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Symmetries of Non-Linear ODEs: Lambda Extensions of the Ising Correlations

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Abstract: This paper provides several illustrations of the numerous remarkable properties of the lambda extensions of the two-point correlation functions of the Ising model, shedding some light on the non-linear ODEs of the Painlevé type they satisfy. We first show that this concept also exists for the factors of the two-point correlation functions focusing, for pedagogical reasons, on two examples, namely $C(0,5)$ and $C(2,5)$ at $\nu = -k$. We then display, in a learn-by-example approach, some of the puzzling properties and structures of these lambda extensions: for an infinite set of (algebraic) values of λ these power series become algebraic functions, and for a finite set of (rational) values of lambda they become D-finite functions, more precisely polynomials (of different degrees) in the complete elliptic integrals of the first and second kind K and E. For generic values of λ these power series are not D-finite, they are differentially algebraic. For an infinite number of other (rational) values of λ these power series are globally bounded series, thus providing an example of an infinite number of globally bounded differentially algebraic series. Finally, taking the example of a product of two diagonal two-point correlation functions, we suggest that many more families of non-linear ODEs of the Painlevé type remain to be discovered on the two-dimensional Ising model, as well as their structures, and in particular their associated lambda extensions. The question of their possible reduction, after complicated transformations, to Okamoto sigma forms of Painlevé VI remains an extremely difficult challenge.

Keywords: Ising two-point correlation functions; lambda extension of correlation functions; sigma form of Painlevé VI; D-finite functions; differentially algebraic functions; globally bounded series

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1. Introduction: Linear versus Non-Linear Symmetry Representations

It is not necessary to underline the fundamental role played by the concept of symmetry in physics [1], or in applied mathematics, and in the foundations for the fundamental theories of modern physics. Symmetries can correspond to continuous or discrete transformations, and are frequently amenable to mathematical formulations such as group representations, with invariant or covariant properties, non-trivial identities, conservation laws, etc.

Integrable models (in dynamical systems, lattice statistical mechanics, quantum field theory, solid state physics, enumerative combinatorics, etc.) play a selected role, since they correspond to situations where one has "enough" (possibly an infinite number of) conserved quantities to solve the problem. We are not going to recall the techniques and tools introduced to achieve that goal (Yang–Baxter equations, Bethe Ansatz, Lax pairs, Schlesinger systems [2], etc.) but we will rather focus on the linear and non-linear differential equations emerging naturally in these problems, and on the corresponding symmetries of these ordinary differential equations. To address that problem we focus, for pedagogical reasons, on the analysis of the two-point correlation functions of a fundamental integrable model, the two-dimensional Ising model [3]. The non-linear ODEs emerging in such an "integrable" framework are highly selected: they have the (fixed critical) Painlevé property, they have algebraic function solutions, etc. This is in (strong) contrast with the generic non-linear ODEs for which more numerical analysis (investigation of the qualitative behavior of non linear ODEs, stability and boundness, etc.) must be performed (see for instance [4–6]).

Some two-point correlation functions $C(M,N)$ of the two-dimensional Ising model can be seen as solutions of *linear* differential equations and, *at the same time*, also as solutions of *non-linear* differential equations, namely Okamoto sigma-forms of Painlevé VI equations. The solutions of these last non-linear ODEs naturally introduce one-parameter families of power series solutions, that are called lambda extensions of the two-point correlation functions.

The two-point correlation functions $C(M,N)$ we consider [7,8] for the special case $\nu = -k$ (or in the isotropic case $\nu = 1$), are *polynomial expressions* of the complete elliptic integrals of the first and second kind K and E: they are solutions of *linear* differential operators with polynomial coefficients, in other words they are *D-finite*; however, when introducing some well-suited log-derivative of these two-point correlation functions (see (4) below), they are also solutions of highly selected non-linear differential equations having the Painlevé property [9], namely Okamoto sigma-forms [10] of Painlevé VI (see (5) below). In other words, they are *differentially algebraic*. A differentially algebraic function [11,12] is a function $f(t)$ solution of a polynomial relation $P(t, f(t), f'(t), \cdots f^{(n)}(t)) = 0$, where $f^{(n)}(t)$ denotes the n-th derivative of $f(t)$ with respect to t. The two-point correlation functions $C(M,N)$ have *at the same time*, a linear (D-finite) description and a *non-linear* (differentially algebraic) description! The question of the analysis of the symmetries of these two *linear and non-linear* ordinary differential equations, and of the symmetries of their solutions naturally pops out. The symmetries of a differential equation and the symmetries of the solutions of the differential equation are two different concepts. It is crucial to note that the *non-linear* ordinary differential equations for the two-point correlation functions $C(M,N)$ correspond to *one closed equation* (see (5) below) where the two integers M and N are parameters in the equation. In contrast, the *linear* differential equations for the $C(M,N)$ correspond to an infinite number of linear differential equations of order (and degree and size) growing with the two integers M and N. Each description (linear versus non-linear) has its own advantages and disadvantages: an infinite number of differential operators to be discovered but they are simply linear, versus one (M, N-dependent) equation encapsulating everything, but it is non-linear. The analysis of the symmetries of the linear differential operators associated with the two-point correlation functions $C(M,N)$ can, for instance, be performed considering the corresponding differential Galois group. Actually, we have seen in previous papers [13] that the linear differential operators emerging in the integrable models are systematically associated with selected differential Galois groups and the operators being homomorphic [13] to their adjoint associated operators. We even have this remarkable property with most of the linear differential operators annihilating diagonals of rational functions [13]. In this Ising case, the linear differential operators are homomorphic [14] to the symmetric N-th power of the order-two linear differential operator annihilating the complete elliptic integrals of the first or second kind K and E. Along this line, some mathematicians could argue that, if a differential Galois group approach of integrability is probably natural, an extension of the concept of differential Galois group for *non-linear* ODEs is certainly hopeless in general [15]. They may even argue (see [15] in Section 6.2) that, even if most of the people that work in integrability consider the families of Painlevé transcendents [16,17] as integrable, their opinion is that, in general, they are non-integrable (at least in the (narrow) Liouville sense [18,19]). Let us recall that the sigma-form of Painlevé VI equations (such as (5) below), are *highly selected non-linear ODEs*:

they have the *fixed critical point property* [20–22] (*Painlevé property*) and can be obtained from isomonodromic deformations of linear differential equations [23,24], which allows us to see these non-linear ODEs as compatibility conditions of a linear Schlesinger system of PDEs. In that case, one could imagine to consider a differential Galois Theory for the underlying Schlesinger system. The purpose of this paper is not to build a differential Galois Theory of Painlevé equations in order to discuss, from a very general mathematical viewpoint the "symmetries" of the non-linear ODEs (such as (5) below) emerging for the $C(M, N)$ Ising two-point correlation functions. On the contrary, in a very pedagogical, learn-by-examples approach, we display a large set of the properties (symmetries, etc.) of the $C(M, N)$ two-point correlation functions, with a focus on the remarkable properties of the *lambda-extensions* solutions of the sigma-form of Painlevé VI non-linear ODEs (such as (5) below). We must also mention the fact that the *lambda extensions* of the two-point correlation functions $C(M, N)$ also verify quadratic *discrete* recursions [25–27] (*lattice recursions* in the two integers M and N), that can be seen as integrable lattice recursions. For pedagogical reasons, we restrict to $C(0, 5)$ and $C(2, 5)$. Then, taking an example of product of two diagonal two-point correlation functions, we suggest that many more families of non-linear ODEs of the Painlevé type remain to be discovered on the two-dimensional Ising model, as well as their structures, and in particular their associated lambda extensions. Finally, we give additional comments and results providing an illustration of a set of remarkable, and sometimes puzzling, properties of the lambda extensions of the Ising two-point correlation functions.

2. Recalls

We revisit, with a pedagogical heuristic motivation, the lambda extensions [14] of some two-point correlation functions $C(M,N)$ of the two-dimensional Ising model. For simplicity, we examine in detail the lambda extensions of a particular low-temperature diagonal correlation function, namely $C(0,5)$ and $C(2,5)$, in order to make crystal clear some structures and subtleties. Note however, that similar structures and results can also be obtained on other two-point correlation functions $C(M,N)$ for the special case $\nu = -k$ studied in [7] where Okamoto sigma-forms of Painlevé VI equations also emerge.

In a previous paper [7], we considered the two-point correlation $C(M,N)$ of spins at sites $(0,0)$ and (M,N), of the anisotropic Ising model defined by the interaction energy

$$\mathcal{E} \;=\; -\sum_{j,k}\{E_v\sigma_{j,k}\sigma_{j+1,k} + E_h\sigma_{j,k}\sigma_{j,k+1}\}, \tag{1}$$

where $\sigma_{j,k} = \pm 1$ is the spin at row j and column k, and where the sum is over all lattice sites. Defining

$$k \;=\; (\sinh 2E_v/k_BT\,\sinh 2E_h/k_BT)^{-1} \qquad \text{and} \qquad \nu \;=\; \frac{\sinh 2E_h/k_BT}{\sinh 2E_v/k_BT}, \tag{2}$$

we found [7] that in the special case

$$\nu \;=\; -k, \tag{3}$$

the correlation $C(M,N)$ (which is the same as the Toeplitz determinants [28] of Forrester–Witte [29] as given in [30]) satisfies an Okamoto sigma-form of the Painlevé VI equation. The condition $\nu = -k$ (as well as the isotropic case $\nu = 1$) is special because it is such that the complete elliptic integrals of the third kind (EllipticPi in Maple), appearing in the anisotropic case, reduce to complete elliptic integrals of the second kind (see Equation (30) in [7]).

For $T < T_c$, $M \le N$ and $\nu = -k$, with $t = k^2$, introducing

$$\sigma \;=\; t\cdot(t-1)\cdot\frac{d\ln C(M,N)}{dt} \;-\;\frac{t}{4}, \tag{4}$$

we have, when $M+N$ is odd, the following Okamoto sigma-form of the Painlevé VI equation [7]:

$$t^2 \cdot (t-1)^2 \cdot \sigma''^2 \; + 4 \cdot \sigma' \cdot (t\sigma' - \sigma) \cdot \Big((t-1) \cdot \sigma' - \sigma\Big)$$
$$-M^2 \cdot (t\sigma' - \sigma)^2 \; - N^2 \cdot \sigma'^2 \; + (M^2+N^2) \cdot \sigma' \cdot (t\sigma' - \sigma) \; = \; 0. \qquad (5)$$

2.1. Two Factors

In this $M+N$ odd, $M \le N$, $M \ne 0$, $\nu = -k$ case, the correlation functions factor into *two* factors. Note that what is written here in Section 2.1 is also true when $M = 0$, with the caveat that g_+ and g_- in (6) factor into two factors (see Section 2.2 below). We write the factorizations of these $C(M,N)$'s as

$$(1-t)^{-1/4} \cdot C(M,N;t) \; = \; g_+(M,N;t) \cdot g_-(M,N;t), \qquad (6)$$

where the two factors $g_\pm$ are *homogeneous polynomials* of the complete elliptic integrals of the first and second kind:

$$\tilde{K}(k) \; = \; \frac{2}{\pi} \cdot K(k) \; = \; {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], k^2\Big),$$
$$\tilde{E}(k) \; = \; \frac{2}{\pi} \cdot E(k) \; = \; {}_2F_1\Big([\frac{1}{2}, -\frac{1}{2}], [1], k^2\Big). \qquad (7)$$

We consider the following logarithmic derivatives of the previous two factors:

$$\sigma_\pm(M,N;t) \; = \; t \cdot (t-1) \cdot \frac{d \ln g_\pm(M,N;t)}{dt}. \qquad (8)$$

The sigma functions have *additive* decompositions, which follow from the multiplicative decompositions (6):

$$\sigma(M,N;t) \; = \; \sigma_+(M,N;t) \; + \sigma_-(M,N;t). \qquad (9)$$

Here we begin with the factorizations (6) of the $C(M,N)$'s with $M+N$ odd, $M \le N$, for miscellaneous values of M and N, and, by use of the methods described in [7] and of the program *guessfunc* of Jay Pantone [31], we find that both $\sigma_+(M,N;t)$ and $\sigma_-(M,N;t)$ in (9) satisfy the *same second-order non-linear differential equation*

$$32\, t^3 \cdot (t-1)^2 \cdot \sigma''^2 \; + 4\, t^2 \cdot (t-1) \cdot \Big(8 \cdot \sigma \; - 8 \cdot (t+1) \cdot \sigma' \; + M^2 - N^2\Big) \cdot \sigma''$$
$$-\Big(8\sigma \; - 16 \cdot t\sigma' \; + M^2 t - N^2 \; + 1 - t\Big) \cdot \Big(8 \cdot t \cdot (t-1) \cdot \sigma'^2 \; - 16\, t \cdot \sigma \cdot \sigma'$$
$$+8 \cdot \sigma^2 \; + (M^2 - N^2) \cdot \sigma\Big) \; = \; 0, \qquad (10)$$

where the prime indicates a derivative with respect to t, and where σ is one of the two log-derivatives (8). Note that this second-order non-linear ODE, which is actually of the Painlevé type, is not of the Okamoto sigma-form of Painlevé VI form, but it can be reduced to such a form using non-trivial transformations (Equations (26), (28) in Section 2 of [8]).

The two solutions (8) of (10), $\sigma_+(M,N;t)$ and $\sigma_-(M,N;t)$, have different boundary conditions. Note that $\sigma_\pm = 0$ *is a selected solution* of (10).

2.2. Four Factors

In [7], we discovered that $C(0,N)$ with N odd and $k = -\nu$, in the low-temperature regime, *factors into four terms* instead of two. The four factors for $C(0,N)$ were presented as

$$C(0,N) \; = \; \text{constant} \cdot (1-t)^{1/2} \cdot t^{(1-N^2)/4} \cdot f_1 f_2 f_3 f_4, \qquad (11)$$

where the factors f_j all vanish at $t = 0$ in such a way to cancel the factor $t^{(1-N^2)/4}$. We normalize the factors f_i in (11) in such a way to extract a factor of $(1-t)^{1/4}$ which is the limiting behavior of $C(0,N)$ as $N \to \infty$, and we impose the condition that *the four new factors satisfy the same non-linear differential equation*. The previous factorization (11) in *four* factors (examples of $g_i(0,N)$'s for $C(0,5)$ and $C(0,7)$ are given in [8]) now reads [8]:

$$(1-t)^{-1/4} \cdot C(0,N) \; = \; g_1(0,N) \cdot g_2(0,N) \cdot g_3(0,N) \cdot g_4(0,N). \tag{12}$$

If one defines

$$\sigma_j \; = \; t \cdot (t-1) \cdot \frac{d \ln g_j(t)}{dt}, \tag{13}$$

the previous factorization (12) in four factors becomes an *additivity property* of the corresponding σ_i's:

$$\sigma(0,\, N) \; = \; \sigma_1(0,\, N) \; + \sigma_2(0,\, N) \; + \sigma_3(0,\, N) \; + \sigma_4(0,\, N). \tag{14}$$

These σ_i's are solutions of the *same* non-linear differential equation of the Painlevé type, which reads:

$$\begin{aligned} & t^2 \cdot (t-1)^2 \cdot \sigma''^2 \; + 4\,\sigma' \cdot (t \cdot \sigma' - \sigma) \cdot \Big((t-1) \cdot \sigma' - \sigma \Big) \\ & + \frac{1}{4} \cdot \Big((N^2+1) \cdot (t-1) \; - t^2 \Big) \cdot \sigma'^2 \; - \frac{1}{2^6} \cdot \Big(16 \cdot (N^2 + 1 - 2\,t) \cdot \sigma + N^2 \cdot t \Big) \cdot \sigma' \\ & \qquad - \frac{1}{4} \cdot \sigma^2 \; + \frac{N^2}{2^6} \cdot \sigma \; - \frac{N^2 \cdot (N^2-3)}{2^{10}} \; = \; 0. \end{aligned} \tag{15}$$

3. α-Extension of the Four Factors f_1, f_2, f_3, f_4 for $C(0,\,5)$

We underlined that the (low-temperature) row correlation functions $C(0,N)$ factor, when is N odd, into four factors (11). These four factors f_i's are each a *homogeneous polynomial* of the complete elliptic functions E and K. Furthermore, one can see that *each of these four factors is a Toeplitz determinant* (see, for instance, Section G.4 of appendix G in [8]).

More specifically, let us revisit the $N = 5$ case detailed in [7] and also [8], where the two-point correlation $C(0,5)$ factors as follows

$$C(0,5) \; = \; \frac{256}{81} \cdot \frac{(1-t)^{1/2}}{t^6} \cdot f_1 \cdot f_2 \cdot f_3 \cdot f_4, \tag{16}$$

where:

$$f_1 \; = \; (2t-1) \cdot \tilde{E} \; + (1-t) \cdot \tilde{K}, \qquad f_2 \; = \; (1+t) \cdot \tilde{E} \; - (1-t) \cdot \tilde{K}, \tag{17}$$

$$f_3 \; = \; (t-2) \cdot \tilde{E} \; + 2 \cdot (1-t) \cdot \tilde{K}, \tag{18}$$

$$f_4 \; = \; 3\,\tilde{E}^2 \; + 2 \cdot (t-2) \cdot \tilde{E}\,\tilde{K} \; + (1-t) \cdot \tilde{K}^2. \tag{19}$$

These exact polynomial expressions in terms of complete elliptic integrals of the first and second kind $\tilde{K}$ and $\tilde{E}$, *actually have some lambda-extensions*. Considering the non-linear ODE's verified by these f_n's one can, by a down-to-earth, order by order expansion of the analytic at $t = 0$ solution, find the series expansion of a *one parameter family of solution* of the non-linear ODE's (we will denote α this parameter), such that $\alpha = 0$ corresponds to the previous exact expressions (17)–(19). The first terms of these α-dependent solutions read:

$$
\begin{aligned}
f_1(\alpha) \;=\; & \frac{3}{2}\,t \;-\frac{9\,t^2}{16} \;-\frac{15\,t^3}{128} \;-\left(\frac{105}{2048}+\frac{15}{1024}\,\alpha\right)\cdot t^4 \;-\left(\frac{945}{32{,}768}+\frac{135}{8192}\,\alpha\right)\cdot t^5 \\
& -\left(\frac{4851}{262{,}144}+\frac{513}{32{,}768}\,\alpha\right)\cdot t^6 \;-\left(\frac{27027}{2{,}097{,}152}+\frac{7497}{524{,}288}\,\alpha\right)\cdot t^7 \\
& -\left(\frac{637{,}065}{67{,}108{,}864}+\frac{434{,}295}{33{,}554{,}432}\,\alpha\right)\cdot t^8 \\
& -\left(\frac{15{,}643{,}485}{2{,}147{,}483{,}648}+\frac{6{,}292{,}455}{536{,}870{,}912}\,\alpha-\frac{105}{536{,}870{,}912}\,\alpha^2\right)\cdot t^9 \;+\;\cdots
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
f_2(\alpha) \;=\; & \frac{3}{2}\,t \;-\frac{3\,t^2}{16} \;-\frac{3\,t^3}{128} \;-\left(\frac{15}{2048}-\frac{15}{1024}\,\alpha\right)\cdot t^4 \;-\left(\frac{105}{32{,}768}-\frac{165}{8192}\,\alpha\right)\cdot t^5 \\
& -\left(\frac{441}{262{,}144}-\frac{723}{32{,}768}\,\alpha\right)\cdot t^6 \;-\left(\frac{2079}{2{,}097{,}152}-\frac{11{,}799}{524{,}288}\,\alpha\right)\cdot t^7 \\
& -\left(\frac{42{,}471}{67{,}108{,}864}-\frac{747{,}927}{33{,}554{,}432}\,\alpha\right)\cdot t^8 \\
& -\left(\frac{920{,}205}{2{,}147{,}483{,}648}-\frac{11{,}692{,}785}{536{,}870{,}912}\,\alpha-\frac{105}{536{,}870{,}912}\,\alpha^2\right)\cdot t^9 \;+\;\cdots
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
f_3(\alpha) \;=\; & -\frac{3}{8}\,t^2 \;-\frac{3\,t^3}{32} \;-\frac{45\,t^4}{1024} \;-\frac{105\,t^5}{4096} \;-\left(\frac{2205}{131{,}072}-\frac{15}{131{,}072}\,\alpha\right)\cdot t^6 \\
& -\left(\frac{6237}{524{,}288}-\frac{135}{524{,}288}\,\alpha\right)\cdot t^7 \;-\left(\frac{297{,}297}{33{,}554{,}432}-\frac{3285}{8{,}388{,}608}\,\alpha\right)\cdot t^8 \\
& -\left(\frac{920{,}205}{134{,}217{,}728}-\frac{16965}{33{,}554{,}432}\,\alpha\right)\cdot t^9 \;+\;\cdots
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
f_4(\alpha) \;=\; & -\frac{3}{8}\,t^2 \;-\frac{3}{16}\,t^3 \;-\frac{129\,t^4}{1024} \;-\frac{195\,t^5}{2048} \;-\left(\frac{5025}{65{,}536}+\frac{15}{131{,}072}\,\alpha\right)\cdot t^6 \\
& -\left(\frac{8421}{131{,}072}+\frac{75}{262{,}144}\,\alpha\right)\cdot t^7 \;-\left(\frac{1{,}856{,}253}{33{,}554{,}432}+\frac{3975}{8{,}388{,}608}\,\alpha\right)\cdot t^8 \\
& -\left(\frac{3{,}260{,}907}{67{,}108{,}864}+\frac{11{,}025}{16{,}777{,}216}\right)\cdot t^9 \;+\;\cdots
\end{aligned}
\tag{23}
$$

Furthermore one sees, on the series expansions of the α-extensions (20)–(23), the following remarkable identities

$$
\begin{aligned}
(1-t)^{1/4}\cdot f_2(\alpha) \;&=\; f_1(1-\alpha), \qquad & (1-t)^{1/4}\cdot f_2(1-\alpha) \;&=\; f_1(\alpha), \\
(1-t)^{1/4}\cdot f_4(\alpha) \;&=\; f_3(1-\alpha), \qquad & (1-t)^{1/4}\cdot f_4(1-\alpha) \;&=\; f_3(\alpha),
\end{aligned}
\tag{24}
$$

and thus:

$$
\begin{aligned}
(1-t)^{1/2}\cdot f_2(\alpha)\cdot f_4(\alpha) \;&=\; f_1(1-\alpha)\cdot f_3(1-\alpha), \\
(1-t)^{1/2}\cdot f_2(1-\alpha)\cdot f_4(1-\alpha) \;&=\; f_1(\alpha)\cdot f_3(\alpha),
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
f_4(\alpha)\cdot f_1(1-\alpha) \;&=\; f_2(\alpha)\cdot f_3(1-\alpha), \\
f_4(1-\alpha)\cdot f_1(\alpha) \;&=\; f_2(1-\alpha)\cdot f_3(\alpha).
\end{aligned}
\tag{26}
$$

In particular one has:

$$f_1(0) \;=\; (2t-1)\cdot \tilde{E} \,+ (1-t)\cdot \tilde{K}, \tag{27}$$

$$f_1(1) \;=\; (1-t)^{1/4}\cdot \Big((1+t)\cdot \tilde{E} \,-(1-t)\cdot \tilde{K}\Big), \tag{28}$$

$$f_2(0) \;=\; (1+t)\cdot \tilde{E} \,-(1-t)\cdot \tilde{K}, \tag{29}$$

$$f_2(1) \;=\; (1-t)^{-1/4}\cdot \Big((2t-1)\cdot \tilde{E} \,+(1-t)\cdot \tilde{K}\Big), \tag{30}$$

$$f_3(0) \;=\; (t-2)\cdot \tilde{E} \,+2\cdot (1-t)\cdot \tilde{K}, \tag{31}$$

$$f_3(1) \;=\; (1-t)^{-1/4}\cdot \Big(3\,\tilde{E}^2 \,+2\cdot (t-2)\cdot \tilde{E}\,\tilde{K} \,+(1-t)\cdot \tilde{K}^2\Big), \tag{32}$$

$$f_4(0) \;=\; 3\,\tilde{E}^2 \,+2\cdot (t-2)\cdot \tilde{E}\,\tilde{K} \,+(1-t)\cdot \tilde{K}^2, \tag{33}$$

$$f_4(1) \;=\; (1-t)^{1/4}\cdot \Big((t-2)\cdot \tilde{E} \,+2\cdot (1-t)\cdot \tilde{K}\Big). \tag{34}$$

It is worth noticing that (in contrast with the λ-extension $C(0,5;\,\lambda)$ see (35) below), the $f_n(\alpha)$'s have *two* different values of the parameter α for which these α-extensions are D-finite, being (homogeneous) polynomials in $\tilde{E}$ and $\tilde{K}$. One remarks with (31) and (32) (or (33) and (34)), that the corresponding polynomials in $\tilde{E}$ and $\tilde{K}$ *are not necessarily of the same degree* in $\tilde{E}$ and $\tilde{K}$.

The λ-extension $C(0,5;\,\lambda)$ solution of the same non-linear ODE verified by $C(0,5)$ (namely (5) for $N=5$) corresponds to the form-factor expansion [14,32], which amounts to seeing this one-parameter family of solutions as a deformation of the $(1-t)^{1/4}$ algebraic solution of the previous non-linear ODE (5) verified by $C(0,5)$:

$$C(0,5;\,\lambda) \;=\; (1-t)^{1/4}\cdot \Big(1 \,+\, \lambda^{2\,n}\cdot \sum_{n=1}^{\infty} f_{0,5}^{2\,n}\Big) \tag{35}$$

$$= \;1 \,-\frac{t}{4} \,-\frac{3\,t^2}{32} \,-\frac{7\,t^3}{128} \,-\frac{77\,t^4}{2048} \,-\frac{231\,t^5}{8192} \,-\Big(\frac{1463}{65{,}536} \,+\, \frac{25}{1{,}048{,}576}\cdot \lambda^2\Big)\cdot t^6$$
$$-\Big(\frac{4807}{262{,}144} \,+\, \frac{275}{4{,}194{,}304}\cdot \lambda^2\Big)\cdot t^7 \,-\Big(\frac{129{,}789}{8{,}388{,}608} \,+\, \frac{123{,}475}{1{,}073{,}741{,}824}\cdot \lambda^2\Big)\cdot t^8 \,+\,\cdots$$

3.1. Deformation around a D-Finite Solution

The λ-extension of the correlation function $C(0,5;\,\lambda)$ can *also* be seen as a μ-deformation of the series of the correlation $C(0,5)$, whose exact expression is given by (16) (with (17)–(19)) in terms of polynomials in $\tilde{E}$ and $\tilde{K}$. This one-parameter μ-family of series expansion which verifies the same non-linear ODE (5) as $C(0,5)$, reads:

$$C(0,5;\,\lambda) \;=\; C(0,5) \,+\mu\cdot G_1(t) \,+\mu^2\cdot G_2(t) \,+\mu^3\cdot G_3(t) \,+\,\cdots$$
$$= \;1 \,-\frac{t}{4} \,-\frac{3\,t^2}{32} \,-\frac{7\,t^3}{128} \,-\frac{77\,t^4}{2048} \,-\frac{231\,t^5}{8192} \,-\Big(\frac{23{,}433}{1{,}048{,}576} \,-\, \frac{25}{1{,}048{,}576}\,\mu\Big)\cdot t^6$$
$$-\Big(\frac{77{,}187}{4{,}194{,}304} \,-\, \frac{275}{4{,}194{,}304}\,\mu\Big)\cdot t^7 \,-\Big(\frac{16{,}736{,}467}{1{,}073{,}741{,}824} \,-\, \frac{123{,}475}{1{,}073{,}741{,}824}\,\mu\Big)\cdot t^8$$
$$-\Big(\frac{57{,}930{,}653}{4{,}294{,}967{,}296} \,-\, \frac{708{,}125}{4{,}294{,}967{,}296}\,\mu\Big)\cdot t^9 \,+\,\cdots \tag{36}$$

The identification of these two power series (35) and (36) corresponds to the simple relation between λ and μ:

$$\lambda^2 \;=\; 1-\mu \qquad\qquad \text{or:} \qquad\qquad \mu \;=\; 1-\lambda^2. \tag{37}$$

This one-parameter series (35), or (36), is in agreement with a α-extension of the four products formula (16)

$$C(0,5;\,\lambda) \;=\; \frac{256}{81}\cdot \frac{(1-t)^{1/2}}{t^6}\cdot f_1(\alpha)\cdot f_2(\alpha)\cdot f_3(\alpha)\cdot f_4(\alpha), \tag{38}$$

if

$$\mu \;=\; 4\cdot\alpha\cdot(1-\alpha) \qquad \text{or:} \qquad \lambda^2 \;=\; \Big(2\,\alpha\,-1\Big)^2, \tag{39}$$

or:

$$\alpha \;=\; \frac{1\,\pm\,\lambda}{2}. \tag{40}$$

Thus, one sees that the $\alpha \leftrightarrow 1-\alpha$ involutive symmetry in the identities (24) amounts to changing the sign of λ: $\lambda \leftrightarrow -\lambda$. The value $\lambda = 1$ (associated with the "physical" correlation functions) corresponds to the two values $\alpha = 0$ and $\alpha = 1$ for which the four factors f_n become polynomials of $\tilde{E}$ and $\tilde{K}$ (not necessarily of the same degree; see, for instance (33), (34)). The value $\lambda = 0$ (associated with the algebraic function $C(0,5;0) = (1-t)^{1/4}$) corresponds to the value $\alpha = 1/2$.

Recalling the usual parametrization [8,14] of the parameter λ, namely $\lambda = \cos(u)$, and the trigonometric identity

$$\cos(u) \;=\; 2\,\cos(u/2)^2\;-1, \tag{41}$$

we see that the parameter α is naturally parameterized as

$$\alpha \;=\; \cos(u/2)^2, \tag{42}$$

the $\alpha \leftrightarrow 1-\alpha$ involutive symmetry in the identities (24) corresponding to the parametrization

$$1-\alpha \;=\; 1-\cos(u/2)^2 \;=\; \sin(u/2)^2, \tag{43}$$

which amounts to changing u into $u \rightarrow u+\pi$ in (42), a transformation that does not change $\lambda^2 = \cos(u)^2$.

3.2. *The Algebraic $\alpha = 1/2$ Case*

One thus sees that the (involutive) symmetry $\alpha \leftrightarrow 1-\alpha$ singles out $\alpha = 1/2$. Along this line, note that, for $\alpha = 1/2$, these α-extensions (20), (21) become *algebraic functions*. One actually has:

$$\begin{aligned} f_1\Big(\frac{1}{2}\Big) \;&=\; \frac{3}{2}\cdot t\cdot(1-t)^{1/16}\cdot\Big(\frac{1+(1-t)^{1/2}}{2}\Big)^{5/4} \\ &=\; \frac{3}{2}\,t\,-\frac{9}{16}\,t^2\,-\frac{15}{128}\,t^3\,-\frac{15}{256}\,t^4\,-\frac{1215}{32{,}768}\,t^5\,-\frac{6903}{262{,}144}\,t^6\;+\cdots \end{aligned} \tag{44}$$

$$\begin{aligned} f_2\Big(\frac{1}{2}\Big) \;&=\; \frac{3}{2}\cdot t\cdot(1-t)^{1/16}\cdot(1-t)^{-1/4}\cdot\Big(\frac{1+(1-t)^{1/2}}{2}\Big)^{5/4} \\ &=\; \frac{3}{2}\,t\,-\frac{3}{16}\,t^2\,-\frac{3}{128}\,t^3\,+\frac{225}{32{,}768}\,t^5\,+\frac{2451}{262{,}144}\,t^6\;+\cdots \end{aligned} \tag{45}$$

The α-extensions (22), (23) for $f_3(\alpha)$ and $f_4(\alpha)$ also become algebraic functions:

$$\begin{aligned} f_3\Big(\frac{1}{2}\Big) \;&=\; -\frac{3}{8}\,t^2\,-\frac{3}{32}\,t^3\,-\frac{45}{1024}\,t^4\,-\frac{105}{4096}\,t^5\,-\frac{4395}{262{,}144}\,t^6\;+\cdots \\ &=\; -\frac{3}{8}\cdot t^2\cdot(1-t)^{1/16}\cdot\Big(\frac{1+(1-t)^{1/2}}{2}\Big)^{-3/4}\cdot\Big(\frac{(1+t^{1/2})^{1/2}\;-(1-t^{1/2})^{1/2}}{t^{1/2}}\Big) \\ &=\; -\frac{3}{8}\cdot t^2\cdot(1-t)^{1/16}\cdot\Big(\frac{1+(1-t)^{1/2}}{2}\Big)^{-3/4}\cdot\Big(2\cdot\frac{(1-(1-t)^{1/2})}{t}\Big)^{1/2}, \end{aligned} \tag{46}$$

$$
\begin{aligned}
f_4\Big(\frac{1}{2}\Big) \;&=\; -\frac{3}{8}\,t^2 \,-\, \frac{3}{16}\,t^3 \,-\, \frac{129}{1024}\,t^4 \,-\, \frac{195}{2048}\,t^5 \,-\, \frac{20,115}{262,144}\,t^6 \,+\, \cdots \\
&=\; -\frac{3}{8}\cdot t^2\cdot (1-t)^{1/16}\cdot (1-t)^{-1/4}\cdot \Big(\frac{1+(1-t)^{1/2}}{2}\Big)^{-3/4} \\
&\qquad \times \Big(\frac{(1+t^{1/2})^{1/2}\,-\,(1-t^{1/2})^{1/2}}{t^{1/2}}\Big) \\
&=\; -\frac{3}{8}\cdot t^2\cdot (1-t)^{1/16}\cdot (1-t)^{-1/4}\cdot \Big(\frac{1+(1-t)^{1/2}}{2}\Big)^{-3/4} \\
&\qquad \times \Big(2\cdot\,\frac{(1-(1-t)^{1/2})}{t}\Big)^{1/2}.
\end{aligned}
\tag{47}
$$

One verifies easily that

$$
f_1\Big(\frac{1}{2}\Big)\cdot f_3\Big(\frac{1}{2}\Big) \;=\; (1-t)^{1/2}\cdot f_2\Big(\frac{1}{2}\Big)\cdot f_4\Big(\frac{1}{2}\Big) \;=\; -\frac{9}{16}\cdot t^3\cdot (1-t)^{1/8}, \tag{48}
$$

$$
f_1\Big(\frac{1}{2}\Big)\cdot f_4\Big(\frac{1}{2}\Big) \;=\; f_2\Big(\frac{1}{2}\Big)\cdot f_3\Big(\frac{1}{2}\Big) \;=\; -\frac{9}{16}\cdot t^3\cdot (1-t)^{-1/8}, \tag{49}
$$

in agreement with the identities (25) and (26).

Do note that $f_1(\alpha)$ and $(1-t)^{1/4}\cdot f_2(\alpha)$, but also $t^{1/4}\cdot f_3(\alpha)$ and also $t^{1/4}\cdot(1-t)^{1/4}\cdot f_4(\alpha)$, *verify the same Okamoto sigma-form of Painlevé VI* (independently of α). Note that the previous algebraic function solutions (44) and (45) are actually such that $f_1(\frac{1}{2})$ and $(1-t)^{1/4}\cdot f_2(\frac{1}{2})$ are not only solutions of the same non-linear ODE but are actually the *same* algebraic function $f_1(\frac{1}{2}) = (1-t)^{1/4}\cdot f_2(\frac{1}{2})$. Similarly (46) and (47) are actually such that $f_3(\frac{1}{2})$ and $(1-t)^{1/4}\cdot f_4(\frac{1}{2})$ are not only solutions of the same non-linear ODE but are actually the *same* algebraic function $f_3(\frac{1}{2}) = (1-t)^{1/4}\cdot f_4(\frac{1}{2})$. For $\alpha = 1/2$ the corresponding λ deduced from (39) is $\lambda = 0$ and the four product formula (38) becomes, with the previous exact algebraic expressions (44)–(47) (and after simplifications):

$$
\begin{aligned}
C(0,5;\,0) \;&=\; \frac{256}{81}\cdot\frac{(1-t)^{1/2}}{t^6}\cdot f_1\Big(\frac{1}{2}\Big)\cdot f_2\Big(\frac{1}{2}\Big)\cdot f_3\Big(\frac{1}{2}\Big)\cdot f_4\Big(\frac{1}{2}\Big) \;=\; (1-t)^{1/4} \\
&=\; 1\,-\frac{1}{4}\,t\,-\frac{3}{32}\,t^2\,-\frac{7}{128}\,t^3\,-\frac{77}{2048}\,t^4\,-\frac{231}{8192}\,t^5\,-\frac{1463}{65536}\,t^6\,+\,\cdots
\end{aligned}
\tag{50}
$$

in agreement with the $\lambda = 0$ evaluation of the form factor expansion (35). Note that, conversely, the identity (50) can be used to find the exact expressions of the products $f_1\,f_4$ and $f_1\,f_3$ evaluated at $\alpha = 1/2$ (see (48) and (49)), when the exact expressions on the f_n's, $n = 1, 2, 3, 4$, are much more involved (see (44)–(47)).

Remark 1. All these calculations are *not* specific of $N = 5$. Similar calculations can be performed for other values of N. Since these calculations become more and more involved, they are not detailed here. Let us just give the expressions of f_1 for different (odd) values of N, in terms of the complete elliptic integrals of the first and second kind $\tilde{K}$ and $\tilde{E}$. These expressions can be compared with expressions (E.2) and (E.13) in Appendix E of [8] but with a different normalization (E.1).

For $N = 5, 7, 9$ the $f_1(N)$ solutions read, respectively:

$$
f_1(N=5) \;=\; (2\,t\,-1)\cdot\,\tilde{E}\,+(1-t)\cdot\,\tilde{K}, \tag{51}
$$

$$
\begin{aligned}
f_1(N=7) \;&=\; -(3\,t+4)\cdot\,(t-1)^2\cdot\,\tilde{K}^2\;\;+2\,(t\,-1)\cdot\,(3\,t^2-7\,t-4)\cdot\,\tilde{E}\,\tilde{K} \\
&\qquad +(11\,t^2-11\,t-4)\cdot\,\tilde{E}^2,
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
f_1(N=9) \;\; = \;\; & (8\,t^2-47\,t+12)\cdot\,(t-1)^2\cdot\,\tilde{K}^2 \\
& -2\cdot\,(t-1)\cdot\,(16\,t^3-63\,t^2+83\,t-12)\cdot\,\tilde{E}\,\tilde{K} \\
& +(32\,t^4-64\,t^3+151\,t^2-119\,t+12)\cdot\,\tilde{E}^2,
\end{aligned}
\tag{53}
$$

We can verify for $N \, = \, 5, \, 9, \, 13, \, \cdots$ that the factor $f_1(N)$ expands as

$$
f_1(N) \;\; = \;\; \lambda_N \cdot\, t^{(N-1)^2/16} \;\; + \;\cdots\,, \tag{54}
$$

when, for $N \, = \, 7, \, 11, \, 15, \, \cdots$ the factor $f_1(N)$ has the expansion:

$$
f_1(N) \;\; = \;\; \mu_N \cdot\, t^{(N+1)^2/16} \;\; + \;\cdots \tag{55}
$$

3.3. Form-Factor Deformation around the Algebraic Function $f_1(1/2)$

Introducing a form-factor β-deformation around the algebraic function (44) (β is the deformation parameter around $\alpha \, = \, 1/2$)

$$
\begin{aligned}
& f_1\Big(\frac{1}{2}\,+\,\beta\Big) \;\; = \\
& \qquad \frac{3}{2}\cdot\,t\cdot\,(1-t)^{1/16}\cdot\,\Big(\frac{1\,+\,(1-t)^{1/2}}{2}\Big)^{5/4} \;\; +\beta\cdot\,G(t) \;\; + \;\;\cdots
\end{aligned}
\tag{56}
$$

and inserting (56) in the non-linear ODE verified by (56), one obtains an *order-three linear differential operator* for the first coefficient $G(t)$.

This order-three linear differential operator has the following solution:

$$
\begin{aligned}
G(t) \;\; = \;\; & \frac{t^2}{64}\cdot\,(1-t)^{1/16}\cdot\,\Big(\frac{1\,+\,(1-t)^{1/2}}{2}\Big)^{1/4}\,\cdot\,P_{E,K} \\
= \;\; & -\frac{15}{1024}\cdot\,t^4\,-\,\frac{135}{8192}\cdot\,t^5\,-\,\frac{513}{32,768}\cdot\,t^6\,-\,\frac{7497}{524,288}\cdot\,t^7\,-\,\frac{434,295}{33,554,432}\cdot\,t^8\;+\;\cdots
\end{aligned}
\tag{57}
$$

where $P_{E,K}$ is a *polynomial in* $\tilde{E}$ *and* $\tilde{K}$:

$$
\begin{aligned}
& P_{E,K} \;\; = \\
& \quad \Big(t-4\;+12\cdot\,(1-t)^{1/2}\Big)\cdot\,{}_2F_1\Big([\frac{3}{2},\frac{3}{2}],[3],t\Big)\;\;-8\cdot\,{}_2F_1\Big([\frac{1}{2},\frac{1}{2}],[2],t\Big)\;\;= \\
& -8\cdot\,\frac{12\cdot\,(t\,-2)\cdot\,(1-t)^{1/2}\;+3\,t^2\;-8\,t\;+8}{t^2}\cdot\,\tilde{K}\;\;-32\cdot\,\frac{t-2+6\cdot\,(1-t)^{1/2}}{t^2}\cdot\,\tilde{E}.
\end{aligned}
\tag{58}
$$

As far as the log-derivative with respect to t is concerned, one obtains

$$
\begin{aligned}
& t\cdot\,(t\,-1)\cdot\,\frac{d}{dt}\ln\Big(f_1\Big(\frac{1}{2}\,+\beta\Big)\Big) \;\; = \;\; \frac{10\cdot\,(1\,-t)^{1/2}\;+27\,t\;-26}{16} \\
& \quad -\,\frac{\beta}{96}\cdot\,\Big(t\cdot\,(1-t)^{1/2}\cdot\,P_{E,K}\;+2\cdot\,t\cdot\,(1-t)\,\cdot\,(1-\,(1-t)^{1/2})\cdot\,\frac{dP_{E,K}}{dt}\Big)\;\;+\;\;\cdots
\end{aligned}
\tag{59}
$$

where the first deformation term is also a polynomial in $\tilde{E}$ and $\tilde{K}$.

4. α-Extensions of the Two Factors F_1, F_2 for $C(2, 5)$

The low-temperature correlation functions $C(M, N)$, at $\nu \, = \, -k$, with $M \, < \, N$, $M + N$ odd, M even but different from 0, factor into the product of, not four terms, but only two terms

$$
C(M,\,N) \;\; = \;\; \rho\cdot\,(1-t)^{1/2}\cdot\,t^{-(N^2-1)/4}\cdot\,F_1(M,N)\cdot\,F_2(M,N). \tag{60}
$$

For instance for $M \, = \, 2$ and $N \, = \, 5$ one has

$$
C(2,\,5) \;\; = \;\; \frac{256}{2025}\cdot\,\frac{(1\,-t)^{1/2}}{t^6}\cdot\,F_1(2,5)\cdot\,F_2(2,5), \tag{61}
$$

where

$$F_1(2,5) \;=\; 2\cdot\,(1-t)\cdot\,(2\,t+1)\cdot\,\tilde{K}^2\;+(7\,t^2-15\,t-4)\cdot\,\tilde{E}\,\tilde{K} \\ +(2\,t^2+13\,t+2)\cdot\,\tilde{E}^2, \tag{62}$$

and:

$$F_2(2,5) \;=\; 5\cdot\,(t-1)^3\cdot\,\tilde{K}^3\;-(11\,t-17)\cdot\,(t-1)^2\cdot\,\tilde{E}\,\tilde{K}^2 \\ +(t-1)\cdot\,(2\,t^2-33\,t+19)\cdot\,\tilde{E}^2\,\tilde{K}\;\;+(7\,t^2-22\,t+7)\cdot\,\tilde{E}^3. \tag{63}$$

The λ-extension $C(2,5;\,\lambda)$ corresponds to a form-factor expansion around the algebraic solution $(1-t)^{1/4}$:

$$C(2,5;\,\lambda) \;=\; (1-t)^{1/4}\cdot\Big(1\,+\,\lambda^{2\,n}\cdot\,\sum_{n=1}^{\infty} f_{0,5}^{2\,n}\Big) \tag{64}$$

$$= \;1\;-\frac{t}{4}\;-\frac{3\,t^2}{32}\;-\frac{7\,t^3}{128}\;-\frac{77\,t^4}{2048}\;-\frac{231\,t^5}{8192}\;-\Big(\frac{1463}{65,536}\,+\,\frac{49}{1,048,576}\cdot\lambda^2\Big)\cdot t^6 \\ -\Big(\frac{4807}{262,144}\,+\,\frac{491}{4,194,304}\cdot\lambda^2\Big)\cdot t^7\;-\Big(\frac{129,789}{8,388,608}\,+\,\frac{205,491}{1,073,741,824}\cdot\lambda^2\Big)\cdot t^8\;+\;\cdots$$

The λ-extension of (61) can *also* be seen as a μ-deformation of the correlation function $C(2,5)$, given by the exact expression (61) with (62) and (63), as a polynomial expression in $\tilde{E}$ and $\tilde{K}$:

$$C(2,\,5;\,\lambda) \;=\; 1\;-\frac{t}{4}\;-\frac{3}{32}\,t^2\;-\frac{7}{128}\,t^3\;-\frac{77}{2048}\,t^4\;-\frac{231}{8192}\,t^5 \\ -\Big(\frac{23,457}{1,048,576}\,-\,\frac{49}{1,048,576}\,\mu\Big)\cdot t^6\;-\Big(\frac{7403}{4,194,304}\,-\,\frac{491}{4,194,304}\,\mu\Big)\cdot t^7 \\ -\Big(\frac{16,818,483}{1,073,741,824}\,-\,\frac{205,491}{1,073,741,824}\,\mu\Big)\cdot t^8 \\ -\Big(\frac{58,337,917}{4,294,967,296}\,-\,\frac{1,115,389}{4,294,967,296}\,\mu\Big)\cdot t^9\;\;+\;\cdots \tag{65}$$

These two series can be seen to identify if one has the following relation between λ and μ:

$$\lambda^2 \;=\; 1-\mu \qquad\qquad \text{or:} \qquad\qquad \mu \;=\; 1-\lambda^2. \tag{66}$$

The α-extension of (62) reads

$$F_1(2,5;\,\alpha) \;=\; -\frac{45}{16}\,t^3\;-\frac{135}{128}\,t^4\;-\frac{1485}{2048}\,t^5\;-\Big(\frac{4545}{8192}+\frac{315}{8192}\,\alpha\Big)\cdot t^6 \\ -\Big(\frac{58,995}{131,072}+\frac{17,955}{262,144}\,\alpha\Big)\cdot t^7\;\;-\Big(\frac{794,745}{2,097,152}+\frac{188,055}{2,097,152}\,\alpha\Big)\cdot t^8 \\ -\Big(\frac{21,971,565}{67,108,864}+\frac{876,645}{8,388,608}\,\alpha\Big)\cdot t^9\;\;+\;\cdots \tag{67}$$

and the α-extension of (63) reads:

$$F_2(2,5;\,\alpha) \;=\; -\frac{45}{16}\,t^3\;+\frac{45}{128}\,t^4\;+\frac{315}{2048}\,t^5\;+\Big(\frac{315}{4096}+\frac{315}{8192}\,\alpha\Big)\cdot t^6 \\ +\Big(\frac{11,655}{262,144}+\frac{12,915}{262,144}\,\alpha\Big)\cdot t^7\;+\Big(\frac{14,805}{524,288}+\frac{106,155}{2,097,152}\,\alpha\Big)\cdot t^8 \\ +\Big(\frac{1,285,515}{67,108,864}+\frac{408,555}{8,388,608}\,\alpha\Big)\cdot t^9\;\;+\;\cdots \tag{68}$$

One thus verifies that relation (61) can be "lambda-extended"

$$C(2,\, 5;\, \lambda) \;\; = \;\; \frac{256}{2025} \cdot \frac{(1\, - t)^{1/2}}{t^6} \cdot F_1(2, 5;\, \alpha) \cdot F_2(2, 5;\, \alpha), \tag{69}$$

provided:

$$\mu \;\; = \;\; 4 \cdot \alpha \cdot (1\, - \alpha) \qquad \text{or:} \qquad \lambda^2 \;\; = \;\; \Big(2\, \alpha\, - 1\Big)^2. \tag{70}$$

Again one verifies the remarkable identities:

$$\begin{aligned} F_2(2, 5;\, \alpha) \;\; &= \;\; (1\, - t)^{1/2} \cdot F_1(2, 5;\, 1\, - \alpha), \\ F_2(2, 5;\, 1\, - \alpha) \;\; &= \;\; (1\, - t)^{1/2} \cdot F_1(2, 5;\, \alpha). \end{aligned} \tag{71}$$

In particular one has:

$$\begin{aligned} F_1(2, 5;\, 0) \;\; = \;\; & 2 \cdot (1\, - t) \cdot (2\, t\, + 1) \cdot \tilde{K}^2 \;\; + (7\, t^2\, - 15\, t\, - 4) \cdot \tilde{E}\, \tilde{K} \\ & + (2\, t^2\, + 13\, t\, + 2) \cdot \tilde{E}^2, \end{aligned} \tag{72}$$

$$\begin{aligned} F_1(2, 5;\, 1) \;\; = \;\; & (1\, - t)^{-1/2} \cdot \Big(5 \cdot (t - 1)^3 \cdot \tilde{K}^3 \;\; - (11\, t\, - 17) \cdot (t\, - 1)^2 \cdot \tilde{E}\, \tilde{K}^2 \\ & + (t\, - 1) \cdot (2\, t^2\, - 33\, t\, + 19) \cdot \tilde{E}^2\, \tilde{K} \;\; + (7\, t^2\, - 22\, t\, + 7) \cdot \tilde{E}^3\Big), \end{aligned} \tag{73}$$

$$\begin{aligned} F_2(2, 5;\, 0) \;\; = \;\; & 5 \cdot (t - 1)^3 \cdot \tilde{K}^3 \;\; - (11\, t\, - 17) \cdot (t\, - 1)^2 \cdot \tilde{E}\, \tilde{K}^2 \\ & + (t\, - 1) \cdot (2\, t^2\, - 33\, t\, + 19) \cdot \tilde{E}^2\, \tilde{K} \;\; + (7\, t^2\, - 22\, t\, + 7) \cdot \tilde{E}^3. \end{aligned} \tag{74}$$

$$\begin{aligned} F_2(2, 5;\, 1) \;\; = \;\; & (1\, - t)^{1/2} \cdot \Big(2 \cdot (1\, - t) \cdot (2\, t\, + 1) \cdot \tilde{K}^2 \;\; + (7\, t^2\, - 15\, t\, - 4) \cdot \tilde{E}\, \tilde{K} \\ & + (2\, t^2\, + 13\, t\, + 2) \cdot \tilde{E}^2\Big). \end{aligned} \tag{75}$$

The series expansions of the previous exact expressions read:

$$\begin{aligned} F_1\Big(2, 5;\, 0\Big) \;\; &= \;\; -\frac{45}{16}\, t^3 \;\; - \frac{135}{128}\, t^4 \;\; - \frac{1485}{2048}\, t^5 \;\; - \frac{4545}{8192}\, t^6 \;\; - \frac{58{,}995}{131{,}072}\, t^7 \;\; + \;\; \cdots \\ F_1\Big(2, 5;\, 1\Big) \;\; &= \;\; -\frac{45}{16}\, t^3 \;\; - \frac{135}{128}\, t^4 \;\; - \frac{1485}{2048}\, t^5 \;\; - \frac{1215}{2048}\, t^6 \;\; - \frac{135{,}945}{262{,}144}\, t^7 \;\; + \;\; \cdots \end{aligned} \tag{76}$$

$$\begin{aligned} F_2\Big(2, 5;\, 0\Big) \;\; &= \;\; -\frac{45}{16}\, t^3 \;\; + \frac{45}{128}\, t^4 \;\; + \frac{315}{2048}\, t^5 \;\; + \frac{315}{4096}\, t^6 \;\; + \frac{11{,}655}{262{,}144}\, t^7 \;\; + \;\; \cdots \\ F_2\Big(2, 5;\, 1\Big) \;\; &= \;\; -\frac{45}{16}\, t^3 \;\; + \frac{45}{128}\, t^4 \;\; + \frac{315}{2048}\, t^5 \;\; + \frac{945}{8192}\, t^6 \;\; + \frac{12{,}285}{131{,}072}\, t^7 \;\; + \;\; \cdots \end{aligned} \tag{77}$$

It is worth noticing that (in contrast with the λ-extension $C(2, 5;\, \lambda)$), the $F_n(2, 5;\, \alpha)$'s have *two* different values of the parameter α for which these α-extensions are D-finite, being (homogeneous) polynomials in $\tilde{E}$ and $\tilde{K}$. One remarks with that the corresponding polynomials in $\tilde{E}$ and $\tilde{K}$ *are not necessarily of the same degree* in $\tilde{E}$ and $\tilde{K}$.

Remark 2. The $\alpha \; = \; 1/2$ algebraic subcase. For $\alpha \; = \; 1/2$ the corresponding λ deduced from (70) is $\lambda \; = \; 0$ and the two product formula (69) becomes

$$C(2,\, 5;\, 0) \;\; = \;\; \frac{256}{2025} \cdot \frac{(1\, - t)^{1/2}}{t^6} \cdot F_1\Big(2, 5;\, \frac{1}{2}\Big) \cdot F_2\Big(2, 5;\, \frac{1}{2}\Big) \;\; = \;\; (1 - t)^{1/4}, \tag{78}$$

in agreement with the expansion (65) evaluated at $\lambda = 0$. Using the identity (71) one obtains

$$F_2\Big(2,5;\,\frac{1}{2}\Big) \;=\; (1-t)^{1/2}\cdot F_1\Big(2,5;\,\frac{1}{2}\Big), \tag{79}$$

which enables to write (78) as:

$$C(2,\,5;\,0) \;=\; \frac{256}{2025}\cdot\frac{1}{t^6}\cdot\Big(F_2\Big(2,5;\,\frac{1}{2}\Big)\Big)^2 \;=\; (1-t)^{1/4}, \tag{80}$$

from which one deduces

$$\begin{aligned} F_2\Big(2,5;\,\frac{1}{2}\Big) \;&=\; -\frac{45}{16}\cdot t^3\cdot(1-t)^{1/8} \\ &=\; -\frac{45}{16}\,t^3 \;+\frac{45}{128}\,t^4 \;+\frac{315}{2048}\,t^5 \;+\frac{1575}{16,384}\,t^6 \;+\frac{36,225}{524,288}\,t^7 \;+\;\cdots \end{aligned} \tag{81}$$

or:

$$\begin{aligned} F_1\Big(2,5;\,\frac{1}{2}\Big) \;&=\; -\frac{45}{16}\cdot t^3\cdot(1-t)^{-3/8} \\ &=\; -\frac{45}{16}\,t^3 \;-\frac{135}{128}\,t^4 \;-\frac{1485}{2048}\,t^5 \;-\frac{9405}{16,384}\,t^6 \;-\frac{253,935}{524,288}\,t^7 \;+\;\cdots \end{aligned} \tag{82}$$

4.1. Form Factor Deformation around the Algebraic Function $F_1(2,5;\,\frac{1}{2})$

Introducing a form-factor β-deformation around the algebraic function (82) (β is the deformation parameter around $\alpha = 1/2$)

$$F_1\Big(2,5;\,\frac{1}{2}+\beta\Big) \;=\; -\frac{45}{16}\cdot t^3\cdot(1-t)^{-3/8} \;+\beta\cdot G(t) \;+\;\cdots \tag{83}$$

and inserting (83) in the non-linear ODE verified by (83), one obtains an *order-three linear differential operator,* which is the direct sum of an order-one linear differential operator and an order-two linear differential operator, yielding the following exact expression for $G(t)$ in (83):

$$\begin{aligned} G(t) \;&=\; -\frac{45}{16}\cdot t^3\cdot(1-t)^{-3/8} \;-\frac{9}{16}\cdot(1-t)^{-3/8}\cdot P_{E,K} \\ &=\; -\frac{315}{8192}\cdot t^6 - \frac{17,955}{262,144}\cdot t^7 - \frac{188,055}{2,097,152}\cdot t^8 - \frac{876,645}{8,388,608}\cdot t^9 - \frac{1,929,015}{16,777,216}\cdot t^{10} \;+\;\cdots \end{aligned} \tag{84}$$

where $P_{E,K}$ is a polynomial in $\tilde{E}$ and $\tilde{K}$:

$$\begin{aligned} P_{E,K} \;&=\; 4\cdot t^2\cdot(t-1)\cdot(t^2-6\,t+16)\cdot\frac{d\tilde{K}}{dt} \;+t^2\cdot(2\,t^2-13\,t+16)\cdot K \\ &=\; t\cdot(t^2-28\,t+32)\cdot\tilde{K} \;-2\cdot(t^2-6\,t+16)\cdot\tilde{E}. \end{aligned} \tag{85}$$

As far as the log-derivative with respect to t is concerned, one obtains

$$\begin{aligned} t\cdot(t-1)\cdot\frac{d}{dt}\ln\Big(F_1\Big(2,5;\,\frac{1}{2}+\beta\Big)\Big) \;&=\; -3\;+\frac{21}{8}\cdot t \\ &+\beta\cdot\frac{t-1}{5\,t^3}\cdot\Big(t\cdot\frac{dP_{E,K}}{dt} - 3\cdot P_{E,K}\Big) \;+\;\cdots \end{aligned} \tag{86}$$

where the first deformation term is also polynomial in $\tilde{E}$ and $\tilde{K}$.

5. Comments and Speculations on the Lambda-Extensions of the Two-Point Correlation Functions

The previous sections provide an illustration of nice involutive symmetries of α-extension solutions of Painlevé-like non-linear ODEs (see (24)). Furthermore, recalling (31), (32), (46) and (33), (34), (47), namely

$$\begin{aligned} f_3(0) \; &= \; (t-2) \cdot \tilde{E} \; + 2 \cdot (1-t) \cdot \tilde{K}, \\ f_3(1) \; &= \; (1-t)^{-1/4} \cdot \Bigl(3\,\tilde{E}^2 \; + 2 \cdot (t-2) \cdot \tilde{E}\,\tilde{K} \; + (1-t) \cdot \tilde{K}^2 \Bigr), \\ f_3\Bigl(\frac{1}{2}\Bigr) \; &= \; -\frac{3}{8} \cdot t^2 \cdot (1-t)^{1/16} \cdot \Bigl(\frac{1+(1-t)^{1/2}}{2} \Bigr)^{-3/4} \cdot \Bigl(2 \cdot \frac{(1-(1-t)^{1/2})}{t} \Bigr)^{1/2}, \end{aligned} \tag{87}$$

and

$$\begin{aligned} f_4(0) \; &= \; 3\,\tilde{E}^2 \; + 2 \cdot (t-2) \cdot \tilde{E}\,\tilde{K} \; + (1-t) \cdot \tilde{K}^2, \\ f_4(1) \; &= \; (1-t)^{1/4} \cdot \Bigl((t-2) \cdot \tilde{E} \; + 2 \cdot (1-t) \cdot \tilde{K} \Bigr), \\ f_4\Bigl(\frac{1}{2}\Bigr) \; &= \; -\frac{3}{8} \cdot t^2 \cdot (1-t)^{1/16} \cdot (1-t)^{-1/4} \cdot \Bigl(\frac{1+(1-t)^{1/2}}{2} \Bigr)^{-3/4} \\ &\quad \times \Bigl(2 \cdot \frac{(1-(1-t)^{1/2})}{t} \Bigr)^{1/2}, \end{aligned} \tag{88}$$

we see that the α-extension $f_3(\alpha)$ (resp. $f_4(\alpha)$) has three different values of the parameter α for which the corresponding α-extensions are D-finite being (homogeneous) polynomials in $\tilde{E}$ and $\tilde{K}$ of *different degree* in $\tilde{E}$ and $\tilde{K}$. It is straightforward to see that $f_3(\alpha)$ (resp. $f_4(\alpha)$) is *not* a linear interpolation of these three D-finite expressions. For generic values of α, $f_3(\alpha)$ (resp. $f_4(\alpha)$) is *not* D-finite, it is differentially algebraic [11,12,33], being solution of a Painlevé-like non-linear ODE. In Section 4.1 of [32] we provide, not a proof, but arguments strongly suggesting that such lambda-extensions are not generically D-finite. Let us now display several remarkable properties of such lambda-extensions.

5.1. Other Remarkable Features of the Lambda-Extensions of the Two-Point Correlation Functions

In fact $\alpha = 1/2$ is not the only value of α for which $f_3(\alpha)$ (resp. $f_4(\alpha)$) becomes an algebraic function. One has an *infinite number* of (algebraic) values of α for which $f_3(\alpha)$ (resp. $f_4(\alpha)$) becomes an algebraic function. This phenomenon is illustrated in detail in [32] in the case of the lambda-extension of the diagonal correlation function $C(1, 1)$, but one has similar results for other non-diagonal two-point correlation functions (at $\nu = -k$), or for factors of the correlation functions such as the $f_i(\alpha)$'s. Recall that diagonal correlation functions depend only on $k = \sqrt{t}$. They are independent of the anisotropic parameter ν.

For pedagogical reasons we restrict our analysis to the low-temperature two-point correlation function $C(1,1)$ and its lambda extension. For instance, the form factor expansion of the lambda extension of this low-temperature correlation function reads

$$C_-(1, 1; \lambda) \; = \; (1-t)^{1/4} \cdot \Bigl(1 + \sum_{n=1}^{\infty} \lambda^{2n} \cdot f_{1,1}^{(2n)} \Bigr), \tag{89}$$

where the first form factors read:

$$f_{1,1}^{(2)} \; = \; \frac{1}{2} \cdot \Bigl(1 \; - 3\,E\,K \; - (t-2) \cdot K^2 \Bigr), \tag{90}$$

$$\begin{aligned} f_{1,1}^{(4)} \; &= \; \frac{1}{24} \cdot \Bigl(9 \; - 30\,\tilde{E}\,\tilde{K} \; - 10 \cdot (t-2) \cdot \tilde{K}^2 \\ &\quad + (t^2 - 6t + 6) \cdot \tilde{K}^4 \; + 15\,\tilde{E}^2\,\tilde{K}^2 \; + 10 \cdot (t-2) \cdot \tilde{E}\,\tilde{K}^3 \Bigr). \end{aligned} \tag{91}$$

For $\lambda = 1$ we must recover, from (89), the well-known expression of the *low-temperature* two-point correlation function $C(1,1) = \tilde{E}$:

$$C_-(1,\,1;\,1) \;=\; E \;=\; 1\,-\frac{1}{4}\cdot t\;-\frac{3}{64}\cdot t^2\;-\frac{5}{256}\cdot t^3\;-\frac{175}{16384}\cdot t^4\;+\;\cdots$$
$$= \;(1-t)^{1/4}\cdot\Big(1\;+\;\sum_{n=1}^{\infty} f_{1,1}^{(2n)}\Big), \tag{92}$$

which corresponds to write the ratio $\tilde{E}/(1-t)^{1/4}$ as an infinite sum of polynomial expressions of $\tilde{E}$ and $\tilde{K}$, thus yielding a non-trivial *infinite sum identity* on the complete elliptic integrals $\tilde{E}$ and $\tilde{K}$.

Since all these lambda extensions are power series in t, we can also try to obtain, order by order, the series expansion of $C_-(1,\,1;\,\lambda)$ from the corresponding non-linear ODE (see (104) below). Recalling [14] the form factor expansion (89), we can either see the series expansion in t as a deformation of the simple algebraic function $(1-t)^{1/4}$, or *more naturally*, see the series expansion of the lambda-extension of the low-temperature two-point correlation function $C_-(1,\,1;\,\lambda)$ as a deformation of the exact expression $C_-(1,1) = \tilde{E}$ (here M denotes a difference to $\lambda^2 = 1$, namely $M = 4\cdot(1-\lambda^2)$):

$$C_-(1,\,1;\,\lambda) \;=\; C_M(1,\,1;\,M)$$
$$= \;\tilde{E}\;\;+M\cdot g_1(t)\;\;+M^2\cdot g_2(t)\;\;+M^3\cdot g_3(t)\;\;+\;\cdots \tag{93}$$

All the $g_n(t)$'s in (93) are also [32] polynomials in $\tilde{E}$ and $\tilde{K}$ (this cannot be deduced straightforwardly from an identification of two representations (95) and (96) of the lambda extension $C_-(1,\,1;\,\lambda)$. This identification yields an infinite number of (infinite sum) non-trivial identities on $\tilde{E}$ and $\tilde{K}$). For instance $g_1(t)$ in (95) reads

$$g_1(t) \;=\; \frac{1}{24}\cdot\tilde{E}\;\;-\frac{1}{8}\cdot\tilde{K}\,\tilde{E}^2\;\;-\frac{t-1}{12}\cdot\tilde{K}^3. \tag{94}$$

Using the sigma-form of Painlevé VI Equation (104) one can find that this expansion (93) reads as a series expansion in the variable t:

$$C_M(1,\,1;\,M) \;=\; 1\;\;-\frac{1}{4}\cdot t\;\;-\Big(\frac{3}{64}+\frac{3}{256}\cdot M\Big)\cdot t^2\;\;-\Big(\frac{5}{256}+\frac{9}{1024}\cdot M\Big)\cdot t^3$$
$$-\Big(\frac{175}{16,384}+\frac{441}{65,536}\cdot M\Big)\cdot t^4\;\;-\Big(\frac{441}{65,536}+\frac{1407}{262,144}\cdot M\Big)\cdot t^5$$
$$-\Big(\frac{4851}{1,048,576}+\frac{9281}{2,097,152}\cdot M-\frac{5}{16,777,216}\cdot M^2\Big)\cdot t^6\;\;+\;\cdots \tag{95}$$

Deformation around an Algebraic Subcase

Recalling that one finds [32] that (95) is actually, for $M = 2$, the series expansion of an *algebraic function* (see (97) below), one can try to write the series (95) as a deformation of this $M = 2$ algebraic function (97)

$$C_\rho(1,1;\rho) \;=\; G_0(t)\;\;+\rho\cdot G_1(t)\;\;+\rho^2\cdot G_2(t)\;\;+\;\cdots \tag{96}$$

where

$$G_0(t) \;=\; (1-t)^{1/16}\cdot\Big(\frac{1\,+(1-t)^{1/2}}{2}\Big)^{3/4}, \tag{97}$$

and where $\rho = M - 2$. Again one can ask whether the $G_n(t)$'s in (96) are D-finite, and, again, polynomials in the complete elliptic integrals $\tilde{E}$ and $\tilde{K}$. This is actually the case. One finds that (96) can be written as

$$\frac{C_\rho(1,1;\rho)}{G_0(t)} \; = \; 1 \; + \rho \cdot \Big(\frac{1}{4} \cdot S_2 \, - \frac{1}{4}\Big) \; + \rho^2 \cdot \Big(\frac{1}{32} \cdot S_3 \, - \frac{1}{16} \cdot S_2 \, + \frac{3}{32}\Big)$$
$$+\rho^3 \cdot \Big(\frac{1}{384} \cdot S_4 \, - \frac{1}{128} \cdot S_3 \, + \frac{13}{384} \cdot S_2 \, - \frac{5}{128}\Big) \; + \; \cdots \tag{98}$$

where

$$S_2 \; = \; \frac{2}{t} \cdot \Big(1 \, - (1-t)^{1/2}\Big) \cdot \tilde{E} \; - \frac{1}{2\,t} \cdot \Big((t-4) \cdot (1-t)^{1/2} \, - (3\,t-4)\Big) \cdot \tilde{K},$$
$$S_3 \; = \; \frac{1}{4} \cdot \Big(6 \cdot (1-t)^{1/2} \, - (t-2)\Big) \cdot \tilde{K}^2 \; - 3\,\tilde{E}\,\tilde{K},$$

$$S_4 \; = \; \frac{3}{t} \cdot \Big((t-4) \cdot (1-t)^{1/2} \, - (3\,t-4)\Big) \cdot \tilde{E}\,\tilde{K}^2 \; - \frac{6}{t} \cdot (1 \, - (1-t)^{1/2}) \cdot \tilde{E}^2\,\tilde{K}$$
$$+ \frac{1}{8\,t} \cdot \Big((t^2 - 28\,t + 48) \cdot (1-t)^{1/2} \, - (21\,t^2 - 68\,t + 48)\Big) \cdot \tilde{K}^3,$$

We thus see the same phenomenon as the one sketched in Section 3.3 for the α-extension $f_1(\alpha)$ and Section 4.1 for the α-extension $F_1(2,5;\,\alpha)$, seen as deformations of algebraic function subcases.

Remark 3. All these $g_n(t)$'s or $G_n(t)$'s are *globally bounded series* [34] (a series with rational coefficients and non-zero radius of convergence is a globally bounded series [34] if it can be recast into a series with integer coefficients with one rescaling $t \rightarrow N\,t$ where N is an integer). This is a consequence of the fact that they are polynomial expressions in $\tilde{E}$ and $\tilde{K}$: they are not only D-finite, they can actually be seen to be *diagonals of rational functions* [34]. We have actually seen, so many times in physics, and in particular in the two-dimensional Ising model, the emergence of globally bounded series as a consequence of the frequent occurrence of *diagonals of rational functions* [34,35] (or n-fold integrals [36]). In contrast the lambda extension $C_{-}(1,\,1;\,\lambda)$ which is an infinite sum of globally bounded series is, at first sight, a differentially algebraic function that has no reason to correspond to a globally bounded series.

5.2. Arithmetic Properties of the Lambda-Extensions and Globally Bounded Series

Let us consider the series expansion (95) for values of the parameter $M \neq 0$ not yielding the previous algebraic function series (i.e., $M \neq 4 \cdot \sin^2(\pi m/n)$ where m and n are integers).

Let us change t into $16\,t$ in the series expansion (95). One obtains the following expansion:

$$1 \; - 4\,t \; - (12 + 3\,M) \cdot t^2 \; - (80 + 36\,M) \cdot t^3 \; - (700 + 441\,M) \cdot t^4$$
$$-(7056 + 5628\,M) \cdot t^5 \; - (77{,}616 + 74{,}248\,M - 5\,M^2) \cdot t^6$$
$$-(906{,}048 + 1{,}004{,}960\,M - 220\,M^2) \cdot t^7 \; - (11{,}042{,}460 + 13{,}877{,}397\,M - 6255\,M^2) \cdot t^8$$
$$-(139{,}053{,}200 + 194{,}712{,}812\,M - 146{,}500\,M^2) \cdot t^9$$
$$-(1{,}796{,}567{,}344 + 2{,}767{,}635{,}832\,M - 3{,}079{,}025\,M^2) \cdot t^{10} \; + \; \cdots \tag{99}$$

For integer values of M one sees, very clearly, that the series (99) becomes a *differentially algebraic series with integer coefficients*. They are solutions of a non-linear ODE, the sigma-form of Painlevé VI. One thus has a first example of an *infinite number of differentially algebraic series with integer coefficients*. As far as integer values of M are concerned we have seen [32] that the lambda extension $C_{-}(1,\,1;\,\lambda)$ is a simple algebraic function for $M = 2, 4$ and slightly more involved algebraic functions for $M = 1, 3$, and corresponds to $\tilde{E}$ for $M = 0$.

These series (99) are, at first sight, *differentially algebraic* [11]: is it possible that such series could become D-finite for selected values integer of M different from $M = 0, 1, 2, 3, 4$?

In Section 4.1 of [32] we give some strong argument to discard, at least for $M = 5$, the possibility that the series expansion (95) (or the series expansion (99)) could be D-finite. It is differentially algebraic.

More generally, one can see that the series expansion (95) (or the series expansion (99)) is a *globally bounded series* when M is *any rational number*. One thus generalizes the quite puzzling result that an *infinite number of* (at first sight, etc.) *differentially algebraic series* can be *globally bounded series*.

Remark 4. Quite often we see the emergence of *globally bounded series* [34] as solutions of D-finite linear differential operators, and more specifically as *diagonals of rational functions* [34,35] (this is related to the so-called Christol's conjecture [37]). The emergence of *globally bounded series* that are not D-finite (not diagonals of rational functions) is more puzzling. It can be tempting to imagine that such *differentially algebraic globally bounded* situations could correspond to particular ratios of D-finite functions (let us recall that *ratios* of D-finite expressions are *not* (generically) D-finite: they are *differentially algebraic* [11]), namely ratios of diagonals of rational functions (or even rational functions of diagonals), or even composition of diagonal of rational functions. Our prejudice is that this is not the case, but discarding these simple scenarios is extremely difficult.

6. More Non-Linear ODEs of the Painlevé Type and More λ-Extensions

In [38], V.V. Mangazeev and A. J. Guttmann derived the following Toda-type recurrence relation for the λ-extension $C(N, N; \lambda)$ of the diagonal correlation functions of the square Ising model (see Equation (6) in [38]):

$$t \cdot \frac{d^2}{dt^2} \ln(C_N) + \frac{d}{dt} \ln(C_N) \; + \frac{N^2}{1-t^2} \; = \; \frac{N^2 - 1/4}{1-t^2} \cdot \frac{C_{N-1} \cdot C_{N+1}}{C_N^2}, \tag{100}$$

where C_N denotes the λ-extensions of the low (resp. high) diagonal correlation functions $C_N = C(N, N)$. Introducing the ratio

$$R_N \; = \; \frac{C_{N-1} \cdot C_{N+1}}{C_N^2} \qquad \text{or:} \qquad P_N \; = \; \frac{N^2 - 1/4}{1-t^2} \cdot \frac{C_{N-1} \cdot C_{N+1}}{C_N^2}, \tag{101}$$

one can easily deduce from (100) (together with the same relation (100) where N is changed into $N-1$ and $N+1$) other relations such as:

$$\begin{aligned} t \cdot \frac{d}{dt}\Big(t \cdot \frac{d \ln(R_N)}{dt}\Big) \; &+ \frac{2}{(1-t)^2} \\ = \; \frac{(N-1)^2 - 1/4}{1-t^2} \cdot R_{N-1} &+ \frac{(N+1)^2 - 1/4}{1-t^2} \cdot R_{N+1} \; - 2 \cdot \frac{N^2 - 1/4}{1-t^2} \cdot R_N, \end{aligned} \tag{102}$$

or:

$$\Big(t \cdot \frac{d}{dt}\Big)^2 \ln(P_N) \; + \frac{2}{1-t} \; = \; P_{N-1} + P_{N+1} - 2\, P_N, \tag{103}$$

Let us now consider, for instance, the low-temperature $T < T_c$ diagonal correlation functions. One knows that they verify the sigma-form of Painlevé VI equation

$$\begin{aligned} &\Big(t \cdot (t-1) \cdot \frac{d^2\sigma}{dt^2}\Big)^2 \\ &= \; N^2 \cdot \Big((t-1) \cdot \frac{d\sigma}{dt} - \sigma\Big)^2 - 4 \cdot \frac{d\sigma}{dt} \cdot \Big((t-1) \cdot \frac{d\sigma}{dt} - \sigma - \frac{1}{4}\Big) \cdot \Big(t \frac{d\sigma}{dt} - \sigma\Big). \end{aligned} \tag{104}$$

with

$$\sigma \; = \; t \cdot (t-1) \cdot \frac{d}{dt} \ln C(N,N) \;\; - \frac{t}{4}. \tag{105}$$

We can rewrite (100) in terms of σ given by (105)

$$\frac{d}{dt} \ln C_N \; = \; \frac{\sigma + \frac{t}{4}}{t \cdot (t-1)}. \tag{106}$$

Relation (100) becomes $\mathcal{L} \, = \, \mathcal{R}$ where

$$\mathcal{L} \; = \; t \cdot \frac{d}{dt} \Big(\frac{\sigma + \frac{t}{4}}{t \cdot (t-1)} \Big) \; + \frac{\sigma + \frac{t}{4}}{t \cdot (t-1)} \; + \frac{N^2}{1-t^2},$$
$$\mathcal{R} \; = \; \frac{N^2 - 1/4}{1-t^2} \cdot \frac{C_{N-1} \cdot C_{N+1}}{C_N^2}. \tag{107}$$

Let us introduce a new sigma corresponding to the product $C_{N-1} \cdot C_{N+1}$

$$\Sigma \; = \; t \cdot (t-1) \cdot \frac{d}{dt} \ln \Big(C_{N-1} \cdot C_{N+1} \Big). \tag{108}$$

Taking a well-suited log-derivatives the previous relation $\mathcal{L} \, = \, \mathcal{R}$ yields

$$t \cdot (t-1) \cdot \frac{d}{dt} \ln \mathcal{L} \; = \; t \cdot (t-1) \cdot \frac{d}{dt} \ln \mathcal{R}, \tag{109}$$

where the RHS of (109) can be written using (105) and (108)

$$\Sigma \;\; - 2\,\sigma \;\; - \frac{5\,t}{2}. \tag{110}$$

Relation (109) becomes

$$8 \cdot t \cdot (t-1)^2 \cdot \sigma'' \;\; + 4 \cdot (t-1) \cdot (t + 4\sigma) \cdot \sigma' \;\; - 16 \cdot \sigma^2 \; + 4 \cdot (4\,N^2 - 1 - t) \cdot \sigma$$
$$+ (4\,N^2 - 1) \cdot t \;\; - 2 \cdot \Big(4\,N^2 - 1 \, + 4 \cdot (t-1) \cdot \sigma' \, - 4\,\sigma \Big) \cdot \Sigma \;\; = \;\; 0. \tag{111}$$

We can now use the non-linear ODE (104) to perform some differential algebra eliminations to eliminate σ and its derivatives in order to obtain a non-linear ODE on Σ. One first eliminates σ'' between (104) and (111), obtaining a (non-linear) relation between σ, σ' and Σ. Performing a derivation of this relation, one obtains a relation between σ, σ', σ'', Σ and Σ'. Again, one eliminates σ'' between this last relation and (111), obtaining a relation between σ, σ', Σ and Σ'. The elimination of σ' using a previous relation gives a relation between σ, Σ and Σ'. A new derivation gives a relation between σ, Σ, Σ' and Σ''. Finally eliminating σ, one obtains a non-linear ODE between Σ, Σ' and Σ''. In other words one can obtain a second-order non-linear ODE on Σ, from the Toda-like relation (100) and the sigma-form of Painlevé VI non-linear ODE (104). This non-linear ODE is too large to be given here. It emerges from a resultant that factors in different spurious terms, a polynomial in Σ, Σ' and Σ'' of degree *six* in Σ'' and another polynomial in Σ, Σ' and Σ'' of degree *twelve* in Σ''. However, it is worth noticing that, again, this non-linear ODE has a *one-parameter* lambda-extension solution. One may conjecture that this new non-linear ODE has again the (fixed critical point) Painlevé property. This (very large) second-order non-linear ODE is *not* quadratic in the second derivative Σ'', in contrast with Okamoto sigma form of Painlevé VI equation. It is of a much higher degree (along this second order but higher degree line let us recall [39]).

The question of the reduction of this quite large non-linear ODE to some Okamoto sigma-form of Painlevé VI, or more generally to second-order non-linear ODE of the Painlevé type [40], is a (challenging) open question. The transformations required to achieve such reduction to the sigma-form of Painlevé VI will correspond to drastic generalizations of the concept of "folding transformations" [41–43]. In the simple case of the reduction of a second-order non-Okamoto non linear ODE to an Okamoto sigma form of Painlevé VI equation, Equations (26), (28) in Section 2 of [8], give some hint of the complexity of such transformations.

Another Non-Linear ODE

If one tries to obtain, more directly, a non-linear ODE on the product of the two diagonal correlation functions $C(N,N) \cdot C(N+2,N+2)$, one can also consider the sigma-form of Painlevé VI Equation (104) together with the definition of sigma (105) and the same equation and definition (104) and (105), but for $N+2$, and obtain by differential algebra eliminations a non-linear ODE on the sum

$$\begin{aligned} \Sigma \, &= \, t \cdot (t-1) \cdot \frac{d}{dt} \ln\Big(C_N \cdot C_{N+2}\Big) \\ &= \, t \cdot (t-1) \cdot \frac{d}{dt} \ln C(N,N) \, + t \cdot (t-1) \cdot \frac{d}{dt} \ln C(N+2,N+2). \end{aligned} \tag{112}$$

which is essentially the sum of the two previous sigmas (Equation (105) for N and for $N+2$). Let us recall (see page 344 in [12,33]) the results on *sums* (but also products, compositions, derivatives, integrals, inverses, etc.) of differentially algebraic functions, showing that these sums are also differentially algebraic functions, and that one also has (see Theorem 2.2 page 345 in [33]) that the order of the non-linear ODE for such sums is less or equal to the sum of the order of the two non-linear ODEs. In our case (112), one expects the order of the non-linear ODE on Σ to be less or equal to $4 = 2+2$ with a prejudice for the generic upper bound being four.

Comment: We thus have, at first sight, two non-linear ODEs on (112): a very large but second-order non-linear ODE obtained by differential algebra eliminations between (104) and (111), and another one, probably also very large but fourth order non-linear ODE. Both equations probably have the fixed critical point Painlevé property. As far as lambda extensions are concerned, we expect the first one to have one-parameter family of power-series analytic at $t = 0$, when we expect two-parameters families of power-series analytic at $t = 0$ (the two lambda parameters for $\sigma(N)$ and $\sigma(N+2)$ are, now, independent). Understanding these different non-linear ODEs occurring on products of two-point correlation functions and their corresponding lambda extensions remains a challenging work-in-progress task.

Remark 5. *Quantum XY chain correlations.* Along this line, it is worth recalling that the emergence of the product $C_{N-1} \cdot C_{N+1}$, or $C(N,N) \cdot C(N+2,N+2)$, is reminiscent of the product $C(N,N) \cdot C(N+1,N+1)$ which is actually the *xx correlation functions of the quantum XY chain in the absence of a magnetic field*. Actually, for the xx correlations of the quantum XY chain, one has (see (2.45a) and (2.45b) in the Lieb, Schultz and Mattis paper [44]) the following relations only valid in the absence of a magnetic field $H = 0$ *i.e., precisely* $\nu = -k$:

$$< \sigma_0^x \, \sigma_{2N}^x > \, = \, C(N,\, N)^2, \tag{113}$$

$$< \sigma_0^x \, \sigma_{2N-1}^x > \, = \, C(N,\, N) \cdot C(N-1,\, N-1). \tag{114}$$

Again, from the previous results, we have a strong incentive to find the non-linear ODEs for the quantum XY chain correlations (114). Note that the non-linear ODE for (113) is obviously an Okamoto sigma-form of Painlevé VI equation similar to (105).

More generally, we have a strong incentive to find non-linear ODEs of the Painlevé type for various families of two-point correlation functions such as the off-diagonal correlations $C(N,\, N+1)$ for which N.Witte showed [45] the existence of a Garnier system for such correlations, and, beyond, $C(N,\, N+2)$, $C(N,\, N+3), \ldots$ correlations. The row correlation function $C(0, N)$ is a tau-function of a Garnier system with five finite singularities, one fixed at the origin: see Corollary 1, pg.7 and Eq.(36), pg. 6 of [46], when $C(N, N+1)$ is more a component of a related isomonodromic system (at least in the description in [45]). Preliminary studies for the row correlation functions $C(0, N)$ seem to indicate that the corresponding non-linear ODEs are drastically more complicated even if N.Witte showed the existence of Garnier systems for these row correlation functions [46].

7. Conclusions

As underlined in the introduction the two-point correlation functions $C(M, N)$ of the 2D Ising model, at $\nu = -k$, can be seen as D-finite functions solutions of *linear* differential equations, but also, at the same time, as solutions of *non-linear* differential equations of the Painlevé type. Around $t = 0$, the other solutions of the linear differential equations are formal series with logarithms (see [14,23]). In contrast, other solutions of the non-linear differential equations of the Painlevé type are one-parameter families of power series analytic at $t = 0$. Such solutions are called lambda-extensions [32]. This paper has tried to provide an illustration of a set of the remarkable properties and structures of such lambda-extensions (resp. α-extensions). The study of non-linear ODEs in the most general framework may look hopeless for mathematicians; however, the square Ising model provides a perfect example of the importance of a *selected set of non-linear ODEs*, namely non-linear ODEs *of the Painlevé type* [47], and we tried to show that the analysis of some of their solutions, the lambda-extensions, is clearly a powerful way to describe these selected non-linear ODEs in a work-in-progress definition of what could be called the "symmetries" of these non-linear ODEs of the Painlevé type.

Although Painlevé equations were introduced from purely mathematical considerations their occurrence in so many domain of physics and theoretical physics is compelling. Let us quote pele mele: particle physics, solid state physics, field theory, lattice statistical mechanics, statistical physics [17], integrable PDE's and their similarity solutions, enumerative combinatorics, Random Matrix Theory [29,48], even Quantum Gravity [49]; the Ising model being, of course, the perfect play ground for these remarkable non-linear ODEs. Unfortunately the compelling evidence of the relevance of these selected non-linear ODEs in physics, is not able to balance the mainstream opinion among pure mathematicians that nothing interesting can be achieved on non-linear problems and that even the word "non-linear" is meaningless [50].

We tried in this paper to show that interesting non-trivial results can be obtained on selected non-linear ODEs.

The exact results sketched in this paper are a strong incentive to obtain more non-linear ODEs, for instance on the correlation functions of XY quantum chain in the absence of magnetic field (which corresponds to the product of two Ising two-point Ising correlation functions $C(N, N) \cdot C(N+1, N+1)$, but also on many more two point off-diagonal correlation functions of the 2D Ising such as $C(N, N+1)$, or $C(N, N+2)$, or $C(N, N+3)$.

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