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# Geometric Properties for a New Class of Analytic Functions Defined by a Certain Operator

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**Abstract:** The aim of this paper is to define and explore a certain class of analytic functions involving the $(p,q)$-Wanas operator related to the Janowski functions. We discuss geometric properties, growth and distortion bounds, necessary and sufficient conditions, the Fekete–Szegö problem, partial sums, and convex combinations for the newly defined class. We solve the Fekete–Szegö problem related to the convolution product and discuss applications to probability distribution.

**Keywords:** holomorphic function; Fekete–Szegö problem; analytic functions; upper bounds; starlike function

**MSC:** 30C45; 33C45; 11B39

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## 1. Introduction

We denote by $\mathcal{A}$ the family of holomorphic functions in $\mathbb{U} = \{t \in \mathbb{C} : |t| < 1\}$, of the form

$$\varsigma(t) = t + \sum_{n=2}^{\infty} a_n t^n. \tag{1}$$

The subfamily of $\mathcal{A}$, consisting of all functions which are also univalent in $\mathbb{U}$, is denoted by $\mathcal{S}$. Let $\varsigma_1$ and $\varsigma_2$ be functions analytic in $\mathbb{U}$. Then, we say that the function $\varsigma_1$ is subordinate to $\varsigma_2$ if there exists a Schwarz function $w(t)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(t)| < 1$ $(t \in \mathbb{U})$, such that $\varsigma_1(t) = \varsigma_2(\omega(t))$ $(t \in \mathbb{U})$. We denote this subordination by

$$\varsigma_1 \prec \varsigma_2 \quad \text{or} \quad \varsigma_1(t) \prec \varsigma_2(t) \quad (t \in \mathbb{U}).$$

In particular, if the function $\varsigma_2$ is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$\varsigma_1(0) = \varsigma_2(0) \quad \textit{and} \quad \varsigma_1(\mathbb{U}) \subset \varsigma_2(\mathbb{U}).$$

The theory of quantum calculus, sometimes called $q$-calculus is equivalent to traditional infinitesimal calculus without the notion of limits. The $q$-calculus of Euler and Jacobi found interesting applications in various areas of mathematics, physics, and engineering sciences. A detailed discussion about the extension of quantum calculus based on two parameters $(p,q)$ [1] was quoted in the theory of special functions by Sahai and Yadav [2]. A further generalization of $q$-calculus is postquantum calculus, and is denoted $(p,q)$-calculus; for more details, see a survey-cum-expository review paper that was recently published by Srivastava [3] and references cited therein. The $(p,q)$-integer was introduced in order to generalize or unify several forms of $q$-oscillator algebras, well known in earlier physics literature related to the representation theory of single parameter quantum algebras [4]. The application of *q-calculus* was initiated by Jackson [5] (also see [6,7]). Kanas and Răducanu [8] have used the fractional *q-calculus operators* to investigate certain classes of functions which are analytic in $\mathbb{U}$.

We briefly recall here the notion of *q-operators* i.e., *q-difference operators*, which play vital role in the theory of hypergeometric series, quantum physics and in the operator theory.

If $\varsigma$ is a holomorphic function, the $(p,q)$-derivative operator is defined by

$$D_{p,q}\varsigma(t) = \frac{\varsigma(pt) - \varsigma(qt)}{(p-q)t} \qquad (t \in \mathbb{U}^* = \mathbb{U} \setminus \{0\}), 0 < q < p <= 1,$$

and

$$D_{p,q}\varsigma(0) = \varsigma'(0).$$

For $\varsigma \in \mathcal{A}$,

$$D_{p,q}\varsigma(t) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n t^{n-1},$$

where

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} \quad (p \neq q),$$

(see for more details [9,10])

$$\lim_{p \to 1^-} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q}.$$

The theory of operators plays a vital role in the development of geometric function theory. Several new operators have been studied systematically from different aspects. A number of integral and differential operators can be described in term of convolution. These operators are helpful to understand the mathematical exploration and geometric configuration of analytic functions. These classes of starlike and convex functions are related to each other by the Alexander relation [11]. Later, Libera introduced an integral operator and showed that these two classes are closed under this operator. Bernardi gave a generalized operator and studied its properties. Ruscheweyh [12], Sălăgean [13], Noor [14], and others [15–28], defined new operators and studied various classes of analytic and univalent functions by generalizing a number of previously known classes and at times discovering new classes of analytic functions that play an important role in the advancement of geometric function theory. Motivated by the aforementioned works, Wanas and Cotîrlă [29] introduced a generalization of some known operators studied in the literature as below:

$$\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q} : \mathcal{A} \longrightarrow \mathcal{A}$$

given by

$$\begin{aligned} \mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}\varsigma(t) &= t + \sum_{n=2}^{\infty} \left( \frac{[\Psi_n(\sigma,\vartheta,\nu)]_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]_{p,q}} \right)^{\delta} a_n t^n, \\ &= t + \sum_{n=2}^{\infty} \frac{[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}} a_n t^n, \end{aligned}$$

where

$$\Psi_n(\sigma,\vartheta,\nu) = \sum_{\tau=1}^{\sigma} \binom{\sigma}{\tau} (-1)^{\tau+1} (\vartheta^{\tau} + n\nu^{\tau}).$$

In particular,

$$\Psi_1(\sigma,\vartheta,\nu) = \sum_{\tau=1}^{\sigma} \binom{\sigma}{\tau} (-1)^{\tau+1} (\vartheta^{\tau} + \nu^{\tau}) \tag{2}$$

$$\Psi_2(\sigma,\vartheta,\nu)=\sum_{\tau=1}^{\sigma}\binom{\sigma}{\tau}(-1)^{\tau+1}(\vartheta^{\tau}+2\nu^{\tau}), \tag{3}$$

where

$$n-1,\sigma\in\mathbb{N},\vartheta\in\mathbb{R},\nu\in\mathbb{R}_0^+ \text{ with } \vartheta+\nu>0,\delta\in\mathbb{N}_0,0<q<p<=1,t\in\mathbb{U}.$$

We recall the following. The class of all the functions $p$ which are holomorphic in $\mathbb{U}$ with the condition

$$\Re(p(t))>0$$

and has the series illustration

$$p(t)=1+\sum_{n=1}^{\infty}c_nt^n, \quad (t\in\mathbb{U}), \tag{4}$$

is denoted by $\mathcal{P}$.

By using the concept of subordination for holomorphic functions (see [30]), Ma and Minda [31] introduced the classes

$$\mathcal{S}^*(\psi)=\left\{\varsigma\in\mathcal{A}:\frac{t\varsigma'(t)}{\varsigma(t)}\prec\psi(t)\right\} \quad\text{and}\quad \mathcal{C}(\psi)=\left\{f\in\mathcal{A}:\frac{(t\varsigma'(t))'}{\varsigma'(t)}\prec\psi(t)\right\},$$

where $\psi\in\mathcal{P}$ with $\psi'(0)>0$ maps $\mathbb{U}$ onto a starlike region with respect to 1 and symmetric with respect to real axis. By choosing $\psi$ to map the unit disc onto some specific regions like parabolas, cardioid, lemniscate of Bernoulli, and the booth lemniscate in the right-half of the complex plane, various interesting subclasses of starlike and convex functions can be obtained. For $-1\le B<A\le 1$, we denote by $\mathcal{S}^*[A,B]$ and by $\mathcal{C}[A,B]$ the class of Janowski starlike functions and Janowski convex functions [32], defined by

$$\mathcal{S}^*[A,B]:=\left\{\varsigma\in\mathcal{A}:\frac{t\varsigma'(t)}{\varsigma(t)}\prec\frac{1+At}{1+Bt},-1\le B<A\le 1\right\}$$

and

$$\mathcal{C}[A,B]:=\left\{f\in\mathcal{A}:\frac{(t\varsigma'(t))'}{\varsigma'(t)}\prec\frac{1+At}{1+Bt},-1\le B<A\le 1\right\},$$

respectively. The class $\mathcal{P}[A,B]$ contains the functions $h$ where $h(0)=1$ if and only if

$$h(t)\prec\frac{1+At}{1+Bt}, \quad -1\le B<A\le 1.$$

Geometrically, $p(t)\in\mathcal{P}[A,B]$ if and only if $p(0)=1$ and $p(\mathbb{U})$ lies inside an open disc centered with center $\frac{1-AB}{1-B^2}$ on the real axis having radius $\frac{A-B}{1-B^2}$ with diameter and points $p_1(-1)=\frac{1-A}{1-B}$ and $p_1(1)=\frac{1+A}{1+B}$. Furthermore, Janowski proved that for a function $p\in\mathcal{P}$, a function $h(t)$ belongs to $\mathcal{P}[A,B]$ if the following relation holds:

$$h(t)=\frac{(A+1)p(t)-(A-1)}{(B+1)p(t)-(B-1)}.$$

Moreover, the class $\mathcal{S}^*[A,B]$ contains the function $\varsigma$ given by the relation (1) if

$$\frac{t\varsigma'(t)}{\varsigma(t)}=\frac{(A+1)p(t)-(A-1)}{(B+1)p(t)-(B-1)},-1\le B<A\le 1. \tag{5}$$

The conic domain $\Omega_k$ $(k \geqq 0)$, is defined in [8,33–36] as follows:

$$\Omega_k = \left\{ \vartheta + i\varphi : \vartheta > k\sqrt{(\vartheta - 1)^2 + \varphi^2} \right\}. \tag{6}$$

For fixed $k$, $\Omega_k$ represents the conic region bounded successively by the imaginary axis ($k = 0$). For $k = 1$, we have a parabola and, for $0 < k < 1$, we have the right-hand branch of hyperbola and for $k > 1$, it represents an ellipse.

The following functions play the role of extremal functions, for these conic regions,

$$p_k(t) = \begin{cases} \mathsf{P}_1(k,t) & (k = 0) \\ \mathsf{P}_2(k,t) & (k = 1) \\ \mathsf{P}_3(k,t) & (0 \leqq k < 1) \\ \mathsf{P}_4(k,t) & (k > 1), \end{cases} \tag{7}$$

where

$$\mathsf{P}_1(k,t) = \frac{1+t}{1-t} = 1 + 2t + 2t^2 + \cdots,$$

$$\mathsf{P}_2(k,t) = 1 + \tfrac{2}{\pi^2}\left(\log \frac{1+\sqrt{t}}{1-\sqrt{t}}\right)^2,$$

$$\mathsf{P}_3(k,t) = 1 + \frac{2}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\operatorname{arctanh}\left(\sqrt{t}\right)\right\}$$

and

$$\mathsf{P}_4(k,t) = 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{e(t)}{\sqrt{\kappa}}} \frac{db}{\sqrt{1-b^2}\sqrt{1-\kappa^2 b^2}}\right) + \frac{1}{k^2-1}$$

$$e(t) = \frac{t - \sqrt{\kappa}}{1 - \sqrt{\kappa}t} \qquad (\forall\, t \in \mathbb{U})$$

and $\kappa \in (0,1)$ is chosen such that $\lambda = \cosh\left(\pi \mathsf{K}'(\kappa)/(4\mathsf{K}(\kappa))\right)$. Here $\mathsf{K}(\kappa)$ is Legendre's complete elliptic integral of first kind and $\mathsf{K}'(\kappa) = \mathsf{K}(\sqrt{1-\kappa^2})$, that is $\mathsf{K}'(\kappa)$ is the complementary integral of $\mathsf{K}(\kappa)$. Assume that

$$p_k(t) = 1 + \wp_1 t + \wp_2 t^2 + \ldots \qquad (\forall\, t \in \mathbb{U}).$$

In [36] it has been shown that, for (7) , one can have

$$\wp_1 = \begin{cases} \dfrac{2N^2}{1-k^2}, & (0 \leqq k < 1) \\ \dfrac{8}{\pi^2}, & (k = 1) \\ \dfrac{\pi^2}{4(\kappa)^2 k^2 (\kappa+1)\sqrt{\kappa}}, & (k > 1) \end{cases} \tag{8}$$

and

$$\wp_2 = \wp_1 D(k), \tag{9}$$

and

$$D(k)=\begin{cases}\dfrac{N^2+2}{3}, & (0\leqq k<1)\\[2ex] \dfrac{2}{3}, & (k=1)\\[2ex] \dfrac{[4K(\kappa)]^2(\kappa^2+6\kappa+1)-\pi^2}{24[K(\kappa)]^2(1+\kappa)\sqrt{\kappa}}, & (k>1)\end{cases} \tag{10}$$

with

$$N=\frac{2}{\pi}\arccos k.$$

**Definition 1.** *The class $k-\mathcal{ST}$ contains the functions $\varsigma$ given by* (1) *if and only if*

$$\frac{t\varsigma'(t)}{\varsigma(t)}\prec p_k(t),\, k\geq 0.$$

**Definition 2** ([37])**.** *The class $k$-$\mathcal{P}[A,B]$ contains the functions $h\in\mathcal{P}$ if and only if*

$$h(t)\prec\frac{(A+1)p_k(t)-(A-1)}{(B+1)p_k(t)-(B-1)},\, k\geq 0.\ -1\leq B<A\leq 1. \tag{11}$$

Geometrically, $h(t)\in k$-$\mathcal{P}[A,B]$ take all values in $\Delta_k[A,B]$, where:

$$\Delta_k[A,B]=\left\{\varpi:\Re\left(\frac{(B-1)\varpi-(A-1)}{(B+1)\varpi-(A+1)}\right)>k\left|\frac{(B-1)\varpi-(A-1)}{(B+1)\varpi-(A+1)}-1\right|\right\},$$

the domain $\Delta_k[A,B]$ represents conic type regions, introduced in [37] and is further generalized by the many authors see for example [38] and the references cited therein.

**Definition 3** ([37])**.** *The class $k$-$\mathcal{S}^*[A,B]$ contains the functions $\varsigma\in\mathcal{A}$ if and only if*

$$\frac{t\varsigma'(t)}{f(z)}\prec\frac{(A+1)p_k(t)-(A-1)}{(B+1)p_k(t)-(B-1)}. \tag{12}$$

Recently, Srivastava [3] gave the mathematical description and implementations of the fractional q-derivative operators and fractional q-calculus in geometric function theory were methodically explored [39–41]. Motivated by the aforementioned works [37] and some classes of q-starlike functions related to the conic region ([38] and references cited therein), in this article, we define a new subclass of Janowski-type starlike functions involving the conic domains by means of the integral operator introduced in [29] as given in Definition 4, and determine sufficient conditions, growth and distortion bounds, convex combination, results on partial sums, and Fekete–Szegö inequality [42].

**Definition 4.** *The class $k$-$\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]$ contains the functions $\varsigma\in\mathcal{A}$ for $-1\leq B<A\leq 1$, if*

$$\Re\left(\frac{(B-1)\Lambda\varsigma(t)-(A-1)}{(B+1)\Lambda\varsigma(t)-(A+1)}\right)\geq k\left|\frac{(B-1)\Lambda\varsigma(t)-(A-1)}{(B+1)\Lambda\varsigma(t)-(A+1)}-1\right|,$$

*where*

$$\Lambda\varsigma(t)=\frac{t(\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}\varsigma(t))'}{\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}\varsigma(t)}.$$

**Lemma 1** ([31]). *If $p \in \mathcal{P}$ has the series expansion given by* (4)*, then*

$$\left|a_3 - \gamma a_2^2\right| \leq 2\max\{1, |2\gamma - 1|\}, \text{ where } \gamma \in \mathbb{C}.$$

## 2. Main Results

**Theorem 1.** *Let $\varsigma$ of the form* (1) *in the class $k$-$\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]$. Then*

$$\sum_{n=2}^{\infty}[2(k+1)(n-1) + |n(1+B) - (1+A)|]\frac{[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}|a_n| < A - B, \tag{13}$$

*where $-1 \leq B < A \leq 1$. For the function of the form* (16) *the result is sharp.*

**Proof.** We suppose that the relation (13) holds true. It is enough to show that

$$k\left|\frac{(B-1)\Lambda\varsigma(t) - (A-1)}{(B+1)\Lambda\varsigma(t) - (A+1)} - 1\right| - \Re\left(\frac{(B-1)\Lambda\varsigma(t) - (A-1)}{(B+1)\Lambda\varsigma(t) - (A+1)} - 1\right) < 1. \tag{14}$$

For this, consider

$$\begin{aligned}
&k\left|\frac{(B-1)\Lambda\varsigma(t) - (A-1)}{(B+1)\Lambda\varsigma(t) - (A+1)} - 1\right| - \Re\left(\frac{(B-1)\Lambda\varsigma(t) - (A-1)}{(B+1)\Lambda\varsigma(t) - (A+1)} - 1\right)\\
&= (k+1)\left|\frac{(B-1)t\left(\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}f(z)\right)' - (A-1)\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}\varsigma(t)}{(B+1)t\left(\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}\varsigma(t)\right)' - (A+1)\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}\varsigma(t)} - 1\right|\\
&= 2(k+1)\left|\frac{\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}\varsigma(t) - t\left(\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}\varsigma(t)\right)'}{(B+1)t\left(\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}f(z)\right)' - (A+1)\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}\varsigma(t)}\right|\\
&= 2(k+1)\left|\frac{\sum_{n=2}^{\infty}(1-n)\frac{[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}a_n t^n}{(B-A)t + \sum_{n=2}^{\infty}[n(1+B) - (1+A)]\frac{[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}a_n t^n}\right|\\
&\leq \frac{2(k+1)\sum_{n=2}^{\infty}(n-1)[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}|a_n|}{(A-B)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q} - \sum_{n=2}^{\infty}|n(1+B) - (1+A)|[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}|a_n|}.
\end{aligned}$$

By using (13), we observe that the above inequality is bounded by 1, and the proof is completed. □

**Example 1.** *For*

$$\varsigma(t) = t + \sum_{n=2}^{\infty}\frac{(A-B)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[2(k+1)(n-1) + |n(1+B) - (1+A)|][\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}x_n t^n \quad (t \in \mathbb{U}),$$

*such that*

$$\sum_{n=2}^{\infty}|x_n| = 1,$$

*we obtain*

$$\begin{aligned}
&\sum_{n=2}^{\infty}[2(n-1)(1+k)+|(1+B)n-(A+1)|]\frac{[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}|a_n| \\
&=\sum_{n=2}^{\infty}[2(n-1)(k+1)+|(B+1)n-(1+A)|]\frac{[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}} \\
&\cdot\frac{(A-B)}{[2(k+1)(1-n)+|n(1+B)-(1+A)|]\frac{[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}}|x_n| \\
&=(A-B)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}\sum_{n=2}^{\infty}|x_n|=(A-B)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}.
\end{aligned}$$

*Hence,* $\varsigma\in k\text{-}\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]$ *and we can see that the result is sharp.*

**Corollary 1.** *If* $\varsigma$ *given by* (1) *be in the class* $k\text{-}\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]$*, then*

$$|a_n|\leqq\frac{(A-B)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[2(n-1)(1+k)+|(B+1)n-(A+1)|][\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}},\tag{15}$$

*and the result is sharp for* $\varsigma_t(t)$ *given by the relation*

$$\varsigma(t)=t+\frac{(A-B)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{\Big[2(k+1)(n-1)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}+|n(1+B)-(1+A)|\Big][\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}t^n.\tag{16}$$

**Theorem 2.** *The class* $k\text{-}\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]$ *is closed under convex combination.*

**Proof.** Let $\varsigma_k(t)\in k\text{-}\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]$, such that

$$\varsigma_k(t)=t+\textstyle\sum_{n=2}^{\infty}a_{n,k}t^n,\qquad k\in\{1,2\}.$$

It is enough to show that

$$rf_1(t)+(1-r)f_2(t)\in k\text{-}\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]\qquad(r\in[0,1]).$$

As

$$rf_1(t)+(1-r)f_2(t)=t+\textstyle\sum_{n=2}^{\infty}[ra_{n,1}+(1-r)a_{n,2}]z^n.$$

Now from Theorem 1, we have

$$
\begin{aligned}
&\sum_{n=2}^{\infty}[2(k+1)(n-1)+|n(1+B)-(1+A)|] \\
&\qquad \cdot \frac{[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}|ra_{n,1}+(1-r)a_{n,2}| \\
&\leq \sum_{n=2}^{\infty}[2(n-1)(1+k)+|(1+B)n-(A+1)|] \\
&\qquad \cdot \frac{[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}[r|a_{n,1}|-(r-1)|a_{n,2}|] \\
&\leq r\sum_{n=2}^{\infty}[2(n-1)(1+k)+|(1+B)n-(A+1)|] \\
&\qquad \cdot \frac{[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}|a_{n,1}|+(1-r) \\
&\cdot \sum_{n=2}^{\infty}[2(k+1)(n-1)+|n(1+B)-(1+A)|]\frac{[\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}|a_{n,2}| \\
&< r(A-B)+(1-r)(A-B)=A-B, \qquad (A>B).
\end{aligned}
$$

Hence,

$$r\varsigma_1(t)+(1-r)f_2(t)\in k\text{-}\mathfrak{S}^{*}_{\vartheta}\,(a,c,A,B).$$

□

**Theorem 3.** *Let $\varsigma \in k\text{-}\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]$, then for $|t|=r$*

$$
\begin{aligned}
r-\frac{(A-B)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{2(k+1)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}+|2B-A+1|[\Psi_2(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}r^2 \quad &\leq \\
|\varsigma(t)|\leq r+\frac{(A-B)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{2(k+1)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}+|2B-A+1|[\Psi_2(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}r^2.&
\end{aligned}
$$

*For the function given by the relation (16) and for $n=2$ the result is sharp.*

**Proof.** Let $\varsigma \in k\text{-}\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]$. By using Theorem 1, we deduce the inequality:

$$
\begin{aligned}
|\varsigma(t)| &\leq |t|+\sum_{n=2}^{\infty}|a_n||t|^n \\
&\leq |t|+|t|^2\sum_{n=2}^{\infty}|a_n| \leq r+\frac{(A-B)}{2(k+1)+|2B-A+1|\dfrac{[\Psi_2(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}}r^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|\varsigma(t)| &\geq |t|-\sum_{n=2}^{\infty}|a_n||t|^n \\
&\geq |t|-|t|^2\sum_{n=2}^{\infty}|a_n| \geq r-\frac{(A-B)}{2(k+1)+|2B-A+1|\dfrac{[\Psi_2(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}}r^2.
\end{aligned}
$$

□

**Theorem 4.** *Let $\varsigma \in k$-$\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]$. For $|t| = r$,*

$$\left|\varsigma'(z)\right| \leq 1 + \frac{2(A-B)}{2(k+1) + |2B - A + 1| \frac{[\Psi_2(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}} r,$$

*and*

$$\left|\varsigma'(t)\right| \geq 1 - \frac{2(A-B)}{2(k+1) + |2B - A + 1| \frac{[\Psi_2(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}} r.$$

*For the function given by the relation (16) and for $n = 2$, the result is sharp.*

**Proof.** The proof is quite similar to Theorem 3, so we omit it. □

## 3. Partial Sums

In [43], Silverman examined partial sums results for a class of convex and starlike functions $\varsigma$ by the form (1) and established through

$$\begin{aligned} \varsigma_1(t) &= t, \\ \varsigma_j(t) &= t + \sum_{n=2}^{j} a_n t^n. \end{aligned}$$

He also proven that the lower bounds on ratios $\Re\left(\frac{\varsigma(t)}{\varsigma_j(t)}\right)$, $\Re\left(\frac{\varsigma_j(t)}{\varsigma(t)}\right)$ have been found to be sharp only when $j = 1$. In [43], Silverman, determined sharpness for all values of "$j$". The lower bounds in question are strictly increasing functions of "$j$". In this section, when the coefficients $\{a_n\}$ are "small", to satisfy (13), we will determine sharp lower bounds for

$$\Re\left(\frac{\varsigma(t)}{\varsigma_j(t)}\right), \ \Re\left(\frac{\varsigma_j(t)}{\varsigma(t)}\right), \ \Re\left(\frac{\varsigma'(t)}{\varsigma_j'(t)}\right) \text{ and } \Re\left(\frac{\varsigma_j'(t)}{\varsigma'(t)}\right).$$

**Theorem 5.** *If $\varsigma$ is given by the relation (1) and satisfies (13), then*

$$\Re\left(\frac{\varsigma(t)}{\varsigma_j(t)}\right) \geq 1 - \frac{1}{m_{j+1}}, \quad \forall t \in \mathbb{U} \tag{17}$$

*and*

$$\Re\left(\frac{\varsigma_j(t)}{\varsigma(t)}\right) \geq \frac{m_{j+1}}{1 + m_{j+1}}, \quad \forall t \in \mathbb{U}, \tag{18}$$

*where*

$$m_j = \frac{\Big[2(k+1)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}(n-1) + |n(1+B) - (1+A)|\Big][\Psi_n(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{(A-B)[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}. \tag{19}$$

*For all the function of the form (16) the result is sharp.*

**Proof.** We can verify that

$$\mathrm{m}_{n+1} > \mathrm{m}_n > 1 \quad \text{where} \quad n > 2.$$

We set in order to prove the relation (17)

$$\mathsf{m}_{j+1}\left[\frac{\varsigma(t)}{\varsigma_j(t)}-\left(1-\frac{1}{\mathsf{m}_{j+1}}\right)\right]=\frac{1+\sum\limits_{n=2}^{j}a_nt^{n-1}+\mathsf{m}_{j+1}\sum\limits_{n=j+1}^{\infty}a_nt^{n-1}}{1+\sum\limits_{n=2}^{j}a_nt^{n-1}}=\frac{h_1(t)+1}{h_2(t)+1}.$$

We now set

$$\frac{h_1(t)+1}{h_2(t)+1}=\frac{1+\varpi(t)}{1-\varpi(t)},$$

and we obtain after simplification that

$$\varpi(t)=\frac{h_1(t)-h_2(t)}{h_1(t)+2+h_2(t)}.$$

We obtain that

$$\varpi(t)=\frac{\mathsf{m}_{j+1}\sum\limits_{n=j+1}^{\infty}a_nt^{n-1}}{2+2\sum\limits_{n=2}^{j}a_nt^{n-1}+\mathsf{m}_{j+1}\sum\limits_{n=j+1}^{\infty}a_nt^{n-1}}.$$

If we apply the trigonometric inequalities, where $|t|<1$, we obtain the following inequality:

$$|\varpi(t)|\leq\frac{\mathsf{m}_{j+1}\sum\limits_{n=j+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{j}|a_n|-\mathsf{m}_{j+1}\sum\limits_{n=j+1}^{\infty}|a_n|}.$$

We can see

$$|\varpi(t)|\leq 1,$$

if and only if

$$2\mathsf{m}_{j+1}\sum_{n=j+1}^{\infty}|a_n|\leq 2-2\sum_{n=2}^{j}|a_n|.$$

It follows that

$$\sum_{n=2}^{j}|a_n|+\mathsf{m}_{j+1}\sum_{n=j+1}^{\infty}|a_n|\leq 1. \tag{20}$$

To prove the relation (17), it suffices to show that the left hand side of (20) is bounded above by the following sum,

$$\sum_{n=2}^{\infty}\mathsf{m}_n|a_n|,$$

relation which is equivalent to

$$\sum_{n=j+1}^{\infty}(\mathsf{m}_n-\mathsf{m}_{j+1})|a_n|+\sum_{n=2}^{j}(\mathsf{m}_n-1)|a_n|\geq 0. \tag{21}$$

From the relation (21), is completed the proof of the relation (17).

Next, in order to prove the relation (18), let

$$(1+\mathsf{m}_{j+1})\left(\frac{\varsigma_j(t)}{\varsigma(t)}-\frac{\mathsf{m}_{j+1}}{1+\mathsf{m}_{j+1}}\right)=\frac{1+\sum\limits_{n=2}^{j}a_nt^{n-1}-\mathsf{m}_{j+1}\sum\limits_{n=j+1}^{\infty}a_nt^{n-1}}{1+\sum\limits_{n=2}^{\infty}a_nt^{n-1}},$$
$$=\frac{1+\varpi(t)}{1-\varpi(t)},$$

where

$$|\varpi(t)|\leq\frac{(1+\mathsf{m}_{j+1})\sum\limits_{n=j+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{j}|a_n|-(\mathsf{m}_{j+1}-1)\sum\limits_{n=j+1}^{\infty}|a_n|}\leq 1. \tag{22}$$

This last inequality in the relation (22) is equivalent to

$$\mathsf{m}_{j+1}\sum_{n=j+1}^{\infty}|a_n|+\sum_{n=2}^{j}|a_n|\leq 1. \tag{23}$$

The left hand side of the relation (23) is bounded above by the sum

$$\sum_{n=2}^{\infty}\mathsf{m}_n|a_n|,$$

so we have completed the proof of the relation (18). □

**Theorem 6.** *If $\varsigma$ is given by the relation (1) and satisfies the relation (13), then*

$$\Re\left(\frac{\varsigma'(t)}{\varsigma_j'(t)}\right)\geq 1-\frac{j+1}{m_{j+1}},\qquad \forall t\in\mathbb{U} \tag{24}$$

*and*

$$\Re\left(\frac{\varsigma_j'(t)}{\varsigma'(t)}\right)\geq\frac{m_{j+1}}{m_{j+1}+j+1},\qquad \forall t\in\mathbb{U}, \tag{25}$$

*where $m_j$ is given by the relation (19), and the result is sharp for all the functions given by the relation (16).*

**Proof.** We will omit the proof of the Theorem 6 because it is similar to that of the Theorem 5. □

## 4. The Fekete–Szegö Problem

**Theorem 7.** *If $\varsigma$ be of the form (1) and $\varsigma\in k\text{-}\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]$, then*

$$\begin{aligned}\left|a_3-\aleph a_2^2\right| &\leq \frac{(A-B)\wp_1[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{4[\Psi_3(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}\\ &\times\ \max\left\{1,\left|\left(\frac{1+B-(A-B)\wp_1-2\wp_2}{2\wp_1}\right)-\aleph\frac{\wp_1(A-B)[\Psi_3(\sigma,\vartheta,\nu)]^{\delta}_{p,q}[\Psi_1(\sigma,\vartheta,\nu)]^{\delta}_{p,q}}{([\Psi_2(\sigma,\vartheta,\nu)]^{\delta}_{p,q})^2}\right|\right\},\end{aligned} \tag{26}$$

*where $\wp_1$ and $\wp_2$ are given by the relations (8) and (9), respectively.*

**Proof.** Let

$$\Lambda\varsigma(t)=\frac{t\left(\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}\varsigma(t)\right)'}{\mathfrak{W}^{\sigma,\delta}_{\vartheta,\nu,p,q}\varsigma(t)},$$

then from (12), we have

$$\Lambda\varsigma(t)\prec\frac{(1+A)p_k(t)-(A-1)}{(1+B)p_k(t)+(1-B)}=\Phi(t). \tag{27}$$

Thus if

$$p_k(t)=1+\wp_1 t+\wp_2 t^2+\cdots,$$

we obtain by simple computation

$$\phi(t)=1+\tfrac{1}{2}(A-B)\wp_1 t+\tfrac{1}{4}(A-B)(2\wp_2-(B+1)\wp_1^2)t^2+\cdots.$$

Now from (27), there exists $h(t)$ which is an analytic function such that

$$h(t)=\frac{1+\phi^{-1}(\Lambda\varsigma(t))}{1-\phi^{-1}(\Lambda\varsigma(t))}=1+d_1t+d_2t^2+\cdots,$$

where

$$\Re(h(t))>0$$

in $\mathbb{U}$. We also have

$$\Lambda\varsigma(t)=\phi\left(\frac{h(t)-1}{1+h(t)}\right),$$

and

$$\frac{t(\Lambda\varsigma(t))'}{\Lambda\varsigma(t)}=1+\frac{[\Psi_2(\sigma,\vartheta,\nu)]^\delta_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^\delta_{p,q}}a_2z+\left(2\frac{[\Psi_3(\sigma,\vartheta,\nu)]^\delta_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^\delta_{p,q}}a_3-[\frac{[\Psi_2(\sigma,\vartheta,\nu)]^\delta_{p,q}}{[\Psi_1(\sigma,\vartheta,\nu)]^\delta_{p,q}}]^2a_2^2\right)t^2+\cdots. \tag{28}$$

Consequently,

$$\begin{aligned}\phi\left(\frac{h(t)-1}{h(t)+1}\right) &= 1+\frac{1}{4}(A-B)\wp_1d_1z\\ &+\frac{1}{4}(A-B)\left[\wp_1d_2+\left(\frac{\wp_2}{2}-\frac{1+B}{4}-\frac{\wp_1}{2}\right)d_1^2\right]z^2+\cdots.\end{aligned} \tag{29}$$

After comparing the (28) and (29), we get

$$a_2=\frac{[\Psi_1(\sigma,\vartheta,\nu)]^\delta_{p,q}}{4[\Psi_2(\sigma,\vartheta,\nu)]^\delta_{p,q}}(A-B)\wp_1d_1, \tag{30}$$

$$a_3=\frac{(A-B)[\Psi_1(\sigma,\vartheta,\nu)]^\delta_{p,q}}{8[\Psi_3(\sigma,\vartheta,\nu)]^\delta_{p,q}}\left(\wp_1d_2+\left(\frac{\wp_2}{2}-\frac{1+B}{4}-\frac{\wp_1}{2}+\frac{A-B}{4}\wp_1^2\right)d_1^2\right). \tag{31}$$

Now by making use of the relations (30) and (31), in conjunction with the Lemma, we have

$$\begin{aligned}\left|a_3-\aleph a_2^2\right| &\leq \frac{\wp_1(A-B)[\Psi_1(\sigma,\vartheta,\nu)]^\delta_{p,q}}{4[\Psi_3(\sigma,\vartheta,\nu)]^\delta_{p,q}}\\ &\times\ \max\left\{1,\left|\left(\frac{1+B-(A-B)\wp_1-2\wp_2}{2\wp_1}\right)-\aleph\frac{(A-B)\wp_1[\Psi_3(\sigma,\vartheta,\nu)]^\delta_{p,q}[\Psi_1(\sigma,\vartheta,\nu)]^\delta_{p,q}}{([\Psi_2(\sigma,\vartheta,\nu)]^\delta_{p,q})^2}\right|\right\},\end{aligned}$$

which is the required result. □

For $\varsigma \in \mathcal{A}$, we take the convolution operator $*$ and introduce the linear operator $\mho : \mathcal{A} \to \mathcal{A}$ defined by

$$\mho\varsigma(t) \quad = \quad \Theta(t) * \varsigma(t) = t + \sum_{n=2}^{\infty} \Theta_n a_n t^n, \tag{32}$$

where $\Theta(t) = t + \sum_{n=2}^{\infty} \Theta_n t^n$.

**Theorem 8.** *If $\varsigma$ is of the form* (1) *and $\varsigma \in k\text{-}\mathfrak{W}\mathfrak{S}^{\sigma,\delta}_{\vartheta,\nu,p,q}[A,B]$, then*

$$\left|a_3 - \hbar a_2^2\right| \leq \frac{(A-B)\wp_1}{4\Theta_3} \max\left\{1, \left|\left(\frac{1+B-\wp_1(A-B)-2\wp_2}{2\wp_2}\right) - \hbar\frac{(A-B)\wp_1\Theta_3}{\Theta_2^2}\right|\right\}, \tag{33}$$

*where $\wp_1$ and $\wp_2$ are given by* (8) *and* (9)*, respectively.*

**Proof.** Proceeding on lines similar to Theorem 7 and using (29)

$$\frac{t(\mho\varsigma(t))'}{\mho\varsigma(t)} = 1 + \Theta_2 a_2 t + \left(2\Theta_3 a_3 - \Theta_2^2 a_2^2\right)t^2 + \cdots. \tag{34}$$

After comparing (34) and (29), we get

$$a_2 = \frac{(A-B)}{4\Theta_2}\wp_1 d_1, \tag{35}$$

$$a_3 = \frac{(A-B)}{8\Theta_3}\left(\wp_1 d_2 + \left(\frac{\wp_2}{2} - \frac{1+B}{4} - \frac{\wp_1}{2} + \frac{A-B}{4}\wp_1^2\right)d_1^2\right). \tag{36}$$

By making use of the relations (35) and (36) and the Lemma, we obtain

$$\left|a_3 - \hbar a_2^2\right| \leq \frac{(A-B)\wp_1}{4\Theta_3} \max\left\{1, \left|\left(\frac{1-(A-B)\wp_1-2\wp_2+B}{2\wp_1}\right) - \hbar\frac{(A-B)\wp_1\Theta_3}{\Theta_2^2}\right|\right\},$$

which is the required result. □

By assuming $\Theta_2 = [\Psi_2(\sigma,\vartheta,\nu)]^{\delta}_{p,q}$ and $\Theta_3 = [\Psi_3(\sigma,\vartheta,\nu)]^{\delta}_{p,q}$, we can easily state results similar to Theorem 7.

We say that a variable $Y$ is said to be Poisson-distributed if it takes the values $0,1,2,3,\cdots$ with probabilities $e^{-d}, d\frac{e^{-d}}{1!}, d^2\frac{e^{-d}}{2!}, d^3\frac{e^{-d}}{3!}, \ldots$ respectively, where $d$ is called the parameter. Thus,

$$P(Y=s) = \frac{d^s e^{-d}}{s!}, \; s = 0,1,2,3,\cdots.$$

In [44], Porwal introduced a power series whose coefficients are probabilities of the Poisson distribution:

$$K(d,t) = t + \sum_{n=2}^{\infty} \frac{d^{n-1}}{(n-1)!} t^n e^{-d}, d > 0, \qquad t \in \mathbb{U}.$$

We know that the radius of convergence of the above series is infinity, by ratio test.

If we take

$$\Theta_2 = de^{-d} \quad \textit{and} \quad \Theta_3 = \frac{d^2}{2}e^{-d}, \tag{37}$$

one can deduce the Fekete–Szegö problem given in Theorem 8 related with Poisson distribution. We establish connections between the geometric function theory and Pascal distribution series (see [45,46]).

We say that a variable $y$ is said to be Pascal distribution if it takes the values $0, 1, 2, 3, \ldots$ with probabilities $(1-q)^d, \frac{qd(1-q)^d}{1!}, \frac{q^2 d(d+1)(1-q)^d}{2!}, \frac{q^3 d(d+1)(d+2)(1-q)^d}{3!}, \ldots$ respectively, where $q$ and $d$ are called the parameter, and thus

$$P(y=k) = \binom{k+d-1}{d-1} \cdot (1-q)^d q^k,$$

where

$$k = 0, 1, 2, 3, \ldots$$

In [46], M. El-Deeb et al. introduced a power series whose coefficients are probabilities of the Pascal distribution

$$\phi_q^d(t) = t + \sum_{n=2}^{\infty} \binom{n+d-2}{d-1} (1-q)^d q^{n-1} t^n, \qquad t \in \mathbb{U}, \tag{38}$$

where $d \geq 1, 0 \leq q \leq 1$, and we note that, by ratio test the radius of convergence of above series is infinity. We consider the operator given in [45]

$$\mathcal{I}_q^d(t) : \mathcal{A} \to \mathcal{A}$$

defined by the convolution or hadamard product

$$\mathcal{I}_q^d \varsigma(t) = \phi_q^d(t) * \varsigma(t) = t + \sum_{n=2}^{\infty} \binom{n+d-2}{d-1} q^{n-1} (1-q)^d a_n t^n, \qquad t \in \mathbb{U}. \tag{39}$$

In particular, by fixing

$$\Theta_2 = \binom{d}{d-1} q(1-q)^d \quad and \quad \Theta_3 = \binom{d+1}{d-1} q^2 (1-q)^d$$

in Theorem 8, we can deduce the Fekete–Szegö problem related to Pascal distribution.

## 5. Conclusions

In the present paper, by using the integral operator introduced in [29], we have defined and studied new subclasses of starlike functions involving the Janowski functions. Furthermore, we have discussed some important geometric properties like necessary and sufficient condition, convex combination, growth and distortion bounds, partial sums, Fekete–Szegö inequality and applications of Poisson and Pascal distribution for this newly defined function subclass. By fixing the parameter, one can define various subclasses of Janowski starlike functions in a conic region and state the analogue results given in Theorems 1–8; we left this exercise for interested readers. We unify and extend various classes of analytic function by defining starlike function by using subordination and the Hadamard product. New extensions were discussed in detail. Furthermore, by replacing the ordinary differentiation with quantum differentiation, we have attempted to discretize some of the well-known results. We believe that this study will motivate a number of researchers to extend this idea for meromorphic functions and class of bi-univalent functions. Moreover, new classes can be defined based on certain probability distribution with special functions. Moreover, by specializing the parameter, our new subclass yields many subclasses of analytic functions which have not been studied so far in association with the integral operator.

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## References

1. Jagannathan, R.; Rao, K.S. Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series. *arXiv* **2006**, arXiv:math/0602613.
2. Sahai, V.; Yadav, S. Representations of two parameter quantum algebras and $p,q$-special functions. *J. Math. Anal. Appl.* **2007**, *335*, 268–279. [CrossRef]
3. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]
4. Chakrabarti, R.; Jagannathan, R. A $(p,q)$-oscillator realization of two-parameter quantum algebras. *J. Phys. A Math. Gen.* **1991**, *24*, L711. [CrossRef]
5. Jackson, F.H. On q-functions and a certain difference operator. *Trans. R. Soc. Edinb.* **1908**, *46*, 253–281. [CrossRef]
6. Aral, A.; Gupta, V.; Agarwal, R.P. *Applications of $q$-Calculus in Operator Theory*; Springer: New York, NY, USA, 2013.
7. Purohit, S.D.; Raina, R.K. Fractional $q$-calculus and certain subclasses of univalent analytic functions. *Mathematica* **2013**, *55*, 62–74.
8. Kanas, S.; Răducanu, D. Some class of analytic functions related to conic domains. *Math. Slovaca* **2014**, *64*, 1183–1196. [CrossRef]
9. Corcino, R.B. On $p,q$-binomial coefficients. *Integers* **2008**, *8*, A29.
10. Sadjang, P.N. On the fundamental theorem of $(p,q)$-calculus and some $(p,q)$-Taylor formulas. *arXiv* **2018**, arXiv:1309.3934.
11. Alexander, J.W. Functions which map the interior of the unit circle upon simple region. *Ann. Math.* **1915**, *17*, 12–22. [CrossRef]
12. Ruscheweyh, S. New criteria for univalent functions. *Proc. Am. Math. Soc.* **1975**, *49*, 109–115. [CrossRef]
13. St. Sălăgean, G. Subclasses of univalent functions. In *Lecture Notes in Mathematics*; Springer: Berlin, Germany, 1983; Volume 1013, pp. 362–372.
14. Noor, K.I.; Riaz, S.; Noor, M.A. On $q$-Bernardi integral operator. *TWMS J. Pure Appl. Math.* **2017**, *8*, 3–11.
15. Cataş, A. A note on subclasses of univalent functions defined by a generalized Sălăgean operator. *Acta Univ. Apulensis* **2006**, *12*, 73–78.
16. Alb Lupaş, A.; Oros, G.I. Fuzzy Differential Subordination and Superordination Results Involving the $q$-Hypergeometric Function and Fractional Calculus Aspects. *Mathematics* **2022**, *10*, 4121. [CrossRef]
17. Srivastava, H.M.; Wanas, A.K.; Srivastava, R. Applications of the $q$-Srivastava-Attiya operator involving a certain family of bi-univalent functions associated with the Horadam polynomials. *Symmetry* **2021**, *13*, 1230. [CrossRef]
18. Breaz, D.; Karthikeyan, K.R.; Umadevi, E.; Senguttuvan, A. Some Properties of Bazilevič Functions Involving Srivastava–Tomovski Operator. *Axioms* **2022**, *11*, 687. [CrossRef]
19. Totoi E.A.; Cotîrlă L.I. Preserving Classes of Meromorphic Functions through Integral Operators. *Symmetry* **2022**, *14*, 1545. [CrossRef]
20. Cho, N.E.; Srivastava, H.M. Argument estimates of certain analytic functions defined by a class of multiplier transformations. *Math. Comput. Model.* **2003**, *37*, 39–49. [CrossRef]
21. Çaglar, M.; Cotîrlă, L.I.; Buyankara, M. Fekete–Szegö inequalities for a new subclass of bi-univalent functions associated with Gegenbauer polynomials. *Symmetry* **2022**, *14*, 1572. [CrossRef]
22. Govindaraj, M.; Sivasubramanian, S. On a class of analytic functions related to conic domains involving $q$-calculus. *Anal. Math.* **2017**, *43*, 475–487. [CrossRef]
23. Jung, I.B.; Kim, Y.C.; Srivastava, H.M. The Hardy space of analytic functions associated with certain one-parameter families of integral operators. *J. Math. Anal. Appl.* **1993**, *176*, 138–147. [CrossRef]
24. Shah, S.A.; Noor, K.I. Study on the $q$-analogue of a certain family of linear operators. *Turk J. Math.* **2019**, *43*, 2707–2714. [CrossRef]
25. Srivastava, H.M.; Attiya, A.A. An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination. *Integral Transform. Spec. Funct.* **2007**, *18*, 207–216. [CrossRef]
26. Swamy, S.R. Inclusion properties of certain subclasses of analytic functions. *Int. Math. Forum* **2012**, *7*, 1751–1760.
27. Wanas, A.K. New differential operator for holomorphic functions. *Earthline J. Math. Sci.* **2019**, *2*, 527–537. [CrossRef]
28. Bernardi, S.D. Convex and starlike univalent functions. *Trans. Am. Math. Soc.* **1969**, *135*, 429–446. [CrossRef]
29. Wanas, A.K.; Cotîrlă, L.I. Initial coefficient estimates and Fekete–Szegö inequalities for new families of bi-univalent functions governed by $(p-q)$-Wanas operator. *Symmetry* **2021**, *13*, 2118. [CrossRef]
30. Miller, S.S.; Mocanu, P.T. *Differential Subordinations: Theory and Applications*; Series on Monographs and Textbooks in Pure and Applied Mathematics; Marcel Dekker Incorporated: New York, NY, USA; Basel, Switzerland, 2000; Volume 225.
31. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–23 June 1992; International Press: Cambridge, MA, USA, 1992; pp. 157–169.

32. Janowski, W. Extremal problems for a family of functions with positive real part and for some related families. *Ann. Pol. Math.* **1970**, *23*, 159–177. [CrossRef]
33. Kanas, S.; Wisniowska, A. Conic regions and *k*-uniformly convexity. *J. Comput. Appl. Math.* **1999**, *105*, 327–336. [CrossRef]
34. Kanas, S.; Wisniowska, A. Conic regions and starlike functions. *Rev. Roum. Math. Pure Appl.* **2000**, *45*, 647–657.
35. Kanas, S.; Srivastava, H.M. Linear operators associated with k-uniformly convex functions. *Integral Transform. Spec. Funct.* **2000**, *9*, 121–132. [CrossRef]
36. Kanas, S. Coefficient estimates in subclasses of the Carathé odary class related to conic domains. *Acta Math. Univ. Comen.* **2005**, *74*, 149–161.
37. Noor, K.I.; Malik, S.N. On coefficient inequalities of functions associated with conic domains. *Comput. Appl. Math.* **2011**, *62*, 2209–2217. [CrossRef]
38. Khan, B.; Srivastava, H.M.; Khan, N.; Darus, M.; Ahmad, Q.Z.; Tahir, M. Applications of Certain Conic Domains to a Subclass of *q*-Starlike Functions Associated with the Janowski Functions. *Symmetry* **2021**, *13*, 574. [CrossRef]
39. Rehman, M.S.U.; Ahmad, Q.Z.; Srivastava, H.M.; Khan, N.; Darus, M.; Khan, B. Applications of higher-order q-derivatives to the subclass of q-starlike functions associated with the Janowski functions. *AIMS Math.* **2020**, *6*, 1110–1125. [CrossRef]
40. Srivastava, H.M.; Khan, B.; Khan, N.; Ahmad, Q.Z. Coefficient Inequalities for q-Starlike Functions Associated with the Janowski Functions. *Hokkaido Math. J.* **2019**, *48*, 407–425. [CrossRef]
41. Srivastava, H.M.; Khan, B.; Khan, N.; Hussain, A.; Khan, N.; Tahir, M. Applications of certain basic (or *q*-) derivatives to subclasses of multivalent Janowski type *q*-starlike functions involving conic domain. *J. Nonlinear Var. Anal.* **2021**, *5*, 531–547.
42. Fekete, M.; Szegö, G. Eine bemerkung uber ungerade schlichte funktionen. *J. Lond. Math. Soc.* **1933**, *2*, 85–89. [CrossRef]
43. Silverman, H. Partial sums of starlike and convex functions. *J. Math. Anal. Appl.* **1997**, *209*, 221–227. [CrossRef]
44. Porwal, S. An application of a Poisson distribution series on certain analytic functions. *J. Complex Anal.* **2014**, *2014*, 984135. [CrossRef]
45. Bulboacă, T.; Murugusundaramoorthy, G. Univalent functions with positive coefficients involving Pascal distribution series. *Commun. Korean Math. Soc.* **2020**, *35*, 867–877.
46. El-Deeb, M.; Bulboacă, T.; Dziok, J. Pascal distribution series connected with certain subclasses of univalent functions. *Kyungpook Math. J.* **2019**, *59*, 301–314.