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Optimal Control for a Class of Riemann-Liouville Fractional Evolution Inclusions

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Abstract: In this paper, under symmetric properties of multivalued operators, the existence of mild solutions as well as optimal control for the nonlocal problem of fractional semilinear evolution inclusions are investigated in abstract spaces. At first, the existence results are proved by applying the theory of operator semigroups and the fixed-point theorem of multivalued mapping. Then the existence theorem on the optimal state-control pair is proved by constructing the minimizing sequence twice. An example is given in the last section as an application of the obtained conclusions.

Keywords: fractional differential inclusion; optimal state-control pair; existence; compact semigroup

MSC: 26A33; 49J20

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Citation: Yang, H.; Ren, Q. Optimal Control for a Class of Riemann-Liouville Fractional Evolution Inclusions. *Symmetry* **2022**, *14*, 248. https://doi.org/10.3390/sym14020248

Academic Editors: Calogero Vetro, Dariusz Wardowski and Bessem Samet

Received: 30 December 2021
Accepted: 19 January 2022
Published: 27 January 2022

1. Introduction

As an important branch of the nonlinear analysis theory, fractional differential inclusions have gained a lot of attention in recent years, because it has wide applications in fluid mechanics, economics, control theory, and so forth (see [1–4] and the references therein). In 2011, Du et al. [5] pointed out that Riemann-Liouville fractional derivatives are more suitable to describe certain characteristics of viscoelastic materials than Caputo ones. Therefore, it is more significant to study Riemann-Liouville fractional differential systems. In 2013, Zhou et al. [6], applying probability density functions and the Laplace transform technique, presented a suitable concept of mild solutions of Riemann-Liouville fractional evolution equations. Additionally, when the C_0-semigroup generated by the linear part is noncompact or compact, Zhou et al. proved existence theorems of mild solutions for Riemann-Liouville fractional Cauchy problems. Pan et al. [7] demonstrated the existence theorems on mild solutions as well as optimal control of the semilinear fractional differential equation

$$\begin{cases} {}^{L}D^{\alpha}x(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad t \in J' := (0, b], \ \alpha \in (0,1), \\ I^{1-\alpha}x(t)|_{t=0} = x_0, \\ u \in U_{ad}, \end{cases} \tag{1}$$

where ${}^{L}D^{\alpha}$ represents the fractional derivative operator of order α in the Riemann-Liouville sense, and $I^{1-\alpha}$ is the $(1-\alpha)$-order Riemann-Liouville fractional integral operator, $A : D(A) \subset X \to X$ generates a compact C_0-semigroup $\{T(t),\ t \geq 0\}$ and X is a reflexive Banach space. Denote by Y another separable reflexive Banach space, in which u takes its values. $B : Y \to X$ is a linear bounded operator and $f : [0, b] \times X \to X$ is Lipschitz-continuous. By utilizing the Schaefer fixed-point theorem and the fractional calculus theory, Pan et al. proved the existence and uniqueness of mild solutions of (1). The existence of optimal state-control pairs was also investigated in the case where the mild solution of (1) is unique.

Kumar in the Ref. [8] demonstrated the existence theorems of mild solutions and optimal control of the semilinear fractional system with fixed delay

$$\begin{cases} {}^{C}D^{\alpha}x(t) = Ax(t) + f(t, x(t-h)) + B(t)u(t), & t \in J := [0, b], \\ x(t) = \varphi(t), & t \in [-h, 0], \end{cases} \tag{2}$$

where ${}^{C}D^{\alpha}$ is the fractional derivative operator of order $\alpha \in (0,1)$ in the Caputo sense. A generates a compact C_0-semigroup $\{T(t),\ t \geq 0\}$ in X. For fixed $t \geq 0$, $B(t) : Y \to X$ is a linear operator. X, Y, and u are defined as above. $f : [0, b] \times X \to X$ is the nonlinear term and $\varphi \in C([-h, 0], X)$. When f is locally Lipschitz-continuous, Kumar studied the existence and uniqueness of mild solutions of (2) by applying the Weissinger fixed-point theorem. Under the case that the mild solution of Equation (2) is unique, he also discussed the existence of optimal control.

In the Ref. [9], Lian et al. were concerned with the existence of mild solutions for the nonlinear fractional differential system in Banach space X

$$\begin{cases} {}^{L}D^{\alpha}x(t) = Ax(t) + f(t, x(t)) + B(t)u(t), & t \in J', \\ I^{1-\alpha}x(t)|_{t=0} = x_0, \\ u \in U_{ad}. \end{cases} \tag{3}$$

They firstly proved the existence results of mild solutions of Equation (3) by using the Schauder fixed-point theorem and the semigroup theory. Then, when f is not Lipschitz-continuous, a new approach was established to investigate the existence of time-optimal pairs without the uniqueness of mild solutions. It is worth noting that all these works consider the case of single-valued mapping. As far as we know, the existence of optimal state-control pairs for Riemann-Liouville fractional evolution inclusions is still rare.

Inspired by the above-mentioned literature, we deal with the existence of mild solutions as well as optimal state-control pairs for fractional evolution inclusions with nonlocal conditions

$$\begin{cases} {}^{L}D^{\alpha}x(t) \in Ax(t) + F(t, x(t)) + B(t)u(t), & t \in J', \ \alpha \in (0,1), \\ I_{0^+}^{1-\alpha}x(0) + g(x) = x_0, \\ u \in U_{ad}, \end{cases} \tag{4}$$

where ${}^{L}D^{\alpha}$ is the α-order Riemann-Liouville fractional derivative operator. $A : D(A) \subset X \to X$ is a densely defined and linear closed operator. It generates a C_0-semigroup $\{T(t),\ t \geq 0\}$ in X. X and Y are (separable) reflexive Banach spaces. u takes values in Y. For fixed $t \geq 0$, $B(t) : Y \to X$ is a linear operator. g denotes the nonlocal function. $F : J \times X \to 2^X \setminus \{\emptyset\}$ is a u.s.c. multi-valued mapping with compact values which satisfies some appropriate conditions. The control $u \in U_{ad}$, U_{ad} will be introduced in Section 2.

The main contributions of this work can be listed as follows:

(i) Under the case that the nonlocal function g is Lipschitz-continuous or completely continuous, the existence of mild solutions of the fractional evolution inclusion (4) is proved by using a fixed-point theorem of multi-valued operators.

(ii) Under the case that the nonlinearity f is not Lipschitz-continuous, the existence of an optimal state-control pair of (4) is obtained by utilizing an approach established in the Ref. [9] when the mild solution is not unique.

It is emphasized that the symmetry of operators plays a key role in the present work. The Lipschitz continuity of the nonlinearity f is not needed in our work. Then, results obtained in this paper extend some existing research, such as that by the Refs. [6–9], and so forth.

2. Preliminaries

Let X be a reflexive Banach space equipped with the norm $\|\cdot\|$. We denote by $C(J,X)$ the continuous function space whose norm is defined by $\|x\|_C = \sup_{t\in J}\|x(t)\|$. Let $L^p(J,X)$ be the p-order Bochner integrable function space with the norm $\|x\|_{L^p} = (\int_0^b \|x(t)\|^p dt)^{\frac{1}{p}}$ for $1 \le p < +\infty$. Let

$$C_{1-\alpha}(J,X) = \{x \in C(J',X)|t^{1-\alpha}x(t) \in C(J,X),\ 0<\alpha<1,\ t\in J\}.$$

Then $C_{1-\alpha}(J,X)$ constitutes a Banach space whose norm is given by $\|x\|_{C_{1-\alpha}} = \sup_{t\in J}\{t^{1-\alpha}\|x(t)\|\}$. Throughout this paper, we assume that the C_0-semigroup $\{T(t),\ t\ge 0\}$, generated by linear operator A, is uniformly bounded in X, which means that there is $M>0$ such that $\sup_{t\in J}\|T(t)\| \le M$. For a Banach space X, let

$P(X) = \{D \subseteq 2^X | D \neq \emptyset\}$,
$P_b(X) = \{D \subseteq P(X)|D$ is a bounded set$\}$,
$P_c(X) = \{D \subseteq P(X)|D$ is a closed set$\}$,
$P_{cv}(X) = \{D \subseteq P(X)|D$ is a convex set$\}$, and
$P_{cp}(X) = \{D \subseteq P(X)|D$ is a compact set$\}$.

If a set, belonging to X, is nonempty convex and closed, then denote it by $P_{c,cv}(X)$, and the other cases are the same. Let $(Y,\|\cdot\|)$ be another separable reflexive Banach space. Denote $\mathcal{L}(Y,X) = \{B|Y\to X$ as a bounded linear operator$\}$. Then $\mathcal{L}(Y,X)$ is a Banach space with an operator norm. Let E be a bounded subset of Y. Assume that the multi-valued mapping $U: J \to P_{c,cv}(Y)$ is graph-measurable and $U(\cdot)\subset E$. Then, U_{ad} is defined by

$$U_{ad} = \{u \in L^p(J,E)|u(t)\in U(t), a.e. t\in J\}, \quad p > \frac{1}{\alpha}.$$

Clearly, (see [10]), $U_{ad} \subset L^p(J,Y)(p>\frac{1}{\alpha})$ is nonempty convex, closed, and bounded.

In this work, we introduce the definition of a mild solution of (4) in the following way (see [6] for more details).

Definition 1. *$x \in C_{1-\alpha}(J,X)$ is said to be a mild solution of (4) if*

(i) $I_{0^+}^{1-\alpha}x(0) + g(x) = x_0$.
(ii) *there exists $f(t)\in F(t,x(t))$ such that*

$$x(t) = t^{\alpha-1}T_\alpha(t)(x_0 - g(x)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f(s)+B(s)u(s)]ds, \quad t\in J', \tag{5}$$

where

$$T_\alpha(t) = \int_0^\infty \alpha\theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \ge 0,$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta\in(0,\infty).$$

$\xi_\alpha(\theta)$, $\theta\in(0,\infty)$ denotes the probability density function, which satisfies

$$\int_0^\infty \xi_\alpha(\theta)d\theta = 1, \qquad \int_0^\infty \theta^v\xi_\alpha(\theta)d\theta = \frac{\Gamma(1+v)}{\Gamma(1+\alpha v)}, \ \ v\in[0,1].$$

Lemma 1 ([11])**.** *The linear operator family $T_\alpha(t)(t\ge 0)$ has properties:*

(i) *for every $t\ge 0$ and any $x\in X$,*

$$\|T_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|.$$

(ii) *for* $t \geq 0$*, the operator* $T_\alpha(t)$ *is strongly continuous.*
(iii) *for* $t > 0$*, if the operator* $T(t)$ *is compact,* $T_\alpha(t)$ *is a compact operator.*

Next, some definitions and basic results of multi-valued mapping are listed (refer to [11,12] for more details).

Definition 2 ([12])**.** *Let* X *and* Z *be two topological spaces, and* $F : X \to P(Z)$ *be a multi-valued mapping.*

(1) *If* $F(x)$ *is convex (closed) in* Z *for all* $x \in X$*, then* F *is said to be convex (closed)-valued.*
(2) *If* $F(D)$ *is relatively compact for every bounded subset* D *of* X*, then* F *is said to be completely continuous.*
(3) *If* $F^{-1}(V) = \{x \in X \mid F(x) \subseteq V\}$ *is an open subset of* X *for every open subset* V *of* Z*, then* F *is said to be upper semi-continuous(u.s.c.) on* X*.*
(4) *If the graph* $G_F = \{(x, y) \in X \times Z \mid y \in F(x)\}$ *is a closed subset of* $X \times Z$*, then* F *is said to be closed.*
(5) *If there is an element* $x \in X$ *satisfying* $x \in F(x)$*, then* F *is said to have a fixed point in* X*.*

Lemma 2 ([11])**.** *If the multi-valued mapping* F *is completely continuous with nonempty compact values, then* F *is u.s.c. if, and only if* F *has a closed graph.*

Lemma 3 ([11])**.** *Let* J *be a compact real interval. Suppose that*

(i) *for each* $x \in X$*,* $F(\cdot, x) : J \to P_{b,c,cv}(X)$ *is measurable and for every* $t \in J$*,* $F(t, \cdot) : X \to P_{b,c,cv}(X)$ *is u.s.c.*

(ii) *for each* $x \in C(J, X)$*, the set* $S_{F,x} = \{f \in L^1(J, X) \mid f(t) \in F(t, x(t)),\ a.e.\ t \in J\}$ *is nonempty.*

If $\mathcal{F}$ *is a linear continuous operator from* $L^1(J, X)$ *to* $C(J, X)$*, then*

$$\mathcal{F} \circ S_F : C(J, X) \to P_{b,c,cv}(C(J, X)),\ x \mapsto (\mathcal{F} \circ S_F)(x) = \mathcal{F}S_{F,x}$$

is a closed graph operator in $C(J, X) \times C(J, X)$*.*

Definition 3 ([13])**.** *A sequence* $\{f_n\}_{n\geq 1} \subset L^1(J, X)$ *is said to be semi-compact if*

(i) *there exists a function* $\omega \in L^1(J, \mathbb{R}^+)$ *such that*

$$\|f_n(t)\| \leq \omega(t),\ a.e.\ t \in J;$$

(ii) *for a.e.* $t \in J$*, the set* $\{f_n(t) \mid n \in \mathbb{N}\}$ *is relatively compact in* X*.*

Lemma 4 ([12])**.** *If a sequence in* $L^1(J, X)$ *is semi-compact, then it is weakly compact in* $L^1(J, X)$*.*

Lemma 5 ([14])**.** *Let* X *be a Banach space and* D *be a compact subset of* X*. Then* $\overline{conv}(D)$ *is compact, where* $\overline{conv}(D)$ *denotes the convex closure of* D*.*

Lemma 6 ([14])**.** *In a normed space, the closure and weak closure of a convex subset are the same.*

Lemma 7 ([11])**.** *Let* $0 < a \leq b$ *and* $\theta \in (0, 1]$*, where we have*

$$\mid a^\theta - b^\theta \mid \leq (b - a)^\theta.$$

To prove our main results, the following two fixed-point theorems concerning multi-valued operators play an important role.

Lemma 8 ([15]). *Let W be a nonempty closed, convex and bounded subset in the Banach space X, and $\Psi : W \to 2^W \setminus \{\emptyset\}$ be a u.s.c. condensing multi-valued mapping. If for every $x \in W$, $\Psi(x)$ is convex and closed in W and $\Psi(W) \subseteq W$, then Ψ has one fixed point in W.*

Lemma 9 ([16]). *Let W be a nonempty subset of X, which is convex, closed and bounded. Suppose that $\Psi : W \to 2^W \setminus \{\emptyset\}$ is u.s.c. with convex and closed values, $\Psi(W) \subseteq W$ and $\Psi(W)$ is a compact set, then Ψ has one fixed-point in W.*

3. Existence of Mild Solutions

In order to prove the main conclusions on the existence of mild solutions of (4), we make the following hypotheses.

(H_1) The semigroup $\{T(t),\ t \geq 0\}$ is a compact semigroup in X.

(H_2) The multi-valued mapping $F : J \times X \to P_{cv,cp}(X)$ satisfies the following hypotheses:

(i) for each $x \in X$, $F(t,x)$ is measurable to t and for every $t \in J$, $F(t,x)$ is u.s.c. to x. For every $x \in X$,

$$S_{F,x} = \{f \in L^1(J,X) \mid f(t) \in F(t,x),\ a.e.\ t \in J\}$$

is nonempty.

(ii) There exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ satisfying $\Lambda := \lim\limits_{r\to\infty} \frac{\psi(r)}{r} < +\infty$ and $m \in L^p(J,\mathbb{R}^+)(p > \frac{1}{\alpha})$ such that

$$\|F(t,x(t))\| := \sup\{\|f(t)\| \mid f(t) \in F(t,x(t)),\ t \in J\} \leq m(t)\psi(\|x\|_{C_{1-\alpha}}).$$

(H_3) The function $g : C_{1-\alpha}(J,X) \to X$ and there is a constant $M_g > 0$ such that

$$\|g(x) - g(y)\| \leq M_g\|x - y\|_{C_{1-\alpha}}, \quad \forall\, x,\ y \in C_{1-\alpha}(J,X).$$

(H_4) $B \in L^\infty(J,\mathcal{L}(Y,X))$, where $L^\infty(J,\mathcal{L}(Y,X))$ is a Banach space with norm $\|\cdot\|_\infty$.

Remark 1. *Combining the definition of U_{ad} with the assumption (H_4), we easily verify that, for all $u \in U_{ad}$, $Bu \in L^p(J,X)$ with $p > \frac{1}{\alpha}$.*

Lemma 10 ([9]). *Let the assumption (H_1) be fulfilled. Then for each $h \in L^p(J,X)$ with $p > \frac{1}{\alpha}$, the operator $\mathcal{B} : L^p(J,X) \to C_{1-\alpha}(J,X)$, given by*

$$(\mathcal{B}h)(\cdot) = \int_0^{\cdot} (\cdot - s)^{\alpha-1}T_\alpha(\cdot - s)h(\cdot)ds,$$

is compact.

Theorem 1. *Assume that (H_1)–(H_4) hold. Then the fractional evolution system (4) possesses one mild solution provided that*

$$M^* := \frac{M}{\Gamma(\alpha)}\Big(M_g + (\frac{p-1}{p\alpha - 1}b)^{\frac{p-1}{p}}\|m\|_{L^p}\Lambda\Big) < 1. \tag{6}$$

Proof of Theorem 1. For $x \in C_{1-\alpha}(J,X)$, we define an operator $\Psi : C_{1-\alpha}(J,X) \to 2^{C_{1-\alpha}(J,X)}$ by

$$\Psi(x) = \Big\{\varphi \mid \varphi(t) = t^{\alpha-1}T_\alpha(t)(x_0 - g(x)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f(s) + B(s)u(s)]ds,\ f \in S_{F,x},\ t \in J'\Big\}.$$

By means of Definition 1, the fixed point of the operator Ψ is equivalent to the mild solution of the system (4). We will prove that Ψ has one fixed point in $C_{1-\alpha}(J,X)$ by applying Lemma 8. The proof will be divided into four steps.

Step 1. We will prove that, for each $x \in C_{1-\alpha}(J,X)$, $\Psi(x)$ is convex.

In fact, if $\varphi_1, \varphi_2 \in \Psi(x)$, there are $f_1, f_2 \in S_{F,x}$ such that

$$\varphi_i(t) = t^{\alpha-1}T_\alpha(t)(x_0 - g(x)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f_i(s) + B(s)u(s)]ds, \quad t \in J', \ (i = 1,2).$$

For any $\lambda \in [0,1]$, we have

$$\lambda\varphi_1(t) + (1-\lambda)\varphi_2(t) = t^{\alpha-1}T_\alpha(t)(x_0 - g(x)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[\lambda f_1(s) + (1-\lambda)f_2(s)]ds$$
$$+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)B(s)u(s)ds.$$

Since F has convex values, it follows that $S_{F,x}$ is convex and $\lambda f_1 + (1-\lambda)f_2 \in S_{F,x}$. Then

$$\lambda\varphi_1 + (1-\lambda)\varphi_2 \in \Psi(x).$$

This fact means that $\Psi(x)$ is convex.

Step 2. We will show that, for each $x \in C_{1-\alpha}(J,X)$, $\Psi(x)$ is closed.

Let $\{z_n\}_{n\geq 1}$ be a sequence in $\Psi(x)$ satisfying $z_n \to z$ as $n \to \infty$. We show that $z \in \Psi(x)$. By the definition of Ψ, there is $\{f_n\}_{n\geq 1} \subset S_{F,x}$ such that

$$z_n(t) = t^{\alpha-1}T_\alpha(t)(x_0 - g(x)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f_n(s) + B(s)u(s)]ds, \quad t \in J'. \tag{7}$$

By $(H_2)(ii)$, we deduce that the sequence $\{f_n\}_{n\geq 1}$ is integral-bounded. Moreover, since $\{f_n(t)\}_{n\geq 1} \subset F(t, x(t))$, it implies that, for *a.e.* $t \in J'$, $\{f_n(t)\}_{n\geq 1}$ is relatively compact in X. Hence, the sequence $\{f_n\}_{n\geq 1}$ is semi-compact in $L^1(J,X)$. According to Lemma 4, $\{f_n\}_{n\geq 1}$ is weakly compact in $L^1(J,X)$. Assume that the sequence $\{f_n\}_{n\geq 1}$ converges weakly to some $f \in L^1(J,X)$. Then by virtue of Lemma 6, there is a subsequence $\{g_n\}_{n\geq 1} \subseteq \overline{co}\{f_n\}_{n\geq 1}$ and g_n converges to f in strong topology. Since $S_{F,x}$ is convex, we obtain that

$$\{g_n\}_{n\geq 1} \subseteq S_{F,x}, \text{ and } f \in S_{F,x}.$$

For fixed $n \geq 1$ and every $t \in J'$, we obtain that

$$\|(t-s)^{\alpha-1}T_\alpha(t-s)f_n(s)\| \leq (t-s)^{\alpha-1}\frac{M}{\Gamma(\alpha)}m(s)\psi(\|x\|_{C_{1-\alpha}})$$

and

$$\int_0^t (t-s)^{\alpha-1}m(s)ds \leq \left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}} b^{\frac{p\alpha-1}{p}}\|m\|_{L^p} < +\infty.$$

Taking $n \to \infty$ on both sides of (7), the Lebesgue-dominated convergence theorem guarantees that

$$z(t) = t^{\alpha-1}T_\alpha(t)(x_0 - g(x)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f(s) + B(s)u(s)]ds.$$

Then $z \in \Psi(x)$.

Step 3. For each $r > 0$, let

$$B_r = \{x \in C_{1-\alpha}(J,X) | \|x\|_{C_{1-\alpha}} \leq r\}.$$

Then, B_r is clearly a nonempty convex, closed and bounded subset in $C_{1-\alpha}(J,X)$. We will prove that $\Psi(B_r) \subseteq B_r$ for some $r > 0$.

If it is not true, for any $r > 0$, there is $x^r \in B_r$ such that $\|\Psi x^r\|_{C_{1-\alpha}} > r$. By assumptions (H_2)–(H_4) and Lemma 1, there is $f^r \in S_{F,x^r}$ such that

$$
\begin{aligned}
r &< \sup_{t\in J}\{t^{1-\alpha}\|\Psi(x^r)(t)\|\} \\
&= \sup_{t\in J}\{t^{1-\alpha}\|t^{\alpha-1}T_\alpha(t)(x_0 - g(x^r)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f^r(s) + B(s)u(s)]ds\|\} \\
&\le \frac{M}{\Gamma(\alpha)}(\|x_0\| + M_g r + \|g(0)\|) + \frac{b^{1-\alpha}M\psi(r)}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}} b^{\frac{p\alpha-1}{p}}\|m\|_{L^p} \\
&\quad + \frac{b^{1-\alpha}M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}} b^{\frac{p\alpha-1}{p}}\|Bu\|_{L^p}.
\end{aligned}
$$

According to the above inequality, we obtain that $M^* \ge 1$, which is a contradiction to (6). Thus, $\Psi(B_r) \subseteq B_r$ for some $r > 0$.

Step 4. We claim that the operator Ψ is u.s.c. and condensing.

Let $\Psi = \Psi_1 + \Psi_2$, where the operators Ψ_1 and Ψ_2 are defined by

$$
\begin{aligned}
(\Psi_1 x)(t) &= t^{\alpha-1}T_\alpha(t)(x_0 - g(x)), \\
(\Psi_2 x) &= \{y \in C_{1-\alpha}(J,X) \mid y(t) = \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f(s) + B(s)u(s)]ds, f \in S_{F,x}, t \in J\}.
\end{aligned}
$$

By Corollary 2.2.1 of [12], we will show that Ψ_1 is a contraction operator and Ψ_2 is completely continuous.

It is easy to check that Ψ_1 is a contraction operator. Since for any $x, y \in B_r$, by (6), we have

$$
\begin{aligned}
\|\Psi_1 x - \Psi_1 y\|_{C_{1-\alpha}} &= \sup_{t\in J}\{t^{1-\alpha}\|t^{\alpha-1}T_\alpha(t)(g(x) - g(y))\|\} \\
&\le \frac{MM_g}{\Gamma(\alpha)}\|x - y\|_{C_{1-\alpha}} \\
&< \|x - y\|_{C_{1-\alpha}}.
\end{aligned}
$$

Next, we will show that Ψ_2 is completely continuous. By assumptions $(H_2), (H_4)$, and Remark 1, we obtain that $f + Bu \in L^p(J,X)$. Thus, in view of the assumption (H_1) and Lemma 10, we deduce the relative compactness of $\Psi_2(B_r)$. Thus, Ψ_2 is completely continuous.

Hence, Ψ is a condensing operator due to Corollary 2.2.1 of [12]. Now, it remains to prove that Ψ_2 has a closed graph.

Suppose $\{x_n\}_{n\ge1} \subset B_r$ with $x_n \to x_*$ as $n \to \infty$, $y_n \in \Psi_2(x_n)$ and $y_n \to y_*$ as $n \to \infty$. We shall show that $y_* \in \Psi_2(x_*)$. It follows from $y_n \in \Psi_2(x_n)$ that there is $f_n \in S_{F,x_n}$ such that

$$
y_n(t) = \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f_n(s) + B(s)u(s)]ds.
$$

We will show that there is $f_* \in S_{F,x_*}$ such that

$$
y_*(t) = \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f_*(s) + B(s)u(s)]ds.
$$

When $t \in J'$, we have

$$
\begin{aligned}
&\|y_n(t) - \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)B(s)u(s)ds - [y_*(t) - \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)B(s)u(s)ds]\| \\
&\to 0, \quad (n \to \infty).
\end{aligned}
$$

Consider an operator $\mathcal{F} : L^1(J, X) \to C_{1-\alpha}(J, X)$ defined by

$$(\mathcal{F}f)(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s) ds.$$

Then $\mathcal{F}$ is a continuous linear operator and $\mathcal{F} \circ S_F$ is a closed-graph operator via Lemma 3. Owing to the definition of $\mathcal{F}$, we know that

$$y_n(t) - \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B(s) u(s) ds \in \mathcal{F}(S_{F,x_n}).$$

By means of $x_n \to x_*, y_n \to y_*$ as $n \to \infty$ and using Lemma 3 again, we obtain that

$$y_*(t) - \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B(s) u(s) ds \in \mathcal{F}(S_{F,x_*}).$$

That is, there is $f_* \in S_{F,x_*}$ such that

$$y_*(t) - \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B(s) u(s) ds = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f_*(s) ds.$$

This fact implies that $y_* \in \Psi_2(x_*)$.

Hence, Ψ_2 has a closed graph. Therefore, Ψ_2 is u.s.c.

Thus, $\Psi = \Psi_1 + \Psi_2$ is condensing and u.s.c. Consequently, Ψ has one fixed point x in B_r due to Lemma 8, and the control system (4) has at least one mild solution. □

Under the case that g is completely continuous in $C_{1-\alpha}(J, X)$, we can also prove an existence theorem of (4).

$(H_3)'$ $g : C_{1-\alpha}(J, X) \to X$ is completely continuous.

Remark 2. *According to $(H_3)'$, $\{g(x) : x \in B_r\}$ is completely bounded. Thus, $\sup\limits_{x \in B_r} \|g(x)\|$ exists and* $\lim\limits_{r \to \infty} \frac{\sup\limits_{x \in B_r} \|g(x)\|}{r} = 0.$

Theorem 2. *Let assumptions (H_1), (H_2), $(H_3)'$, and (H_4) hold. If*

$$\frac{M}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha - 1} b\right)^{\frac{p-1}{p}} \|m\|_{L^p} \Lambda < 1,$$

then the fractional evolution system (4) has one mild solution in B_r.

Proof of Theorem 2. We only prove that $\Psi : B_r \to 2^{B_r} \setminus \{\emptyset\}$ is u.s.c. Because $g : C_{1-\alpha}(J, X) \to X$ is completely continuous, we easily obtain that Ψ_1 is completely continuous. Combining this fact with the complete continuity of Ψ_2, Ψ is completely continuous. By using the similar proof of Theorem 1, we deduce that Ψ has a closed graph. Furthermore, Ψ is u.s.c. owing to the fact that Ψ has compact values. Therefore, by applying Lemma 9, we conclude that Ψ has one fixed point x in B_r. This x is the mild solution of (4). □

Let the following condition is satisfied:

$(H_2)'$ $F : J \times X \to P_{cv,cp}(X)$ satisfies the following conditions:

(i) For each $x \in X$, $F(t, x)$ is measurable to t and for every $t \in J$, $F(t, x)$ is u.s.c. to x. For every $x \in X$,

$$S_{F,x} = \{f \in L^1(J, X) \mid f(t) \in F(t, x),\ a.e.\ t \in J\}$$

is nonempty.

(ii) There exists a function $m \in L^p(J, \mathbb{R}^+) (p > \frac{1}{\alpha})$ and a constant $\rho > 0$ such that

$$\|F(t, x)\| := \sup\{\|f(t)\| \mid f(t) \in F(t, x),\ t \in J\} \le m(t) + \rho t^{1-\alpha} \|x\|.$$

Then it is obvious that the assumption (H_2) is fulfilled. Hence, by Theorems 1 and 2, we can obtain the following corollaries.

Corollary 1. *Let $(H_1),(H_2)'$, (H_3) and (H_4) hold. If*

$$\rho < \frac{\Gamma(1+\alpha) - \alpha M M_g}{Mb}, \tag{8}$$

then the fractional evolution system (4) has one mild solution.

Corollary 2. *Let $(H_1),(H_2)'$, $(H_3)'$ and (H_4) hold. If*

$$\rho < \frac{\Gamma(1+\alpha)}{Mb}, \tag{9}$$

then the fractional evolution system (4) has one mild solution.

Remark 3. *In our existence results, we apply the fixed-point theorems of multi-valued mapping to prove existence theorems of the considered system when the nonlinearity f is not Lipschitz-continuous. Hence, our results partly extend [7–9].*

4. Existence of Optimal Control

In this part, we will demonstrate the existence of the optimal state-control pair of (4). Under the assumption that g is completely continuous or Lipschitz-continuous in $C_{1-\alpha}(J,X)$, we will utilize the technique established in the Ref. [9] to study the existence of optimal state-control pair of (4). By constructing minimizing sequences twice, we delete the Lipschitz continuity of the nonlinear term f, which is extensively used as an essential assumption in existing papers (see [7,8]), and without the uniqueness of mild solutions, we prove the existence of optimal state-control pair of (4). Hence, our results improve and generalize some related works.

Lemma 11. *Let the assumptions of Corollary 1 be fulfilled, and there is $k>0$ such that $\|g(x)\| \le k$ for every $x \in C_{1-\alpha}(J,X)$. Then, for fixed $u \in U_{ad}$, there is $R>0$ such that $\|x^u\|_{C_{1-\alpha}} \le R$, where x^u is the mild solution of (4) associated with $u \in U_{ad}$.*

Proof of Lemma 11. Since x^u is the mild solution of (4) corresponding to $u \in U_{ad}$, then there exists $f_{x^u} \in S_{F,x^u}$ such that

$$x^u(t) = t^{\alpha-1}T_\alpha(t)(x_0 - g(x^u)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f_{x^u}(s) + B(s)u(s)]ds, \quad t \in J'.$$

For any $t \in J'$, by $(H_2)',(H_4)$ and Lemma 1, we have

$$\begin{aligned} t^{1-\alpha}\|x^u(t)\| &= t^{1-\alpha}\|t^{\alpha-1}T_\alpha(t)(x_0 - g(x^u)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f_{x^u}(s) + B(s)u(s)]ds\| \\ &\le \|T_\alpha(t)(x_0 - g(x^u))\| + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)[f_{x^u}(s) + B(s)u(s)]\|ds \\ &\le \frac{M}{\Gamma(\alpha)}[\|x_0\| + k + (\frac{(p-1)b}{p\alpha-1})^{\frac{p-1}{p}}(\|m\|_{L^p} + \|Bu\|_{L^p})] \\ &\quad + \frac{M\rho b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{1-\alpha}\|x^u(s)\|ds. \end{aligned}$$

By employing Corollary 2 of [17], we get that

$$t^{1-\alpha}\|x^u(t)\| \le \frac{M}{\Gamma(\alpha)}[\|x_0\| + k + (\frac{(p-1)b}{p\alpha-1})^{\frac{p-1}{p}}(\|m\|_{L^p} + \|Bu\|_{L^p})]E_\alpha(Mb\rho) := R, \tag{10}$$

where $E_\alpha(\kappa) = \sum\limits_{n=0}^{\infty} \frac{\kappa^n}{\Gamma(n\alpha+1)}$ is the Mittag-Leffler function. Taking the supremum on both sides of (10), we have

$$\|x^u\|_{C_{1-\alpha}} = \sup_{t\in J}\{t^{1-\alpha}\|x^u(t)\|\} \le R.$$

This completes the proof of Lemma 11. $\square$

When g is not Lipschitz-continuous, by Corollary 2, similarly to Lemma 11, we deduce the following lemma.

Lemma 12. *Let the assumptions of Corollary 2 be fulfilled, and there is $k > 0$ such that $\|g(x)\| \le k$ for every $x \in C_{1-\alpha}(J, X)$. Then, for fixed $u \in U_{ad}$, there is $R > 0$ such that $\|x^u\|_{C_{1-\alpha}} \le R$, where x^u is the mild solution of (4) associated with $u \in U_{ad}$.*

Denote $\mathcal{S}(u) := \{x^u \in B_R : \ x^u \in \Psi(x^u)$ as the mild solution of (4) associated with $u \in U_{ad}$ in $B_R\}$ and $\mathcal{A}_{ad} := \{(x^u, u) : \ u \in U_{ad}, \ x^u \in \mathcal{S}(u)\}$. We call $\mathcal{A}_{ad}$ the set of admissible state-control pairs.

To consider the optimal control problem of (4), we investigate the limited Lagrange problem (P):

Seek a pair $(\bar{x}^{u_0}, u_0) \in \mathcal{A}_{ad}$ such that

$$\mathcal{J}(\bar{x}^{u_0}, u_0) := \inf\{\mathcal{J}(x^u, u) | (x^u, u) \in \mathcal{A}_{ad}\}, \quad \forall u \in U_{ad}, \tag{11}$$

where $\mathcal{J}(x^u, u)$ is the integral cost function given by

$$\mathcal{J}(x^u, u) = \int_0^b \mathcal{L}(t, x^u(t), u(t))dt.$$

If a pair $(\bar{x}^{u_0}, u_0) \in \mathcal{A}_{ad}$ satisfies the formula (11), then the limited Lagrange problem (P) is solvable. In this case, we call the pair $(\bar{x}^{u_0}, u_0) \in \mathcal{A}_{ad}$ the optimal state-control pair of (4).

To study the limited Lagrange problem (P), let $\mathcal{L} : J \times X \times Y \to \mathbb{R} \cup \{\infty\}$ satisfy the following condition:

(H_L): (i) $\mathcal{L} : J \times X \times Y \to \mathbb{R} \cup \{\infty\}$ is Borel measurable;

(ii) For each $x \in X$ and $a.e. t \in J$, $\mathcal{L}(t, x, \cdot)$ is convex on Y;

(iii) For $a.e. t \in J$, $\mathcal{L}(t, \cdot, \cdot)$ is sequentially lower semi-continuous on $X \times Y$;

(iv) There are constants $c_1 \ge 0, c_2 > 0$ and a function $\eta \in L^1(J, \mathbb{R}^+)$ such that

$$\mathcal{L}(t, x, u) \ge \eta(t) + c_1\|x\| + c_2\|u\|_Y^p, \quad \forall t \in J, x \in X, \ u \in Y.$$

Theorem 3. *Let (H_1), $(H_2)'$, (H_3), (H_4) and (H_L) hold. Moreover, the inequality (8) is satisfied and there is $k > 0$ such that $\|g(x)\| \le k$ for every $x \in C_{1-\alpha}(J, X)$. Then the limited Lagrange problem (P) has one optimal state-control pair. That is, there is one pair $(\bar{x}^{u_0}, u_0) \in \mathcal{A}_{ad}$ such that*

$$\mathcal{J}(\bar{x}^{u_0}, u_0) \le \mathcal{J}(x^u, u), \quad (x^u, u) \in \mathcal{A}_{ad}.$$

Proof of Theorem 3. For fixed $u \in U_{ad}$, let $\mathcal{J}(u) := \inf\limits_{x^u \in \mathcal{S}(u)} \mathcal{J}(x^u, u)$. The proof is completed in two steps.

Step 1. We will show that there is $\bar{x}^u \in \mathcal{S}(u)$ such that

$$\mathcal{J}(\bar{x}^u, u) = \inf_{x^u \in \mathcal{S}(u)} \mathcal{J}(x^u, u) = \mathcal{J}(u).$$

We suppose that $\mathcal{S}(u)$ has infinite elements. if $\inf\limits_{x^u \in \mathcal{S}(u)} \mathcal{J}(x^u, u) = +\infty$, there is nothing to prove. Thus, let $\mathcal{J}(u) = \inf\limits_{x^u \in \mathcal{S}(u)} \mathcal{J}(x^u, u) < +\infty$. Owing to (H_L)(iv), we have $\mathcal{J}(u) >$

$-\infty$. Employing the definition of the infimum, we get a sequence $\{x_n^u\}_{n\geq 1} \subseteq \mathcal{S}(u)$ such that

$$\lim_{n\to\infty} \mathcal{J}(x_n^u, u) = \mathcal{J}(u).$$

We will first prove that for fixed $u \in U_{ad}$, $\{x_n^u\}_{n\geq 1}$ is relatively compact in $C_{1-\alpha}(J, X)$. For fixed $n \geq 1$, since $(x_n^u, u) \in \mathcal{A}_{ad}$, there exists $f_{x_n^u} \in S_{F,x_n^u}$ such that

$$\begin{aligned} x_n^u(t) &= t^{\alpha-1}T_\alpha(t)(x_0 - g(x_n^u)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f_{x_n^u}(s) + B(s)u(s)]ds \\ &:= \mathcal{I}_1 x_n^u(t) + \mathcal{I}_2 x_n^u(t), \quad \forall t \in J'. \end{aligned} \tag{12}$$

From Lemma 10, $\{\mathcal{I}_2 x_n^u\}_{n\geq 1}$ is relatively compact in $C_{1-\alpha}(J, X)$.

In the following, the relative compactness of $\{\mathcal{I}_1 x_n^u\}_{n\geq 1}$ in $C_{1-\alpha}(J, X)$ is proved.

(i) We verify that $\{\cdot^{1-\alpha}\mathcal{I}_1 x_n^u\}_{n\geq 1}$ is uniformly bounded for every $n \geq 1$. In view of Lemma 1, we have

$$\|T_\alpha(t)(x_0 - g(x_n^u))\| \leq \frac{M}{\Gamma(\alpha)}(\|x_0\| + k),$$

that is,

$$\|\cdot^{1-\alpha}\mathcal{I}_1 x_n^u\| \leq \frac{M}{\Gamma(\alpha)}(\|x_0\| + k).$$

Therefore, $\{\cdot^{1-\alpha}\mathcal{I}_1 x_n^u\}_{n\geq 1}$ is uniformly bounded.

(ii) We show that $\{\cdot^{1-\alpha}\mathcal{I}_1 x_n^u\}_{n\geq 1}$ is equi-continuous for every $n \geq 1$.

When $t_1 \equiv 0$ and $0 < t_2 \leq b$, by the strong continuity of $T_\alpha(t)$ for $t \geq 0$ (see Lemma 1), we have

$$\|T_\alpha(t_2)(x_0 - g(x_n^u)) - T_\alpha(0)(x_0 - g(x_n^u))\| \to 0 \quad (t_2 \to 0).$$

When $0 < t_1 < t_2 \leq b$, by using the strong continuity of $T_\alpha(t)(t \geq 0)$ again, we have

$$\|T_\alpha(t_2)(x_0 - g(x_n^u)) - T_\alpha(t_1)(x_0 - g(x_n^u))\| \to 0 \quad (t_2 - t_1 \to 0).$$

These facts imply that $\{\cdot^{1-\alpha}\mathcal{I}_1 x_n^u\}_{n\geq 1}$ is equi-continuous for every $n \geq 1$.

(iii) For each $n \geq 1$, we prove that $V(t) = \{\omega(t) \mid \omega(t) = t^{1-\alpha}\mathcal{I}_1 x_n^u(t)\}_{n\geq 1}$ is relatively compact in X.

The relative compactness of $V(0)$ in X is obvious. Next, we prove the case of $t > 0$.

Let $0 < t \leq b$, for any $\delta > 0$, define $V^\delta(t) = \{\omega^\delta(t)\}_{n\geq 1}$, where

$$\begin{aligned} \omega^\delta(t) &= \alpha \int_\delta^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)(x_0 - g(x_n^u))d\theta \\ &= \alpha T(t^\alpha\delta) \int_\delta^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta - t^\alpha\delta)(x_0 - g(x_n^u))d\theta. \end{aligned}$$

Owing to the compactness of $T(t^\alpha\delta)$ for $t^\alpha\delta > 0$, the set $V^\delta(t)$ is relatively compact in X.

Moreover, we get that

$$\begin{aligned} \|\omega(t) - \omega^\delta(t)\| &= \|\alpha \int_0^\delta \theta\xi_\alpha(\theta)T(t^\alpha\theta)(x_0 - g(x_n^u))d\theta\| \\ &\leq \alpha M(\|x_0\| + k) \int_0^\delta \theta\xi_\alpha(\theta)d\theta \\ &\to 0 \quad (\delta \to 0). \end{aligned}$$

Hence, for $t \in J'$, the set $V(t)$ is relatively compact in X due to the fact that the relatively compact set $V^\delta(t)$ is arbitrarily close to it in X. Thanks to the Arzela-Ascoli theorem, $\{\cdot^{1-\alpha}\mathcal{I}_1 x_n^u\}_{n\geq 1}$ is a relatively compact subset in $C(J, X)$, which means that $\{\mathcal{I}_1 x_n^u\}_{n\geq 1}$ is a relatively compact subset in $C_{1-\alpha}(J, X)$, Thus, $\{x_n^u\}_{n\geq 1}$ is a relatively compact subset in

$C_{1-\alpha}(J, X)$ for fixed $u \in U_{ad}$. Without loss of generality, for fixed $u \in U_{ad}$, let $\lim_{n\to\infty} x_n^u \to \bar{x}^u$. The assumption $(H_2)'$ and Lemma 1 yield that

$$\|(t-s)^{\alpha-1}T_\alpha(t-s)[f_{x_n^u}(s) + B(s)u(s)]\| \leq \frac{M}{\Gamma(\alpha)}(t-s)^{\alpha-1}[m(s) + \rho R + \|B(s)u(s)\|]$$

and

$$\int_0^t (t-s)^{\alpha-1}[m(s) + B(s)u(s)]ds \leq (\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}} b^{\frac{p\alpha-1}{p}}[\|m\|_{L^p} + \|Bu\|_{L^p}] < +\infty.$$

Since the operator Ψ has a closed graph, taking $n \to \infty$ on both sides of (12), by the continuity of g, we deduce that

$$\bar{x}^u(t) = t^{\alpha-1}T_\alpha(t)(x_0 - g(\bar{x}^u)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f_{\bar{x}^u}(s) + B(s)u(s)]ds,$$

where $f_{\bar{x}^u} \in S_{F,\bar{x}^u}$. This fact yields that $\bar{x}^u \in \mathcal{S}(u)$.

It follows from (H_L) and the Balder theorem [18] that

$$\begin{aligned}
\mathcal{J}(u) &= \lim_{n\to\infty} \mathcal{J}(x_n^u, u) \\
&= \lim_{n\to\infty} \int_0^b \mathcal{L}(t, x_n^u(t), u(t))dt \\
&\geq \int_0^b \mathcal{L}(t, \bar{x}^u(t), u(t))dt \\
&= \mathcal{J}(\bar{x}^u, u) \\
&\geq \mathcal{J}(u).
\end{aligned}$$

Therefore, $\mathcal{J}(\bar{x}^u, u) = \mathcal{J}(u) = \inf_{x^u \in \mathcal{S}(u)} \mathcal{J}(x^u, u)$, which implies that, for each $u \in U_{ad}$, $\mathcal{J}(x^u, u)$ attains its minimum at $\bar{x}^u \in \mathcal{S}(u)$.

Step 2. We will prove that there is $u_0 \in U_{ad}$, satisfying $\mathcal{J}(u_0) = \inf_{u\in U_{ad}} \mathcal{J}(u)$.

Let $\inf_{u\in U_{ad}} \mathcal{J}(u) < +\infty$. By Step 1, we have $\inf_{u\in U_{ad}} \mathcal{J}(u) > -\infty$. According to the definition of infimum, there is $\{u_n\}_{n\geq 1} \subseteq U_{ad}$ satisfying $\lim_{n\to\infty} \mathcal{J}(u_n) = \inf_{u\in U_{ad}} \mathcal{J}(u)$. Since

$$\{u_n\}_{n\geq 1} \subseteq U_{ad} \subset L^p(J, Y)\ (p > \frac{1}{\alpha})$$

is bounded and $L^p(J, Y)$ $(p > \frac{1}{\alpha})$ is a reflexive Banach space, it follows that there is a subsequence, still denoted by $\{u_n\}_{n\geq 1}$, such that

$$u_n \xrightarrow{w} u_0 \quad (n \to \infty),$$

for some $u_0 \in L^p(J, Y)$. By utilizing the closedness and convexity of U_{ad}, we obtain that $u_0 \in U_{ad}$.

For every $n \geq 1$, by Step 1, we can find $\bar{x}^{u_n} \in \mathcal{S}(u_n)$ satisfying $\mathcal{J}(\bar{x}^{u_n}, u_n) = \mathcal{J}(u_n)$. Therefore, $(\bar{x}^{u_n}, u_n) \in \mathcal{A}_{ad}$ and satisfies

$$\begin{aligned}
\bar{x}^{u_n}(t) &= t^{\alpha-1}T_\alpha(t)(x_0 - g(\bar{x}^{u_n})) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f_{\bar{x}^{u_n}}(s) + B(s)u_n(s)]ds \\
&= \mathcal{P}_1\bar{x}^{u_n}(t) + \mathcal{P}_2\bar{x}^{u_n}(t),
\end{aligned} \tag{13}$$

where $f_{\bar{x}^{u_n}} \in S_{F,\bar{x}^{u_n}}$. By employing the technique used in Step 1, we know that $\{\mathcal{P}_1\bar{x}^{u_n}(\cdot)\}_{n\geq 1}$ is relatively compact in $C_{1-\alpha}(J,X)$. By Lemma 10, $\{\mathcal{P}_2\bar{x}^{u_n}(\cdot)\}_{n\geq 1}$ is relatively compact in $C_{1-\alpha}(J,X)$. Consequently, $\{\bar{x}^{u_n}\}_{n\geq 1}$ is relatively compact in $C_{1-\alpha}(J,X)$.

Without loss of generality, we suppose that there is a subsequence of $\{\bar{x}^{u_n}\}_{n\geq 1}$, labeled by itself, satisfying $\lim_{n\to\infty} \bar{x}^{u_n} = \bar{x}^{u_0}$. Hence, taking $n\to\infty$ in (13), since the operator Ψ has a closed graph, it follows from the continuity of g that

$$\bar{x}^{u_0}(t) = t^{\alpha-1}T_\alpha(t)(x_0 - g(\bar{x}^{u_0})) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f_{\bar{x}^{u_0}}(s) + B(s)u_0(s)]ds,$$

where $f_{\bar{x}^{u_0}} \in S_{F,\bar{x}^{u_0}}$. This fact implies that $(\bar{x}^{u_0}, u_0) \in \mathcal{A}_{ad}$ is an admissible state-control pair. According to the Balder theorem [18] and (H_L), we obtain that

$$\begin{aligned}
\inf_{u\in U_{ad}} \mathcal{J}(u) = \lim_{n\to\infty} \mathcal{J}(u_n) &= \lim_{n\to\infty} \mathcal{J}(\bar{x}^{u_n}, u_n) \\
&= \lim_{n\to\infty} \int_0^b \mathcal{L}(t, \bar{x}^{u_n}(t), u_n(t))dt \\
&\geq \int_0^b \mathcal{L}(t, \bar{x}^{u_0}(t), u_0(t))dt \\
&= \mathcal{J}(\bar{x}^{u_0}, u_0) = \mathcal{J}(u_0) \\
&\geq \inf_{u\in U_{ad}} \mathcal{J}(u).
\end{aligned}$$

Thus,

$$\mathcal{J}(u_0) = \inf_{u\in U_{ad}} \mathcal{J}(u).$$

Therefore, we have

$$\mathcal{J}(\bar{x}^{u_0}, u_0) = \inf_{u\in U_{ad}} \mathcal{J}(u) = \inf_{u\in U_{ad}} \inf_{x\in S(u)} \mathcal{J}(x^u, u).$$

That is, the limited Lagrange problem (P) has one optimal state-control pair $(\bar{x}^{u_0}, u_0)$ in $\mathcal{A}_{ad}$. □

Similarly, we can prove the following theorem when g is completely continuous.

Theorem 4. *Let (H_1), $(H_2)'$, $(H_3)'$, (H_4) and (H_L) hold. Moreover, the inequality (9) is satisfied, and there is $k>0$ such that $\|g(x)\| \leq k$ for every $x \in C_{1-\alpha}(J,X)$. Then the limited Lagrange problem (P) has one optimal state-control pair.*

Remark 4. *In Theorems 3 and 4, we demonstrate the existence of optimal state-control pair of (4) when the nonlinearity f is not Lipschitz-continuous. By constructing the minimizing sequence twice, we prove that the limited Lagrange problem (P) has one optimal state-control pair without the uniqueness of mild solutions of (4). The obtained theorems extend the main results of [7,8].*

Remark 5. *In the present work, by using the fixed-point theorems of multivalued mapping, the existence theorem on mild solutions as well as optimal controls are investigated for (4) under the assumption that g is completely continuous or Lipschitz-continuous. The obtained results are natural improvements of [9,14].*

5. An Application

Example 1. *We consider the fractional partial differential inclusion*

$$\begin{cases} {}^{L}D^{\frac{2}{3}}x(t,y) \in \partial_y^2 x(t,y) + \widehat{F}(t,x(t,y)) + \int_0^\pi q(y,\tau)u(\tau,t)d\tau, \quad t \in (0,1],\ y \in [0,\pi], \\ x(t,0) = x(t,\pi) = 0, \quad t \in (0,1], \\ I^{\frac{1}{3}}x(t,y)|_{t=0} + \sum_{i=0}^{n} k_g x(t_i,y) = x_0(y), \quad y \in [0,\pi], \end{cases} \tag{14}$$

where ${}^{L}D^{\frac{2}{3}}$ *stands for the* $\frac{2}{3}$*-order fractional derivative operator in a Riemann-Liouville sense, and* $I^{\frac{1}{3}}$ *represents the* $\frac{1}{3}$*-order Riemann-Liouville fractional integral operator,* $q \in C([0,\pi] \times [0,\pi], \mathbb{R})$*,* $k_g\mu(y) = \int_0^\pi k(y,\tau)\mu(\tau)d\tau,\ \mu \in L^2([0,\pi],\mathbb{R}), 0 < t_0 < t_1 < \cdots < t_n \leq 1.$

Let $X = Y = L^2([0,\pi],\mathbb{R})$. We define $A : D(A) \subset X \to X$ as follows:

$$Ax = \frac{\partial^2 x}{\partial y^2},$$

where $D(A) = \{x \in X \mid\ x'' \in X,\ x(0) = x(\pi) = 0, x \text{ and } x' \text{ are absolutely continuous}\}$. From [9], A generates a compact analytic semigroup $\{T(t), t \geq 0\}$ in X. This means that (H_1) holds. Let

$$\mathcal{J}(x,u) = \int_0^1 \int_0^\pi |x(t,y)|^2 dydt + \int_0^1 \int_0^\pi |u(t,y)|^2 dydt.$$

For any $t \in [0,1]$, let

$$\begin{aligned} x(t)(y) &:= x(t,y), \\ B(t)u(t)(y) &:= \int_0^\pi q(y,\tau)u(\tau,t)d\tau, \\ F(t,x(t))(y) &:= \widehat{F}(t,x(t,y)), \\ g(x)(y) &:= \sum_{i=0}^{n} k_g x(t_i,y). \end{aligned}$$

Then the differential inclusion (14) can be transformed into the form of abstract fractional evolution inclusion (4) and

$$\mathcal{J}(x,u) = \int_0^1 (\|u(t)\|^2 + \|x(t)\|^2)dt.$$

Now we take

$$M_g = (n+1)(\int_0^\pi \int_0^\pi k^2(y,\tau)d\tau dy)^{\frac{1}{2}}.$$

Then, the assumption (H_3) is satisfied. Let multi-valued mapping $\widehat{F}(t,x(t,y))$ satisfy the following condition:

(P) The multivalued mapping $\widehat{F}(t,\vartheta) : [0,1] \times X \to P_{cv,cp}(X)$ is satisfied:

(i) For every $\vartheta \in X$, $\widehat{F}$ is measurable to t and for each $t \in [0,1]$, $\widehat{F}$ is u.s.c. to ϑ. For every $\vartheta \in X$,

$$S_{F,\vartheta} = \{f \in L^1([0,1],X) | f(t) \in \widehat{F}(t,\vartheta),\ t \in [0,1]\}$$

is nonempty.

(ii) There are $m \in L^2([0,1],\mathbb{R}^+)$ and $0 < \rho < \Gamma(\frac{5}{3}) - \frac{2}{3}M_g$ such that

$$\|\widehat{F}(t,x(t,y))\| \leq m(t) + \rho t^{1-\alpha}\|x\|.$$

Thus, the assumption $(H_2)'$ is satisfied. According to Theorem 3, let (H_L) hold, then the fractional partial differential inclusion (14) has at least one mild solution, and the corresponding Lagrange problem (P) has one optimal state-control pair.

Remark 6. *Clearly, if we take $g(x)(y) := \sum_{i=0}^{n} C_i x(t_i, y)$ where $C_i > 0$ are constants for $i = 1, 2, \cdots, n$, then the assumption (H_3) is satisfied with $M_g = \sum_{i=0}^{n} C_i$.*

Remark 7. *In applications, we can give the multi-valued mapping $\widehat{F}(t, x(t, y))$ the specific expression, which satisfies the assumption (P). Then the assumption $(H_2)'$ can be satisfied.*

6. Conclusions

In this work, we first proved the existence theorem on mild solutions of (4) by using the theory of operator semigroups and fixed-point theorems of multi-valued mapping. Then, by constructing the minimizing sequence twice, the existence theorem on optimal state-control pairs is also obtained. It is worth emphasizing that we delete the uniqueness of mild solutions, which is an essential assumption in some existing papers. Hence, our work improves some of the existing literature. If the Riemann-Liouville fractional evolution inclusions involve time delays, it is difficult to prove the existence of mild solutions as well as the optimal control because the Riemann-Liouville fractional derivative is singular at $t = 0$. It is a valuable topic which we will study in the future.

Author Contributions: All authors contributed equally in writing this paper. All authors have read and agreed to the published version of the manuscript.

Funding: The research is supported by the National Natural Science Function of China (No. 12061062).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data can be made available upon request.

Acknowledgments: The authors thanks the anonymous reviewers for their valuable and helpful comments on improving this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Henderson, J.; Ouahab, A. Fractional functional differential inclusions with finite delay. *Nonlinear Anal.* **2009**, *70*, 2091–2105. [CrossRef]
2. Cernea, A. On a fractional integro-differential inclusion. *Electron. J. Qual. Theory Differ. Equ.* **2014**, *25*, 1–11. [CrossRef]
3. Ahmad, B.; Ntouyas, S.K. Fractional differential inclusions with fractional separated boundary conditions. *Fract. Calc. Appl. Anal.* **2012**, *15*, 362–382. [CrossRef]
4. Chaudhary, R.; Reich, S. Extremal mild solutions to fractional delay integro-differential equations with non-instantaneous impulses. *Appl. Anal.* **2021**. [CrossRef]
5. Du, M.L.; Wang, Z.H. Initialized fractional differential equations with Riemann-Liouville fractional-order derivative. *Eur. Phys. J. Spec. Top.* **2011**, *193*, 49–60. [CrossRef]
6. Zhou, Y.; Zhang, L.; Shen, X.H. Existence of mild solutions for fractional evolution equations. *J. Integral Equ. Appl.* **2013**, *25*, 557–586. [CrossRef]
7. Pan, X.; Li, X.W.; Zhao, J. Solvability and optimal controls of semilinear Riemann-Liouville fractional differential equations. *Abstr. Appl. Anal.* **2014**, *2014*, 216919. [CrossRef] [PubMed]
8. Kumar, S. Mild solution and fractional optimal control of semilinear system with fixed delay. *J. Optim. Theory Appl.* **2017**, *174*, 108–121. [CrossRef]
9. Lian, T.T.; Fan, Z.B.; Li, G. time-optimal controls for fractional differential systems with Riemann-Liouville derivatives. *Fract. Calc. Appl. Anal.* **2018**, *21*, 1524–1541. [CrossRef]
10. Hu, S.; Papageorgiou, N.S. *Handbook of Multivalued Analysis: Volume II: Applications*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2013.

11. Wang, J.R.; Zhou, Y. Existence and controllability results for fractional semilinear differential inclusions. *Nonlinear Anal.* **2011**, *12*, 3642–3653. [CrossRef]
12. Kamenskii, M.; Obukhovskii, V.; Zecca, P. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*; De Gruyter: Berlin, Germany, 2001.
13. Alsarori, N.A.; Ghadle, K.P. On the mild solution for nonlocal impulsive fractional semilinear differential inclusion in Banach spaces. *J. Math. Model.* **2018**, *6*, 239–258. [CrossRef]
14. Chang, Y.K.; Pei, Y.T.; Ponce, R. Existence and optimal controls for fractional stochastic evolution equations of Sobolev type via fractional resolvent operators. *J. Optim. Theory Appl.* **2019**, *182*, 558–572. [CrossRef]
15. Deimling K. *Multivalued Differential Equations*; De Gruyter: Berlin, Germany, 1992.
16. Bohnenblust, H.F.; Karlin, S. *On a Theorem of Ville, in: Contributions to the Theory of Games*; Princeton University Press: Princeton, NJ, USA, 1950; pp. 155–160.
17. Ye, H.P.; Gao, J.M.; Ding, Y.S. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **2007**, *328*, 1075–1081. [CrossRef]
18. Balder, E.J. Necessary and sufficient conditions for L_1-strong-weak lower semicontinuity of integral functional. *Nonlinear Anal.* **1987**, *11*, 1399–1404. [CrossRef]