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A Method for Performing the Symmetric Anti-Difference Equations in Quantum Fractional Calculus

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Abstract: In this paper, we develop theorems on finite and infinite summation formulas by utilizing the $\mathfrak{q}$ and $(\mathfrak{q}, \mathfrak{h})$ anti-difference operators, and also we extend these core theorems to $\mathfrak{q}_{(\alpha)}$ and $(\mathfrak{q}, \mathfrak{h})_\alpha$ difference operators. Several integer order theorems based on $\mathfrak{q}$ and $\mathfrak{q}_{(\alpha)}$ difference operator have been published, which gave us the idea to derive the fractional order anti-difference equations for $\mathfrak{q}$ and $\mathfrak{q}_{(\alpha)}$ difference operators. In order to develop the fractional order anti-difference equations for $\mathfrak{q}$ and $\mathfrak{q}_{(\alpha)}$ difference operators, we construct a function known as the quantum geometric and alpha-quantum geometric function, which behaves as the class of geometric series. We can use this function to convert an infinite summation to a limited summation. Using this concept and by the gamma function, we derive the fractional order anti-difference equations for $\mathfrak{q}$ and $\mathfrak{q}_{(\alpha)}$ difference operators for polynomials, polynomial factorials, and logarithmic functions that provide solutions for symmetric difference operator. We provide appropriate examples to support our results. In addition, we extend these concepts to the $(\mathfrak{q}, \mathfrak{h})$ and $(\mathfrak{q}, \mathfrak{h})_\alpha$ difference operators, and we derive several integer and fractional order theorems that give solutions for the mixed symmetric difference operator. Finally, we plot the diagrams to analyze the $(\mathfrak{q}, \mathfrak{h})$ and $(\mathfrak{q}, \mathfrak{h})_\alpha$ difference operators for verification.

Keywords: $\mathfrak{q}$ and $(\mathfrak{q}, \mathfrak{h})$ difference operators; quantum geometric function; alpha quantum geometric function; gamma functions and fractional order sum

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1. Introduction

The study of calculus without limits is nowadays known as quantum calculus. Jackson's work [1] sheds light on the invention of q-calculus, often known as quantum calculus, while in 1908, Euler and Jacobi had already developed this type of calculus. The field of q-calculus emerged as a link between mathematics and physics. Numerous mathematical fields, including combinatorics, orthogonal polynomials, number theory, fundamental hyper-geometric functions, as well as other sciences, including mechanics, quantum theory, and the theory of relativity, make extensive use of it.

Most of the basic facts of quantum calculus are covered in the book by Kac and Cheung [2]. Quantum calculus is a branch within the mathematical topic of time scales calculus. The q-differential equations are typically defined on a time scale set T_q, where q is the scale index. Time scales offer a unifying framework for investigating the dynamic equations. The majority of the fundamental theory in the calculus of time scales was compiled in the text by Bohner and Peterson [3].

Though quantum calculus plays a major role in physics, engineers and mathematics also show interest in fractional q-difference equations and q-calculus. The main focus of developing q-difference equations is to characterize some unique physical processes and other areas. Some of the topics that have been developed and investigated in conjunction with the creation of the q-calculus theory include the q-Gamma, q-Laplace transform, q-Taylor expansion, q-Beta functions, q-integral transforms theory, q-Mittag Leffler functions, and others. Refer to the articles [4–12] for additional information on fractional and q-calculus equations with q-differentials. The study of fractional q-calculus is still in its early stages when compared to the classical fractional calculus.

In recent years, there has been some research on the uniqueness and existence of solutions to fractional q-calculus. In [13], the authors suggested a technique for solving several linear fractional q-differential equations that involves corresponding integer order equations. Abdeljawad et al. demonstrated the uniqueness of a nonlinear delay Caputo fractional q-difference system initial value problem in [14] by employing a new extended form of the discrete fractional q-Gronwall inequality, whereas the author provided the applications in [15,16]. By utilizing the Banach's contraction mapping concept and by using the p-Laplacian operator, the authors of [17] showed that the Caputo q-fractional boundary value problem has a unique solution. [18] The contraction mapping principle was used by Ren et al. They also used traditional fixed point theorems to prove that numerous positive solutions exist under certain conditions. In [19], Zhang et al. provided the uniqueness and existence of solutions to the Caputo fractional q-differential equations, and also in [20] they considered the possibility of a singular solution in the q-integral space. The authors in [21] provided the applications of quantum calculus to impulsive difference equations on finite intervals. The applications of the q-calculus to the problem of a falling body in a resisting medium have been given in [22]. Later, the authors in [23] developed the $\mathfrak{q}$-symmetric derivative, which is defined as $(\mathfrak{u}(\mathfrak{q}\mathfrak{k}) - \mathfrak{u}(\mathfrak{q}^{-1}\mathfrak{k}))/(\mathfrak{q}\mathfrak{k} - \mathfrak{q}^{-1}\mathfrak{k})$.

The q-differential operator is then extended to q-difference operator. The q-difference operator was proposed by the authors [24] in 2014 and is defined as $d_q\mathfrak{u}(\mathfrak{k}) = \mathfrak{u}(\mathfrak{k}\mathfrak{q}) - \mathfrak{u}(\mathfrak{k})$, and the oscillation of q difference equation was discussed in [25]. In [26], the authors suggested the $d_{\mathfrak{q}_{(\alpha)}}$ operator by defining $d_{q_{(\alpha)}}\mathfrak{u}(\mathfrak{k}) = \mathfrak{u}(\mathfrak{k}\mathfrak{q}) - \alpha\mathfrak{u}(\mathfrak{k})$. In 2022, the authors in [27] developed the $\mathfrak{q}$-symmetric difference operator, that is $\mathfrak{D}_\mathfrak{q}\mathfrak{u}(\mathfrak{k}) = \mathfrak{u}(\mathfrak{k}\mathfrak{q}) - \mathfrak{u}(\mathfrak{k}\mathfrak{q}^{-1})$, which is the combination of forward and backward $\mathfrak{q}$-difference operator. Here, the authors developed the theorems for integer order using the q-difference operator that generates a solution for the q-symmetric difference operator. This motivates us to develop the fractional order theorems for q-symmetric difference operator. In addition, we have extended this q-symmetric difference operator to (q,h)-symmetric operator which is defined as $\mathfrak{D}_\mathfrak{q}^\mathfrak{h}\mathfrak{u}(\mathfrak{k}) = \mathfrak{u}(\mathfrak{k}\mathfrak{q} + \mathfrak{h}) - \mathfrak{u}(\mathfrak{k}\mathfrak{q}^{-1} - \mathfrak{h})$, and its alpha (q,h)-symmetric operator is defined as $\mathfrak{D}_{\mathfrak{q}^\mathfrak{h}_{(\alpha)}}\mathfrak{u}(\mathfrak{k}) = \mathfrak{u}(\mathfrak{k}\mathfrak{q} + \mathfrak{h}) - \alpha\mathfrak{u}(\mathfrak{k}\mathfrak{q}^{-1} - \mathfrak{h})$. Throughout this paper, we concentrate only on the development of fractional order $\mathfrak{q}$ and $(\mathfrak{q},\mathfrak{h})$ anti-difference equations, and we have extended these core theorems to $\mathfrak{q}_{(\alpha)}$ and $(\mathfrak{q},\mathfrak{h})_\alpha$ fractional anti-difference equations. Those findings will provide fractional order solution for the $(\mathfrak{q},\mathfrak{h})$ and $(\mathfrak{q},\mathfrak{h})_\alpha$ symmetric difference operator. Here, the findings are based only on the delta operator. One can do the same for the nabla operator.

This is how the paper is structured. The Introduction is the focus of Section 1. In Section 2, we discuss the preliminaries of $\mathfrak{q}$ and $\mathfrak{q}_{(\alpha)}$ difference operator. In Sections 3–5, we develop the integer and fractional order theorems for q, $\mathfrak{q}_{(\alpha)}$, $(\mathfrak{q},\mathfrak{h})$, and $(\mathfrak{q},\mathfrak{h})_\alpha$ difference operators. The conclusion is covered in Section 6.

2. Preliminaries

In this section, we discuss the basic definitions of $\mathfrak{q}$ and $\mathfrak{q}_{(\alpha)}$ difference operators and their inverse operators. Here, for any $\mathfrak{a} \in \mathfrak{R}$, we define an infinite set $\mathfrak{T}_\mathfrak{q} = \{\mathfrak{a}, \mathfrak{a}\mathfrak{q}^{\pm 1}, \mathfrak{a}\mathfrak{q}^{\pm 2}, \dots\}$ such that if $\mathfrak{k} \in \mathfrak{T}_\mathfrak{q}$, then $\mathfrak{k}\mathfrak{q}^{\pm 1} \in \mathfrak{T}_\mathfrak{q}$, where $0 \neq \mathfrak{k} \in \mathfrak{R}$, $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$ and $\mathfrak{R} = (-\infty, \infty)$.

Definition 1 ([25])**.** *Let $\mathfrak{u} : \mathfrak{T}_\mathfrak{q} \to \mathfrak{R}$ and $1 \neq \mathfrak{q} > 0 \in \mathfrak{R}$. The $\mathfrak{q}$ and $\mathfrak{q}_{(\alpha)}$ difference operator ($\mathfrak{q}$ and $\mathfrak{q}_{(\alpha)}$-symmetric difference operator), denoted as d_q and $d_{q_{(\alpha)}}$, on $\mathfrak{u}(\mathfrak{k})$ are, respectively, defined as*

$$d_q\mathfrak{u}(\mathfrak{k}) \ = \ \mathfrak{u}(\mathfrak{k}\mathfrak{q}) \ - \ \mathfrak{u}(\mathfrak{k}),\ \mathfrak{k} \in \mathfrak{T}_\mathfrak{q} \tag{1}$$

and

$$d_{q_{(\alpha)}}\mathfrak{u}(\mathfrak{k}) = \mathfrak{u}(\mathfrak{k}\mathfrak{q}) - \alpha\mathfrak{u}(\mathfrak{k}),\ 0 \neq \alpha \in \mathfrak{R},\ \mathfrak{k} \in \mathfrak{T}_{\mathfrak{q}}. \tag{2}$$

Definition 2 ([25]). *If there exists a function* $\mathfrak{v} : \mathfrak{T}_{\mathfrak{q}} \to \mathfrak{R}$ *such that* $d_q\mathfrak{u}(\mathfrak{k}) = \mathfrak{v}(\mathfrak{k})$, *then its inverse* $\mathfrak{q}$ *and* $\mathfrak{q}_{(\alpha)}$ *difference operator, denoted as* I_q *and* $I_{q_{(\alpha)}}$ *are, respectively, defined as*

$$\mathfrak{v}(\mathfrak{k}) = I_q\mathfrak{u}(\mathfrak{k}) + \mathfrak{c},\ I_q = d_q^{-1} \tag{3}$$

and

$$\mathfrak{v}(\mathfrak{k}) = I_{q_{(\alpha)}}\mathfrak{u}(\mathfrak{k}) + \mathfrak{c},\ I_{q_{(\alpha)}} = d_{q_{(\alpha)}}^{-1}, \tag{4}$$

where $\mathfrak{c}$ *is a constant.*

Definition 3 ([27]). *Let* $\mathfrak{n} \in \mathfrak{N}$ *and* $\mathfrak{k}, \mathfrak{q}$ *be any real number. Then, the* $\mathfrak{q}$*-polynomial falling factorial function of* $\mathfrak{k}_{\mathfrak{q}}^{(\mathfrak{n})}$ *is defined as*

$$\mathfrak{k}_{\mathfrak{q}}^{(\mathfrak{n})} = \mathfrak{k}\Big(\prod_{\mathfrak{r}=1}^{\mathfrak{n}-1}(\mathfrak{k} - \mathfrak{q}^{\mathfrak{r}})\Big). \tag{5}$$

Lemma 1 ([27]). *The power rule for* $\mathfrak{q}$ *and* $\mathfrak{q}_{(\alpha)}$ *difference operator is as follows:*

1. *If* $\mathfrak{n} \in \mathfrak{N}$ *and* $\mathfrak{q} \neq 1$, *then for* $\mathfrak{k} \in \mathfrak{R}$,

$$d_q\mathfrak{k}_{\mathfrak{q}}^{(\mathfrak{n})} = (\mathfrak{q}^{\mathfrak{n}} - 1)\mathfrak{k}_{\mathfrak{q}}^{(\mathfrak{n})} \text{ and } I_q\mathfrak{k}_{\mathfrak{q}}^{(\mathfrak{n})} = \mathfrak{k}_{\mathfrak{q}}^{(\mathfrak{n})}/(\mathfrak{q}^{\mathfrak{n}} - 1) + \mathfrak{c}. \tag{6}$$

2. *If* $\mathfrak{n} \in \mathfrak{N}$, $\mathfrak{q}^{\mathfrak{n}} \neq \alpha$ *and* $\alpha \in \mathfrak{R}$, *then for* $\mathfrak{k} \in \mathfrak{R}$,

$$d_{q_{(\alpha)}}\mathfrak{k}_{\mathfrak{q}}^{(\mathfrak{n})} = (\mathfrak{q}^{\mathfrak{n}} - \alpha)\mathfrak{k}_{\mathfrak{q}}^{(\mathfrak{n})} \text{ and } I_{q_{(\alpha)}}\mathfrak{k}_{\mathfrak{q}}^{(\mathfrak{n})} = \mathfrak{k}_{\mathfrak{q}}^{(\mathfrak{n})}/(\mathfrak{q}^{\mathfrak{n}} - \alpha) + \mathfrak{c}. \tag{7}$$

Lemma 2 ([27]). *Let* $\mathfrak{u}, \mathfrak{v} : \mathfrak{T}_{\mathfrak{q}} \to \mathfrak{R}$ *and* $\mathfrak{q} \neq 1 \in \mathfrak{R}$. *The product rule of* $\mathfrak{q}$ *and* $\mathfrak{q}_{(\alpha)}$ *difference operator is, respectively, defined by*

$$I_q\{\mathfrak{u}(\mathfrak{k})\mathfrak{v}(\mathfrak{k})\} = \mathfrak{u}(\mathfrak{k})I_q\mathfrak{v}(\mathfrak{k}) - I_q\big\{I_q\mathfrak{v}(\mathfrak{k}\mathfrak{q})d_q\mathfrak{u}(\mathfrak{k})\big\}. \tag{8}$$

and

$$I_{q_{(\alpha)}}\{\mathfrak{u}(\mathfrak{k})\mathfrak{v}(\mathfrak{k})\} = \mathfrak{u}(\mathfrak{k})I_{q_{(\alpha)}}\mathfrak{v}(\mathfrak{k}) - I_{q_{(\alpha)}}\Big\{I_{q_{(\alpha)}}\mathfrak{v}(\mathfrak{k}\mathfrak{q})d_q\mathfrak{u}(\mathfrak{k})\Big\},\ \alpha \in \mathfrak{R}. \tag{9}$$

Result 1 ([27]). *Let* $\mathfrak{k}, \alpha \in \mathfrak{R}$ *and* $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$. *Then,*

$$I_q(1) = \log(\mathfrak{k})/\log(\mathfrak{q}) \text{ and } I_{q_{(\alpha)}}(1) = 1/(1 - \alpha). \tag{10}$$

Result 2 ([27]). *If* $1 \neq \mathfrak{q} > 0$ *and* $\alpha < 1 \in \mathfrak{R}$, *then for* $\mathfrak{k} \in (0, \infty)$, *we have*

$$I_{q_{(\alpha)}}\mathfrak{log}(\mathfrak{k}) = (\mathfrak{log}(\mathfrak{k})/(1 - \alpha)) - (\mathfrak{log}(\mathfrak{q})/(1 - \alpha)^2) \tag{11}$$

and

$$I_{q_{(\alpha)}}\mathfrak{log}(\mathfrak{k}/\mathfrak{q}^{\mathfrak{n}}) = \big(\mathfrak{log}(\mathfrak{k}/\mathfrak{q}^{\mathfrak{n}})/(1 - \alpha)\big) - (\mathfrak{log}(\mathfrak{q})/(1 - \alpha)^2),\ \mathfrak{n} \in \mathfrak{R}. \tag{12}$$

3. Fundamental Theorems for $\mathfrak{q}$ and $\mathfrak{q}_{(\alpha)}$ Symmetric Difference Operator

In this section, we present some basic notions of polynomial factorial function and gamma function. Then, we use the $\mathfrak{q}$ and $\mathfrak{q}_{(\alpha)}$ difference operator and its inverse operators to derive fundamental theorems.

Definition 4 ([28]). *Let* $\mathfrak{k} \in \mathfrak{R}$ *and* $\mathfrak{n} \in \mathfrak{N}$. *Then, the falling factorial function is defined as*

$$\mathfrak{k}^{(\mathfrak{n})} = \prod_{\mathfrak{r}=1}^{\mathfrak{n}}(\mathfrak{k} - (\mathfrak{r} - 1)). \tag{13}$$

For $0 < \nu \in \mathfrak{R}$ and $\mathfrak{k} \in \mathfrak{R}$, the generalized gamma function is

$$\mathfrak{k}^{(\nu)} = \Gamma(\mathfrak{k}+1)/\Gamma(\mathfrak{k}-\nu+1), \tag{14}$$

where $\mathfrak{k}+1$ and $\mathfrak{k}-\nu+1$ is non-equal to zero or a negative integer.

Lemma 3 ([29]). *For the first $\mathfrak{n}$ natural numbers, the $\mathfrak{x}^{th}$ power polynomial factorial is*

$$\sum_{\mathfrak{z}=1}^{\mathfrak{n}-1} \mathfrak{z}^{(\mathfrak{x})} = (\mathfrak{n}+1)^{(\mathfrak{x}+1)}/(\mathfrak{x}+1),\ \mathfrak{n} \in \mathfrak{N}. \tag{15}$$

3.1. Fundamental Theorems for $\mathfrak{q}$ Operator

Using the $\mathfrak{q}$ symmetric difference operator, we develop a few theorems for integer order ($\mathfrak{x}$-th order) and fractional order (ν-th order) sums.

Definition 5. *Let $\mathfrak{s}, \mathfrak{k} \in \mathfrak{R}$, $\mathfrak{q} \in \mathfrak{R}-\{0,1\}$ such that $\mathfrak{s} \in \mathfrak{T}_{\mathfrak{q}}$ and $\mathfrak{u} : \mathfrak{T}_{\mathfrak{q}} \to \mathfrak{R}$ be a function. Then, the quantum geometric function (or $\mathfrak{q}$-geometric function) is defined as*

$$\sum_{\mathfrak{r}=\mathfrak{k}}^{\infty} \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}}) = \frac{[\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}})]^2}{\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}}) - \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+1})}, \tag{16}$$

if $\sum_{\mathfrak{r}=\mathfrak{k}}^{\infty} \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}})$ is convergent.

Lemma 4. *If $\mathfrak{s}, \mathfrak{t} \in \mathfrak{R}$, $\mathfrak{q} \in \mathfrak{R}-\{0,1\}$, and $\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\mathfrak{j}})$ is convergent, then*

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}) = \frac{[\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2})]^2}{\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2}) - \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+3})} \tag{17}$$

Proof. The proof completes by replacing $\mathfrak{k}$ by $\mathfrak{k}+1$ and $\mathfrak{r}$ by $\mathfrak{r}+1$ in Definition 5. □

Lemma 5. *If $\mathfrak{x} \in \mathfrak{N}$ and assuming the conditions given in Lemma 4, then*

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \frac{(\mathfrak{r}+\mathfrak{x}-1)^{(\mathfrak{x}-1)}}{(\mathfrak{x}-1)!} \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\mathfrak{x}}) = \frac{(\mathfrak{x}-1)!\big[((\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1})\big]^2}{(\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-1)}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1}) - (\mathfrak{k}+\mathfrak{x}+1)^{(\mathfrak{x}-1)}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+2})}. \tag{18}$$

Proof. Equation (17) can be represented as

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} ((\mathfrak{r}+0)^{(0)}/0!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}) = \frac{\big[((\mathfrak{k}+1)^{(0)}/0!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2})\big]^2}{((\mathfrak{k}+1)^{(0)}/0!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2}) - ((\mathfrak{k}+2)^{(0)}/0!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+3})}. \tag{19}$$

From (19), one can easily find the next term as

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} ((\mathfrak{r}+1)^{(1)}/1!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+2}) = \frac{\big[((\mathfrak{k}+2)^{(1)}/1!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+3})\big]^2}{((\mathfrak{k}+2)^{(1)}/1!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+3}) - ((\mathfrak{k}+3)^{(1)}/1!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+4})}.$$

Similarly, we can find

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \frac{(\mathfrak{r}+\mathfrak{x}-2)^{(\mathfrak{x}-2)}}{(\mathfrak{x}-2)!} \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\mathfrak{x}-1}) = \frac{(\mathfrak{x}-2)!\big[((\mathfrak{k}+\mathfrak{x}-1)^{(\mathfrak{x}-2)}/(\mathfrak{x}-2)!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}})\big]^2}{(\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-2)}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}}) - (\mathfrak{k}+\mathfrak{x})^{(1)}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1})}. \tag{20}$$

Hence, the proof completes by replacing $\mathfrak{x}$ by $\mathfrak{x}+1$ in Equation (20). □

Theorem 1. *Let* $\mathfrak{u}, \mathfrak{v} : \mathfrak{T}_\mathfrak{q} \to \mathfrak{R}$, $\mathfrak{k} \in \mathfrak{N}$, $\mathfrak{s} \in \mathfrak{R}$ *and* $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$. *Then, the anti-difference principle of* $\mathfrak{q}$ *difference operator is given by*

$$I_q \mathfrak{u}(\mathfrak{k})\Big|_{\mathfrak{k}=\mathfrak{s}} - \frac{\left[\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2})\right]^2}{\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2}) - \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+3})} = \sum_{\mathfrak{r}=0}^{\mathfrak{k}} \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}). \tag{21}$$

Proof. Since $I_q \mathfrak{u}(\mathfrak{k}) = \mathfrak{v}(\mathfrak{k})$, we can deduce that to $\mathfrak{u}(\mathfrak{k}) = d_q \mathfrak{v}(\mathfrak{k})$. Therefore, $\mathfrak{u}(\mathfrak{k}) = \mathfrak{v}(\mathfrak{k}\mathfrak{q}) - \mathfrak{v}(\mathfrak{k})$ and

$$\mathfrak{v}(\mathfrak{k}\mathfrak{q}) = \mathfrak{u}(\mathfrak{k}) + \mathfrak{v}(\mathfrak{k}). \tag{22}$$

When $\mathfrak{k}$ is substituted for $(\mathfrak{k}/\mathfrak{q})$ in Equation (22), we obtain

$$\mathfrak{v}(\mathfrak{k}) \;=\; \mathfrak{u}(\mathfrak{k}/\mathfrak{q}) \;+\; \mathfrak{v}(\mathfrak{k}/\mathfrak{q}). \tag{23}$$

Once again, by changing $\mathfrak{k}$ to $(\mathfrak{k}/\mathfrak{q})$ in Equation (23), we obtain

$$\mathfrak{v}(\mathfrak{k}/\mathfrak{q}) = \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^2) + \mathfrak{v}(\mathfrak{k}/\mathfrak{q}^2). \tag{24}$$

Substituting Equation (24) in Equation (23), we obtain

$$\mathfrak{v}(\mathfrak{k}) = \mathfrak{u}(\mathfrak{k}/\mathfrak{q}) + \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^2) + \mathfrak{v}(\mathfrak{k}/\mathfrak{q}^2). \tag{25}$$

Proceeding like this up to $\mathfrak{n}$ times, we obtain

$$\mathfrak{v}(\mathfrak{k}) = \sum_{\mathfrak{r}=1}^{\mathfrak{n}} \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^{\mathfrak{r}}) + \mathfrak{v}(\mathfrak{k}/\mathfrak{q}^{\mathfrak{n}}).$$

Applying $\lim\limits_{\mathfrak{n}\to\infty}$ in the previous equation and assuming $\mathfrak{v}(0) = \mathfrak{u}(0) = 0$, we obtain

$$\mathfrak{v}(\mathfrak{k}) = \mathfrak{u}(\mathfrak{k}/\mathfrak{q}) + \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^2) + \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^3) + \ldots + \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^{\mathfrak{r}+1}) + \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^{\mathfrak{r}+2}) + \ldots \tag{26}$$

Replacing $'\mathfrak{k}'$ by $'\mathfrak{s}'$ and $'\mathfrak{r}'$ by $'\mathfrak{k}'$ in (26), we obtain

$$\mathfrak{v}(\mathfrak{s}) = \sum_{\mathfrak{r}=0}^{\mathfrak{k}} \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}) + \sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}). \tag{27}$$

From Equation (3), we arrive at

$$I_q \mathfrak{u}(\mathfrak{k})\Big|_{\mathfrak{k}=\mathfrak{s}} - \sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}) = \sum_{\mathfrak{r}=0}^{\mathfrak{k}} \mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}).$$

Finally, the proof completes by substituting Equation (17) in the previous equation. □

Theorem 2. *Let* $\mathfrak{u}, \mathfrak{v} : \mathfrak{T}_\mathfrak{q} \to \mathfrak{R}$, $\mathfrak{z} \in \mathfrak{N}$, $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$ *and* $\mathfrak{s}, \mathfrak{k} \in \mathfrak{R}$. *Thus, for the* $\mathfrak{q}$ *difference operator, the higher order anti-difference principle is given by*

$$I_q^{\mathfrak{z}} \mathfrak{u}(\mathfrak{k})\Big|_{\mathfrak{k}=\mathfrak{s}} - \frac{(\mathfrak{z}-1)!\left[((\mathfrak{k}+\mathfrak{z})^{(\mathfrak{z}-1)}/(\mathfrak{z}-1)!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{z}+1})\right]^2}{((\mathfrak{k}+\mathfrak{z})^{(\mathfrak{z}-1)}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{z}+1}) - (\mathfrak{k}+\mathfrak{z}+1)^{(\mathfrak{z}-1)}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{z}+2}))}$$
$$= \sum_{\mathfrak{r}=0}^{\mathfrak{k}} ((\mathfrak{r}+\mathfrak{z}-1)^{(\mathfrak{z}-1)}/(\mathfrak{z}-1)!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\mathfrak{z}}). \tag{28}$$

Proof. Theorem 1 provides the proof for $\mathfrak{z} = 1$.
If we apply the I_q operator on both sides of Equation (26), we obtain

$$I_q^2 \mathfrak{u}(\mathfrak{k}) = I_q \mathfrak{u}(\mathfrak{k}/\mathfrak{q}) + I_q \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^2) + I_q \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^3) + I_q \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^4) + I_q \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^5) + \ldots$$

Replacing the right side of the aforementioned equation by (26), we obtain

$$I_q^2 u(\mathfrak{k}) = \sum_{\mathfrak{r}=2}^{\infty} u(\mathfrak{k}/q^{\mathfrak{r}}) + \sum_{\mathfrak{r}=3}^{\infty} u(\mathfrak{k}/q^{\mathfrak{r}}) + \sum_{\mathfrak{r}=4}^{\infty} u(\mathfrak{k}/q^{\mathfrak{r}}) + \sum_{\mathfrak{r}=5}^{\infty} u(\mathfrak{k}/q^{\mathfrak{r}}) + \sum_{\mathfrak{r}=6}^{\infty} u(\mathfrak{k}/q^{\mathfrak{r}}) + \dots,$$

which implies

$$I_q^2 u(\mathfrak{k}) = u(\mathfrak{k}/q^2) + 2u(\mathfrak{k}/q^3) + 3u(\mathfrak{k}/q^4) + \dots + (\mathfrak{r}+1)u(\mathfrak{k}/q^{\mathfrak{r}+2}) + \dots \quad (29)$$

Replacing $'\mathfrak{k}'$ by $'\mathfrak{s}'$ and $'\mathfrak{r}'$ by $'\mathfrak{k}'$ in Equation (29), we arrive at

$$I_q^2 u(\mathfrak{k})\Big|_{\mathfrak{k}=\mathfrak{s}} = u(\mathfrak{s}/q^2) + 2u(\mathfrak{s}/q^3) + 3u(\mathfrak{s}/q^4) + \dots + (\mathfrak{k}+1)u(\mathfrak{s}/q^{\mathfrak{k}+2}) + \dots$$

Therefore,

$$I_q^2 u(\mathfrak{k})\Big|_{\mathfrak{k}=\mathfrak{s}} - \sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} (\mathfrak{r}+1)u(\mathfrak{s}/q^{\mathfrak{r}+2}) = \sum_{\mathfrak{r}=0}^{\mathfrak{k}} (\mathfrak{r}+1)u(\mathfrak{s}/q^{\mathfrak{r}+2}).$$

Once again, by using the I_q operator on both sides of the expression (29), we obtain

$$I_q^3 u(\mathfrak{k}) = I_q u(\mathfrak{k}/q^2) + 2I_q u(\mathfrak{k}/q^3) + 3I_q u(\mathfrak{k}/q^4) + 4I_q u(\mathfrak{k}/q^5) + 5I_q u(\mathfrak{k}/q^6) + \dots$$

Inserting Equation (26) in each term of the right side of the previous equation, we obtain

$$I_q^3 u(\mathfrak{k}) = \sum_{\mathfrak{r}=3}^{\infty} u(\mathfrak{k}/q^{\mathfrak{r}}) + 2\sum_{\mathfrak{r}=4}^{\infty} u(\mathfrak{k}/q^{\mathfrak{r}}) + 3\sum_{\mathfrak{r}=5}^{\infty} u(\mathfrak{k}/q^{\mathfrak{r}}) + 4\sum_{\mathfrak{r}=6}^{\infty} u(\mathfrak{k}/q^{\mathfrak{r}}) + \dots$$

The above equation will be written as

$$I_q^3 u(\mathfrak{k}) = u(\mathfrak{k}/q^3) + (1+2)u(\mathfrak{k}/q^4) + (1+2+3)u(\mathfrak{k}/q^5) + (1+2+3+4)u(\mathfrak{k}/q^6) + \dots$$

By Lemma 3 for $\mathfrak{m} = 1$, the above equation becomes

$$I_q^3 u(\mathfrak{k}) = (2^{(2)}/2)u(\mathfrak{k}/q^2) + (3^{(2)}/2)u(\mathfrak{k}/q^3) + \dots + ((\mathfrak{r}+2)^{(2)}/2)u(\mathfrak{k}/q^{\mathfrak{r}+3}) + \dots \quad (30)$$

Replacing $'\mathfrak{k}'$ by $'\mathfrak{s}'$ and $'\mathfrak{r}'$ by $'\mathfrak{k}'$ in Equation (30), we obtain

$$I_q^3 u(\mathfrak{s}) - \sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} ((\mathfrak{r}+2)^{(2)}/2)u(\mathfrak{s}/q^{\mathfrak{r}+3}) = \sum_{\mathfrak{r}=0}^{\mathfrak{k}} ((\mathfrak{r}+2)^{(2)}/2)u(\mathfrak{s}/q^{\mathfrak{r}+3}).$$

Similarly, the fourth inverse will be

$$I_q^4 u(\mathfrak{s}) - \sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} ((\mathfrak{r}+3)^{(3)}/3!)u(\mathfrak{s}/q^{\mathfrak{r}+4}) = \sum_{\mathfrak{r}=0}^{\mathfrak{k}} ((\mathfrak{r}+3)^{(3)}/3!)u(\mathfrak{s}/q^{\mathfrak{r}+4}).$$

Following the similar manner, we obtain the general term as

$$I_q^{\mathfrak{x}} u(\mathfrak{s}) - \sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} ((\mathfrak{r}+\mathfrak{x}-1)^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)u(\mathfrak{s}/q^{\mathfrak{r}+\mathfrak{x}}) = \sum_{\mathfrak{r}=0}^{\mathfrak{k}} ((\mathfrak{r}+\mathfrak{x}-1)^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)u(\mathfrak{s}/q^{\mathfrak{r}+\mathfrak{x}}).$$

Hence, by Lemma 5, we obtain (28). □

Example 1. *Taking $u(k) = k_q^{(2)}$ and $\mathfrak{x} = 3$ in Equation (28), we obtain*

$$I_q^3 k_q^{(2)}\Big|_{k=s} - \frac{[((k+3)^{(2)}/2!)(s/q^{k+4})_q^{(2)}]^2}{((k+3)^{(2)}/2!)(s/q^{k+4})_q^{(2)} - ((k+4)^{(2)}/2!)(s/q^{k+5})_q^{(2)}}$$

$$= \sum_{r=0}^{k} ((r+2)^{(2)}/2!)(s/q^{r+3})_q^{(2)}. \quad (31)$$

Using Equation (1), we arrive at $I_q k_q^{(2)} = k_q^{(2)}(q^2-1)$*. Then, it is easy to find*

$$I_q k_q^{(2)} = k_q^{(2)}/(q^2-1).$$

Similarly, applying the I_q *operator on the function* $k_q^{(2)}$ *for x times, we obtain*

$$I_q^{\mathfrak{x}} k_q^{(2)} = k_q^{(2)}/(q^2-1)^{\mathfrak{x}}.$$

Taking $s=8$, $k=3$, *and* $q=4$ *in Equation (31), we arrive at*

$$I_q^3 k_4^{(2)}\Big|_{k=8} = ((s)(s-4))/(4^2-1)^3\Big|_{s=8} = ((8)(8-4))/(16-1)^3. \tag{32}$$

Next, the second term of Equation (31) becomes

$$\frac{\left[((6)^{(2)}/2!)(8/4^7)((8-4)/4^7)\right]^2}{\left[((6)^{(2)}/2!)(8/4^7)((8-4)/4^7)\right]-\left[((7)^{(2)}/2!)(8/4^8)((8-4)/4^8)\right]}. \tag{33}$$

The right side of Equation (31) becomes

$$\sum_{r=0}^{7}((r+2)^{(2)}/2!)(s/q^{r+3})_q^{(2)} = 32/(4^3)^2+96/(4^4)^2+192/(4^5)^2+320/(4^6)^2. \tag{34}$$

Hence, by substituting Equations (32)–(34) in Equation (31), we obtain the result.

The following Definition 6 is the generalized version for Definition 5.

Definition 6. *Let* $\mathfrak{s}\in\mathfrak{R}$, $\mathfrak{k}\in\mathfrak{N}$, $\mathfrak{q}\in\mathfrak{R}-\{0,1\}$ *and* $\sum\limits_{\mathfrak{r}=\mathfrak{k}+1}^{\infty}(\Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1))\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\nu})$ *be convergent such that* $\mathfrak{s}\in\mathfrak{T}_{\mathfrak{q}}$ *and* $\mathfrak{u}:\mathfrak{T}_{\mathfrak{q}}\to\mathfrak{R}$ *be a function. Then, for* $\nu>0$*, the generalized quantum geometric function (or generalized* $\mathfrak{q}$*-geometric function) is defined as*

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty}(\Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1))\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\nu}) = \frac{\left[\mathscr{A}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+1})\right]^2}{\mathscr{A}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+1})-\mathscr{B}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+2})}, \tag{35}$$

where $\mathscr{A}=\Gamma(\mathfrak{k}+\nu+1)/\Gamma(\mathfrak{k}+2)\Gamma(\nu)$ *and* $\mathscr{B}=\Gamma(\mathfrak{k}+\nu+2)/\Gamma(\mathfrak{k}+3)\Gamma(\nu)$.

The following Theorem 3 is the generalized version for Theorem 2.

Theorem 3 (Generalized $\mathfrak{q}$ difference equation)**.** *Let* $\mathfrak{u},\mathfrak{v}:\mathfrak{T}_{\mathfrak{q}}\to\mathfrak{R}$, $\mathfrak{s}$, $\in\mathfrak{R}$, $\mathfrak{k}\in\mathfrak{N}$ *and* $\mathfrak{q}\in\mathfrak{R}-\{0,1\}$*. Then, the* ν*-th order (or real order) anti-difference principle of* $\mathfrak{q}$ *difference operator is given by*

$$I_q^{\nu}\mathfrak{u}(\mathfrak{k})\Big|_{\mathfrak{k}=\mathfrak{s}} - \frac{\Gamma(\nu)\left[(\Gamma(\mathfrak{k}+\nu+1)/\Gamma(\mathfrak{k}+2)\Gamma(\nu))\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+1})\right]^2}{\Gamma(\mathfrak{k}+\nu+1)/\Gamma(\mathfrak{k}+2)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+1})-\Gamma(\mathfrak{k}+\nu+2)/\Gamma(\mathfrak{k}+3)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+2})}$$
$$= (1/\Gamma(\nu))\sum_{\mathfrak{r}=0}^{\mathfrak{k}}(\Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1))\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\nu}). \tag{36}$$

Proof. If Theorem 2 is extending to any real order ($\nu>0$), then we obtain

$$I_q^{\nu}\mathfrak{u}(\mathfrak{k})\Big|_{\mathfrak{k}=\mathfrak{s}} - \frac{(\nu-1)!\left[((\mathfrak{k}+\nu)^{(\nu-1)}/(\nu-1)!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+1})\right]^2}{(\mathfrak{k}+\nu)^{(\nu-1)}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+1})-(\mathfrak{k}+\nu+1)^{(\nu-1)}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+2})}$$
$$= \sum_{\mathfrak{r}=0}^{\mathfrak{k}}((\mathfrak{r}+\nu-1)^{(\nu-1)}/(\nu-1)!)\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\nu}). \tag{37}$$

Since the polynomial factorial does not exists for non-integer values, by using Equation (14), we take $(\mathfrak{k}+\nu)^{(\nu-1)} = (\Gamma(\mathfrak{k}+\nu+1)/\Gamma(\mathfrak{k}+2))$, $(\mathfrak{k}+\nu+1)^{(\nu-1)} = \Gamma(\mathfrak{k}+\nu+2)/\Gamma(\mathfrak{k}+3))$ and $(\mathfrak{k}+\nu-1)^{(\nu-1)} = \Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1)$ in (37). Hence, the proof completes. □

Example 2. *Taking $u(k) = k^2$ and $\nu = 2.7$ in Equation (36), we obtain*

$$I_q^{2.7}k^2\Big|_{k=s} - \frac{\Gamma(\nu)\left[(\Gamma(k+\nu+1)/\Gamma(k+2)\Gamma(\nu))(s/q^{k+3.7})^2\right]^2}{\Gamma(k+\nu+1)/\Gamma(k+2)(s/q^{k+3.7})^2 - \Gamma(k+\nu+2)/\Gamma(k+3)(s/q^{k+4.7})^2}$$
$$= (1/\Gamma(\nu))\sum_{r=0}^{k}(\Gamma(r+\nu)/\Gamma(r+1))u(s/q^{r+\nu}). \tag{38}$$

Using Equation (1) and then applying the I_q operator on the function k_q^2 for $\mathfrak{x}$ times, we obtain

$$I_q^{\mathfrak{x}}k_q^2 = k^2/(q^2-1)^{\mathfrak{x}}. \tag{39}$$

For any real $\nu > 0$, Equation (39) becomes

$$I_q^{\nu}k^2 = k^2/(q^2-1)^{\nu}.$$

Taking $s = 8.1$, $k = 6$ and $q = 3.2$ in Equation (38), we arrive

$$(8.1)^2/(3.2^2-1)^{2.7} - \frac{\Gamma(2.7)\left[(\Gamma(9.7)/\Gamma(8)\Gamma(2.7))(8.1/(3.2)^{9.7})^2\right]^2}{(\Gamma(9.7)/\Gamma(8))(8.1/(3.2)^{9.7})^2 - (\Gamma(10.7)/\Gamma(9))(8.1/(3.2)^{10.7})^2}. \tag{40}$$

The right side of Equation (38) becomes

$$\sum_{r=0}^{7}((r+2)^{(2)}/2!)(8.1/(3.2)^{r+3})^2_{3.2} = (1/\Gamma(2.7))\left[\Gamma(2.7)\left(8.1/(3.2)^{2.7}\right)^2 + \Gamma(3.7)/\left(8.1/(3.2)^{3.7}\right)^2\right.$$
$$+(\Gamma(4.7)/2!)/\left(8.1/(3.2)^{4.7}\right)^2 + (\Gamma(5.7)/3!)/\left(8.1/(3.2)^{5.7}\right)^2 + (\Gamma(6.7)/4!)/\left(8.1/(3.2)^{6.7}\right)^2$$
$$\left.+(\Gamma(7.7)/5!)/\left(8.1/(3.2)^{7.7}\right)^2 + (\Gamma(8.7)/6!)/\left(8.1/(3.2)^{8.7}\right)^2\right]. \tag{41}$$

Hence, substituting Equations (40) and (41) in Equation (38), we obtain the result.

3.2. Fundamental Theorems for $\mathfrak{q}_{(\alpha)}$ Operator

By utilizing the $\mathfrak{q}_{(\alpha)}$ symmetric difference operator, we developed theorems for integer order (or $\mathfrak{m}$-th order) and the fractional order (ν-th order). Here, the difference operator $\mathfrak{q}_{(\alpha)}$ changes to the $\mathfrak{q}$-difference operator if $\alpha = 1$.

Definition 7. *Let $\mathfrak{s}, \mathfrak{k} \in \mathfrak{R}$, $\alpha \in \mathfrak{R} > 0$, $\mathfrak{q} \in \mathfrak{R} - \{0\}$ and if $\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\mathfrak{j}})$ be convergent such that $\mathfrak{s} \in \mathfrak{T}_{\mathfrak{q}}$ and $\mathfrak{u} : \mathfrak{T}_{\mathfrak{q}} \to \mathfrak{R}$ is a function. Then the alpha-quantum geometric function (or $\mathfrak{q}_{(\alpha)}$-geometric function) is defined as*

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}) = \frac{\left[\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2})\right]^2}{\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2}) - \alpha^{\mathfrak{k}+2}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+3})}. \tag{42}$$

Lemma 6. *Consider the conditions given in Definition 7. If $\mathfrak{x} \in \mathfrak{N}$, then*

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \frac{(\mathfrak{r}+\mathfrak{x}-1)^{(\mathfrak{x}-1)}}{(\mathfrak{x}-1)!}\alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\mathfrak{x}}) = \frac{(\mathfrak{x}-1)!\left[((\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1})\right]^2}{(\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-1)}\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1}) - (\mathfrak{k}+\mathfrak{x}+1)^{(\mathfrak{x}-1)}\alpha^{\mathfrak{k}+2}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+2})}. \tag{43}$$

Proof. Equation (42) can be written as

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} ((\mathfrak{r}+0)^{(0)}/0!)\alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}) = \frac{(0!)\left[((\mathfrak{k}+1)^{(0)}/0!)\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2})\right]^2}{(\mathfrak{k}+1)^{(0)}\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2}) - (\mathfrak{k}+2)^{(0)}\alpha^{\mathfrak{k}+2}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+3})}. \tag{44}$$

From (44), one can easily find the next term as

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty}((\mathfrak{r}+1)^{(1)}/1!)\alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+2}) = \frac{(1!)\big[((\mathfrak{k}+2)^{(1)}/1!)\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+3})\big]^2}{(\mathfrak{k}+2)^{(1)}\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+3}) - (\mathfrak{k}+3)^{(1)}\alpha^{\mathfrak{k}+2}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+4})}.$$

Similarly, we can find

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty}\frac{(\mathfrak{r}+\mathfrak{x}-2)^{(\mathfrak{x}-2)}}{(\mathfrak{x}-2)!}\alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\mathfrak{x}-1}) = \frac{(\mathfrak{x}-2)!\big[((\mathfrak{k}+\mathfrak{x}-1)^{(\mathfrak{x}-2)}/(\mathfrak{x}-2)!)\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}})\big]^2}{(\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-2)}\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}-1}) - (\mathfrak{k}+\mathfrak{x})^{(1)}\alpha^{\mathfrak{k}+2}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1})}. \tag{45}$$

Hence, the proof completes by replacing $\mathfrak{x}$ by $\mathfrak{x}+1$ in Equation (45). □

Theorem 4. *Let* $\mathfrak{u},\mathfrak{v}:\mathfrak{T}_\mathfrak{q}\to\mathfrak{R}$, $\mathfrak{s},\alpha\in\mathfrak{R}$, $\mathfrak{k}\in\mathfrak{N}$ *and* $\mathfrak{q}\in\mathfrak{R}-\{0,1\}$. *Then, the anti-difference principle of* $\mathfrak{q}_{(\alpha)}$ *difference operator is defined as*

$$I_{q_{(\alpha)}}\mathfrak{u}(\mathfrak{s}) - \frac{\big[\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2})\big]^2}{\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+2}) - \alpha^{\mathfrak{k}+2}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+3})} = \sum_{\mathfrak{r}=0}^{\mathfrak{k}}\alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}). \tag{46}$$

Proof. Since $I_{q_{(\alpha)}}\mathfrak{u}(\mathfrak{k}) = \mathfrak{v}(\mathfrak{k})$, we have $\mathfrak{u}(\mathfrak{k}) = d_{q_{(\alpha)}}\mathfrak{v}(\mathfrak{k})$.

Now, following the similar steps from Equation (22) to Equation (27) and using (2), we arrive at

$$I_{q_{(\alpha)}}\mathfrak{u}(\mathfrak{s}) - \sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty}\alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}) = \sum_{\mathfrak{r}=0}^{\mathfrak{k}}\alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+1}).$$

Hence, the proof completes by inserting Equation (42) in the previous equation. □

Theorem 5. *Let* $\mathfrak{u},\mathfrak{v}:\mathfrak{T}_\mathfrak{q}\to\mathfrak{R}$, $\mathfrak{s},\alpha\in\mathfrak{R}$, $\mathfrak{x},\mathfrak{k}\in\mathfrak{N}$ *and* $\mathfrak{q}\in\mathfrak{R}-\{0,1\}$. *Then, the higher order* $\mathfrak{q}_{(\alpha)}$ *anti-difference principle is thus given by*

$$\begin{aligned} I^{\mathfrak{x}}_{q_{(\alpha)}}\mathfrak{u}(\mathfrak{s}) - \frac{(\mathfrak{x}-1)!\big[((\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1})\big]^2}{(\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-1)}\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1}) - (\mathfrak{k}+\mathfrak{x}+1)^{(\mathfrak{x}-1)}\alpha^{\mathfrak{k}+2}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+2})} \\ = \sum_{\mathfrak{r}=0}^{\mathfrak{k}}((\mathfrak{r}+\mathfrak{x}-1)^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\mathfrak{x}}). \end{aligned} \tag{47}$$

Proof. The proof is similar to Theorem 2 using Lemma 6 and Equation (2). □

Example 3. *If* $u(k) = \log(k)$, *then Equation* (47) *becomes*

$$\begin{aligned} I^{\mathfrak{x}}_{q_{(\alpha)}}log(s) - \frac{(\mathfrak{x}-1)!\big[((k+\mathfrak{x})^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\alpha^{k+1}log(s/q^{k+\mathfrak{x}+1})\big]^2}{(k+\mathfrak{x})^{(\mathfrak{x}-1)}\alpha^{k+1}log(s/q^{k+\mathfrak{x}+1}) - (k+\mathfrak{x}+1)^{(\mathfrak{x}-1)}\alpha^{k+2}log(s/q^{k+\mathfrak{x}+2})} \\ = \sum_{r=0}^{k}((r+\mathfrak{x}-1)^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\alpha^{r}log(s/q^{r+\mathfrak{x}}). \end{aligned} \tag{48}$$

From Equation (11), *we obtain*

$$I_{q_{(\alpha)}}log(k) = (log(k)/(1-\alpha)) - (log(q)/(1-\alpha)^2). \tag{49}$$

Now, applying the $I_{q_{(\alpha)}}$ *operator on the function* $log(k)$ *for* $\mathfrak{x}$ *times in* (49), *we obtain*

$$I^{\mathfrak{x}}_{q_{(\alpha)}}log(k) = (log(k)/(1-\alpha)^{\mathfrak{x}}) - \mathfrak{x}(log(q)/(1-\alpha)^{\mathfrak{x}+1}). \tag{50}$$

Taking $\mathfrak{x}=2$, $s=7$, $k=3$, $q=5$ *and* $\alpha=0.05$ *in Equation* (48), *we arrive at*

$$(log(7)/(0.05)^2) - 2(log(5)/(0.05)^3) - \frac{\big[(5)(0.05)^4log(7/5^6)\big]^2}{((5)(0.05)^4log(7/5^6)) - ((6)(0.05)^5log(7/5^7))}. \tag{51}$$

The right side of Equation (48) becomes

$$\sum_{r=0}^{3}(r+1)^{(1)}(0.5)^r log(s/q^{r+2}) = log(7/5^2) + 2(0.05)log(7/5^3)$$
$$+ 3(0.05)^2 log(7/5^4) + 4(0.05)^3 log(7/5^5). \quad (52)$$

Hence, substituting Equations (51) and (52) in Equation (48), we obtain the result.

The following Definition 8 is the generalized version for Definition 7.

Definition 8. *Let* $\mathfrak{s} \in \mathfrak{R}$, $\mathfrak{k} \in \mathfrak{N}$, $\alpha \in \mathfrak{R} > 0$, $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$ *and if* $\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty}(\Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1))$ $\alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\nu})$ *is convergent such that* $\mathfrak{s} \in \mathfrak{T}_{\mathfrak{q}}$ *and* $\mathfrak{u} : \mathfrak{T}_{\mathfrak{q}} \to \mathfrak{R}$ *be a function. Then, for* $\nu > 0$, *the generalized quantum geometric function (or generalized* $\mathfrak{q}$*-geometric function) is defined as*

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty}(\Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1))\alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\nu}) = \frac{\left[\mathscr{A}\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+1})\right]^2}{\mathscr{A}\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+1}) - \mathscr{B}\alpha^{\mathfrak{k}+2}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+2})}, \quad (53)$$

where $\mathscr{A} = \Gamma(\mathfrak{k}+\nu+1)/\Gamma(\mathfrak{k}+2)\Gamma(\nu)$ *and* $\mathscr{B} = \Gamma(\mathfrak{k}+\nu+2)/\Gamma(\mathfrak{k}+3)\Gamma(\nu)$.

The following Theorem 6 is the generalized version for Theorem 5.

Theorem 6. *(Generalized* $\mathfrak{q}_{(\alpha)}$ *difference equation) Let* $\mathfrak{u}, \mathfrak{v} : \mathfrak{T}_{\mathfrak{q}} \to \mathfrak{R}$, $\nu, \mathfrak{s}, \alpha \in \mathfrak{R}$, $\mathfrak{k} \in \mathfrak{N}$ *and* $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$, *Then, the* ν*-th order (real order) anti-difference principle of* $\mathfrak{q}_{(\alpha)}$ *difference operator is given by*

$$I^{\nu}_{q_{(\alpha)}}\mathfrak{u}(\mathfrak{s}) - \frac{(\Gamma(\nu))\left[(\Gamma(\mathfrak{k}+\nu+1)/\Gamma(\mathfrak{k}+2)\Gamma(\nu))\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+1})\right]^2}{(\Gamma(\mathfrak{k}+\nu+1)/\Gamma(\mathfrak{k}+2))\alpha^{\mathfrak{k}+1}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+1}) - (\Gamma(\mathfrak{k}+\nu+2)/\Gamma(\mathfrak{k}+3))\alpha^{\mathfrak{k}+2}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{k}+\nu+2})}$$
$$= \sum_{\mathfrak{r}=0}^{\mathfrak{k}}(\Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1)\Gamma(\nu))\alpha^{\mathfrak{r}}\mathfrak{u}(\mathfrak{s}/\mathfrak{q}^{\mathfrak{r}+\nu}). \quad (54)$$

Proof. The proof is similar to Theorem 3 using Equations (4) and (53). □

Example 4. *Taking* $u(k) = \log(k)$ *and* $\nu = 1.5$ *in Equation (54), we obtain* $I^{1.5}_{q_{(\alpha)}} logk\Big|_{k=s}$

$$-\frac{(\Gamma(1.5))\left[(\Gamma(k+2.5)/\Gamma(k+2)\Gamma(1.5))\alpha^{k+1}log(s/q^{k+2.5})\right]^2}{(\Gamma(k+2.5)/\Gamma(k+2)\Gamma(1.5))\alpha^{k+1}log(s/q^{k+2.5}) - (\Gamma(k+3.5)/\Gamma(k+3))\alpha^{k+2}log(s/q^{k+3.5})}$$
$$= \sum_{r=0}^{k}(\Gamma(r+1.5)/\Gamma(r+1)\Gamma(1.5))\alpha^r log(s/q^{r+1.5}). \quad (55)$$

Extending Equation (50) to any real number, Equation (50) becomes

$$I^{\nu}_{q_{(\alpha)}} log(k) = (log(k)/(1-\alpha)^{\nu}) - \nu(log(q)/(1-\alpha)^{\nu+1}). \quad (56)$$

Taking $s = 3.5$, $k = 2$, $q = 2.5$, *and* $\alpha = 0.1$ *in Equation (56), we arrive at*

$$I^{1.5}_{q_{(\alpha)}} log(k)\Big|_{k=3.5} = (log(3.5)/(0.9)^{1.5}) - (1.5)(log(2.5)/(0.9)^{2.5}). \quad (57)$$

Next, the second term of Equation (55) becomes

$$\frac{(\Gamma(1.5))\left[(\Gamma(4.5)/\Gamma(4)\Gamma(1.5))(0.1)^3 log(3.5/(2.5)^{4.5})\right]^2}{(\Gamma(4.5)/\Gamma(4))(0.1)^3 log(3.5/(2.5)^{4.5}) - (\Gamma(5.5)/\Gamma(5))(0.1)^4 log(3.5/(2.5)^{5.5})}. \quad (58)$$

The right side of Equation (55) becomes

$$\sum_{r=0}^{2}(\Gamma(r+1.5)/\Gamma(r+1)\Gamma(1.5))(0.1)^r log(3.5/(2.5)^{r+1.5}) = (1/\Gamma(1))log(3.5/(2.5)^{1.5})$$
$$(1.5/\Gamma(2))(0.1)log(3.5/(2.5)^{2.5}) + ((1.5)(2.5)/\Gamma(3))(0.1)^2 log(3.5/(2.5)^{3.5}). \quad (59)$$

Hence, substituting Equations (57)–(59) in Equation (55), we obtain the result.

The integer and fractional order anti-difference equations developed in this section provides the solution for $\mathfrak{q}$ and $\mathfrak{q}_{(\alpha)}$ symmetric difference operators.

4. Mixed Symmetric Difference Operator

In this section, we derive some fundamental theorems using $(\mathfrak{q},\mathfrak{h})$ difference operator and its inverse operators. Here, we introduce the infinite set $\mathscr{M}_h^q = \{\mathfrak{k}, \mathfrak{k}\mathfrak{q}+\mathfrak{h}, \mathfrak{k}\mathfrak{q}^2+2\mathfrak{h}, \ldots\}$ satisfying the condition that for any $\mathfrak{k} \in \mathscr{M}_h^q$ implies $\mathfrak{k}\mathfrak{q}^{\pm 1} \pm \mathfrak{h} \in \mathscr{M}_h^q$ for any fixed number $0 \neq \mathfrak{k} \in \mathfrak{R}$. One can refer the $\mathfrak{h}$-difference operator in [30].

Definition 9. *Let* $\mathfrak{u} : \mathscr{M}_h^q \to \mathfrak{R}$ *be a function. Then, the* $(\mathfrak{q},\mathfrak{h})$ *difference operator (mixed symmetric difference operator), denoted by* $\underset{(q,h)}{\Delta}$ *is defined as*

$$\underset{(q,h)}{\Delta} \mathfrak{u}(\mathfrak{k}) = \mathfrak{u}(\mathfrak{k}\mathfrak{q}+\mathfrak{h}) - \mathfrak{u}(\mathfrak{k}), \ \mathfrak{k} \in \mathscr{M}_h^q. \quad (60)$$

Definition 10. *Let* $\mathfrak{h}, \mathfrak{q}, \mathfrak{k} \in \mathfrak{R}$ *and* $\mathfrak{n} \in \mathfrak{N}$. *The* $(\mathfrak{q},\mathfrak{h})$ *polynomial factorial function* $\mathfrak{k}_{\mathfrak{q},\mathfrak{h}}^{(\mathfrak{n})}$ *is defined as*

$$\mathfrak{k}_{\mathfrak{q},\mathfrak{h}}^{(\mathfrak{n})} = \mathfrak{k}\prod_{\mathfrak{r}=1}^{\mathfrak{n}-1}(\mathfrak{k} - (\mathfrak{q}^{\mathfrak{r}} + \mathfrak{r}\mathfrak{h})). \quad (61)$$

Lemma 7. *If* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$ *and* $0 \neq \mathfrak{h} \in \mathfrak{R}$. *Then, the product rule of* $(\mathfrak{q},\mathfrak{h})$ *difference operator is obtained as*

$$\overset{-1}{\underset{(q,h)}{\Delta}}\{\mathfrak{u}(\mathfrak{k})\mathfrak{v}(\mathfrak{k})\} = \mathfrak{u}(\mathfrak{k})\overset{-1}{\underset{(q,h)}{\Delta}}\mathfrak{v}(\mathfrak{k}) - \overset{-1}{\underset{(q,h)}{\Delta}}\{\overset{-1}{\underset{(q,h)}{\Delta}}\mathfrak{v}(\mathfrak{k}\mathfrak{q}+\mathfrak{h})\underset{(q,h)}{\Delta}\mathfrak{u}(\mathfrak{k})\}. \quad (62)$$

Proof. Applying the operator $\underset{(q,h)}{\Delta}$ on the function $\mathfrak{u}(\mathfrak{k})\mathfrak{v}(\mathfrak{k})$ and then adding and subtracting the term $\mathfrak{u}(\mathfrak{k})\mathfrak{w}(\mathfrak{k}\mathfrak{q}+\mathfrak{h})$, we obtain

$$\overset{-1}{\underset{(q,h)}{\Delta}}\{\mathfrak{u}(\mathfrak{k})\mathfrak{v}(\mathfrak{k})\} = \mathfrak{w}(\mathfrak{k}\mathfrak{q}+\mathfrak{h})\underset{(q,h)}{\Delta}\mathfrak{u}(\mathfrak{k}) + \mathfrak{u}(\mathfrak{k})\underset{(q,h)}{\Delta}\mathfrak{w}(\mathfrak{k})$$

Thus, the proof completes by taking $\underset{(q,h)}{\Delta}\mathfrak{w}(\mathfrak{k}) = \mathfrak{v}(\mathfrak{k})$ and $\overset{-1}{\underset{(q,h)}{\Delta}}\mathfrak{v}(\mathfrak{k}) = \mathfrak{w}(\mathfrak{k})$. □

Property 1. *Some of the properties of* $(\mathfrak{q},\mathfrak{h})$ *difference operator are given below:*

(i) *If* $\mathfrak{q} = 1$, *then* (60) *becomes* $\mathfrak{h}$*-difference operator.*
(ii) *If* $\mathfrak{h} = 0$, *then* (60) *becomes* $\mathfrak{q}$*-difference operator.*
(iii) *If* $\mathfrak{q} > 1$ *and* $\mathfrak{h} > 0$, *then we say* (60) *as* $(\mathfrak{q},\mathfrak{h})$*-difference operator.*
(iv) *The solution does not exist if we take* $\mathfrak{q} = 1$ *and* $\mathfrak{h} = 0$ *simultaneously.*

4.1. Integer Order Theorems

Here, we develop several theorems for integer order ($\mathfrak{r}$-th order) using the $(\mathfrak{q},\mathfrak{h})$ difference operator.

Theorem 7. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{k} \in \mathfrak{R}$, $\mathfrak{n} \in \mathfrak{N}$, $\mathfrak{q} \in \mathfrak{R} - \{\mathfrak{o}, \mathfrak{1}\}$ *and* $\mathfrak{h} \neq \mathfrak{o} \in \mathfrak{R}$. *Then, the anti-difference principle of* $(\mathfrak{q}, \mathfrak{h})$ *operator is given by*

$$\underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{j}=\mathfrak{o}}^{\mathfrak{n}-\mathfrak{1}} \mathfrak{q}^{\mathfrak{j}}\big) \Big/ \mathfrak{q}^{\mathfrak{n}}\Big) = \sum_{\mathfrak{r}=\mathfrak{o}}^{\mathfrak{n}-\mathfrak{1}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{s}=\mathfrak{o}}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{s}}\big) \Big/ \mathfrak{q}^{\mathfrak{r}+\mathfrak{1}}\Big). \tag{63}$$

Proof. Since $\underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}(\mathfrak{k}) = \mathfrak{v}(\mathfrak{k})$, we have

$$\mathfrak{u}(\mathfrak{k}) = \underset{(q,h)}{\Delta} \mathfrak{v}(\mathfrak{k}). \tag{64}$$

From Definition 9, Equation (64) becomes

$$\mathfrak{u}(\mathfrak{k}) = \mathfrak{v}(\mathfrak{k}\mathfrak{q} + \mathfrak{h}) - \mathfrak{v}(\mathfrak{k}).$$

The above equation can be represented as

$$\mathfrak{v}(\mathfrak{q}\mathfrak{k} + \mathfrak{h}) = \mathfrak{u}(\mathfrak{k}) + \mathfrak{v}(\mathfrak{k}). \tag{65}$$

Replacing $\mathfrak{k}$ by $\mathfrak{k}/\mathfrak{q}$ in Equation (65), we obtain $\mathfrak{v}\big(\mathfrak{q}(\mathfrak{k}/\mathfrak{q}) + \mathfrak{h}\big) = \mathfrak{u}(\mathfrak{k}/\mathfrak{q}) + \mathfrak{v}(\mathfrak{k}/\mathfrak{q})$ which implies

$$\mathfrak{v}(\mathfrak{k} + \mathfrak{h}) = \mathfrak{u}(\mathfrak{k}/\mathfrak{q}) + \mathfrak{v}(\mathfrak{k}/\mathfrak{q}). \tag{66}$$

Replacing $\mathfrak{k}$ by $\mathfrak{k} - \mathfrak{h}$ in (66), we obtain $\mathfrak{v}(\mathfrak{k} - \mathfrak{h} + \mathfrak{h}) = \mathfrak{u}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) + \mathfrak{v}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q})$ which implies

$$\mathfrak{v}(\mathfrak{k}) = \mathfrak{u}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) + \mathfrak{v}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}). \tag{67}$$

Replacing $\mathfrak{k}$ by $(\mathfrak{k} - \mathfrak{h})/\mathfrak{q}$ in Equation (67), we arrive at

$$\mathfrak{v}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) = \mathfrak{u}\big([((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) - \mathfrak{h}]/\mathfrak{q}\big) + \mathfrak{v}\big([((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) - \mathfrak{h}]/\mathfrak{q}\big),$$

which gives $\mathfrak{v}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) = \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} - \mathfrak{q}\mathfrak{h})/\mathfrak{q}^2\big) + \mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} - \mathfrak{q}\mathfrak{h})/\mathfrak{q}^2\big)$.
The aforementioned equation can be written as

$$\mathfrak{v}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) = \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=\mathfrak{o}}^{\mathfrak{1}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big) + \mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=\mathfrak{o}}^{\mathfrak{1}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big). \tag{68}$$

Now, substituting Equation (68) in Equation (67), we obtain

$$\mathfrak{v}(\mathfrak{k}) = \mathfrak{u}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=\mathfrak{o}}^{\mathfrak{1}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big) + \mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=\mathfrak{o}}^{\mathfrak{1}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big). \tag{69}$$

Again, replacing $\mathfrak{k}$ by $(\mathfrak{k} - \mathfrak{h} - \mathfrak{q}\mathfrak{h})/\mathfrak{q}^2$ in Equation (67), we obtain

$$\mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} - \mathfrak{q}\mathfrak{h})/\mathfrak{q}^2\big) = \mathfrak{u}\big([((\mathfrak{k} - \mathfrak{h} - \mathfrak{q}\mathfrak{h})/\mathfrak{q}^2) - \mathfrak{h}]/\mathfrak{q}\big) + \mathfrak{v}\big([((\mathfrak{k} - \mathfrak{h} - \mathfrak{q}\mathfrak{h})/\mathfrak{q}^2) - \mathfrak{h}]/\mathfrak{q}\big),$$

which is the same as

$$\mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=\mathfrak{o}}^{\mathfrak{1}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}\big) = \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=\mathfrak{o}}^{\mathfrak{2}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) + \mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=\mathfrak{o}}^{\mathfrak{2}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big). \tag{70}$$

Substituting Equation (70) in Equation (69), we obtain

$$\mathfrak{v}(\mathfrak{k}) = \mathfrak{u}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=\mathfrak{o}}^{\mathfrak{1}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=\mathfrak{o}}^{\mathfrak{2}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) + \mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=\mathfrak{o}}^{\mathfrak{2}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big). \tag{71}$$

Similarly, replacing $\mathfrak{k}$ by $(\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3$ in Equation (67), we obtain

$$\mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) = \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big) + \mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big). \tag{72}$$

Substituting Equation (72) in Equation (71), we obtain

$$\begin{aligned}\mathfrak{v}(\mathfrak{k}) = {} & \mathfrak{u}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) \\ & + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big) + \mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big).\end{aligned} \tag{73}$$

Similarly, again replacing $\mathfrak{k}$ by $(\mathfrak{k} - \mathfrak{h}(\mathfrak{q}^3 + \mathfrak{q}^2 + \mathfrak{q} + 1))/\mathfrak{q}^4$ in Equation (67), and then substituting Equation (67) in Equation (73), we arrive at

$$\begin{aligned}\mathfrak{v}(\mathfrak{k}) = {} & \mathfrak{u}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) \\ & + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{4} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^5\big) + \mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{4} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^5\big).\end{aligned} \tag{74}$$

Proceeding in a similar manner for n times, we obtain the general term as

$$\begin{aligned}\mathfrak{v}(\mathfrak{k}) = {} & \mathfrak{u}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big) \\ & + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{4} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^5\big) + \ldots + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big) + \mathfrak{v}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big).\end{aligned} \tag{75}$$

If $\overset{-1}{\underset{(q,h)}{\Delta}} \mathfrak{u}(\mathfrak{k}) = \mathfrak{v}(\mathfrak{k})$, then (75) becomes

$$\begin{aligned}\overset{-1}{\underset{(q,h)}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \overset{-1}{\underset{(q,h)}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big) = {} & \mathfrak{u}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big) \\ & + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big) + \ldots + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big),\end{aligned}$$

which completes the proof. □

Corollary 1. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{k} \in \mathfrak{R}$, $\mathfrak{n} \in \mathfrak{N}$, $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$ *and if* $\mathfrak{h} = 0$, *then Equation* (63) *becomes*

$$\overset{-1}{\underset{(q,0)}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \overset{-1}{\underset{(q,0)}{\Delta}} \mathfrak{u}\big(\mathfrak{k}/\mathfrak{q}^{\mathfrak{n}}\big) = \sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{u}\big(\mathfrak{k}/\mathfrak{q}^{\mathfrak{r}+1}\big). \tag{76}$$

Corollary 2. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{k} \in \mathfrak{R}$, $\mathfrak{h} \in \mathfrak{R} - \{0\}$, $\mathfrak{n} \in \mathfrak{N}$ *and if* $\mathfrak{q} = 1$, *then Equation* (63) *becomes*

$$\overset{-1}{\underset{(1,h)}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \overset{-1}{\underset{(1,h)}{\Delta}} \mathfrak{u}(\mathfrak{k} - \mathfrak{n}\mathfrak{h}) = \sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{u}(\mathfrak{k} - (\mathfrak{r} + 1)\mathfrak{h}). \tag{77}$$

Remark 1. *The operators* $\overset{-1}{\underset{(q,0)}{\Delta}}$ *and* $\overset{-1}{\underset{(1,h)}{\Delta}}$ *are the first order q and h difference operators, respectively. That is,* $\overset{-1}{\underset{(q,0)}{\Delta}} = \overset{-1}{\underset{q}{\Delta}}$ *and* $\overset{-1}{\underset{(1,h)}{\Delta}} = \overset{-1}{\underset{h}{\Delta}}$.

Theorem 8. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \rightarrow \mathfrak{R}$, $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$, $\mathfrak{h} \neq 0 \in \mathfrak{R}$, $\mathfrak{x}, \mathfrak{n} \in \mathfrak{N}$ *and* $\mathfrak{k} \in \mathfrak{R}$. *Then, the higher order ($\mathfrak{x}$-th order) of* $(\mathfrak{q}, \mathfrak{h})$ *difference equation is given by*

$$\underset{(q,h)}{\overset{-\mathfrak{x}}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \sum_{\mathfrak{d}=0}^{\mathfrak{x}-1} (\mathfrak{n}^{(\mathfrak{d})}/\mathfrak{d}!) \underset{(q,h)}{\overset{-(\mathfrak{x}-\mathfrak{d})}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{j}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{j}}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big) = \sum_{\mathfrak{r}=\mathfrak{x}-1}^{\mathfrak{n}-1} (\mathfrak{r}^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big). \tag{78}$$

Proof. Theorem 7 provides the proof for $\mathfrak{x} = 1$.

When we apply $\underset{(q,h)}{\overset{-1}{\Delta}}$ to both sides of Equation (63), we obtain

$$\underset{(q,h)}{\overset{-2}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \underset{(q,h)}{\overset{-2}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{j}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{j}}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big) = \underset{(q,h)}{\overset{-1}{\Delta}} \Big[\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big)\Big]. \tag{79}$$

The right side of Equation (79) becomes

$$\begin{aligned}\underset{(q,h)}{\overset{-1}{\Delta}} \Big[\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big)\Big] &= \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0} \mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}\Big) + \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0}^{1} \mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^2\Big) \\ &+ \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0}^{2} \mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^3\Big) + \ldots + \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big),\end{aligned}$$

which gives

$$\begin{aligned}\underset{(q,h)}{\overset{-1}{\Delta}} \Big[\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big)\Big] &= \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}\big) + \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big) \\ + \underset{(\mathfrak{q},\mathfrak{h})}{\overset{-1}{\mathfrak{d}}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) &+ \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big) + \ldots + \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}}/\mathfrak{q}^{\mathfrak{n}}\big).\end{aligned} \tag{80}$$

Replacing $\mathfrak{k}$ by $(\mathfrak{k} - \mathfrak{h})/\mathfrak{q}$, $(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2$, $(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2, \ldots$ in Equation (75) and then substituting Equation (75) on the right side of Equation (80), we obtain

$$\begin{aligned}\underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}\big) &= \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) \\ +\mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big) &+ \ldots + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big) + \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big). \\ \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big) &= \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big) \\ +\mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{4} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^5\big) &+ \ldots + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big) + \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big). \\ \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) &= \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big) + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{4} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^5\big) \\ +\mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{5} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^6\big) &+ \ldots + \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big) + \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big).\end{aligned}$$

Similarly, we can easily find the other terms such as $\underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big)$, $\underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{4} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^5\big), \ldots$ and so on. Substituting all the above terms in the right side of Equation (80), we obtain

$$\begin{aligned}\underset{(q,h)}{\overset{-1}{\Delta}} \Big[\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big)\Big] &= \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\big) + 2\mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\big) \\ &+3\mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\big) + 4\mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{4} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^5\big) + \ldots \\ &+(\mathfrak{n}-1)\mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big) + \mathfrak{n} \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}\big((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\big).\end{aligned}$$

Now, inserting all the above equations in Equation (79), we obtain

$$\overset{-2}{\underset{(q,h)}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \overset{-2}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big) - \mathfrak{n} \overset{-1}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$
$$= \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{2}\Big) + 2\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{2}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{3}\Big) + \ldots + (\mathfrak{n}-1)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big),$$

which is the same as

$$\overset{-2}{\underset{(q,h)}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \overset{-2}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big) - \mathfrak{n} \overset{-1}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$
$$= \sum_{\mathfrak{r}=1}^{\mathfrak{n}-1} \mathfrak{r}\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}}\mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big). \tag{81}$$

Again, applying $\overset{-1}{\underset{(q,h)}{\Delta}}$ on both sides of Equation (81) and then inserting Equation (75) in the right side of Equation (81), we arrive at

$$\overset{-3}{\underset{(q,h)}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \overset{-3}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big) - \mathfrak{n} \overset{-2}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$
$$= \big(\sum_{\mathfrak{p}=1} \mathfrak{p}\big)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{2}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{3}\Big) + \big(\sum_{\mathfrak{p}=1}^{2} \mathfrak{p}\big)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{3}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{4}\Big)$$
$$+ \big(\sum_{\mathfrak{p}=1}^{3} \mathfrak{p}\big)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{4}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{5}\Big) + \ldots + \big(\sum_{\mathfrak{p}=1}^{\mathfrak{n}-2} \mathfrak{p}\big)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$
$$+ \big(\sum_{\mathfrak{p}=1}^{\mathfrak{n}-1} \mathfrak{p}\big) \overset{-1}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big).$$

Putting $\mathfrak{r} = 1$ in corollary 3 and then substituting in the above equation, it yields

$$\overset{-3}{\underset{(q,h)}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \overset{-3}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big) - \mathfrak{n} \overset{-2}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$
$$-(\mathfrak{n}^{(2)}/2!) \overset{-1}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big) = \sum_{\mathfrak{r}=2}^{\mathfrak{n}-1} (\mathfrak{r}^{(2)}/2!)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}}\mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big).$$

Similarly, the fourth inverse will be

$$\overset{-4}{\underset{(q,h)}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \overset{-4}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big) - \mathfrak{n} \overset{-3}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$
$$= \sum_{\mathfrak{p}=2} (\mathfrak{p}^{(2)}/2)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{3}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{4}\Big) + \sum_{\mathfrak{p}=2}^{3} (\mathfrak{p}^{(2)}/2)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{4}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{5}\Big)$$
$$+ \sum_{\mathfrak{p}=2}^{4} (\mathfrak{p}^{(2)}/2)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{5}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{6}\Big) + \ldots + \sum_{\mathfrak{p}=2}^{\mathfrak{n}-2} (\mathfrak{p}^{(2)}/2)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$
$$+ \sum_{\mathfrak{p}=2}^{\mathfrak{n}-1} (\mathfrak{p}^{(2)}/2) \overset{-1}{\underset{(\mathfrak{q},\mathfrak{h})}{\eth}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$

Putting $\mathfrak{m} = 2$ in Corollary 3 and then substituting in the above equation, we obtain

$$\overset{-4}{\underset{(q,h)}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \overset{-4}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big) - \mathfrak{n} \overset{-3}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$
$$-(\mathfrak{n}^{(3)}/3!) \overset{-2}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big) - (\mathfrak{n}^{(3)}/3!) \overset{-1}{\underset{(q,h)}{\Delta}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^{j}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$
$$= \sum_{\mathfrak{r}=3}^{\mathfrak{n}-1} (\mathfrak{r}^{(3)}/3!)\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}}\mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big).$$

Proceeding like this up to $\mathfrak{m}$ times, we obtain the general form as

$$\underset{(q,h)}{\overset{-\mathfrak{x}}{\Delta}}\mathfrak{u}(\mathfrak{k})-\underset{(q,h)}{\overset{-\mathfrak{x}}{\Delta}}\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{j}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{j}}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)-(\mathfrak{n}^{(1)}/1!)\underset{(q,h)}{\overset{-(\mathfrak{x}-1)}{\Delta}}\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{j}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{j}}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$

$$-(\mathfrak{n}^{(2)}/2!)\underset{(q,h)}{\overset{-(\mathfrak{x}-2)}{\Delta}}\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{j}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{j}}\big)/\mathfrak{q}^{\mathfrak{n}}\Big)-\ldots-(\mathfrak{n}^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\underset{(q,h)}{\overset{-1}{\Delta}}\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{j}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{j}}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$

$$=\sum_{\mathfrak{r}=\mathfrak{x}-1}^{\mathfrak{n}-1}(\mathfrak{r}^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}}\mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big),$$

which completes the proof. □

Corollary 3. *Let* $\mathfrak{u},\mathfrak{v}:\mathscr{M}_h^q\rightarrow\mathfrak{R}$, $\mathfrak{k},\mathfrak{n}\in\mathfrak{R}$, $\mathfrak{q}\in\mathfrak{R}-\{0\}$, $\mathfrak{x}\in\mathfrak{N}$ *and if* $\mathfrak{h}=0$, *then Equation* (78) *becomes*

$$\underset{(q,0)}{\overset{-\mathfrak{x}}{\Delta}}\mathfrak{u}(\mathfrak{k})-\sum_{\mathfrak{r}=0}^{\mathfrak{x}-1}(\mathfrak{n}^{(\mathfrak{d})}/\mathfrak{d}!)\underset{(q,0)}{\overset{-(\mathfrak{x}-\mathfrak{d})}{\Delta}}\mathfrak{u}\big(\mathfrak{k}/\mathfrak{q}^{\mathfrak{n}}\big)=\sum_{\mathfrak{r}=\mathfrak{x}-1}^{\mathfrak{n}-1}(\mathfrak{r}^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}\big(\mathfrak{k}/\mathfrak{q}^{\mathfrak{r}+1}\big). \tag{82}$$

Corollary 4. *Let* $\mathfrak{u},\mathfrak{v}:\mathscr{M}_h^q\rightarrow\mathfrak{R}$, $\mathfrak{k},\mathfrak{n}\in\mathfrak{R}$, $\mathfrak{h}\in\mathfrak{R}-\{0\}$, $\mathfrak{x}\in\mathfrak{N}$ *and if* $\mathfrak{q}=1$, *then Equation* (78) *becomes*

$$\underset{(1,h)}{\overset{-\mathfrak{x}}{\Delta}}\mathfrak{u}(\mathfrak{k})-\sum_{\mathfrak{d}=0}^{\mathfrak{x}-1}(\mathfrak{n}^{(\mathfrak{d})}/\mathfrak{d}!)\underset{(1,h)}{\overset{-(\mathfrak{x}-\mathfrak{d})}{\Delta}}\mathfrak{u}(\mathfrak{k}-\mathfrak{n}\mathfrak{h})=\sum_{\mathfrak{r}=\mathfrak{x}-1}^{\mathfrak{n}-1}(\mathfrak{r}^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}(\mathfrak{k}-(\mathfrak{r}+1)\mathfrak{h}). \tag{83}$$

Corollary 5. *Let* $\mathfrak{u},\mathfrak{v}:\mathscr{M}_h^q\rightarrow\mathfrak{R}$, $\mathfrak{h}\neq0\in\mathfrak{R}$, $\mathfrak{q}\in\mathfrak{R}-\{0,1\}$, $\mathfrak{n},\mathfrak{x}\in\mathfrak{N}$ *and* $\mathfrak{k}\in\mathfrak{R}$. *Then, the* $\mathfrak{m}$*-th order of* $(\mathfrak{q},\mathfrak{h})$ *difference equation is given by*

$$\underset{(q,h)}{\overset{-\mathfrak{x}}{\Delta}}\mathfrak{u}(\mathfrak{k})-\sum_{\mathfrak{d}=\mathfrak{n}-\mathfrak{x}}^{\mathfrak{n}-1}\mathfrak{n}^{(\mathfrak{r}-\mathfrak{n}+\mathfrak{x})}/(\mathfrak{r}-\mathfrak{n}+\mathfrak{x})!\underset{(q,h)}{\overset{-(\mathfrak{n}-\mathfrak{r})}{\Delta}}\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{j}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{j}}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$

$$=\sum_{\mathfrak{r}=0}^{\mathfrak{n}-\mathfrak{x}}(\mathfrak{x}+\mathfrak{r}-1)^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{x}+\mathfrak{r}-1}\mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{x}+\mathfrak{r}}\Big). \tag{84}$$

Proof. The proof completes by replacing
$\sum_{\mathfrak{d}=0}^{\mathfrak{x}-1}\frac{\mathfrak{n}^{(\mathfrak{d})}}{\mathfrak{d}!}\underset{(q,h)}{\overset{-(\mathfrak{x}-\mathfrak{d})}{\Delta}}\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{j}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{j}}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$ by $\sum_{\mathfrak{d}=\mathfrak{n}-\mathfrak{x}}^{\mathfrak{n}-1}\frac{\mathfrak{n}^{(\mathfrak{r}-\mathfrak{n}+\mathfrak{x})}}{(\mathfrak{r}-\mathfrak{n}+\mathfrak{x})!}\underset{(q,h)}{\overset{-(\mathfrak{n}-\mathfrak{r})}{\Delta}}\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{j}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{j}}\big)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$
and
$\sum_{\mathfrak{r}=\mathfrak{x}-1}^{\mathfrak{n}-1}\frac{\mathfrak{r}^{(\mathfrak{x}-1)}}{(\mathfrak{x}-1)!}\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}}\mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big)$ by $\sum_{\mathfrak{r}=\mathfrak{x}-1}^{\mathfrak{n}-1}\frac{\mathfrak{r}^{(\mathfrak{x}-1)}}{(\mathfrak{x}-1)!}\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}}\mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big)$
in Equation (78). □

Theorem 9. *Let* $\mathfrak{u},\mathfrak{v}:\mathscr{M}_h^q\rightarrow\mathfrak{R}$, $\mathfrak{x},\mathfrak{k}\in\mathfrak{N}$ *and* $\mathfrak{q},\mathfrak{h}\in\mathfrak{R}-\{0\}$. *Then, the* $\mathfrak{x}$*-th order anti-difference principle of* $(\mathfrak{q},\mathfrak{h})$ *operator for infinite series is given by*

$$\underset{(q,h)}{\overset{-\mathfrak{x}}{\Delta}}\mathfrak{u}(\mathfrak{k})=\sum_{\mathfrak{r}=0}^{\infty}((\mathfrak{x}+\mathfrak{r}-1)^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}\Big(\big(\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{x}+\mathfrak{r}-1}\mathfrak{q}^{\mathfrak{s}}\big)\Big/\mathfrak{q}^{\mathfrak{x}+\mathfrak{r}}\Big). \tag{85}$$

Proof. Taking $\lim_{\mathfrak{n}\to\infty}$ in Equation (78) and assuming $\underset{(q,h)}{\overset{-\mathfrak{x}}{\Delta}}\mathfrak{u}(0)=0$, we arrive at Equation (85). □

Corollary 6. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{s} \in \mathfrak{R}$, $\mathfrak{k} \in \mathfrak{N}$, $\mathfrak{q}, \mathfrak{h} \in \mathfrak{R} - \{0, 1\}$ *be a real number, and if the series* $\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{r}} \mathfrak{q}^{j})\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big)$ *is convergent, then*

$$\underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}(\mathfrak{s}) - \sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{r}} \mathfrak{q}^{j})\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big) = \sum_{\mathfrak{r}=0}^{\mathfrak{k}} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{r}} \mathfrak{q}^{j})\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big). \tag{86}$$

Proof. Taking $\lim_{n\to\infty}$ in Equation (75) and assuming $v(0) = 0 = u(0)$, then

$$\underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}(\mathfrak{k}) = \mathfrak{u}((\mathfrak{k} - \mathfrak{h})/\mathfrak{q}) + \mathfrak{u}((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2) + \mathfrak{u}((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3)$$

$$+\mathfrak{u}((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4) + \mathfrak{u}((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{4} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^5) + \ldots\ldots$$

$$+\mathfrak{u}((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{p}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{p}})/\mathfrak{q}^{\mathfrak{r}+1}) + \mathfrak{u}((\mathfrak{k} - \mathfrak{h}\sum_{\mathfrak{p}=0}^{\mathfrak{r}+1} \mathfrak{q}^{\mathfrak{p}})/\mathfrak{q}^{\mathfrak{r}+2}) + \ldots \tag{87}$$

Replacing $'\mathfrak{k}'$ by $'\mathfrak{s}'$ and $'\mathfrak{r}'$ by $'\mathfrak{k}'$ in (87), we obtain

$$\underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}(\mathfrak{s}) = \mathfrak{u}((\mathfrak{s} - \mathfrak{h})/\mathfrak{q}) + \mathfrak{u}((\mathfrak{s} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2) + \mathfrak{u}((\mathfrak{s} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3)$$

$$+ \ldots + \mathfrak{u}((\mathfrak{s} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{k}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{k}+1}) + \mathfrak{u}((\mathfrak{s} - \mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{k}+1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{k}+2}) + \ldots,$$

which is the same as

$$\underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}(\mathfrak{s}) = \sum_{\mathfrak{r}=0}^{\mathfrak{k}} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{r}} \mathfrak{q}^{j})\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big) + \sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{r}} \mathfrak{q}^{j})\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big). \tag{88}$$

Now, the proof completes by shifting the infinite series term of (88) to the left side. □

Definition 11. *Let* $\mathfrak{s}, \mathfrak{k} \in \mathfrak{R}$, $\mathfrak{h} \in \mathfrak{R} > 0$, $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$, *and if* $\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{r}} \mathfrak{q}^{j})\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big)$ *is convergent such that* $\mathfrak{s} \in \mathscr{M}_h^q$ *and* $\mathfrak{u} : \mathscr{M}_h^q \to \mathfrak{R}$ *be a function. Then the quantum geometric function (or* $\mathfrak{q}$*-geometric function) on* $(\mathfrak{q}, \mathfrak{h})$ *operator is defined as*

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{r}} \mathfrak{q}^{j})\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big) = \frac{\Big[\mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1} \mathfrak{q}^{j})\Big/\mathfrak{q}^{\mathfrak{k}+2}\Big)\Big]^2}{\mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1} \mathfrak{q}^{j})\Big/\mathfrak{q}^{\mathfrak{k}+2}\Big) - \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{k}+2} \mathfrak{q}^{j})\Big/\mathfrak{q}^{\mathfrak{k}+3}\Big)}. \tag{89}$$

The following Theorem 10 is the finite series formula for the $(\mathfrak{q}, \mathfrak{h})$ difference operator derived from infinite series.

Theorem 10. *Assuming the conditions given in Corollary 6, then the first order anti-difference principle of* $(\mathfrak{q}, \mathfrak{h})$ *difference operator is given by*

$$\underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{u}(\mathfrak{s}) - \frac{[\mathfrak{u}((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1} \mathfrak{q}^{j})\big/\mathfrak{q}^{\mathfrak{k}+2})]^2}{\mathfrak{u}((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1} \mathfrak{q}^{j})\big/\mathfrak{q}^{\mathfrak{k}+2}) - \mathfrak{u}((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{k}+2} \mathfrak{q}^{j})\big/\mathfrak{q}^{\mathfrak{k}+3})}$$

$$= \sum_{\mathfrak{r}=0}^{\mathfrak{k}} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h}\sum_{j=0}^{\mathfrak{r}} \mathfrak{q}^{j})\Big/\mathfrak{q}^{\mathfrak{r}+1}\Big). \tag{90}$$

Proof. The proof completes by substituting Equation (89) in (86). □

Theorem 11. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{s} \in \mathfrak{R}$, $\mathfrak{q}, \mathfrak{h} \in \mathfrak{R} - \{0\}$ *and* $\mathfrak{x}, \mathfrak{k} \in \mathfrak{N}$. *Then, the higher order of* $(\mathfrak{q}, \mathfrak{h})$ *difference operator is given by* $\underset{(q,h)}{\overset{-\mathfrak{x}}{\Delta}} \mathfrak{u}(\mathfrak{s})$

$$-\frac{(\mathfrak{x}-1)![((\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1}\Big)]^2}{(\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-1)}\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1}\Big)-(\mathfrak{k}+\mathfrak{x}+1)^{(\mathfrak{x}-1)}\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+2}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+2}\Big)}$$

$$=\sum_{\mathfrak{r}=0}^{\mathfrak{k}}((\mathfrak{r}+\mathfrak{x}-1)^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{r}}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{r}+\mathfrak{x}}\Big). \tag{91}$$

Proof. From Equation (85), we have

$$\underset{(q,h)}{\overset{-\mathfrak{x}}{\Delta}}\mathfrak{u}(\mathfrak{k}) = \frac{(\mathfrak{x}-1)^{(\mathfrak{x}-1)}}{(\mathfrak{x}-1)!}\mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{x}-1}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{x}}\Big)+\frac{\mathfrak{x}^{(\mathfrak{x}-1)}}{(\mathfrak{x}-1)!}\mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{x}}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{x}+1}\Big)$$
$$+\ldots+\frac{(\mathfrak{x}-(\mathfrak{r}-1))^{(\mathfrak{x}-1)}}{(\mathfrak{x}-1)!}\mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{x}-(\mathfrak{r}-1)}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{x}+\mathfrak{r}}\Big)$$
$$+\frac{(\mathfrak{x}-\mathfrak{r})^{(\mathfrak{x}-1)}}{(\mathfrak{x}-1)!}\mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{x}-\mathfrak{r}}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{x}+\mathfrak{r}+1}\Big)+\ldots$$

Replacing $'\mathfrak{k}'$ by $'\mathfrak{s}'$ and $'\mathfrak{r}'$ by $'\mathfrak{k}'$, the above equation becomes

$$\underset{(q,h)}{\overset{-\mathfrak{x}}{\Delta}}\mathfrak{u}(\mathfrak{s}) = \sum_{\mathfrak{y}=\mathfrak{x}-1}^{\mathfrak{x}+\mathfrak{k}-1}\frac{\mathfrak{y}^{(\mathfrak{x}-1)}}{(\mathfrak{x}-1)!}\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{y}}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{y}+1}\Big)+\sum_{\mathfrak{y}=\mathfrak{x}+\mathfrak{k}}^{\infty}\frac{\mathfrak{y}^{(\mathfrak{x}-1)}}{(\mathfrak{x}-1)!}\mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{y}}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{y}+1}\Big).$$

Interchanging the terms $\sum_{\mathfrak{y}=\mathfrak{x}-1}^{\mathfrak{x}+\mathfrak{k}-1}(\mathfrak{y}^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}(\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{y}}\mathfrak{q}^j)$ by $\sum_{\mathfrak{r}=0}^{\mathfrak{k}}((\mathfrak{x}+\mathfrak{r}-1)^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}(\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{x}+\mathfrak{r}-1}\mathfrak{q}^j/\mathfrak{q}^{\mathfrak{x}+\mathfrak{r}})$ and $\sum_{\mathfrak{y}=\mathfrak{x}+\mathfrak{k}}^{\infty}(\mathfrak{y}^{(\mathfrak{x}-1)}(\mathfrak{x}-1)!)\mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{y}}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{y}+1}\Big)$ by $\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty}((\mathfrak{x}+\mathfrak{r}-1)^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}(\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{x}+\mathfrak{r}-1}\mathfrak{q}^j/\mathfrak{q}^{\mathfrak{x}+\mathfrak{r}})$, and then using Equation (89) for $\mathfrak{x}$-th order, the above equation becomes

$$\sum_{\mathfrak{y}=\mathfrak{k}+1}^{\infty}(\mathfrak{y}^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{y}}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{y}+1}\Big)=$$

$$\frac{(\mathfrak{x}-1)![((\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}\mathfrak{q}^j)/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1}\Big)]^2}{(\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-1)}\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}\mathfrak{q}^j)/\mathfrak{q}^{\mathfrak{k}+\mathfrak{x}+1}\Big)-(\mathfrak{k}+\mathfrak{x}+1)^{(m-1)}\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+2}\mathfrak{q}^j)/\mathfrak{q}^{\mathfrak{k}+3}\Big)},$$

which completes the proof. □

4.2. Fractional order Theorems

In this section, we develop fractional order anti-difference principle from its integer order given in Definition 11, by which we derive fundamental theorems of quantum fractional calculus. For $\nu > 0$, we obtain

$$\sum_{\mathfrak{r}=\infty}^{\mathfrak{k}+1}(\Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1))\mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{r}}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{r}+\nu}\Big)=\frac{\mathscr{A}\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{k}+\nu+1}\Big)]^2}{\mathscr{A}\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{k}+\nu+1}\Big)-\mathscr{B}\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+2}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{k}+\nu+2}\Big)}, \tag{92}$$

where $\mathscr{A} = (\Gamma(\mathfrak{k}+\nu+1)/\Gamma(\nu)\Gamma(\mathfrak{k}+2))$ and $\mathscr{B} = (\Gamma(\mathfrak{k}+\nu+2)/\Gamma(\nu)\Gamma(\mathfrak{k}+3))$.

Theorem 12. *Let* $\mathfrak{u},\mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{h} \neq 0 \in \mathfrak{R}$, $\mathfrak{q} \in \mathfrak{R} - \{0,1\}$, $\mathfrak{k},\nu \in \mathfrak{R}$ *and* $\mathfrak{n} \in \mathfrak{N}$. *Then, the* ν*-th order of* $(\mathfrak{q},\mathfrak{h})$ *difference equation is given by*

$$\underset{(q,h)}{\overset{-\nu}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \sum_{\mathfrak{d}=\mathfrak{n}-\nu}^{\mathfrak{n}-1} (\Gamma(\mathfrak{n}+1)/\Gamma(2\mathfrak{n}-\mathfrak{d}-\nu+1)\Gamma(\mathfrak{d}-\mathfrak{n}+\nu-1)) \underset{(q,h)}{\overset{-(\mathfrak{n}-\mathfrak{d})}{\Delta}} \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{n}}\Big)$$
$$= (1/\Gamma(\nu))\sum_{\mathfrak{r}=0}^{\mathfrak{n}-\nu}(\Gamma(\nu+\mathfrak{r})/\Gamma(\mathfrak{r}+1))\mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{s}=0}^{\nu+\mathfrak{r}-1}\mathfrak{q}^{\mathfrak{s}})\Big/\mathfrak{q}^{\nu+\mathfrak{r}}\Big). \tag{93}$$

Proof. When generalizing the integer order to real order ($\mathfrak{m} > 0 \in \mathfrak{R} = \nu$) in Equation (84), we obtain

$$\underset{(q,h)}{\overset{-\nu}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \sum_{\mathfrak{d}=\mathfrak{n}-\nu}^{\mathfrak{n}-1} \frac{\mathfrak{n}^{(\mathfrak{r}-\mathfrak{n}+\nu)}}{(\mathfrak{r}-\mathfrak{n}+\nu)!} \underset{(q,h)}{\overset{-(\mathfrak{n}-\mathfrak{r})}{\Delta}} \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{n}-1}\mathfrak{q}^j)/\mathfrak{q}^{\mathfrak{n}}\Big)$$
$$= \sum_{r=0}^{\mathfrak{n}-\nu} \frac{(\nu+\mathfrak{r}-1)^{(\nu-1)}}{(\nu-1)!} \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{s}=0}^{\nu+\mathfrak{r}-1}\mathfrak{q}^{\mathfrak{s}})/\mathfrak{q}^{\nu+\mathfrak{r}}\Big).$$

Now, the proof completes by (14), that is, $\mathfrak{n}^{(\mathfrak{r}-\mathfrak{n}+\mathfrak{m})} = (\Gamma(\mathfrak{n}+1)/\Gamma(2\mathfrak{n}-\mathfrak{d}-\nu+1))$ and $(\mathfrak{m}+\mathfrak{r}-1)^{(\mathfrak{m}-1)} = (\Gamma(\nu+\mathfrak{r})/\Gamma(\mathfrak{r}+1))$ in Equation (84). □

Theorem 13. *Let* $\mathfrak{u},\mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{h} \in \mathfrak{R} - \{0\}$, $\mathfrak{q} \in \mathfrak{R} - \{0,1\}$, $\mathfrak{k} \in \mathfrak{N}$ *and* $\mathfrak{s},\nu \in \mathfrak{R}$. *Then, the* ν*-th order (fractional or real order) of* $(\mathfrak{q},\mathfrak{h})$ *difference equation is given by*

$$\underset{(q,h)}{\overset{-\nu}{\Delta}} \mathfrak{u}(\mathfrak{s}) - \frac{\Gamma(\nu)\Big[(\Gamma(\mathfrak{k}+\nu+1)/\Gamma(\nu)\Gamma(\mathfrak{k}+2))\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{k}+\nu+1}\Big)\Big]^2}{\frac{\Gamma(\mathfrak{k}+\nu+1)}{\Gamma(\mathfrak{k}+2)}\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{k}+\nu 1}\Big) - \frac{\Gamma(\mathfrak{k}+\nu+2)}{\Gamma(\mathfrak{k}+3)}\mathfrak{u}\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+2}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{k}+\nu+2}\Big)}$$
$$= (1/\Gamma(\nu))\sum_{\mathfrak{r}=0}^{\mathfrak{k}}(\Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1))\mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{r}}\mathfrak{q}^j)\Big/\mathfrak{q}^{\mathfrak{r}+\nu}\Big). \tag{94}$$

Proof. From Equation (14), we obtain $(\mathfrak{k}+\nu)^{(\nu-1)} = (\Gamma(\mathfrak{k}+\nu+1)/\Gamma(\mathfrak{k}+2))$ and $(\mathfrak{k}+\nu+1)^{(\nu-1)} = (\Gamma(\mathfrak{k}+\nu+2)/\Gamma(\mathfrak{k}+3))$. Thus, by generalizing the integer order ($\mathfrak{m}$-th order) of Equations (89) and (91) to any real order (ν-th order), the proof is complete. □

Result 3. *For finding the fractional difference equation in* $(\mathfrak{q},\mathfrak{h})$ *difference operator for infinite series, we should know about the behavior of* $\sum_{j=0}^{\mathfrak{s}}\mathfrak{q}^j$ *series.*

(1) If $\mathfrak{s}$ *is odd and* $\mathfrak{q} \in \mathfrak{R}$*, then*

$$\sum_{j=0}^{\mathfrak{s}}\mathfrak{q}^j = (1+\mathfrak{q}^2+\mathfrak{q}^4+\ldots+\mathfrak{q}^{\mathfrak{s}})(1+\mathfrak{q}) = \sum_{j=0}^{\mathfrak{s}/2}\mathfrak{q}^{2j}(1+\mathfrak{q}). \tag{95}$$

(2) If $\mathfrak{s}$ *is even and* $\mathfrak{q} \in \mathfrak{R}$*, then*

$$\sum_{j=0}^{\mathfrak{s}}\mathfrak{q}^j = (1+\mathfrak{q}^2+\mathfrak{q}^4+\ldots+\mathfrak{q}^{\mathfrak{s}})(1+\mathfrak{q}) = \sum_{j=0}^{(\mathfrak{s}-2)/2}\mathfrak{q}^{2j}(1+\mathfrak{q})+\mathfrak{q}^{\mathfrak{s}}. \tag{96}$$

Theorem 14. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\nu \in \mathfrak{R}$, $\mathfrak{k} \in \mathfrak{R}$, $\mathfrak{h} \in \mathfrak{R} - \{0\}$ *and* $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$ *such that* $(\nu + \mathfrak{r} - 1)/2 \in \mathfrak{N}$ *and* $(\nu + \mathfrak{r} - 3)/2 \in \mathfrak{N}$ *. Then the* ν*-th order of* $(\mathfrak{q}, \mathfrak{h})$ *difference operator for infinite series is given by*

$$\underset{(q,h)}{\overset{-\nu}{\Delta}} \mathfrak{u}(\mathfrak{k}) = (1/\Gamma(\nu)) \sum_{\mathfrak{r}=0}^{\infty} (\Gamma(\mathfrak{r} + \nu)/\Gamma(\mathfrak{r} + 1)) \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{s}=0}^{(\nu+\mathfrak{r}-1)/2} \mathfrak{q}^{2\mathfrak{r}}(1 + \mathfrak{q})\big) \Big/ \mathfrak{q}^{\nu+\mathfrak{r}} \Big). \quad (97)$$

and

$$\underset{(q,h)}{\overset{-\nu}{\Delta}} \mathfrak{u}(\mathfrak{k}) = (1/\Gamma(\nu)) \sum_{\mathfrak{r}=0}^{\infty} (\Gamma(\mathfrak{r} + \nu)/\Gamma(\mathfrak{r} + 1)) \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{s}=0}^{(\nu+\mathfrak{r}-3)/2} \mathfrak{q}^{2\mathfrak{r}}(1 + \mathfrak{q}) + \mathfrak{q}^{\nu+\mathfrak{r}-1}\big) \Big/ \mathfrak{q}^{\nu+\mathfrak{r}} \Big). \quad (98)$$

Proof. The proof completes by generalizing Theorem 9 and Result 3 to any real order ($\nu \in \mathfrak{R}$) and by (14). □

5. Mixed Alpha Symmetric Difference Operator

In this section, we develop fundamental theorems using $(\mathfrak{q}, \mathfrak{h})_\alpha$ difference operator and its inverse operators. If we take $\alpha = 1$, then the $(\mathfrak{q}, \mathfrak{h})_\alpha$ difference equation will become $(\mathfrak{q}, \mathfrak{h})$ difference equation.

5.1. Integer Order Theorems

Here, we develop certain theorems for integer order ($\mathfrak{m}$-th order) using the $(\mathfrak{q}, \mathfrak{h})_\alpha$ difference operator.

Definition 12. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$ *be a function and* $\alpha \in \mathfrak{R}$. *Then,* $(\mathfrak{q}, \mathfrak{h})_\alpha$ *difference operator (mixed alpha symmetric operator) is defined as*

$$\underset{(q,h)_\alpha}{\Delta} \mathfrak{u}(\mathfrak{k}) = \mathfrak{u}(\mathfrak{k}\mathfrak{q} + \mathfrak{h}) - \alpha \mathfrak{u}(\mathfrak{k}), \ \mathfrak{k} \in \mathfrak{R}. \quad (99)$$

Remark 2. *If* $\alpha = 1$*, then Equation* (99) *becomes* $(\mathfrak{q}, \mathfrak{h})$*-difference operator.*

Lemma 8. *If* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$, $0 \neq \mathfrak{h} \in \mathfrak{R}$ *and* $\alpha \in \mathfrak{R}$. *Then, the product rule of* $(\mathfrak{q}, \mathfrak{h})_\alpha$ *difference operator is obtained as*

$$\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} \{\mathfrak{u}(\mathfrak{k})\mathfrak{v}(\mathfrak{k})\} = \mathfrak{u}(\mathfrak{k}) \underset{(q,h)_\alpha}{\overset{-1}{\Delta}} \mathfrak{v}(\mathfrak{k}) - \underset{(q,h)_\alpha}{\overset{-1}{\Delta}} \{ \underset{(q,h)}{\overset{-1}{\Delta}} \mathfrak{v}(\mathfrak{k}\mathfrak{q} + \mathfrak{h}) \underset{(q,h)}{\Delta} \mathfrak{u}(\mathfrak{k}) \}. \quad (100)$$

Proof. The proof is similar to Lemma 7 by using the $\underset{(q,h)_\alpha}{\Delta}$ operator. □

Property 2. *Some of the properties of* $(\mathfrak{q}, \mathfrak{h})_\alpha$ *difference operator is given below:*

(i) *If* $\mathfrak{q} = 1$*, then* (99) *becomes* $\mathfrak{h}(\alpha)$*-difference operator.*
(ii) *If* $\mathfrak{h} = 0$*, then* (99) *becomes* $\mathfrak{q}_{(\alpha)}$*-difference operator.*
(iii) *If* $\mathfrak{q} > 1$, $\mathfrak{h} > 0$ *and* $\alpha \in \mathfrak{R}$*, then we say* (99) *as* $(\mathfrak{q}, \mathfrak{h})_\alpha$*-difference operator.*
(iv) *The solution does not exist if we take* $\mathfrak{q} = 1$, $\mathfrak{h} = 0$ *and* $\alpha = 1$ *simultaneously.*

Theorem 15. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{k}, \alpha \in \mathfrak{R}$, $\mathfrak{n} \in \mathfrak{N}$, $\mathfrak{h} \in \mathfrak{R} - \{0\}$ *and* $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$*. Then, the anti-difference principle of* $(\mathfrak{q}, \mathfrak{h})_\alpha$ *difference operator is given by*

$$\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \underset{(q,h)_\alpha}{\overset{-1}{\Delta}} \alpha^{\mathfrak{n}} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{j}}\big) \Big/ \mathfrak{q}^{\mathfrak{n}} \Big) = \sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} {}^{\mathfrak{r}}\mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{s}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{s}}\big) \Big/ \mathfrak{q}^{\mathfrak{r}+1} \Big). \quad (101)$$

Proof. Following similar steps from (64) to (74) in Theorem 7 using Equation (99), we obtain the general form as

$$\mathfrak{v}(\mathfrak{k}) = \mathfrak{u}((\mathfrak{k}-\mathfrak{h})/\mathfrak{q}) + \alpha \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{1}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\Big) + \alpha^2 \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{2}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\Big) + \alpha^3 \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{3}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\Big)$$
$$+\alpha^4 \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{4}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^5\Big) + \ldots + \alpha^{\mathfrak{n}-1} \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\Big) + \alpha^{\mathfrak{n}} \mathfrak{v}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\Big) \quad . \tag{102}$$

If $\overset{-1}{\underset{(q,h)_\alpha}{\Delta}} \mathfrak{u}(\mathfrak{k}) = \mathfrak{v}(\mathfrak{k})$, then (102) becomes

$$\overset{-1}{\underset{(q,h)_\alpha}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \alpha^{\mathfrak{n}} \overset{-1}{\underset{(q,h)_\alpha}{\Delta}} \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\Big) = \mathfrak{u}((\mathfrak{k}-\mathfrak{h})/\mathfrak{q}) + \alpha \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{1}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2\Big)$$
$$+\alpha^2 \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{2}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3\Big) + \alpha^3 \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{3}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4\Big) + \alpha^4 \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{4}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^5\Big)$$
$$+ \ldots + \alpha^{\mathfrak{n}-1} \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{r}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{n}}\Big),$$

which completes the proof. □

Corollary 7. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{k}, \alpha \in \mathfrak{R}$, $\mathfrak{n} \in \mathfrak{N}$, $\mathfrak{q} \in \mathfrak{R} - \{0\}$, *and if* $\mathfrak{h} = 0$, *then Equation* (101) *becomes*

$$\overset{-1}{\underset{(q,0)_\alpha}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \overset{-1}{\underset{(q,0)_\alpha}{\Delta}} \alpha^{\mathfrak{n}} \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^{\mathfrak{n}}) = \sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \alpha^{\mathfrak{r}} \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^{\mathfrak{r}+1}). \tag{103}$$

Corollary 8. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{k}, \alpha \in \mathfrak{R}$, $\mathfrak{n} \in \mathfrak{N}$, $\mathfrak{h} \in \mathfrak{R} - \{0\}$, *and if* $\mathfrak{q} = 1$, *then Equation* (101) *becomes*

$$\overset{-1}{\underset{(1,h)_\alpha}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \overset{-1}{\underset{(1,h)_\alpha}{\Delta}} \alpha^{\mathfrak{n}} \mathfrak{u}(\mathfrak{k} - \mathfrak{n}\mathfrak{h}) = \sum_{\mathfrak{r}=0}^{\mathfrak{n}-1} \alpha^{\mathfrak{r}} \mathfrak{u}(\mathfrak{k} - (\mathfrak{r}+1)\mathfrak{h}). \tag{104}$$

Remark 3. *The operators* $\overset{-1}{\underset{(q,0)_\alpha}{\Delta}}$ *and* $\overset{-1}{\underset{(1,h)_\alpha}{\Delta}}$ *are the first order* $q_{(\alpha)}$ *and* $h(\alpha)$ *difference operators, respectively. That is,* $\overset{-1}{\underset{(q,0)_\alpha}{\Delta}} = \overset{-1}{\underset{q_{(\alpha)}}{\Delta}}$ *and* $\overset{-1}{\underset{(1,h)_\alpha}{\Delta}} = \overset{-1}{\underset{h(\alpha)}{\Delta}}$.

Theorem 16. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{k}, \alpha \in \mathfrak{R}$, $\mathfrak{h} \in \mathfrak{R} - \{0\}$, $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$, *and* $\mathfrak{n}, \mathfrak{x} \in \mathfrak{N}$. *Then, the higher order of* $(\mathfrak{q}, \mathfrak{h})_\alpha$ *difference operator is given by*

$$\overset{-\mathfrak{x}}{\underset{(q,h)_\alpha}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \sum_{\mathfrak{d}=0}^{\mathfrak{x}-1} \frac{\mathfrak{n}^{(\mathfrak{d})}}{\mathfrak{d}!} \alpha^{\mathfrak{n}-\mathfrak{d}} \overset{-(\mathfrak{x}-\mathfrak{d})}{\underset{(q,h)_\alpha}{\Delta}} \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{j}=0}^{\mathfrak{n}-1}\mathfrak{q}^{\mathfrak{j}})/\mathfrak{q}^{\mathfrak{n}}\Big) = \sum_{\mathfrak{r}=\mathfrak{x}-1}^{\mathfrak{n}-1} \frac{\mathfrak{r}^{(\mathfrak{x}-1)}}{(\mathfrak{x}-1)!} \alpha^{\mathfrak{r}-(\mathfrak{x}-1)} \mathfrak{u}\Big((\mathfrak{k}-\mathfrak{h}\sum_{\mathfrak{s}=0}^{\mathfrak{r}}\mathfrak{q}^{\mathfrak{s}})/\mathfrak{q}^{\mathfrak{r}+\mathfrak{x}}\Big). \tag{105}$$

Proof. The proof is similar to Theorem 8 by applying the $\overset{-1}{\underset{(q,h)_\alpha}{\Delta}}$ operator repeatedly on both sides of Equation (101). □

Corollary 9. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{k}, \alpha \in \mathfrak{R}$, $\mathfrak{q} \in \mathfrak{R} - \{0\}$, $\mathfrak{x}, \mathfrak{n} \in \mathfrak{N}$, *and if* $\mathfrak{h} = 0$, *then Equation* (105) *becomes*

$$\overset{-\mathfrak{x}}{\underset{(q,0)_\alpha}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \sum_{\mathfrak{r}=0}^{\mathfrak{x}-1} (\mathfrak{n}^{(\mathfrak{d})}/\mathfrak{d}!) \alpha^{\mathfrak{n}-\mathfrak{d}} \overset{-(\mathfrak{x}-\mathfrak{d})}{\underset{(q,0)_\alpha}{\Delta}} \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^{\mathfrak{n}}) = \sum_{\mathfrak{r}=\mathfrak{x}-1}^{\mathfrak{n}-1} (\mathfrak{r}^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!) \alpha^{\mathfrak{r}-(\mathfrak{x}-1)} \mathfrak{u}(\mathfrak{k}/\mathfrak{q}^{\mathfrak{r}+1}). \tag{106}$$

Corollary 10. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{k}, \alpha \in \mathfrak{R}$, $\mathfrak{h} \in \mathfrak{R} - \{0\}$, $\mathfrak{x}, \mathfrak{n} \in \mathfrak{N}$ *and if* $\mathfrak{q} = 1$, *then Equation* (105) *becomes*

$$\overset{-\mathfrak{x}}{\underset{(1,h)_\alpha}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \sum_{\mathfrak{d}=0}^{\mathfrak{x}-1} (\mathfrak{n}^{(\mathfrak{d})}/\mathfrak{d}!) \alpha^{\mathfrak{n}-\mathfrak{d}} \overset{-(\mathfrak{x}-\mathfrak{d})}{\underset{(1,h)_\alpha}{\Delta}} \mathfrak{u}(\mathfrak{k}-\mathfrak{n}\mathfrak{h}) = \sum_{\mathfrak{r}=\mathfrak{x}-1}^{\mathfrak{n}-1} (\mathfrak{r}^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!) \alpha^{\mathfrak{r}-(\mathfrak{x}-1)} \mathfrak{u}(\mathfrak{k}-(\mathfrak{r}+1)\mathfrak{h}). \tag{107}$$

Corollary 11. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \Re$, $\mathfrak{h} \neq 0 \in \Re$, $\mathfrak{q} \in \Re - \{0, 1\}$, $\alpha, \mathfrak{k} \in \Re$, *and* $\mathfrak{n}, \mathfrak{x} \in \mathfrak{N}$. *Then, the* $\mathfrak{x}$*-th order of* $(\mathfrak{q}, \mathfrak{h})_\alpha$ *difference equation is given by*

$$\underset{(q,h)_\alpha}{\overset{-\mathfrak{x}}{\Delta}} \mathfrak{u}(\mathfrak{k}) - \sum_{\mathfrak{d}=\mathfrak{n}-\mathfrak{x}}^{\mathfrak{n}-1} (\mathfrak{n}^{(\mathfrak{r}-\mathfrak{n}+\mathfrak{x})} \alpha^{\mathfrak{n}-\mathfrak{d}} / (\mathfrak{r}-\mathfrak{n}+\mathfrak{x})!) \underset{(q,h)_\alpha}{\overset{-(n-r)}{\Delta}} \mathfrak{u}\Big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{j}}) \Big/ \mathfrak{q}^{\mathfrak{n}}\Big)$$
$$= \sum_{\mathfrak{r}=0}^{\mathfrak{n}-\mathfrak{x}} ((\mathfrak{x}+\mathfrak{r}-1)^{(\mathfrak{x}-1)} / (\mathfrak{x}-1)!) \alpha^{\mathfrak{r}-(\mathfrak{x}-1)} \mathfrak{u}\Big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{s}=0}^{\mathfrak{x}+\mathfrak{r}-1} \mathfrak{q}^{\mathfrak{s}}) \Big/ \mathfrak{q}^{\mathfrak{x}+\mathfrak{r}}\Big). \quad (108)$$

Proof. The proof completes by replacing $\sum_{\mathfrak{d}=0}^{\mathfrak{x}-1} (\mathfrak{n}^{(\mathfrak{d})} / \mathfrak{d}!) \alpha^{\mathfrak{n}-\mathfrak{d}} \underset{(q,h)_\alpha}{\overset{-(\mathfrak{x}-\mathfrak{d})}{\Delta}} \mathfrak{u}\Big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{j}}) \Big/ \mathfrak{q}^{\mathfrak{n}}\Big)$ by $\sum_{\mathfrak{d}=\mathfrak{n}-\mathfrak{x}}^{\mathfrak{n}-1} (\mathfrak{n}^{(\mathfrak{r}-\mathfrak{n}+\mathfrak{x})} / (\mathfrak{r}-\mathfrak{n}+\mathfrak{x})!) \alpha^{\mathfrak{n}-\mathfrak{d}} \underset{(q,h)_\alpha}{\overset{-(\mathfrak{n}-\mathfrak{r})}{\Delta}} \mathfrak{u}\Big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{n}-1} \mathfrak{q}^{\mathfrak{j}}) \Big/ \mathfrak{q}^{\mathfrak{n}}\Big)$ and $\sum_{\mathfrak{r}=\mathfrak{x}-1}^{\mathfrak{n}-1} (\mathfrak{r}^{(\mathfrak{x}-1)} / (\mathfrak{x}-1)!) \alpha^{\mathfrak{r}-(\mathfrak{x}-1)} \mathfrak{u}\Big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{s}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{s}}) \Big/ \mathfrak{q}^{\mathfrak{r}+1}\Big)$ by $\sum_{\mathfrak{r}=0}^{\mathfrak{n}-\mathfrak{x}} ((\mathfrak{x}+\mathfrak{r}-1)^{(\mathfrak{x}-1)} / (\mathfrak{x}-1)!) \alpha^{\mathfrak{r}-(\mathfrak{x}-1)} \mathfrak{u}\Big((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{s}=0}^{\mathfrak{x}+\mathfrak{r}-1} \mathfrak{q}^{\mathfrak{s}}) \Big/ \mathfrak{q}^{\mathfrak{x}+\mathfrak{r}}\Big)$ in Equation (105). □

Corollary 12. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \Re$, $\mathfrak{q} \in \Re - \{0, 1\}$, $\mathfrak{h} \in \Re - \{0\}$, $\mathfrak{k} \in \mathfrak{N}$, *and* $\mathfrak{s}, \alpha \in \Re$. *If* $\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \alpha^{\mathfrak{r}} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{j}}) \Big/ \mathfrak{q}^{\mathfrak{r}+1}\Big)$ *is convergent, then*

$$\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} \mathfrak{u}(\mathfrak{s}) - \sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \alpha^{\mathfrak{r}} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{j}}) \Big/ \mathfrak{q}^{\mathfrak{r}+1}\Big) = \sum_{\mathfrak{r}=0}^{\mathfrak{k}} \alpha^{\mathfrak{r}} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{j}}) \Big/ \mathfrak{q}^{\mathfrak{r}+1}\Big). \quad (109)$$

Proof. Taking $\lim_{\mathfrak{n}\to\infty}$ in Equation (102) and assuming $\mathfrak{v}(0) = 0 = \mathfrak{u}(0)$, then

$$\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} \mathfrak{u}(\mathfrak{k}) = \mathfrak{u}((\mathfrak{k}-\mathfrak{h})/\mathfrak{q}) + \alpha \mathfrak{u}((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2) + \alpha^2 \mathfrak{u}((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3)$$
$$+ \alpha^3 \mathfrak{u}((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{3} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^4) + \alpha^4 \mathfrak{u}((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{4} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^5) + \ldots + \alpha^{\mathfrak{r}} \mathfrak{u}((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{p}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{p}})/\mathfrak{q}^{\mathfrak{r}+1})$$
$$+ \alpha^{\mathfrak{r}+1} \mathfrak{u}((\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{p}=0}^{\mathfrak{r}+1} \mathfrak{q}^{\mathfrak{p}})/\mathfrak{q}^{\mathfrak{r}+2}) + \ldots \quad (110)$$

Replacing '$\mathfrak{k}$' by '$\mathfrak{s}$' and '$\mathfrak{r}$' by '$\mathfrak{k}$' in (110), we obtain

$$\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} \mathfrak{u}(\mathfrak{s}) = \mathfrak{u}((\mathfrak{s}-\mathfrak{h})/\mathfrak{q}) + \alpha \mathfrak{u}((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^2) + \alpha^2 \mathfrak{u}((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{2} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^3)$$
$$+ \ldots + \alpha^{\mathfrak{k}} \mathfrak{u}((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{\mathfrak{k}} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{k}+1}) + \alpha^{\mathfrak{k}+1} \mathfrak{u}((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{r}=0}^{\mathfrak{k}+1} \mathfrak{q}^{\mathfrak{r}})/\mathfrak{q}^{\mathfrak{k}+2}) + \ldots,$$

which completes the proof. □

Definition 13. *Let* $\mathfrak{s} \in \Re$, $\mathfrak{k} \in \mathfrak{N}$, $\alpha, \mathfrak{h} \in \Re > 0$, $\mathfrak{q} \in \Re - \{0, 1\}$, *and if* $\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \alpha^{\mathfrak{r}} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{j}}) \Big/ \mathfrak{q}^{\mathfrak{r}+1}\Big)$ *is convergent such that* $\mathfrak{s} \in \mathscr{M}_h^q$ *and* $\mathfrak{u} : \mathscr{M}_h^q \to \Re$ *be a function. Then, the alpha-quantum geometric function (or* $\mathfrak{q}$*-alpha geometric function) on* $(\mathfrak{q}, \mathfrak{h})_\alpha$ *operator is defined as*

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty} \alpha^{\mathfrak{r}} \mathfrak{u}\Big((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{j}}) \Big/ \mathfrak{q}^{\mathfrak{r}+1}\Big) = \frac{\Big[\alpha^{\mathfrak{k}+1} u\Big((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{k}+1} \mathfrak{q}^{\mathfrak{j}}) \Big/ \mathfrak{q}^{\mathfrak{k}+2}\Big)\Big]^2}{\alpha^{\mathfrak{k}+1} u\Big((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{k}+1} \mathfrak{q}^{\mathfrak{j}}) \Big/ \mathfrak{q}^{\mathfrak{k}+2}\Big) - \alpha^{\mathfrak{k}+2} u\Big((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{k}+2} \mathfrak{q}^{\mathfrak{j}}) \Big/ \mathfrak{q}^{\mathfrak{k}+3}\Big)}. \quad (111)$$

Theorem 17. *Consider the conditions given in Corollary 12 and Definition 13. Then,*

$$\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} u(\mathfrak{s}) - \frac{[\alpha^{\mathfrak{k}+1}u\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}q^j)\Big/q^{\mathfrak{k}+2}\Big)]^2}{\alpha^{\mathfrak{k}+1}u\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}q^j)\Big/q^{\mathfrak{k}+2}\Big) - \alpha^{\mathfrak{k}+2}u\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+2}q^j)\Big/q^{\mathfrak{k}+3}\Big)} = \sum_{\mathfrak{r}=0}^{\mathfrak{k}}\alpha^{\mathfrak{r}}u\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{r}}q^j)\Big/q^{\mathfrak{r}+1}\Big). \tag{112}$$

Proof. The proof completes by substituting Equation (111) in (109). □

Theorem 18. *Let* $u, v : \mathscr{M}_h^q \to \mathfrak{R}$, $q \in \mathfrak{R} - \{0, 1\}$, $\mathfrak{h} \in \mathfrak{R} - \{0\}$, $\mathfrak{x} \in \mathfrak{N}$, *and* $\mathfrak{k}, \alpha \in \mathfrak{R}$. *Then, the higher order anti-difference principle for the infinite series is given by*

$$\underset{(q,h)_\alpha}{\overset{-\mathfrak{x}}{\Delta}} u(\mathfrak{k}) = \sum_{\mathfrak{r}=\mathfrak{x}-1}^{\infty}((\mathfrak{r}+\mathfrak{x}-1)^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!)\alpha^{\mathfrak{r}-(\mathfrak{x}-1)}u\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{y}}q^j)\Big/q^{\mathfrak{r}+\mathfrak{x}}\Big). \tag{113}$$

Proof. Taking $\lim_{n\to\infty}$ in Equation (105) and assuming $\underset{(q,h)_\alpha}{\overset{-\mathfrak{x}}{\Delta}} u(0) = 0$, we arrive at Equation (113). □

Theorem 19. *Let* $u, v : \mathscr{M}_h^q \to \mathfrak{R}$, $\alpha, \mathfrak{s} \in \mathfrak{R}$, $q \in \mathfrak{R} - \{0, 1\}$, $\mathfrak{h} \in \mathfrak{R} - \{0\}$, *and* $\mathfrak{k}, \mathfrak{x} \in \mathfrak{N}$. *Then, the higher order of* $(q, h)_\alpha$ *difference operator is given by*

$$\underset{(q,h)_\alpha}{\overset{-\mathfrak{x}}{\Delta}} u(\mathfrak{s}) - \frac{[\mathscr{A}\alpha^{\mathfrak{k}+1}u\big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}q^j)/q^{\mathfrak{k}+\mathfrak{x}+1}\big)]^2}{\mathscr{A}\alpha^{\mathfrak{k}+1}u\big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}q^j)/q^{\mathfrak{k}+\mathfrak{x}+1}\big) - \mathscr{B}\alpha^{\mathfrak{k}+2}u\big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+2}q^j)/q^{\mathfrak{k}+\mathfrak{x}+2}\big)} = \sum_{\mathfrak{r}=0}^{\mathfrak{k}}((\mathfrak{r}+\mathfrak{x}-1)^{(\mathfrak{x}-1)}\alpha^{\mathfrak{r}}/(\mathfrak{x}-1)!)u\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{r}}q^j)\Big/q^{\mathfrak{r}+\mathfrak{x}}\Big), \tag{114}$$

where $\mathscr{A} = (\mathfrak{k}+\mathfrak{x})^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!$ *and* $\mathscr{B} = (\mathfrak{k}+\mathfrak{x}+1)^{(\mathfrak{x}-1)}/(\mathfrak{x}-1)!$.

Proof. Applying the proof of Theorem 11 in Equation (113), we obtain Equation (114). □

5.2. Generalized Theorems for $(\mathfrak{q}, \mathfrak{h})_\alpha$ *Difference Operators*

In this section, we develop fractional order anti-difference theorems from its integer order given in Definition 13, from which we derive fundamental theorems of alpha quantum fractional calculus. For $\nu > 0$, we obtain

$$\sum_{\mathfrak{r}=\mathfrak{k}+1}^{\infty}(\Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1))\alpha^{\mathfrak{k}+1}u\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{\mathfrak{r}}q^j)\Big/q^{\mathfrak{r}+\nu}\Big) = \frac{\mathscr{A}\alpha^{\mathfrak{k}+1}u\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}q^j)\Big/q^{\mathfrak{k}+\nu+1}\Big)]^2}{\mathscr{A}\alpha^{\mathfrak{k}+1}u\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+1}q^j)\Big/q^{\mathfrak{k}+\nu+1}\Big) - \mathscr{B}\alpha^{\mathfrak{k}+2}u\Big((\mathfrak{s}-\mathfrak{h}\sum_{j=0}^{\mathfrak{k}+2}q^j)\Big/q^{\mathfrak{k}+\nu+2}\Big)}, \tag{115}$$

where $\mathscr{A} = (\Gamma(\mathfrak{k}+\nu+1)/\Gamma(\nu)\Gamma(\mathfrak{k}+2))$ and $\mathscr{B} = (\Gamma(\mathfrak{k}+\nu+2)/\Gamma(\nu)\Gamma(\mathfrak{k}+3))$.

Theorem 20. *Let* $u, v : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{h} \neq 0 \in \mathfrak{R}$, $q \in \mathfrak{R} - \{0, 1\}$, $\mathfrak{k}, \nu, \alpha \in \mathfrak{R}$, *and* $n \in \mathfrak{N}$. *Then, the* ν*-th order of* $(\mathfrak{q}, \mathfrak{h})$ *difference equation is given by*

$$\underset{(q,h)_\alpha}{\overset{-\nu}{\Delta}} u(\mathfrak{k}) - \sum_{\mathfrak{d}=n-\nu}^{n-1}\frac{\Gamma(n+1)\alpha^{n-\mathfrak{d}}}{\Gamma(2n-\mathfrak{d}-\nu+1)\Gamma(\mathfrak{d}-n+\nu-1)}\underset{(q,h)_\alpha}{\overset{-(n-\mathfrak{d})}{\Delta}} u\Big((\mathfrak{k}-\mathfrak{h}\sum_{j=0}^{n-1}q^j)\Big/q^n\Big)$$

$$= (1/\Gamma(\nu)) \sum_{\mathfrak{r}=0}^{\mathfrak{n}-\nu} (\Gamma(\nu+\mathfrak{r})/\Gamma(\mathfrak{r}+1))\alpha^{\mathfrak{r}-(\nu-1)} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{s}=0}^{\nu+\mathfrak{r}-1} \mathfrak{q}^{\mathfrak{s}}\big) \Big/ \mathfrak{q}^{\nu+\mathfrak{r}} \Big). \tag{116}$$

Proof. The proof follows from Corollary 11, Theorem 12 and by Equation (14) using the $(\mathfrak{q}, \mathfrak{h})_\alpha$ difference operator. □

Theorem 21. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\mathfrak{q} \in \mathfrak{R} - \{0, 1\}$, $\mathfrak{h} \in \mathfrak{R} - \{0\}$, $\alpha, \mathfrak{s} \in \mathfrak{R}$, $\mathfrak{k} \in \mathfrak{N}$, *and* $\nu \in \mathfrak{R}$. *Then, the* ν*-th order of* $(\mathfrak{q}, \mathfrak{h})_\alpha$ *difference operator is given by*

$$\underset{(q,h)_\alpha}{\overset{-\nu}{\Delta}} \mathfrak{u}(\mathfrak{s}) - \frac{[\mathscr{A}\alpha^{\mathfrak{k}+1} \mathfrak{u}\big((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{k}+1} \mathfrak{q}^{\mathfrak{j}})/\mathfrak{q}^{\mathfrak{k}+\nu+1}\big)]^2}{\mathscr{A}\alpha^{\mathfrak{k}+1} \mathfrak{u}\big((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{k}+1} \mathfrak{q}^{\mathfrak{j}})/\mathfrak{q}^{\mathfrak{k}+\nu+1}\big) - \mathscr{B}\alpha^{\mathfrak{k}+2} \mathfrak{u}\big((\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{k}+2} \mathfrak{q}^{\mathfrak{j}})/\mathfrak{q}^{\mathfrak{k}+\nu+2}\big)}$$

$$= \sum_{\mathfrak{r}=0}^{\mathfrak{k}} (\Gamma(\mathfrak{r}+\nu)/\Gamma(\nu)\Gamma(\mathfrak{r}+1))\alpha^{\mathfrak{r}} \mathfrak{u}\Big(\big(\mathfrak{s} - \mathfrak{h} \sum_{\mathfrak{j}=0}^{\mathfrak{r}} \mathfrak{q}^{\mathfrak{j}}\big) \Big/ \mathfrak{q}^{\mathfrak{r}+\nu} \Big), \tag{117}$$

where $\mathscr{A} = \Gamma(\mathfrak{k}+\nu+1)/\Gamma(\nu)\Gamma(\mathfrak{k}+2)$ *and* $\mathscr{B} = \Gamma(\mathfrak{k}+\nu+2)/\Gamma(\nu)\Gamma(\mathfrak{k}+3)$.

Proof. The proof follows from Theorem 19, Theorem 13, and by Equation (14). □

Theorem 22. *Let* $\mathfrak{u}, \mathfrak{v} : \mathscr{M}_h^q \to \mathfrak{R}$, $\alpha, \nu, \mathfrak{k} \in \mathfrak{R}$ *and* $\mathfrak{q}, \mathfrak{h} \in \mathfrak{R} - \{0\}$ *such that* $(\nu+\mathfrak{r}-1)/2$ *and* $(\nu+\mathfrak{r}-3)/2$ *are natural numbers. Then, the* ν*-th of* $(\mathfrak{q}, \mathfrak{h})_\alpha$ *difference operator for infinite series is given by*

$$\underset{(q,h)}{\overset{-\nu}{\Delta}} \mathfrak{u}(\mathfrak{k}) = (1/\Gamma(\nu)) \sum_{\mathfrak{r}=0}^{\infty} (\Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1))\alpha^{\mathfrak{r}-\nu+1} u\Big(\big(\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{s}=0}^{(\nu+\mathfrak{r}-1)/2} \mathfrak{q}^{2\mathfrak{r}}(1+\mathfrak{q})\big) \Big/ \mathfrak{q}^{\nu+\mathfrak{r}} \Big). \tag{118}$$

and

$$\underset{(q,h)}{\overset{-\nu}{\Delta}} \mathfrak{u}(\mathfrak{k}) = (1/\Gamma(\nu)) \sum_{\mathfrak{r}=0}^{\infty} (\Gamma(\mathfrak{r}+\nu)/\Gamma(\mathfrak{r}+1))\alpha^{\mathfrak{r}-\nu+1} \mathfrak{u}\Big(\big(\mathfrak{k} - \mathfrak{h} \sum_{\mathfrak{s}=0}^{(\nu+\mathfrak{r}-3)/2} \mathfrak{q}^{2\mathfrak{r}}(1+\mathfrak{q}) + \mathfrak{q}^{\nu+\mathfrak{r}-1}\big) \Big/ \mathfrak{q}^{\nu+\mathfrak{r}} \Big). \tag{119}$$

Proof. The proof completes by generalizing Theorem 18 and Result 3 to any real order ($\nu \in \mathfrak{R}$) and by (14). □

The integer and fractional order $(\mathfrak{q}, \mathfrak{h})$ and $(\mathfrak{q}, \mathfrak{h})_\alpha$ anti-difference equation acts as the solution for mixed symmetric difference operator and mixed alpha symmetric difference operator. One can do the same for the nabla operator.

6. Results and Discussion

The value analysis of the difference operators $(\mathfrak{q}, \mathfrak{h})$ and $(\mathfrak{q}, \mathfrak{h})_\alpha$ will be looked at in this section.

Example 5. *Fixing the values* $s = 8.3$ *and* $k = 50$, *then Figure 1 shows that for any* $\nu > 0 \in R$, *the values of the* (q, h) *difference equation is dropping over time, indicating that it will converge. Figure 2 demonstrates that if the* ν *and* α *value increases, then the values of the* $(q, h)_\alpha$ *difference operator progressively increase and then eventually decrease, which says that it will converge.*

Example 5 gives the general solution for Theorems 13 and 21 for any real q and h values. As a result, we can easily predict the value stability for (q, h) and $(q, h)_\alpha$ operators.

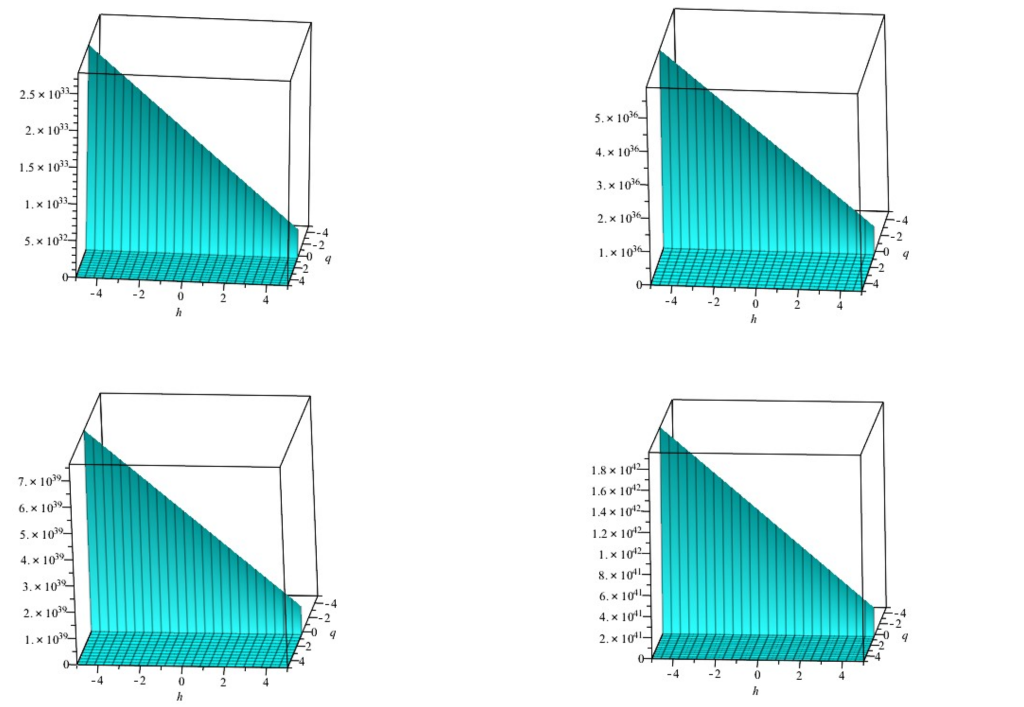

Figure 1. Solution for Theorem 13 with ν values 0.2, 1.3, 2.7, and 3.9, where q and h vary from -4 to 4.

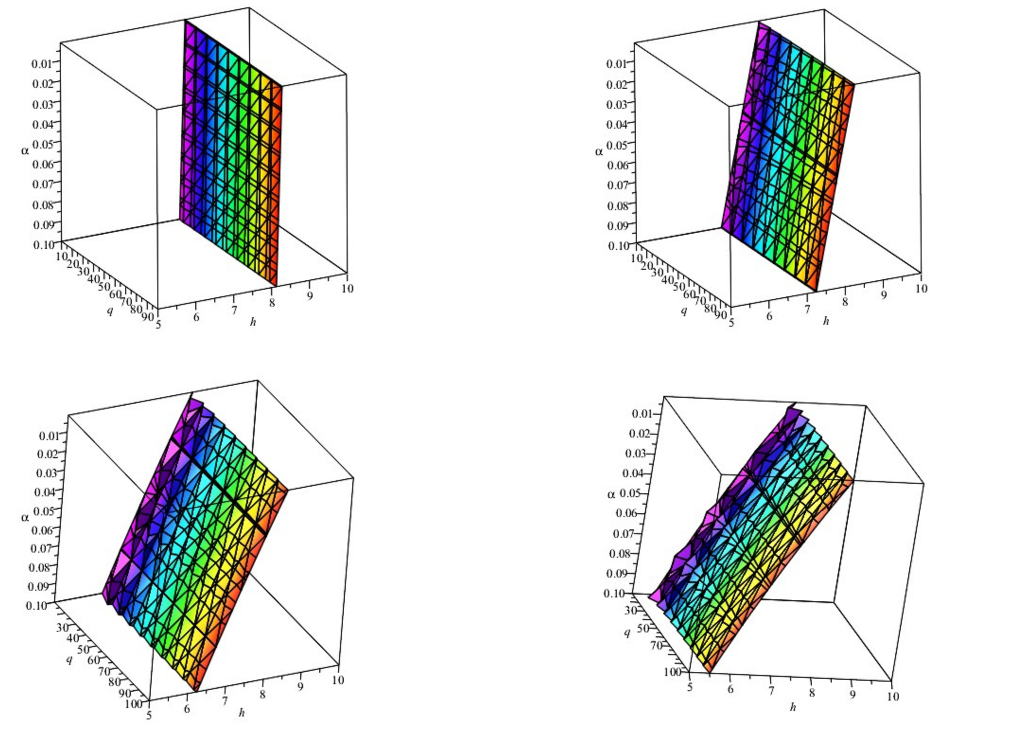

Figure 2. Solution for Theorem 21 with ν values 0.2, 1.3, 2.7, and 3.9, where q varies from 10 to 100, h varies from 5 to 10, and α varies from 1×10^{-2} to 0.1.

7. Conclusions

In this research work, we have developed several integer and fractional order anti-difference equations for both $\mathfrak{q}$ and $(\mathfrak{q}, \mathfrak{h})$ operators and its alpha difference operators. In addition, we have derived fundamental theorems using $\mathfrak{q}_\alpha$ and $(\mathfrak{q}, \mathfrak{h})_\alpha$ operators and their inverses for both integer and fractional order. Finally, our results are verified with numerical examples and discussed with graphs. This study will result in applications for transforming the infinite series difference equation to the finite series equation. The future work of this paper is finding the polynomials and polynomial factorial functions for the $(\mathfrak{q}, \mathfrak{h})$ and $(\mathfrak{q}, \mathfrak{h})_\alpha$ difference operator and its inverse operators. In addition, we will extend this paper to Fibonacci quantum fractional calculus.

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