k(\vartheta_1-\rho_1+k)} \\ &+|\chi_1|\frac{(\psi(\Xi_1)-\psi(a))^{\frac{\vartheta_1+\mu_1}{k}}}{\Gamma_k(\vartheta_1+\mu_1+k)} + \frac{(\psi(\Xi_1)-\psi(a))^{\frac{\vartheta_1}{k}}}{\Gamma_k(\vartheta_1+k)}\Bigg].\end{aligned}$$

and

$$\begin{aligned}\Delta_2 =& \frac{(\psi(T)-\psi(a))^{\frac{\vartheta_2}{k}}}{\Gamma_k(\vartheta_2+k)} + \frac{(\psi(T)-\psi(a))^{\frac{\phi_k}{k}-1}}{|\hat{\Lambda}_2|\Gamma_k(\vartheta_2+k)}\Bigg[|\varpi_2|\frac{(\psi(\Xi_2)-\psi(a))^{\frac{\vartheta_2-\rho}{k}}}{\Gamma_k(\vartheta_2-\rho_2+k)} \\ &+|\chi_2|\frac{(\psi(\Xi_2)-\psi(a))^{\frac{\vartheta_2+\mu_2}{k}}}{\Gamma_k(\vartheta_2+\mu_2+k)} + \frac{(\psi(\Xi_2)-\psi(a))^{\frac{\vartheta_2}{k}}}{\Gamma_k(\vartheta_2+k)}\Bigg],\end{aligned}$$

in the following, we present the hypotheses that support verifying the possibility of the existence of the solution for system (1).

($\mathcal{A}_1$) Let $\mathcal{B}_1, \mathcal{B}_2 : [a,T] \times (\mathcal{R}_e)^2 \to \mathcal{R}_e$ satisfy Carathéodory conditions.

($\mathcal{A}_2$) $\exists\ \mathcal{Y}_{\mathcal{B}_1}, \mathcal{Y}_{\mathcal{B}_2} \in \mathcal{L}^1[a,T] \times (\mathcal{R}_e)_+$ and $\exists\ \mathfrak{H}_{\mathcal{B}_1}, \mathfrak{H}_{\mathcal{B}_2} : (\mathcal{R}_e)_+ \to (\mathcal{R}_e)_+$ such that $\forall\ \varsigma \in [a,T], \forall(\mathcal{W}, \mathcal{S} \in \widehat{\mathcal{E}})$ we have

$$\begin{aligned}||\mathcal{B}_1(\varsigma, \mathcal{W}, \mathcal{S})||_\infty \le \mathcal{Y}_{\mathcal{B}_1}(\varsigma)\mathfrak{H}_{\mathcal{B}_1}(||\mathcal{W}||_\infty + ||\mathcal{S}||_\infty),\\ ||\mathcal{B}_2(\varsigma, \mathcal{W}, \mathcal{S})||_\infty \le \mathcal{Y}_{\mathcal{B}_2}(\varsigma)\mathfrak{H}_{\mathcal{B}_2}(||\mathcal{W}||_\infty + ||\mathcal{S}||_\infty),\end{aligned}$$

here $\mathfrak{H}_{\mathcal{B}_1}, \mathfrak{H}_{\mathcal{B}_2}$ are nondecreasing continuous functions.

($\mathcal{A}_3$) Let $\mathcal{G} \subset \widehat{\mathcal{E}} \times \widehat{\mathcal{E}}$, assumed to be bounded, and

$$\begin{aligned}\mathcal{Y}(\mathcal{B}_1, (\varsigma, \mathcal{G})) \le \mathcal{Y}_{\mathcal{B}_1}(\varsigma)\mathcal{Y}(\mathcal{G}),\\ \mathcal{Y}(\mathcal{B}_2, (\varsigma, \mathcal{G})) \le \mathcal{Y}_{\mathcal{B}_2}(\varsigma)\mathcal{Y}(\mathcal{G}).\end{aligned}$$

Theorem 2. *Assume that the assumptions* ($\mathcal{A}_1$), ($\mathcal{A}_2$), *and* ($\mathcal{A}_3$) *hold. If*

$$\max\{\mathcal{Y}^*_{\mathcal{B}_1}\Delta_1, \mathcal{Y}^*_{\mathcal{B}_2}\Delta_2\} < 1, \tag{22}$$

where $\mathcal{Y}^*_{\mathcal{B}_i} = \sup_{a\le\varsigma\le T} \mathcal{Y}^*_{\mathcal{B}_i}(\varsigma),\ \ \forall_i = 1,2,$ *then the system of fractional differential equations given by (1) has at least one solution on* $[a,T]$.

Proof. Define the operator $\mathcal{M} : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}$ as:

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_1(\mathcal{W},\mathcal{S})(\varsigma) \\ \mathcal{M}_2(\mathcal{W},\mathcal{S})(\varsigma) \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned}\mathcal{M}_1(\mathcal{W},\mathcal{S})(\varsigma) =& \frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_1 \Gamma_k(\phi_k)} \Big[\varpi_1 {}^k\mathfrak{I}^{\vartheta_1-\rho_1;\psi}\mathcal{B}_1(\Xi_1,\mathcal{W}(\Xi_1),\mathcal{S}(\Xi_1)) \\ &+\chi_1 {}^k\mathfrak{I}^{\mu_1+\vartheta_1;\psi}\mathcal{B}_1(\varrho_1,\mathcal{W}(\varrho_1),\mathcal{S}(\varrho_1)) - {}^k\mathfrak{I}^{\vartheta_1;\psi}\mathcal{B}_1(T,\mathcal{W}(T),\mathcal{S}(T))\Big] \\ &+ {}^k\mathfrak{I}^{\vartheta_1;\psi}\mathcal{B}_1(\varsigma,\mathcal{W}(\varsigma),\mathcal{S}(\varsigma)),\end{aligned} \tag{24}$$

and

$$\begin{aligned}\mathcal{M}_2(\mathcal{W},\mathcal{S})(\varsigma) =& \frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_2 \Gamma_k(\phi_k)} \Big[\varpi_2 {}^k\mathfrak{I}^{\vartheta_2-\rho_2;\psi}\mathcal{B}_2(\Xi_2,\mathcal{W}(\Xi_2),\mathcal{S}(\Xi_2)) \\ &+\chi_2 {}^k\mathfrak{I}^{\mu_2+\vartheta_2;\psi}\mathcal{B}_2(\varrho_2,\mathcal{W}(\varrho_2),\mathcal{S}(\varrho_2)) - {}^k\mathfrak{I}^{\vartheta_2;\psi}\mathcal{B}_2(T,\mathcal{W}(T),\mathcal{S}(T))\Big] \\ &+ {}^k\mathfrak{I}^{\vartheta_2;\psi}\mathcal{B}_2(\varsigma,\mathcal{W}(\varsigma),\mathcal{S}(\varsigma)), \quad \varsigma \in [a,T],\end{aligned} \tag{25}$$

define the operator equation

$$(\mathcal{W},\mathcal{S}) = \mathcal{M}(\mathcal{W},\mathcal{S}), \tag{26}$$

observe that both systems given in (1) and (26) are equivalent, that is, by showing the existence of the solution to the defined operator in Equation (26), we show the existence of a solution to system (1).

Next, we define the closed, bounded, convex set in $\hat{\mathcal{E}}$ given by $\mathcal{G}_\delta = \{(\mathcal{W},\mathcal{S}) \in \hat{\mathcal{E}} : ||(\mathcal{W},\mathcal{S})||_{\hat{\mathcal{E}}} \leq \delta,\ \delta > 0\}$ with

$$\delta \geq \mathcal{Y}^*_{\mathcal{B}_1}\Delta_1\mathfrak{H}_{\mathcal{B}_1}(\delta) + \mathcal{Y}^*_{\mathcal{B}_2}\Delta_2\mathfrak{H}_{\mathcal{B}_2}(\delta),$$

to facilitate the proof of Theorem (2). We present the proof in four steps.

Step 1: We show that $\mathcal{M}\mathcal{G}_\delta \subset \mathcal{G}_\delta$. Let $\tau \in [a,T]$ and $\forall(\mathcal{W},\mathcal{S}) \in \mathcal{G}_\delta$; we have

$$\begin{aligned}||\mathcal{M}_1(\mathcal{W},\mathcal{S})||_\infty \leq& \frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_1 \Gamma_k(\phi_k)} \Big[\varpi_1 {}^k\mathfrak{I}^{\vartheta_1-\rho_1;\psi}||\mathcal{B}_1(\Xi_1,\mathcal{W}(\Xi_1),\mathcal{S}(\Xi_1)))||_\infty \\ &+\chi_1 {}^k\mathfrak{I}^{\mu_1+\vartheta_1;\psi}||\mathcal{B}_1(\varrho_1,\mathcal{W}(\varrho_1),\mathcal{S}(\varrho_1))||_\infty - {}^k\mathfrak{I}^{\vartheta_1;\psi}||\mathcal{B}_1(T,\mathcal{W}(T),\mathcal{S}(T))||_\infty\Big] \\ &+ {}^k\mathfrak{I}^{\vartheta_1;\psi}||\mathcal{B}_1(\varsigma,\mathcal{W}(\varsigma),\mathcal{S}(\varsigma))||_\infty,\end{aligned} \tag{27}$$

and

$$\begin{aligned}||\mathcal{M}_1(\mathcal{W},\mathcal{S})||_\infty \leq& \frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_2 \Gamma_k(\phi_k)} \Big[\varpi_2 {}^k\mathfrak{I}^{\vartheta_2-\rho_2;\psi}||\mathcal{B}_2(\Xi_2,\mathcal{W}(\Xi_2),\mathcal{S}(\Xi_2))||_\infty \\ &+\chi_2 {}^k\mathfrak{I}^{\mu_2+\vartheta_2;\psi}||\mathcal{B}_2(\varrho_2,\mathcal{W}(\varrho_2),\mathcal{S}(\varrho_2))||_\infty - {}^k\mathfrak{I}^{\vartheta_2;\psi}||\mathcal{B}_2(T,\mathcal{W}(T),\mathcal{S}(T))||_\infty\Big] \\ &+ {}^k\mathfrak{I}^{\vartheta_2;\psi}||\mathcal{B}_2(\varsigma,\mathcal{W}(\varsigma),\mathcal{S}(\varsigma))||_\infty, \quad \varsigma \in [a,T];\end{aligned} \tag{28}$$

using $(\mathcal{A}_2)$, $\forall \varsigma \in [a,T]$, we have

$$\begin{aligned}||\mathcal{B}_1(\varsigma),\mathcal{W}(\varsigma),\mathcal{S}(\varsigma)||_\infty \leq& \mathcal{Y}^*_{\mathcal{B}_1}(\varsigma)\mathfrak{H}_{\mathcal{B}_1}(||\mathcal{W}(\varsigma)||_\infty + ||\mathcal{S}(\varsigma)||_\infty) \\ \leq& \mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}(\delta),\end{aligned}$$

that is

$$\begin{aligned}||\mathcal{M}_1(\mathcal{W},\mathcal{S})||_\infty \leq &\frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_1\Gamma_k(\phi_k)}\Big[\varpi_1{}^k\mathfrak{I}^{\vartheta_1-\rho_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}||\mathcal{B}_1(\Xi_1,\mathcal{W}(\Xi_1),\mathcal{S}(\Xi_1)))||_\infty \\ &+\chi_1{}^k\mathfrak{I}^{\mu_1+\vartheta_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}||\mathcal{B}_1(\varrho_1,\mathcal{W}(\varrho_1),\mathcal{S}(\varrho_1))||_\infty - {}^k\mathfrak{I}^{\vartheta_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}||\mathcal{B}_1(T,\mathcal{W}(T),\mathcal{S}(T))||_\infty\Big] \\ &+{}^k\mathfrak{I}^{\vartheta_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}||\mathcal{B}_1(\varsigma,\mathcal{W}(\varsigma),\mathcal{S}(\varsigma))||_\infty, \\ &\leq \mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}(\delta).\end{aligned} \tag{29}$$

Similarly,

$$\begin{aligned}||\mathcal{M}_2(\mathcal{W},\mathcal{S})||_\infty \leq &\frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_2\Gamma_k(\phi_k)}\Big[\varpi_2{}^k\mathfrak{I}^{\vartheta_2-\rho_2;\psi}\mathcal{Y}^*_{\mathcal{B}_2}\mathfrak{H}_{\mathcal{B}_2}||\mathcal{B}_2(\Xi_2,\mathcal{W}(\Xi_2),\mathcal{S}(\Xi_2))||_\infty \\ &+\chi_2{}^k\mathfrak{I}^{\mu_2+\vartheta_2;\psi}\mathcal{Y}^*_{\mathcal{B}_2}\mathfrak{H}_{\mathcal{B}_2}||\mathcal{B}_2(\varrho_2,\mathcal{W}(\varrho_2))||_\infty \\ &-{}^k\mathfrak{I}^{\vartheta_2;\psi}\mathcal{Y}^*_{\mathcal{B}_2}\mathfrak{H}_{\mathcal{B}_2}||\mathcal{B}_2(T,\mathcal{W}(T),\mathcal{S}(T))||_\infty\Big] \\ &+{}^k\mathfrak{I}^{\vartheta_2;\psi}\mathcal{Y}^*_{\mathcal{B}_2}\mathfrak{H}_{\mathcal{B}_2}||\mathcal{B}_2(\varsigma,\mathcal{W}(\varsigma),\mathcal{S}(\varsigma))||_\infty \\ &\leq \mathcal{Y}^*_{\mathcal{B}_2}\mathfrak{H}_{\mathcal{B}_2}(\delta).\end{aligned} \tag{30}$$

(29) and (30) yield

$$\begin{aligned}||\mathcal{M}(\mathcal{W},\mathcal{S})||_{\hat{\mathcal{E}}} =&||\mathcal{M}_1(\mathcal{W},\mathcal{S})||_\infty + ||\mathcal{M}_2(\mathcal{W},\mathcal{S})||_\infty \\ \leq&\mathcal{Y}^*_{\mathcal{B}_1}\Delta_1\mathfrak{H}_{\mathcal{B}_1}(\delta)+\mathcal{Y}^*_{\mathcal{B}_2}\Delta_2\mathfrak{H}_{\mathcal{B}_2}(\delta) \\ \leq&\delta,\end{aligned} \tag{31}$$

that is, $\mathcal{M}\mathcal{G}_\delta \subset \mathcal{G}_\delta$.

Step 2: We show that the operator $\mathcal{M}$ is continuous. Indeed, define the sequence

$$\{\mathcal{V}_n = (\mathcal{W}_n,\mathcal{S}_n)\} \in \mathcal{G}_\delta; \text{ we show that } \mathcal{V}_n \to \mathcal{V} = (\mathcal{W},\mathcal{S}) \text{ as } n \to \infty,$$

because of hypothesis $\mathcal{A}_1$, it is clear that

$$\mathcal{B}_1(\cdot,\mathcal{W}_n(\cdot),\mathcal{S}_n(\cdot)) \to \mathcal{B}_1(\cdot,\mathcal{W}(\cdot),\mathcal{S}(\cdot)) \text{ as } n \to \infty,$$

recalling ($\mathcal{A}_2$), we deduce that

$${}^k\mathfrak{I}^{\varphi_1;\psi}||\mathcal{B}_1(s,\mathcal{W}_n(s),\mathcal{S}_n(s))-\mathcal{B}_1(s,\mathcal{W}(s),\mathcal{S}(s))||_\infty \leq \mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}(\delta){}^k\mathfrak{I}^{\varphi_1;\psi}, \tag{32}$$

additionally, using the function's Lebesgue dominated convergence theorem and the fact that

$$\mathcal{X} \to \mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}(\delta){}^k\mathfrak{I}^{\varphi_1;\psi}, \tag{33}$$

is Lebesgue integrable on $[a,T]$, we get

$$\begin{aligned}||\mathcal{M}_1(\mathcal{W},\mathcal{S})||_\infty \leq &\Bigg\{\frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_1\Gamma_k(\phi_k)}\Big[\varpi_1{}^k\mathfrak{I}^{\vartheta_1-\rho_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}||\mathcal{B}_1(s,\mathcal{W}_n(s),\mathcal{S}_n(s))-\mathcal{B}_1(s,\mathcal{W}(s),\mathcal{S}(s))||_\infty \\ &+\chi_1{}^k\mathfrak{I}^{\mu_1+\vartheta_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}||\mathcal{B}_1(s,\mathcal{W}_n(s),\mathcal{S}_n(s))-\mathcal{B}_1(s,\mathcal{W}(s),\mathcal{S}(s))||_\infty \\ &-{}^k\mathfrak{I}^{\vartheta_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}||\mathcal{B}_1(s,\mathcal{W}_n(s),\mathcal{S}_n(s))-\mathcal{B}_1(s,\mathcal{W}(s),\mathcal{S}(s))||_\infty\Big] \\ &+{}^k\mathfrak{I}^{\vartheta_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}||\mathcal{B}_1(s,\mathcal{W}_n(s),\mathcal{S}_n(s))-\mathcal{B}_1(s,\mathcal{W}(s),\mathcal{S}(s))||_\infty,\Bigg\} \to 0 \text{ as } n \to \infty,\end{aligned} \tag{34}$$

that is,

$$||\mathcal{M}_1(\mathcal{W}_n,\mathcal{S}_n)(\varsigma)-\mathcal{M}_1(\mathcal{W},\mathcal{S})(\varsigma)||_\infty\to 0 \text{ as } n\to\infty\ \forall\ \varsigma\in[a,T],$$

then

$$||\mathcal{M}_1(\mathcal{W}_n,\mathcal{S}_n)-\mathcal{M}_1(\mathcal{W},\mathcal{S})||_\infty\to 0 \text{ as } n\to\infty, \tag{35}$$

which shows the continuity of $\mathcal{M}_1$.

Similarly,

$$||\mathcal{M}_2(\mathcal{W}_n,\mathcal{S}_n)-\mathcal{M}_2(\mathcal{W},\mathcal{S})||_\infty\to 0 \text{ as } n\to\infty. \tag{36}$$

(35) and (36) yield

$$||\mathcal{M}(\mathcal{W}_n,\mathcal{S}_n)-\mathcal{M}(\mathcal{W},\mathcal{S})||_{\widehat{\mathcal{E}}}\to 0 \text{ as } n\to\infty, \tag{37}$$

and by getting (37), we conclude that the operator $\mathcal{M}$ is continuous.

Step 3: We show that $\mathcal{M}$ is equicontinuous.

Let $\varsigma_1,\varsigma_2\in[a,T]$ and $\forall(\mathcal{W},\mathcal{S})\in\mathcal{G}_\delta$; then

$$\begin{aligned}
&||\mathcal{M}_1(\mathcal{W})(\varsigma_2)-\mathcal{M}_1(\mathcal{W})(\varsigma_1)||_\infty\\
\le&\frac{1}{\Gamma_k(\vartheta_1)}\bigg|\int_a^{\varsigma_1}\psi'(s)[(\psi(\varsigma_2)-\psi(s))^{\frac{\vartheta_1}{k}-1}-(\psi(\varsigma_1)-\psi(s))^{\frac{\vartheta_1}{k}-1}]\mathcal{B}_1(s,\mathcal{W}(s))ds\\
&+\int_{\varsigma_2}^{\varsigma_1}\psi'(s)(\psi(\varsigma_2)-\psi(s))^{\frac{\vartheta_1}{k}-1}\mathcal{B}_1(s,\mathcal{W}(s))ds\bigg|\\
&+\frac{(\psi(\varsigma_2)-\psi(a))^{\frac{\phi_k}{k}-1}-(\psi(\varsigma_1)-\psi(a))^{\frac{\phi_k}{k}-1}}{|\hat{\Lambda}_1|\Gamma_k(\phi_k)}\Big[\varpi_1{}^k\mathfrak{J}^{\vartheta_1-\rho_1;\psi}|\mathcal{B}_1(\Xi_1,\mathcal{W}(\Xi_1),\mathcal{S}(\Xi_1))|\\
&+\chi_1{}^k\mathfrak{J}^{\mu_1+\vartheta_1;\psi}|\mathcal{B}_1(\varrho_1,\mathcal{W}(\varrho_1),\mathcal{S}(\varrho_1))|+{}^k\mathfrak{J}^{\vartheta_1;\psi}|\mathcal{B}_1(T,\mathcal{W}(T),\mathcal{S}(T))|\Big]
\end{aligned}\tag{38}$$

$$\begin{aligned}
\le&\ \mathcal{Y}^*_{\mathcal{B}_1}\mathfrak{H}_{\mathcal{B}_1}(\delta)\times\frac{1}{\Gamma_k(\vartheta_1)}\bigg|\int_a^{\varsigma_1}\psi'(s)[(\psi(\varsigma_2)-\psi(s))^{\frac{\vartheta_1}{k}-1}-(\psi(\varsigma_1)-\psi(s))^{\frac{\vartheta_1}{k}-1}]ds\\
&+\int_{\varsigma_2}^{\varsigma_1}\psi'(s)(\psi(\varsigma_2)-\psi(s))^{\frac{\vartheta_1}{k}-1}ds\bigg|\\
&+\frac{(\psi(\varsigma_2)-\psi(a))^{\frac{\phi_k}{k}-1}-(\psi(\varsigma_1)-\psi(a))^{\frac{\phi_k}{k}-1}}{|\hat{\Lambda}_1|\Gamma_k(\phi_k)}\Big[\varpi_1{}^k\mathfrak{J}^{\vartheta_1-\rho_1;\psi}|\mathcal{B}_1(\Xi_1,\mathcal{W}(\Xi_1),\mathcal{S}(\Xi_1))|\\
&+\chi_1{}^k\mathfrak{J}^{\mu_1+\vartheta_1;\psi}|\mathcal{B}_1(\varrho_1,\mathcal{W}(\varrho_1),\mathcal{S}(\varrho_1))|+{}^k\mathfrak{J}^{\vartheta_1;\psi}|\mathcal{B}_1(T,\mathcal{W}(T),\mathcal{S}(T))|\Big]\to 0\ as\ \varsigma_2\to\varsigma_1,
\end{aligned}\tag{39}$$

in a similar manner, we have

$$\begin{aligned}
&||\mathcal{M}_2(\mathcal{W},\mathcal{S})(\varsigma_2)-\mathcal{M}_2(\mathcal{W},\mathcal{S})(\varsigma_1)||_\infty\\
\le&\frac{1}{\Gamma_k(\vartheta_2)}\bigg|\int_a^{\varsigma_1}\psi'(s)[(\psi(\varsigma_2)-\psi(s))^{\frac{\vartheta_2}{k}-1}-(\psi(\varsigma_1)-\psi(s))^{\frac{\vartheta_2}{k}-1}]\mathcal{B}_1(s,\mathcal{W}(s))ds\\
&+\int_{\varsigma_2}^{\varsigma_1}\psi'(s)(\psi(\varsigma_2)-\psi(s))^{\frac{\vartheta_2}{k}-1}\mathcal{B}_2(s,\mathcal{W}(s))ds\bigg|\\
&+\frac{(\psi(\varsigma_2)-\psi(a))^{\frac{\phi_k}{k}-1}-(\psi(\varsigma_1)-\psi(a))^{\frac{\phi_k}{k}-1}}{|\hat{\Lambda}_2|\Gamma_k(\phi_k)}\Big[\varpi_2{}^k\mathfrak{J}^{\vartheta_2-\rho_2;\psi}|\mathcal{B}_2(\Xi_2,\mathcal{W}(\Xi_2),\mathcal{S}(\Xi_2))|\\
&+\chi_2{}^k\mathfrak{J}^{\mu_2+\vartheta_2;\psi}|\mathcal{B}_2(\varrho_2,\mathcal{W}(\varrho_2),\mathcal{S}(\varrho_2))|-{}^k\mathfrak{J}^{\vartheta_2;\psi}|\mathcal{B}_2(T,\mathcal{W}(T),\mathcal{S}(T))|\Big]
\end{aligned}\tag{40}$$

$$\le\ \mathcal{Y}^*_{\mathcal{B}_2}\mathfrak{H}_{\mathcal{B}_2}(\delta)\times\frac{1}{\Gamma_k(\vartheta_1)}\bigg|\int_a^{\varsigma_1}\psi'(s)[(\psi(\varsigma_2)-\psi(s))^{\frac{\vartheta_2}{k}-1}-(\psi(\varsigma_1)-\psi(s))^{\frac{\vartheta_2}{k}-1}]ds$$

$$
\begin{aligned}
&+ \int_{\varsigma_2}^{\varsigma_1} \psi'(s)(\psi(\varsigma_2)-\psi(s))^{\frac{\vartheta_2}{k}-1}ds\Big| \\
&+ \frac{(\psi(\varsigma_2)-\psi(a))^{\frac{\phi_k}{k}-1}-(\psi(\varsigma_1)-\psi(a))^{\frac{\phi_k}{k}-1}}{|\hat{\Lambda}_2|\Gamma_k(\phi_k)}\Big[\varpi_2{}^k\mathfrak{J}^{\vartheta_2-\rho_2;\psi}|\mathcal{B}_2(\Xi_2,\mathcal{W}(\Xi_2),\mathcal{S}(\Xi_2))| \\
&+\chi_2{}^k\mathfrak{J}^{\mu_2+\vartheta_2;\psi}|\mathcal{B}_2(\varrho_2,\mathcal{W}(\varrho_2),\mathcal{S}(\varrho_2))| - {}^k\mathfrak{J}^{\vartheta_2;\psi}|\mathcal{B}_2(T,\mathcal{W}(T),\mathcal{S}(T))|\Big] \to 0 \ \ as \ \ \varsigma_2\to\varsigma_1. \qquad (41)
\end{aligned}
$$

It is notable that (38) and (40) are both free of $(\mathcal{W},\mathcal{S})\in\mathcal{G}_\delta$ and because of what is obtained from (38) and (40), we obtain the equicontinuity of operator $\mathcal{M}$.

Step 4: Finally, we let $\Phi=\Phi_1\cap\Phi_2;\Phi_1,\Phi_2\subset\mathcal{G}_\delta$. Furthermore, Φ_1 and Φ_2 are assumed to be bounded and equicontinuous.

We show that

$$
\Phi_1\subset\overline{conv}(\mathcal{M}_1(\Phi_1)\cup\{o\}) \ \text{ and } \ \Phi_2\subset\overline{conv}(\mathcal{M}_1(\Phi_1)\cup\{o\}),
$$

thus, the functions

$$
\begin{aligned}
\Pi_1(\varsigma)&=k(\Phi_1(\varsigma)),\\
\Pi_2(\varsigma)&=k(\Phi_2(\varsigma)),
\end{aligned}
$$

are continuous on $[a,T]$. By Kuratowski's Lemma (4) and ($\mathcal{A}_3$), we write

$$
\begin{aligned}
\Pi_1(\varsigma)=&k(\Phi_1(\varsigma))\\
&\le k(\overline{conv}(\mathcal{M}_1(\Phi_1)\cup\{o\}))\\
&\le k(\mathcal{M}_1\Phi_1(\varsigma)) \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
\le& k\Bigg\{\frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_1\Gamma_k(\phi_k)}\Big[\varpi_1{}^k\mathfrak{J}^{\vartheta_1-\rho_1;\psi}||\mathcal{B}_1(\Xi_1,\mathcal{W}(\Xi_1),\mathcal{S}(\Xi_1)))|| \\
&+\chi_1{}^k\mathfrak{J}^{\mu_1+\vartheta_1;\psi}||\mathcal{B}_1(\varrho_1,\mathcal{W}(\varrho_1),\mathcal{S}(\varrho_1))|| - {}^k\mathfrak{J}^{\vartheta_1;\psi}||\mathcal{B}_1(T,\mathcal{W}(T),\mathcal{S}(T))||\Big] \qquad (43)\\
&+{}^k\mathfrak{J}^{\vartheta_1;\psi}||\mathcal{B}_1(\varsigma,\mathcal{W}(\varsigma),\mathcal{S}(\varsigma))||,:(\mathcal{W},\mathcal{S})\in\Phi\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\le& k\Bigg\{\frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_1\Gamma_k(\phi_k)}\Big[\varpi_1{}^k\mathfrak{J}^{\vartheta_1-\rho_1;\psi}\mathcal{B}_1(s,\Phi_1(s)) \\
&+\chi_1{}^k\mathfrak{J}^{\mu_1+\vartheta_1;\psi}\mathcal{B}_1(s,\Phi_1(s)) - {}^k\mathfrak{J}^{\vartheta_1;\psi}\mathcal{B}_1(s,\Phi_1(s))\Big] \qquad (44)\\
&+{}^k\mathfrak{J}^{\vartheta_1;\psi}\mathcal{B}_1(s,\Phi_1(s)),:(\mathcal{W},\mathcal{S})\in\Phi\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\le& k\Bigg\{\frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_1\Gamma_k(\phi_k)}\Big[\varpi_1{}^k\mathfrak{J}^{\vartheta_1-\rho_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\Delta_1k(s,\Phi_1(s)) \\
&+\chi_1{}^k\mathfrak{J}^{\mu_1+\vartheta_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\Delta_1k(s,\Phi_1(s)) - {}^k\mathfrak{J}^{\vartheta_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\Delta_1k(s,\Phi_1(s))\Big]\\
&+{}^k\mathfrak{J}^{\vartheta_1;\psi}\mathcal{Y}^*_{\mathcal{B}_1}\Delta_1k(s,\Phi_1(s)),:(\mathcal{W},\mathcal{S})\in\Phi\Bigg\}
\end{aligned}
$$

$$
\le \mathcal{Y}^*_{\mathcal{B}_1}\Delta_1||\Pi_1||_\infty,
$$

that is,

$$
||\Pi_1||_\infty \le \mathcal{Y}^*_{\mathcal{B}_1}\Delta_1||\Pi_1||_\infty,
$$

but it was assumed $\max\{\mathcal{Y}^*_{\mathcal{B}_1}\Delta_1, \mathcal{Y}^*_{\mathcal{B}_2}\Delta_2\} < 1$, which yields $||\Pi_1||_\infty = 0$, so $\Pi_1(\varsigma) = 0, \forall\varsigma \in [a, T]$.

Similarly, we get $\Pi_2(\varsigma) = 0, \forall\varsigma \in [a, T]$.

Consequently $k(\Phi(\varsigma)) \leq k(\Phi_1(\varsigma)) = 0$ and $k(\Phi(\varsigma)) \leq k(\Phi_1(\varsigma)) = 0$, implying that $\Phi(\varsigma)$ is relatively compact in $\widehat{\mathcal{E}} \times \widehat{\mathcal{E}}$, and based on the Arzila–Ascoli theorem, we obtain that Φ is relatively compact in $\mathcal{G}_\delta$.

By all the results obtained from the four steps, Mönch's fixed point theorem applies. Hence, $\mathcal{M}$ has a fixed point $(\mathcal{W}, \mathcal{S})$ on $\mathcal{G}_\sigma$. □

4. Stability Results for the Problem

Define the operators $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathcal{C}([a, T], \mathcal{R}_e) \times \mathcal{C}([a, T], \mathcal{R}_e) \to \mathcal{C}([a, T], \mathcal{R}_e)$, such that

$$\begin{cases} {}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_1,\varphi_1;\psi}\mathcal{W}(\varsigma) - \mathcal{B}_1(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma)) = \mathcal{Z}_1(\mathcal{W}, \mathcal{S})(\varsigma), & \varsigma \in (a, T], \ \subset_1 \in (1, 2], \\ {}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_2,\varphi_2;\psi}\mathcal{S}(\varsigma) - \mathcal{B}_2(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma)) = \mathcal{Z}_2(\mathcal{W}, \mathcal{S})(\varsigma), \end{cases}$$

with

$$||\mathcal{Z}_1, (\mathcal{W}, \mathcal{S})|| \leq \tau_1, \quad ||\mathcal{Z}_2, (\mathcal{W}, \mathcal{S})|| \leq \tau_2 \forall \tau_1, \tau_2 > 0. \tag{45}$$

Definition 9. *The coupled system (1) is said to be stable in the Ulam–Hyers sense, if* $\exists\, \mathcal{X}_1, \mathcal{X}_2 > 0$ *and there is a unique solution* $(\mathcal{W}, \mathcal{S}) \in \mathcal{C}([a, T], \mathcal{R}_e) \times \mathcal{C}([a, T], \mathcal{R}_e)$ *of problem (1) with*

$$||(\mathcal{W}, \mathcal{S}) - (\hat{\mathcal{W}}, \hat{\mathcal{S}})|| \leq \mathcal{X}_1\tau_1 + \mathcal{X}_2\tau_2$$

$\forall\ (\hat{\mathcal{W}}, \hat{\mathcal{S}}) \in \mathcal{C}([a, T], \mathcal{R}_e) \times \mathcal{C}([a, T], \mathcal{R}_e)$.

Theorem 3. *Suppose that Theorem 2's assumptions hold. Then, the boundary value problem (1) is Ulam–Hyers stable.*

Proof. Let $(\mathcal{W}, \mathcal{S}) \in \mathcal{C}([a, T], \mathcal{R}_e) \times \mathcal{C}([a, T], \mathcal{R}_e)$ be the solutions of problems (16) and (17). Let $(\hat{\mathcal{W}}, \hat{\mathcal{S}})$ be any solution satisfying (45).

$$\begin{cases} {}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_1,\varphi_1;\psi}\mathcal{W}(\varsigma) = \mathcal{B}_1(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma)) + \mathcal{Z}_1(\mathcal{W}, \mathcal{S})(\varsigma), \\ {}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_2,\varphi_2;\psi}\mathcal{S}(\varsigma) = \mathcal{B}_2(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma)) + \mathcal{Z}_2(\mathcal{W}, \mathcal{S})(\varsigma), \end{cases}$$

$\forall\ \varsigma \in [a, T]$.

Therefore,

$$\begin{aligned} \hat{\mathcal{W}}(\varsigma) =& \mathcal{M}_1(\hat{\mathcal{W}}, \hat{\mathcal{S}})(\varsigma) + \Bigg\{ \frac{(\psi(\varsigma) - \psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_1 \Gamma_k(\phi_k)} \Big[\varpi_1 {}^k\mathfrak{I}^{\vartheta_1-\rho_1;\psi}(\mathcal{B}_1(\Xi_1, \mathcal{W}(\Xi_1), \mathcal{S}(\Xi_1)))_\infty \\ &+ \chi_1 {}^k\mathfrak{I}^{\mu_1+\vartheta_1;\psi}(\mathcal{B}_1(\varrho_1, \mathcal{W}(\varrho_1), \mathcal{S}(\varrho_1)))_\infty - {}^k\mathfrak{I}^{\vartheta_1;\psi}(\mathcal{B}_1(T, \mathcal{W}(T), \mathcal{S}(T)))_\infty \Big] \\ &+ {}^k\mathfrak{I}^{\vartheta_1;\psi}(\mathcal{B}_1(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma)))_\infty \Bigg\}, \end{aligned}$$

and it follows that

$$\begin{aligned} |\mathcal{M}_1(\hat{\mathcal{W}}, \hat{\mathcal{S}})(\varsigma) - \hat{\mathcal{W}}(\varsigma)| \leq& \Bigg\{ \frac{(\psi(\varsigma) - \psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_1 \Gamma_k(\phi_k)} \Big[\varpi_1 {}^k\mathfrak{I}^{\vartheta_1-\rho_1;\psi}(\mathcal{B}_1(\Xi_1, \mathcal{W}(\Xi_1), \mathcal{S}(\Xi_1)))_\infty \\ &+ \chi_1 {}^k\mathfrak{I}^{\mu_1+\vartheta_1;\psi}(\mathcal{B}_1(\varrho_1, \mathcal{W}(\varrho_1), \mathcal{S}(\varrho_1)))_\infty - {}^k\mathfrak{I}^{\vartheta_1;\psi}(\mathcal{B}_1(T, \mathcal{W}(T), \mathcal{S}(T)))_\infty \Big] \\ &+ {}^k\mathfrak{I}^{\vartheta_1;\psi}(\mathcal{B}_1(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma)))_\infty \Bigg\}, \end{aligned}$$

$$\leq \left\{ \frac{(\psi(T)-\psi(a))^{\frac{\vartheta_1}{k}}}{\Gamma_k(\vartheta_1+k)} + \frac{(\psi(T)-\psi(a))^{\frac{\phi_k}{k}-1}}{|\hat{\Lambda}_1|\Gamma_k(\vartheta_1+k)} \left[|\varpi_1| \frac{(\psi(\Xi_1)-\psi(a))^{\frac{\vartheta_1-\rho}{k}}}{\Gamma_k(\vartheta_1-\rho_1+k)} \right. \right.$$
$$\left. \left. + |\chi_1| \frac{(\psi(\Xi_1)-\psi(a))^{\frac{\vartheta_1+\mu_1}{k}}}{\Gamma_k(\vartheta_1+\mu_1+k)} + \frac{(\psi(\Xi_1)-\psi(a))^{\frac{\vartheta_1}{k}}}{\Gamma_k(\vartheta_1+k)} \right] \right\} \tau_1$$
$$\leq \Delta_1 \tau_1,$$

and again

$$|\mathcal{M}_2(\hat{\mathcal{W}},\hat{\mathcal{S}})(\varsigma) - \hat{\mathcal{S}}(\varsigma)| \leq \left\{ \frac{(\psi(\varsigma)-\psi(a))^{\frac{\phi_k}{k}-1}}{\hat{\Lambda}_2 \Gamma_k(\phi_k)} \Big[\varpi_2 {}^k\mathfrak{J}^{\vartheta_2-\rho_2;\psi}(\mathcal{B}_2(\Xi_2,\mathcal{W}(\Xi_2),\mathcal{S}(\Xi_2)))_\infty \right.$$
$$+ \chi_2 {}^k\mathfrak{J}^{\mu_2+\vartheta_2;\psi}(\mathcal{B}_2(\varrho_2,\mathcal{W}(\varrho_2),\mathcal{S}(\varrho_2)))_\infty - {}^k\mathfrak{J}^{\vartheta_2;\psi}(\mathcal{B}_2(T,\mathcal{W}(T),\mathcal{S}(T)))_\infty \Big]$$
$$\left. + {}^k\mathfrak{J}^{\vartheta_2;\psi}(\mathcal{B}_2(\varsigma,\mathcal{W}(\varsigma),\mathcal{S}(\varsigma)))_\infty \right\},$$
$$\leq \left\{ \frac{(\psi(T)-\psi(a))^{\frac{\vartheta_2}{k}}}{\Gamma_k(\vartheta_2+k)} + \frac{(\psi(T)-\psi(a))^{\frac{\phi_k}{k}-1}}{|\hat{\Lambda}_2|\Gamma_k(\vartheta_2+k)} \left[|\varpi_2| \frac{(\psi(\Xi_2)-\psi(a))^{\frac{\vartheta_2-\rho}{k}}}{\Gamma_k(\vartheta_2-\rho_2+k)} \right. \right.$$
$$\left. \left. + |\chi_2| \frac{(\psi(\Xi_2)-\psi(a))^{\frac{\vartheta_2+\mu_2}{k}}}{\Gamma_k(\vartheta_2+\mu_2+k)} + \frac{(\psi(\Xi_2)-\psi(a))^{\frac{\vartheta_2}{k}}}{\Gamma_k(\vartheta_2+k)} \right] \right\} \tau_2$$
$$\leq \Delta_2 \tau_2,$$

$$|(\mathcal{W},\mathcal{S}) - (\hat{\mathcal{W}},\hat{\mathcal{S}})| \leq \Delta_1\tau_1 + \Delta_2\tau_2,$$

thus, operator $\mathcal{M}$, which is given by (24) and (25), can be extracted from the fixed point property as follows:

$$\begin{aligned} |\mathcal{W}(\varsigma) - \mathcal{W}*(\varsigma)| =& |\mathcal{W}(\varsigma) - \mathcal{M}_1(\mathcal{W}*,\mathcal{S}*)(\varsigma) + \mathcal{M}_1(\mathcal{W}*,\mathcal{S}*)(\varsigma) - \mathcal{W}*(\varsigma)| \\ \leq& |\mathcal{M}_1(\mathcal{W},\mathcal{S})(\varsigma) - \mathcal{M}_1(\mathcal{W}*,\mathcal{S}*)(\varsigma)| + |\mathcal{M}_1(\mathcal{W}*,\mathcal{S}*)(\varsigma) - \mathcal{W}*(\varsigma)| \\ \leq& ((\Delta_1\vartheta_1 + \Delta_1\hat{\vartheta}_1) + (\Delta_1\vartheta_2 + \Delta_1\hat{\vartheta}_2))||(\mathcal{W},\mathcal{S}) - (\mathcal{W}*,\mathcal{S}*)|| \\ &+ \Delta_1\hat{\tau}_1 + \Delta_1\hat{\tau}_2. \end{aligned} \tag{46}$$

$$\begin{aligned} |\mathcal{S}(\varsigma) - \mathcal{S}*(\varsigma)| =& |\mathcal{S}\varsigma) - \mathcal{M}_2(\mathcal{W}*,\mathcal{S}*)(\varsigma) + \mathcal{M}_2(\mathcal{W}*,\mathcal{S}*)(\varsigma) - \mathcal{S}*(\varsigma)| \\ \leq& |\mathcal{M}_2(\mathcal{W},\mathcal{S})(\varsigma) - \mathcal{M}_2(\mathcal{W}*,\mathcal{S}*)(\varsigma)| + |\mathcal{M}_2(\mathcal{W}*,\mathcal{S}*)(\varsigma) - \mathcal{S}*(\varsigma)| \\ \leq& ((\Delta_2\vartheta_1 + \Delta_2\hat{\vartheta}_1) + (\Delta_2\vartheta_2 + \Delta_2\hat{\vartheta}_2))||(\mathcal{W},\mathcal{S}) - (\mathcal{W}*,\mathcal{S}*)|| \\ &+ \Delta_2\hat{\tau}_1 + \Delta_2\hat{\tau}_2, \end{aligned} \tag{47}$$

from the above Equations (46) and (47), it follows that

$$\begin{aligned} ||(\mathcal{W},\mathcal{S}) - (\mathcal{W}*,\mathcal{S}*)|| \leq& \frac{(\Delta_1+\Delta_2)\hat{\tau}_1 + (\Delta_1+\Delta_2)\hat{\tau}_2}{1 - ((\Delta_1+\Delta_2)(\vartheta_1+\vartheta_2) + (\Delta_1+\Delta_2)(\hat{\vartheta}_1+\hat{\vartheta}_2))} \\ \leq& \mathcal{V}_1\hat{\tau}_1 + \mathcal{V}_2\hat{\tau}_2, \end{aligned}$$

with

$$\mathcal{V}_1 = \frac{(\Delta_1+\Delta_2)}{1 - ((\Delta_1+\Delta_2)(\vartheta_1+\vartheta_2) + (\Delta_1+\Delta_2)(\hat{\vartheta}_1+\hat{\vartheta}_2))}$$

$$\mathcal{V}_2 = \frac{(\Delta_1 + \Delta_2)}{1 - ((\Delta_1 + \Delta_2)(\vartheta_1 + \vartheta_2) + (\Delta_1 + \Delta_2)(\hat{\vartheta}_1 + \hat{\vartheta}_2))}.$$

this demonstrates that the problem (1) is Ulam–Hyers stable. □

5. Example

An applied example is used to support the result presented in this work as follows Let $\mathcal{W}_0 = \{\mathcal{W}_1, \mathcal{W}_2, \cdots \mathcal{W}_n, \cdots\}$, it clear that $\mathcal{W}_0$ is a Banach space with $||\mathcal{W}||_\infty = \sup_{n\geq 1} |\mathcal{W}_n|$. Consider the following boundary value problem:

Example 1.

$$\begin{cases} {}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_1,\varphi_1;\psi}\mathcal{W}(\varsigma) = \mathcal{B}_1(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma)),\ \varsigma \in (a,T],\ k > 0, 1 < \vartheta_1 \leq 2,\ \varphi_1 \in [0,1], \\ {}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_2,\varphi_2;\psi}\mathcal{S}(\varsigma) = \mathcal{B}_2(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma)),\ \varsigma \in (a,T],\ k > 0, 1 < \vartheta_2 \leq 2,\ \varphi_2 \in [0,1], \\ \mathcal{W}(a) = 0,\ \ \mathcal{W}(T) = \varpi_1{}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_1,\varphi_1;\psi}\mathcal{W}(\Xi_1) + \chi_1{}^{k}\mathfrak{J}^{\mu_1;\psi}\mathcal{W}(\varrho_1), \\ \mathcal{S}(a) = 0,\ \ \mathcal{S}(T) = \varpi_2{}^{k,\mathcal{H}}\mathcal{D}^{\vartheta_2,\varphi_2;\psi}\mathcal{W}(\Xi_2) + \chi_2{}^{k}\mathfrak{J}^{\mu_2;\psi}\mathcal{W}(\varrho_2), \end{cases} \tag{48}$$

where $k = \frac{5}{3}$, $\vartheta_1 = 8/5$, $\vartheta_1 = 7/5$, $\varphi_1 = 2/5, \varphi_2 = 3/5$ $a = 1/5, T = 5/3$, $\Xi_1 = 4/7$, $\Xi_2 = 3/5$, $\varrho_1 = 5/8, \varrho_2 = 6/5$, $\chi_1 = 3/51, \chi_2 = 4/51, \varpi_2 = 2/51, \varpi_1 = 1/51$, $\phi_k = 35/15$,

$$\mathcal{B}_1(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma)) = \left\{\frac{|\mathcal{W}(\varsigma)|}{(\varsigma+9)(1+|\mathcal{W}(\varsigma)|)} + \frac{1}{27(1+|\mathcal{S}^2(\varsigma)|)} + \frac{1}{81}\right\},$$
$$\mathcal{B}_2(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma)) = \left\{\frac{\sin(2\pi|\mathcal{W}(\varsigma))|}{40\pi} + \frac{1}{10\sqrt{\varsigma+4}} + \frac{|\mathcal{S}(\varsigma)|}{10(1+|\mathcal{W}(\varsigma)|)}\right\},$$

and $\forall \varsigma \in [a,T]$ *with* $\{\mathcal{W}_n\}_{n\geq 1}, \{\mathcal{S}_n\}_{n\geq 1} \in \mathcal{W}_0$, *the hypothesis* $\mathcal{A}_2$ *of Theorem 2 is verified. Moreover,*

$$\begin{aligned} ||\mathcal{B}_1(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma))||_\infty &\leq \left\|\left\{\frac{|\mathcal{W}(\varsigma)|}{(\varsigma+9)(1+|\mathcal{W}(\varsigma)|)} + \frac{1}{27(1+|\mathcal{S}^2(\varsigma)|)} + \frac{1}{81}\right\}\right\|_\infty, \\ &\leq \frac{1}{(\varsigma+9)}(||\mathcal{W}|| + 1) \\ &= \mathcal{Y}_{\mathcal{B}_1}(\varsigma)\mathfrak{H}_{\mathcal{B}_1}(||\mathcal{W}||_\infty) \end{aligned}$$

and similarly,

$$\begin{aligned} ||\mathcal{B}_2(\varsigma, \mathcal{W}(\varsigma), \mathcal{S}(\varsigma))||_\infty &\leq \left\|\left\{\frac{\sin(2\pi|\mathcal{W}(\varsigma))|}{40\pi} + \frac{1}{10\sqrt{\varsigma+4}} + \frac{|\mathcal{S}(\varsigma)|}{10(1+|\mathcal{S}(\varsigma)|)}\right\}\right\|_\infty, \\ &\leq \frac{1}{10}(||\mathcal{S}|| + 1) \\ &= \mathcal{Y}_{\mathcal{B}_2}(\varsigma)\mathfrak{H}_{\mathcal{B}_2}(||\mathcal{S}||_\infty), \end{aligned}$$

this implies condition $\mathcal{A}_2$ *of Theorem 2 is verified.*

Next,

$$\mathcal{Y}(\mathcal{B}_1, (\varsigma, \mathcal{G})) \leq \mathcal{Y}_{\mathcal{B}_1}(\varsigma)\mathcal{Y}(\mathcal{G}),$$
$$\mathcal{Y}(\mathcal{B}_2, (\varsigma, \mathcal{G})) \leq \mathcal{Y}_{\mathcal{B}_2}(\varsigma)\mathcal{Y}(\mathcal{G}),$$

where in our case, we have $\mathcal{Y}_{\mathcal{B}_1}(\varsigma) = \frac{1}{\varsigma+9}$, $\mathcal{Y}_{\mathcal{B}_2}(\varsigma) = \frac{\varsigma}{10}$, *and the next two inequalities guarantee that the condition* $(\mathcal{A}_2)$ *of Theorem 2 is valid.*

Finally, we calculate

$$\mathcal{Y}^*_{\mathcal{B}_1}(\varsigma) = \frac{1}{10},$$

$$\Delta_1 \leq \frac{(\psi(T)-\psi(a))^{\frac{\vartheta_1}{k}}}{\Gamma_k(\vartheta_1+k)} + \frac{(\psi(T)-\psi(a))^{\frac{\phi_k}{k}-1}}{|\hat{\Lambda}_1|\Gamma_k(\vartheta_1+k)}\Bigg[|\varpi_1|\frac{(\psi(\Xi_1)-\psi(a))^{\frac{\vartheta_1-\rho}{k}}}{\Gamma_k(\vartheta_1-\rho_1+k)} + |\chi_1|\frac{(\psi(\Xi_1)-\psi(a))^{\frac{\vartheta_1+\mu_1}{k}}}{\Gamma_k(\vartheta_1+\mu_1+k)} + \frac{(\psi(\Xi_1)-\psi(a))^{\frac{\vartheta_1}{k}}}{\Gamma_k(\vartheta_1+k)}\Bigg] \approx 0.356723562 \tag{49}$$

and

$$\mathcal{Y}^*_{\mathcal{B}_2}(\varsigma) = \frac{2}{10},$$

$$\Delta_2 \leq \frac{(\psi(T)-\psi(a))^{\frac{\vartheta_2}{k}}}{\Gamma_k(\vartheta_2+k)} + \frac{(\psi(T)-\psi(a))^{\frac{\phi_k}{k}-1}}{|\hat{\Lambda}_2|\Gamma_k(\vartheta_2+k)}\Bigg[|\varpi_2|\frac{(\psi(\Xi_2)-\psi(a))^{\frac{\vartheta_2-\rho}{k}}}{\Gamma_k(\vartheta_2-\rho_2+k)} + |\chi_2|\frac{(\psi(\Xi_2)-\psi(a))^{\frac{\vartheta_2+\mu_2}{k}}}{\Gamma_k(\vartheta_2+\mu_2+k)} + \frac{(\psi(\Xi_2)-\psi(a))^{\frac{\vartheta_2}{k}}}{\Gamma_k(\vartheta_2+k)}\Bigg] \approx 0.69703560 \tag{50}$$

then, $\max\{\Delta_1\mathcal{Y}_{\mathcal{B}_1}(\varsigma), \Delta_2\mathcal{Y}_{\mathcal{B}_2}(\varsigma)\} = \max\{0.356723562, 0.13940712\} = 0.13940712 < 1$. *This shows that all requirements of Theorem 2 are satisfied,* $(\mathcal{W}, \mathcal{S}) \in \mathcal{C}([a,T], \mathcal{S}_0) \times \mathcal{C}([a,T], \mathcal{S}_0)$.

6. Conclusions

We studied a system of Hilfer (k, ψ)-fractional differential equations, with nonlocal boundary conditions and (k, ψ)-Hilfer derivatives and integrals. We used Mönch's fixed point theorem, Carathéaodory's conditions, and Kuratowski's measure of noncompactness to introduce the results in this work. In addition, the stability of solutions to the system (1) was verified via the Ulam–Hyers stability technique. Adding to this, the following conclusions are drawn from this study:

1 According to [27], if $k = 1$, the Hilfer (k,ψ)-fractional operator becomes a Hilfer ψ-fractional operator.
2 According to [31], if $\psi(\varsigma) = \varsigma$ and $k = 1$, the Hilfer (k,ψ)-fractional operator becomes a Hilfer fractional operator.
3 For $\psi(\varsigma) = \varsigma$ and $\varphi_i = 0, i = 1,2$, the Hilfer (k,ψ)-fractional system becomes a Riemann–Liouville (k,ψ)-fractional system.
4 For $\psi(\varsigma) = \varsigma$and $\varphi_i = 1, i = 1,2$, the Hilfer (k,ψ)-fractional system becomes a Caputo (k,ψ)-fractional system. Furthermore, the solution form in the types of systems mentioned above can be used to study the positive solution and its asymmetry in greater depth. We conclude that our results are novel and can be viewed as an expansion of the qualitative analysis of fractional differential equations. For those interested in this subject, this system can be studied using different fractional derivatives such as Katugampula or ψ-Caputo and these results can be used in practical applications in various subjects such as the predator–prey model.

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References

1. Kolmanovskii, V.; Myshkis, A. *Introduction to the Theory and Applications of Functional Differential 158 Equations*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2013; Volume 463.
2. Awadalla, M.; Noupoue, Y.Y.Y.; Asbeh, K.A.; Ghiloufi, N. Modeling Drug Concentration Level in Blood Using Fractional Differential Equation Based on Psi-Caputo Derivative. *J. Math.* **2022**, *2022*, 1–8. [CrossRef]
3. Qureshi, S.; Yusuf, A.; Shaikh, A. A.; Inc, M.; Baleanu, D. Fractional modeling of blood ethanol concentration system with real data application. *Chaos: Interdiscip. J. Nonlinear Sci.* **2019**, *29*, 013143. [CrossRef]
4. Yameni Noupoue, Y.Y.; Tandogdu, Y.; Awadalla, M. On numerical techniques for solving the fractional logistic differential equation. *Adv. Differ. Equ.* **2019**, *2019*, 1–13. [CrossRef]
5. Manigandan, M.; Muthaiah, S.; Nandhagopal, T.; Vadivel, R.; Unyong, B.; Gunasekaran, N. Existence results for coupled system of nonlinear differential equations and inclusions involving sequential derivatives of fractional order. *AIMS Math.* **2022**, *7*, 723–755. [CrossRef]
6. Ahmad, B.; Nieto, J.J. Sequential fractional differential equations with three-point boundary conditions. *Comput. Math. Appl.* **2012**, *64*, 3046–3052. [CrossRef]
7. Matar, M.M.; Amra, I.A.; Alzabut, J. Existence of solutions for tripled system of fractional differential equations involving cyclic permutation boundary conditions. *Bound. Value Probl.* **2020**, *2020*, 1–13. [CrossRef]
8. Su, X. Boundary value problem for a coupled system of nonlinear fractional differential equations. *Appl. Math. Lett.* **2021**, *22*, 64–69. [CrossRef]
9. Subramanian, M.; Manigandan, M.; Tung, C.; Gopal, T.N.; Alzabut, J. On system of nonlinear coupled differential equations and inclusions involving Caputo-type sequential derivatives of fractional order. *J. Taibah Univ. Sci.* **2022**, *16*, 1–23. [CrossRef]
10. Hamoud, A.A. Existence and uniqueness of solutions for fractional neutral volterra-Fredholm integrodifferential equations. *Adv. Theory Nonlinear Anal. Its Appl.* **2020**, *4*, 321–331.
11. Khan, H.; Tunc, C.; Chen, W.; Khan, A. Existence theorems and Hyers-Ulam stability for a class of hybrid fractional differential equations with p-Laplacian operator. *J. Appl. Anal. Comput.* **2018**, *8*, 1211–1226.
12. Ferraoun, S.; Dahmani, Z. Existence and stability of solutions of a class of hybrid fractional differential equations involving RL-operator. *J. Interdiscip. Math.* **2020**, *23*, 885–903. [CrossRef]
13. Elaiw, A.A.; Awadalla, M.M.; Manigandan, M.; Abuasbeh, K. A novel implementation of Mönch's fixed point theorem to a system of nonlinear Hadamard fractional differential equations. *Fractal Fract.* **2022**, *6*, 586. [CrossRef]
14. Al-Sadi, W.; Huang, Z.Y.; Alkhazzan, A. Existence and stability of a positive solution for nonlinear hybrid fractional differential equations with singularity. *J. Taibah Univ. Sci.* **2019**, *13*, 951–960. [CrossRef]
15. Subramanian, M.; Manigandan, M.; Gopal, T.N. Fractional differential equations involving Hadamard fractional derivatives with nonlocal multi-point boundary conditions. *Discontinuity Nonlinearity Complex.* **2020**, *9*, 421–431. [CrossRef]
16. Awadalla, M.; Abuasbeh, K.; Subramanian, M.; Manigandan, M. On a system of ψ-Caputo hybrid fractional differential equations with Dirichlett boundary conditions. *Mathematics* **2022**, *10*, 1681. [CrossRef]
17. Al-khateeb, A.; Zureigat, H.; Alaa, Z.O.; Bawaneh, S. Ulam-Hyers Stability and Uniqueness for Nonlinear Sequential Fractional Differential Equations Involving Integral Boundary Conditions. *Fractal Fract.* **2021**, *5*, 235. [CrossRef]
18. Rus, I.A. Ulam stabilities of ordinary differential equations in a Banach space. *Carpathian J. Math.* **2010**, *2010*, 103–107.
19. Manigandan, M.; Subramanian, M.; Nandha Gopal, T.; Unyong, B. Existence and Stability Results for a Tripled System of the Caputo Type with Multi-Point and Integral Boundary Conditions. *Fractal Fract.* **2022**, *6*, 285. [CrossRef]
20. Kucche, K.D.; Mali, A.D. On the nonlinear (k, ψ)-Hilfer fractional differential equations. *Chaos Solitons Fractals* **2021**, *152*, 111335. [CrossRef]
21. Jarad, F.; Harikrishnan, S.; Shah, K.; Kanagarajan, K. Existence and stability results to a class of fractional random implicit differential equations involving a generalized Hilfer fractional derivative. *Discret. Contin. Dyn. Syst.-S* **2020**, *13*, 723. [CrossRef]
22. Mubeen, S.; Habibullah, G.M. k-Fractional integrals and application. *Int. J. Contemp. Math. Sci* **2012**, *7*, 89–94.
23. Diaz, R.; Pariguan, E. On Hypergeometric Functions and Pochhammer K-symbol. *Divulg. Matemticas* **2007**, *15*, 179–192.
24. Dorrego, G.A. An Alternative Definition for the k-Riemann Liouville Fractional Derivative. *Appl. Math. Sci.* **2015**, *10*, 481–491. [CrossRef]
25. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
26. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [CrossRef]
27. Sousa, J.V.D.C.; De Oliveira, E.C. On the ψ-Hilfer fractional derivative. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *60*, 72–91. [CrossRef]
28. Zhang, S. Existence of solution for a boundary value problem of fractional order. *Acta Math. Sci.* **2006**, *26*, 220–228. [CrossRef]
29. Derbazi, C.; Baitiche, Z. Coupled Systems of ψ -Caputo Differential Equations with Initial Conditions in Banach Spaces. *Mediterr. J. Math.* **2020**, *17*, 1–13. [CrossRef]

30. Mönch, H. Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nonlinear Anal. Theory, Methods Appl.* **1980**, *4*, 985–999. [CrossRef]
31. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.