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Product of Hessians and Discriminant of Critical Points of Level Function Attached to Sphere Arrangement

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Abstract: We state the product formulae of the values of the levels of functions at critical points involved in asymptotic behaviors of hypergeometric integrals associated with symmetric arrangements of three-dimensional spheres. We show, in an explicit way, how the product of the Hessian, regarding the level functions at all critical points, is related to the behavior of its critical points. We also state two conjectures concerning the same problem associated with general hypersphere arrangements.

Keywords: hypergeometric integrals; sphere arrangements; asymptotic behavior; critical points; norms of Hessian; discriminants; interpolation curves

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1. Introduction

For a given $\alpha_{j0} \in \mathbb{R}$ and $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn}) \in \mathbb{R}^n$ $(j = 1, 2, \ldots, n+1)$, let f_j be real quadratic polynomials in $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ specified by

$$f_j(x) := (x, x) + 2(\alpha_j, x) + \alpha_{j0} = |x + \alpha_j|^2 - |\alpha_j|^2 + \alpha_{j0},$$

where $(x, y) := \sum_{\nu=1}^n x_\nu y_\nu$ and $|x|^2 := (x, x)$ for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Let O_j be the point $-\alpha_j \in \mathbb{R}^n$, which is the center of the hypersphere $\{x \in \mathbb{R}^n \mid f_j(x) = 0\}$. The radius $r_i > 0$ of S_i and the distance $\rho_{jk} > 0$ between O_j and O_k are given by

$$r_j^2 = -\alpha_{j0} + |\alpha_j|^2 \quad \text{and} \quad \rho_{jk}^2 = |\alpha_j - \alpha_k|^2,$$

respectively. In this paper we assume that the points $O_1, \ldots, O_{n+1} \in \mathbb{R}^n$ make an n-simplex, so that without loss of generality, we may assume the following:

$$\alpha_{j\nu} = 0 \quad (1 \le j \le n+1,\ n-j+1 < \nu \le n) \quad \text{and} \quad \alpha_{j,n-j+1} > 0 \quad (1 \le j \le n), \tag{1}$$

i.e.,

$$\begin{aligned}
O_1 &= -\alpha_1 = -(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1,n-2}, \alpha_{1,n-1}, \alpha_{1n}), & \alpha_{1n} &> 0,\\
O_2 &= -\alpha_2 = -(\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2,n-2}, \alpha_{2,n-1}, 0), & \alpha_{2,n-1} &> 0,\\
O_3 &= -\alpha_3 = -(\alpha_{31}, \alpha_{32}, \ldots, \alpha_{3,n-2}, 0, 0), & \alpha_{3,n-2} &> 0,\\
&\vdots & &\vdots\\
O_n &= -\alpha_n = -(\alpha_{n1}, 0, \ldots, 0, 0, 0), & \alpha_{n1} &> 0,\\
O_{n+1} &= -\alpha_{n+1} = (0, 0, \ldots, 0, 0, 0).
\end{aligned}$$

Here, we consider the $n-1$ dimensional hyperspheres $f_j(x) = 0$ in $\mathbb{C}^n$, i.e., we define S_j as

$$S_j = \{x \in \mathbb{C}^n \mid f_j(x) = 0\}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_{n+1}) \in \mathbb{R}^{n+1}$ let $\Phi(x) = \Phi(x_1, \ldots, x_n)$ be a multiplicative meromorphic function on $\mathbb{C}^n$ specified by

$$\Phi(x) := \prod_{j=1}^{n+1} f_j(x)^{\lambda_j}.$$

We set $X := \mathbb{C}^n - \bigcup_{j=1}^{n+1} S_j$. For $0 \le r \le n$, we denote by $\Omega^r = \Omega^r(X, \star \bigcup_{j=1}^{n+1} S_j)$ the space of rational r-forms on X whose singularities all lie in the set $\bigcup_{j=1}^{n+1} S_j$. For the complex

$$\Omega^{\cdot} : 0 \to \Omega^0 \xrightarrow{\nabla} \Omega^1 \xrightarrow{\nabla} \Omega^2 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n \xrightarrow{\nabla} 0,$$

where $\nabla : \Omega^r \to \Omega^{r+1}$ is the covariant derivation given by

$$\nabla\psi := d\psi + d\log\Phi \wedge \psi \quad (\psi \in \Omega^r),$$

the rth twisted de Rham cohomology $H^r_\nabla(X, \Omega^{\cdot})$ is defined by

$$H^r_\nabla(X, \Omega^{\cdot}) := \mathrm{Ker}(\nabla : \Omega^r \to \Omega^{r+1}) / \mathrm{Im}(\nabla : \Omega^{r-1} \to \Omega^r).$$

See [1,2] for more details. For $\varphi(x)dx_1 \wedge \cdots \wedge dx_n \in \Omega^n$ as a representative in $H^n_\nabla(X, \Omega^{\cdot})$, the hypergeometric integral associated with $\Phi(x)$ over an n-twisted cycle $\mathfrak{z}$ is defined as

$$\mathcal{J}_\lambda(\varphi; \mathfrak{z}) := \int_{\mathfrak{z}} \Phi(x)\varphi(x)dx_1 \wedge \cdots \wedge dx_n = \int_{\mathfrak{z}} \varphi(x) \prod_{j=1}^{n+1} f_j(x)^{\lambda_j} dx_1 \wedge \cdots \wedge dx_n.$$

For an arbitrary integer N, we put $\lambda = N\mu + \lambda'$, where $\mu = (\mu_1, \ldots, \mu_{n+1}) \in \mathbb{Z}^{n+1}$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_{n+1}) \in \mathbb{R}^{n+1}$ are fixed. When $\varphi(x)$ is independent of N, we are interested in the asymptotic behavior of the following integral as $N \to \infty$ in the direction μ:

$$\mathcal{J}_{N\mu+\lambda'}(\varphi; \mathfrak{z}) = \int_{\mathfrak{z}} e^{NF(x)} \varphi(x) \prod_{j=1}^{n+1} f_j(x)^{\lambda'_j} dx_1 \wedge \cdots \wedge dx_n,$$

where

$$F(x) = \sum_{j=1}^{n+1} \mu_j \log f_j(x).$$

For the real valued level function $\mathfrak{Re}F$ corresponding to $|e^{NF(x)}| = e^{N\mathfrak{Re}F(x)}$, the singularity of the gradient flow of $\boldsymbol{v} = \mathrm{grad}\, \mathfrak{Re}F$ in X coincides with the set of its critical points given by

$$\mathcal{C} := \{x \in X \,|\, dF(x) = \sum_{j=1}^{n+1} \mu_j d\log f_j(x) = 0\}. \tag{2}$$

By definition dF is explicitly written as

$$dF = \sum_{j=1}^{n+1} \mu_j d\log f_j = \sum_{\nu=1}^{n} \Big(\sum_{j=1}^{n+1} \frac{\mu_j}{f_j} \frac{\partial f_j}{\partial x_\nu} \Big) dx_\nu = \sum_{\nu=1}^{n} \Big(\sum_{j=1}^{n+1} \mu_j \frac{2(x_\nu + \alpha_{j\nu})}{f_j} \Big) dx_\nu.$$

In this paper, we assume that the direction μ of the asymptotic behavior is specifically fixed as

$$\mu = \mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}^{n+1}.$$

Then, the set $\mathcal{C}$ of critical points given by (2) is rewritten as

$$\mathcal{C} = \{x \in X \,|\, G_1(x) = G_2(x) = \cdots = G_n(x) = 0\}, \tag{3}$$

where

$$G_\nu(x) := \frac{1}{2}\frac{\partial F}{\partial x_\nu} = \sum_{j=1}^{n+1} \frac{x_\nu + \alpha_{j\nu}}{f_j(x)} \qquad (\nu = 1, 2, \ldots, n).$$

The functions $G_\nu(x)$ $(\nu = 1, 2, \ldots, n)$ play an important role in describing the asymptotic behavior of $J_{N\mathbf{1}+\lambda'}(\varphi)$ for large N (see Theorems 4.9 and 4.10 in [1]). The Hessian of F is defined by $\mathrm{Hess}(F) := \det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{1\le i,j\le n}$, and is expressed as the Jacobian of $G_\nu(x)$ $(\nu = 1, 2, \ldots, n)$, i.e.,

$$\frac{1}{2^n}\mathrm{Hess}(F) = \frac{\partial(G_1, G_2, \ldots, G_n)}{\partial(x_1, x_2, \ldots, x_n)}. \tag{4}$$

According to the method of steepest decent (saddle-point method), if an n-twisted cycle $\mathfrak{z}$ includes the critical point (saddle point) $\mathbf{c} \in \mathcal{C}$ which gives the maximal value of $\mathfrak{Re}F$ on $\mathfrak{z}$, then the asymptotic behavior of $\mathcal{J}_{N\mathbf{1}+\lambda'}(\varphi; \mathfrak{z})$ $(N \to \infty)$ is expressed as

$$\mathcal{J}_{N\mathbf{1}+\lambda'}(\varphi; \mathfrak{z}) \sim \Phi(\mathbf{c})\varphi(\mathbf{c})\sqrt{\frac{\pi^n}{(-N)^n \mathrm{Hess}(F)|_{x=\mathbf{c}}}} \quad (N \to \infty). \tag{5}$$

If S_j $(1 \le j \le n+1)$ are located as general position in $\mathbb{C}^n$, for generic exponents $\lambda = (\lambda_1, \ldots, \lambda_{n+1}) \in \mathbb{R}^{n+1}$ the dimension of the nth cohomology $H^n_\nabla(X, \Omega^\cdot) = \Omega^n / \nabla\Omega^{n-1}$ as a $\mathbb{C}$-linear space is known to be $\kappa = 2^{n+1} - 1$, i.e., $\dim_{\mathbb{C}} H^n_\nabla(X, \Omega^\cdot) = \kappa$ (see [3,4] for hypergeometric integrals associated with hypersphere arrangements). The basis of $H^n_\nabla(X, \Omega^\cdot)$ can generally be chosen as an NBC (non-broken circuit) basis for a commutative algebra associated with hypersphere arrangement (see [5]). The number κ also coincides with the absolute value of the Euler number of X. It also equals the number of the critical points of the function F specified by (3) provided that they are non-degenerate and different from each other. We denote by $\mathbf{c}_j$ $(1 \le j \le \kappa)$ all of the critical points (real or imaginary) in X, i.e., $\mathcal{C} = \{\mathbf{c}_j \,|\, 1 \le j \le \kappa\}$. For a rational function φ on X, we denote by $\mathcal{N}(\varphi)$ the product of the critical values at all points in $\mathcal{C}$, i.e.,

$$\mathcal{N}(\varphi) := \prod_{j=1}^{\kappa} \varphi(\mathbf{c}_j),$$

which is called the *norm* of φ. Here, we state our first claim as follows.

Conjecture 1. *Suppose that S_j $(1 \le j \le n+1)$ are located as general position in $\mathbb{C}^n$. Then,*

$$\mathcal{N}(\mathrm{Hess}(F)) \neq 0$$

if and only if every critical point in $\mathcal{C}$ is different from each other.

Remark 1. *When Conjecture 1 holds true, if the n-dimensional stable Lagrangian cycles $\mathfrak{z}_j$ include $\mathbf{c}_j \in \mathcal{C}$ as their limiting points, respectively, then by (5) the pairing*

$$\left(\mathcal{J}_{N\mathbf{1}+\lambda'}(\varphi_i; \mathfrak{z}_j)\right)_{i,j=1}^{\kappa},$$

where $\varphi_i(x)dx_1 \wedge \cdots \wedge dx_n \in \Omega^n$ are representatives in $H^n_\nabla(X, \Omega^\cdot)$, satisfies the following asymptotic behavior

$$\det\left(\mathcal{J}_{N\mathbf{1}+\lambda'}(\varphi_i; \mathfrak{z}_j)\right)_{i,j=1}^{\kappa} \sim \frac{\prod_{j=1}^{\kappa} \Phi(\mathbf{c}_j)\pi^{n/2}}{\sqrt{(-N)^n \mathcal{N}(\mathrm{Hess}(F))}} \det\left(\varphi_i(\mathbf{c}_j)\right)_{i,j=1}^{\kappa} \quad (N \to \infty),$$

which gives a criterion for $\mathbb{C}$-linear independence of the set $\{\mathcal{J}_\lambda(\varphi_i; \mathfrak{z}) \,|\, 1 \le i \le \kappa\}$. This is a rough explanation as to why we consider Conjecture 1.

In this paper, one of our aims is to confirm Conjecture 1 when $n = 3$ for a special pyramid $\triangle O_1O_2O_3O_4$. The result is stated in Theorem 4. For this purpose, we need to compute $\mathcal{N}(f_j)$ and $\mathcal{N}(\sum_{j=1}^{n+1} f_j^{-1})$. In order to state the explicit expressions of $\mathcal{N}(f_j)$ and $\mathcal{N}(\sum_{j=1}^{n+1} f_j^{-1})$ we introduce the *Cayley–Menger determinants* as follows.

Consider the $(n+3)\times(n+3)$ symmetric matrix $B = (b_{ij})_{i,j=0,\star,1,2,\dots,n+1}$, whose entries are given by $b_{00} = 0$, $b_{\star\star} = 0$, $b_{0\star} = b_{0j} = 1$ $(1 \le j \le n+1)$, $b_{1\star} = r_j^2$ $(1 \le j \le n+1)$, $b_{ij} = \rho_{ij}^2$ $(1 \le i,j \le n+1)$, i.e.,

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & r_1^2 & r_2^2 & r_3^2 & r_4^2 & \cdots & r_{n+1}^2 \\ 1 & r_1^2 & 0 & \rho_{12}^2 & \rho_{13}^2 & \rho_{14}^2 & \cdots & \rho_{1,n+1}^2 \\ 1 & r_2^2 & \rho_{21}^2 & 0 & \rho_{23}^2 & \rho_{24}^2 & \cdots & \rho_{2,n+1}^2 \\ 1 & r_3^2 & \rho_{31}^2 & \rho_{32}^2 & 0 & \rho_{34}^2 & \cdots & \rho_{3,n+1}^2 \\ 1 & r_4^2 & \rho_{41}^2 & \rho_{42}^2 & \rho_{43}^2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \rho_{n,n+1}^2 \\ 1 & r_{n+1}^2 & \rho_{n+1,1}^2 & \rho_{n+1,2}^2 & \rho_{n+1,3}^2 & \cdots & \rho_{n+1,n}^2 & 0 \end{pmatrix}.$$

The Cayley–Menger determinants are defined as the minors of the matrix of B. See [4].

Definition 1. *Denote by $\rho_{\star j} = \rho_{j\star}$ the radius r_j for $j \in \{1,2,\dots,n+1\}$ or 0 for $j = \star$. The determinant*

$$B\begin{pmatrix} 0 & J \\ 0 & K \end{pmatrix} = B\begin{pmatrix} 0\, j_1 \cdots j_p \\ 0\, k_1 \cdots k_p \end{pmatrix} := \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & \rho_{j_1k_1}^2 & \cdots & \rho_{j_1k_p}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{j_pk_1}^2 & \cdots & \rho_{j_pk_p}^2 \end{vmatrix}$$

is called the Cayley–Menger determinant, where $J = \{j_1,\dots,j_p\}$ and $K = \{k_1,\dots,k_p\}$ denote two subsets of the indices in $\{\star,1,\dots,n+1\}$. We simply write $B(0\,J)$ instead of $B\begin{pmatrix} 0 & J \\ 0 & J \end{pmatrix}$. Notice that $B(0j) = -1$, $B(0\star j) = 2r_j^2 > 0$, $B(0jk) = 2\rho_{jk}^2 > 0$ and

$$B(0\star j\,k) = -(\rho_{jk}+r_j-r_k)(\rho_{jk}-r_j+r_k)(-\rho_{jk}+r_j+r_k)(\rho_{jk}+r_j+r_k),$$
$$B(0\,j\,k\,l) = -(\rho_{jk}+\rho_{jl}-\rho_{kl})(\rho_{jk}-\rho_{jl}+\rho_{kl})(-\rho_{jk}+\rho_{jl}+\rho_{kl})(\rho_{jk}+\rho_{jl}+\rho_{kl}).$$

Using the Cayley–Menger determinants, the latter assumption of (1) is rewritten as

$$\prod_{k=1}^{n-j+1} \alpha_{n+1-k,k} = \sqrt{\frac{(-1)^{n-j} B(0\,j\,j{+}1 \cdots n{+}1)}{2^{n-j+1}}} > 0$$

for $j = 1,2,\dots,n$.

Throughout this paper, we suppose the condition

$$(\mathcal{H}_0): \qquad B(0\,J) \neq 0 \quad \text{and} \quad B(0\star J) \neq 0$$

for $J = \{j_1 < j_2 < \cdots < j_p\} \subset \{1,2,\dots,n+1\}$. The condition $(\mathcal{H}_0)$ gives the moduli space of arrangement of n dimensional real hyperspheres in general position in $\mathbb{C}^n$.

Denote D_j the n dimensional real open ball with boundary $\mathfrak{Re}S_j$ in $\mathbb{R}^n$, where $\mathfrak{Re}S_j = \{x \in \mathbb{R}^n \mid f_j(x) = 0\}$. One sees that

$$(-1)^{|J|}B(0J) > 0 \quad \text{for all non-empty } J \subset \{1,2,\ldots,n+1\},$$

where $|J|$ is the cardinality of J. In general every real critical point lies in $\bigcup_{j=1}^{n+1} D_j$ or the real simplex $\triangle O_1 \ldots O_{n+1}$. If further

$$(-1)^{|J|}B(0\star J) < 0 \quad \text{for all non-empty } J \subset \{1,2,\ldots,n+1\},$$

then every intersection $(\bigcap_{j\in J} D_j) \cap (\bigcap_{k\in J^c} \overline{D}_k^c)$ is not empty, where $\overline{D}_k^c$ means the complement of the closure $\overline{D}_k$. There exists a unique critical point of $\mathfrak{Re}F$ (and so of F) there.

We now state the other claim of ours for the explicit forms of $\mathcal{N}(f_j)$ and $\mathcal{N}(\sum_{j=1}^{n+1} f_j^{-1})$ using the Cayley–Menger determinants.

Conjecture 2. *For $j = 1,\ldots,n+1$ let $I_{\hat{j}}$ be the set $\{1,2,\ldots,n+1\} - \{j\}$. Under the condition $(\mathcal{H}_0)$, the norms $\mathcal{N}(f_j)$ and $\mathcal{N}(\sum_{j=1}^{n+1} f_j^{-1})$ are expressed as*

$$\mathcal{N}(f_j) = \frac{B(0\star j)}{2(n+1)^{2^n}} \prod_{p=1}^{n} \prod_{\substack{K\subset I_{\hat{j}} \\ |K|=p}} \frac{B(0\star jK)}{B(0K)} \quad (1 \le j \le n+1), \tag{6}$$

$$\mathcal{N}\Big(\sum_{j=1}^{n+1} \frac{1}{f_j}\Big) = 2(n+1)^{2^{n+1}-1} \prod_{p=1}^{n+1} \prod_{\substack{K\subset I_{\hat{j}} \\ |K|=p}} \frac{B(0K)}{B(0\star K)}. \tag{7}$$

Remark 2. *We call φ the* unit *relative to the set of all critical points $\mathcal{C}$ if $\mathcal{N}(\varphi)$ does not vanish under the condition $(\mathcal{H}_0)$. In this sense f_j and $\sum f_j^{-1}$ are all units if Conjecture 2 holds for them.*

Remark 3. *If $n = 2$, (6) of Conjecture 2 implies*

$$\begin{aligned}
\mathcal{N}(f_1) &= \frac{2r_1^2}{2\cdot 3^4} B(0\star 12)B(0\star 13)\frac{B(0\star 123)}{2\rho_{23}^2},\\
\mathcal{N}(f_2) &= \frac{2r_2^2}{2\cdot 3^4} B(0\star 12)B(0\star 23)\frac{B(0\star 123)}{2\rho_{13}^2},\\
\mathcal{N}(f_3) &= \frac{2r_3^2}{2\cdot 3^4} B(0\star 13)B(0\star 23)\frac{B(0\star 123)}{2\rho_{12}^2},
\end{aligned}$$

which have been confirmed under the situation $r_1 = r_2 = r_3$. See Theorems 5.19 in [6]. Moreover, using these formulae, consequently $\mathcal{N}(\mathrm{Hess}(F))$ is also obtained explicitly when $\triangle O_1O_2O_3$ is an arbitrary isosceles triangle under $r_1 = r_2 = r_3$. Thus, Conjecture 1 is confirmed when $n = 2$, $r_1 = r_2 = r_3$ and $\rho_{12} = \rho_{13}$. See also Corollary 7.16 in [6] for details.

Remark 4. *If $n = 3$, (6) and (7) of Conjecture 2 are written as*

$$\begin{aligned}
\mathcal{N}(f_1) = \frac{2r_1^2}{2\cdot 4^8} &\frac{B(0\star 12)}{-1}\frac{B(0\star 13)}{-1}\frac{B(0\star 14)}{-1}\\
&\times \frac{B(0\star 123)}{2\rho_{23}^2}\frac{B(0\star 124)}{2\rho_{24}^2}\frac{B(0\star 134)}{2\rho_{34}^2}\frac{B(0\star 1234)}{B(0234)},
\end{aligned}$$

$$\mathcal{N}\Big(\sum_{j=1}^{4}\frac{1}{f_j}\Big) = \frac{2\cdot 4^{15}}{2^4(r_1r_2r_3r_4)^2}\frac{B(01234)}{B(0\star 1234)} \prod_{1\le i<j<k\le 4}\frac{B(0\,ijk)}{B(0\star ijk)} \prod_{1\le j<k\le 4}\frac{2\rho_{jk}^2}{B(0\star jk)}.$$

In Section 5 , we shall prove Conjecture 2 in a special pyramid case ($n = 3$) when the base triangle $\triangle O_1O_2O_3$ is regular and each edge length ρ_{j4} ($1 \le j \le 3$) and each radius r_j ($1 \le j \le 4$) are all equal respectively (see Corollary 2).

This paper is organized as follows. In Section 2 we confirm that Conjecture 2 holds for the very special case, where $\triangle O_1 \dots O_{n+1}$ is the regular simplex and all hyperspheres S_j have the same radius. The result is stated as Theorem 1. From Section 3 to Section 7, we discuss three-dimensional case. In Section 3, we introduce a special coordinate system (denoted by $t = (t_1, t_2, t_3) \in \tilde{X} \subset \mathbb{C}^3$) attached to a tetrahedron, i.e., the fundamental three-dimensional simplex $\triangle O_1O_2O_3O_4$, and by means of the projective map $\iota: \tilde{X} \to X; t \mapsto x$ we transfer the terms relative to $\mathcal{C}$ to those of $\tilde{\mathcal{C}} = \iota^{-1}\mathcal{C} = \{t \in \tilde{X} \,|\, \tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3 = 0\}$, where $\tilde{g}_j$ are polynomials in t of degree 3 given by (48). In particular, we call t_1 the *basic parameter*, and a rational curve $t_2 = \omega_2(t_1), t_3 = \omega_3(t_1)$ passing through specified points in $\tilde{\mathcal{C}}$, which we call the *interpolation curve* of those points in $\tilde{\mathcal{C}}$, plays an important role in this paper. In Section 4, we restrict ourselves to a special symmetric case when $\triangle O_1O_2O_3O_4$ is a pyramid with an axis of symmetry whose base triangle $\triangle O_1O_2O_3$ is regular and all spheres have the same radius. The critical points are classified into typical four parts $\tilde{\mathcal{C}}_j$ ($1 \le j \le 4$). In Section 5, under the assumption $\rho_{12} \ne \rho_{14}$, for each $\tilde{\mathcal{C}}_j$ the interpolation curve $\omega(t_1) = (t_1, \omega_2(t_1), \omega_3(t_1)) \in \tilde{X}$ is still significant, and we calculate the norms of several linear functions on $\tilde{X}$ solving the defining equation $\psi_j(t_1) = 0$ of $\tilde{\mathcal{C}}_j$, where $\psi_j(t_1)$ is the *characteristic function* of $\tilde{\mathcal{C}}_j$ defined by $\psi_j(t_1) := \tilde{g}_1(\omega(t_1))$. Using these norms, we evaluate $\mathcal{N}(f_j)$ and $\mathcal{N}(\sum_{j=1}^4 f_j^{-1})$ and, thus, prove Conjecture 2 for our symmetric special case. See Corollary 2. In Section 6, we consider the other case, $\rho_{12} = \rho_{14}$, i.e., the case where $\triangle O_1O_2O_3O_4$ is the regular tetrahedron. The results in this section compensate for those in Section 5. In Section 7, under the same constraint as Section 5, we shall show the explicit formula for the norm $\mathcal{N}(\mathrm{Hess}(F))$ of the Hessian of the level function F relative to the critical points $\mathcal{C}$. The formula is expressed in terms of the *discriminant* associated with $\mathcal{C}$ (or equivalently $\tilde{\mathcal{C}}$), see the invariants Δ_2, Δ_3 and Δ_4 in Theorem 4. Consequently, we also prove Corollary 8, which is Conjecture 1 for our symmetric special case. The method of proving Theorem 4 and Corollary 8 can be regarded as a generalization of the Routh–Hurwitz scheme to a case of several variables. This scheme is stated in terms of Hankel matrices and a system of resultants related to a pair of polynomials in a single variable (see Chapter XV in [7] or Chapter X in [8], for example).

We note in passing that there is an analogy between the notions "differente", "discriminant" in the theory of algebraic numbers, and the ones "Hessian", "norm of Hessian" in our present situation, respectively (see [9–13] for general definition of "discriminant" of algebraic numbers, algebraic functions, or more generally commutative algebra).

In [14], there is an interesting argument on zero points of coquaternionic polynomials using characteristic polynomials, which enable to linearize the problem by Euclidean algorithm. Moreover, our argument goes along the similar line in a more complicated situation.

2. Configuration of Critical Points in the Case of Regular Simplex

In this section, we consider the very special case when $\triangle O_1 \dots O_{n+1}$ is a regular simplex and all hyperspheres S_j have the same radius:

$$\rho_{jk}^2 = \rho^2 \quad (1 \le j < k \le n+1), \qquad r_j^2 = r^2 \quad (1 \le j \le n+1). \tag{8}$$

In this case, all of the critical points can be explicitly described. The total number of critical points is equal to $2^{n+1} - 1$.

Denote by $I = \{1, 2, \dots, n+1\}$. For the set $J = \{j_1, \dots, j_p\} \subset I$, let W_J be the central point of each $(|J| - 1)$-dimensional face $\triangle O_J := \triangle O_{j_1} \dots O_{j_p}$ defined by

$$W_J := \frac{1}{|J|} \sum_{j \in J} O_j,$$

where $|J|$ denotes the size of J. In particular, we see that $W_j = O_j$ $(1 \le j \le n+1)$ and we simply denote by W the center W_I of $\triangle O_1 \dots O_{n+1}$. For $J \subset I$ we denote by J^c the complement $I - J$. For $\emptyset \subsetneq J \subsetneq I$ let $\mathfrak{l}(W_J, W_{J^c})$ be the straight line passing through two points W_J and W_{J^c}, which is parameterized by

$$\mathfrak{l}(W_J, W_{J^c}) : x = (1-\tau)W_J + \tau W_{J^c} \quad (-\infty < \tau < \infty). \tag{9}$$

An arbitrary line $\mathfrak{l}(W_J, W_{J^c})$ passes through the center $W = W_N$ of $\triangle O_1 \dots O_{n+1}$, so that

$$\{W\} = \bigcap_{\emptyset \subsetneq J \subsetneq I} \mathfrak{l}(W_J, W_{J^c}).$$

Symmetry argument shows that every linear p-dimensional real affine subspace $\mathfrak{p}_{J,J^c}$ spanned by the real p-simplex $\triangle O_{j_1} \dots O_{j_p} W_{J^c}$ is preserved by the vector field grad $\mathfrak{Re}F$. In particular the real straight line $\mathfrak{l}(W_J, W_{J^c})$ is a trajectory of grad $\mathfrak{Re}F$.

We now consider the critical points on $\mathfrak{l}(W_J, W_{J^c})$ for $\emptyset \subsetneq J \subsetneq I$.

Lemma 1. *Suppose that $|J| = p$ $(1 \le p \le n)$. Then f_j on $\mathfrak{l}(W_J, W_{J^c})$ as a function of τ is expressed as*

$$\begin{aligned} f_j(x) &= f_j((1-\tau)W_J + \tau W_{J^c}) \\ &= \begin{cases} \left(\dfrac{n+1}{p(n+1-p)}\tau^2 + \dfrac{p-1}{p} \right) \dfrac{\rho^2}{2} - r^2 & \text{if} \quad j \in J, \\ \left(\dfrac{n+1}{p(n+1-p)}(\tau-1)^2 + \dfrac{n-p}{n+1-p} \right) \dfrac{\rho^2}{2} - r^2 & \text{if} \quad j \in J^c. \end{cases} \end{aligned} \tag{10}$$

Proof. Without loss of generality we may assume that $J = \{1, 2, \dots, p\}$ and $J^c = \{p+1, \dots, n+1\}$ $(1 \le p \le n)$. From (9), for $x \in \mathfrak{l}(W_J, W_{J^c})$ we have

$$x + \alpha_j = (1-\tau)W_J + \tau W_{J^c} + \alpha_j = (1-\tau) \sum_{k=1}^{p} \frac{\alpha_j - \alpha_k}{p} + \tau \sum_{l=p+1}^{n+1} \frac{\alpha_j - \alpha_l}{n+1-p},$$

so that we have

$$\begin{aligned} |x + \alpha_j|^2 &= \frac{(1-\tau)^2}{p^2} \Big| \sum_{k=1}^{p} (\alpha_j - \alpha_k) \Big|^2 + \frac{\tau^2}{(n+1-p)^2} \Big| \sum_{l=p+1}^{n+1} (\alpha_j - \alpha_k) \Big|^2 \\ &\quad + \frac{(1-\tau)\tau}{p(n+1-p)} \sum_{k=1}^{p} \sum_{l=p+1}^{n+1} 2(\alpha_j - \alpha_k, \alpha_j - \alpha_l) \\ &= \frac{(1-\tau)^2}{p^2} \Big\{ \sum_{k=1}^{p} |\alpha_j - \alpha_k|^2 + \sum_{1 \le k < l \le p} 2(\alpha_j - \alpha_k, \alpha_j - \alpha_l) \Big\} \\ &\quad + \frac{\tau^2}{(n+1-p)^2} \Big\{ \sum_{l=p+1}^{n+1} |\alpha_j - \alpha_l|^2 + \sum_{p+1 \le k < l \le n+1} 2(\alpha_j - \alpha_k, \alpha_j - \alpha_l) \Big\} \\ &\quad + \frac{(1-\tau)\tau}{p(n+1-p)} \sum_{k=1}^{p} \sum_{l=p+1}^{n+1} 2(\alpha_j - \alpha_k, \alpha_j - \alpha_l). \end{aligned} \tag{11}$$

Since $\triangle O_1 \dots O_{n+1}$ is regular, we have

$$\begin{aligned} |\alpha_j - \alpha_k|^2 &= \rho^2 \quad (j \ne k), \\ 2(\alpha_j - \alpha_k, \alpha_j - \alpha_l) &= 2\rho^2 \cos(\pi/3) = \rho^2 \quad (j \ne k, j \ne l, k \ne l). \end{aligned}$$

Therefore, if $1 \leq j \leq p$, then (11) implies that

$$
\begin{aligned}
|x+\alpha_j|^2 &= \frac{(1-\tau)^2}{p^2}\left\{(p-1)\rho^2 + \frac{(p-1)(p-2)}{2}\rho^2\right\} + \frac{\tau^2}{(n+1-p)^2}\left\{(n+1-p)\rho^2\right. \\
&\quad \left. + \frac{(n+1-p)(n-p)}{2}\rho^2\right\} + \frac{(1-\tau)\tau}{p(n+1-p)}(p-1)(n+1-p)\rho^2 \\
&= \left\{\frac{(1-\tau)^2}{2}\frac{p-1}{p} + \frac{\tau^2}{2}\frac{n+2-p}{n+1-p} + (1-\tau)\tau\frac{p-1}{p}\right\}\rho^2 \\
&= \left(\frac{n+1}{p(n+1-p)}\tau^2 + \frac{p-1}{p}\right)\frac{\rho^2}{2},
\end{aligned}
$$

so that $f_j(x) = |x+\alpha_j|^2 - r^2$ coincides with (10). In the same way as above, if $p+1 \leq j \leq n+1$, then we see that (11) implies (10). $\square$

By Lemma 1 F on $\mathfrak{l}(W_J, W_{J^c})$ is expressed as

$$
F = \sum_{j=1}^{n+1} \log f_j = \sum_{j \in J} \log f_j + \sum_{k \in J^c} \log f_k = p \log f_j + (n+1-p)\log f_k \quad (j \in J,\ k \in J^c),
$$

so that dF on $\mathfrak{l}(W_J, W_{J^c})$ is written as

$$
\begin{aligned}
dF &= p\,d\log f_j + (n+1-p)\,d\log f_k = \left(p f_k \frac{df_j}{d\tau} + (n+1-p) f_j \frac{df_k}{d\tau}\right)\frac{d\tau}{f_j f_k} \\
&= \frac{(n+1)^3\rho^4}{2(n+1-p)^2 p^2}\left(\tau - \frac{n+1-p}{n+1}\right)\left\{\tau^2 - \frac{2p}{n+1}\tau + \frac{n+1-p}{n+1}\left(p-1-2p\frac{r^2}{\rho^2}\right)\right\}\frac{d\tau}{f_j f_k}.
\end{aligned}
$$

The critical points on $\mathfrak{l}(W_J, W_{J^c})$ correspond to the solutions τ of the equation $dF = 0$, which is equivalent to

$$
\left(\tau - \frac{n+1-p}{n+1}\right)\left\{\tau^2 - \frac{2p}{n+1}\tau + \frac{n+1-p}{n+1}\left(p-1-2p\frac{r^2}{\rho^2}\right)\right\} = 0. \tag{12}
$$

The point $x = (1-\tau)W_J + \tau W_{J^c}$ on $\mathfrak{l}(W_J, W_{J^c})$ for $\tau = \frac{n+1-p}{n+1}$ coincides with W. The other two points on $\mathfrak{l}(W_J, W_{J^c})$ differ from W and satisfy the quadratic equation

$$
\tau^2 - \frac{2p}{n+1}\tau + \frac{n+1-p}{n+1}\left(p-1-2p\frac{r^2}{\rho^2}\right) = 0. \tag{13}
$$

The discriminant of this quadratic equation is given by $\mathfrak{D}_p/(n+1)^2$, where

$$
\mathfrak{D}_p = p^2 - (n+1-p)(n+1)\left(p-1-2p\frac{r^2}{\rho^2}\right), \tag{14}
$$

which satisfies

$$
\mathfrak{D}_p = \mathfrak{D}_{n+1-p}.
$$

Denote by τ_1, τ_2 the two solutions of (13) such that $\tau_1 < \tau_2$ if $\mathfrak{D}_p > 0$. We denote by Q_J and Q_{J^c} the corresponding two points in $\mathfrak{l}(W_J, W_{J^c})$ to τ_1 and τ_2, respectively. In addition to W all these points in $\mathfrak{l}(W_J, W_{J^c})$ $(\emptyset \subsetneq J \subsetneq I)$ give all the critical points of F in X. One can prove the following proposition.

Proposition 1. *The number of critical points is equal to $2^{n+1}-1$. All the critical points of F lie on one of the straight lines $\mathfrak{l}(W_J, W_{J^c})$ $(\emptyset \subsetneq J \subsetneq I)$. Suppose that r satisfies*

$$\begin{cases} \dfrac{n-2}{2(n+1)}\rho^2 < r^2 & \text{if } n \text{ is odd}, \\ \dfrac{n^3-4n-4}{2n(n+1)(n+2)}\rho^2 < r^2 < \dfrac{n-2}{2(n+1)}\rho^2 \ \text{ or } \ \dfrac{n-2}{2(n+1)}\rho^2 < r^2 & \text{if } n \text{ is even}. \end{cases} \tag{15}$$

Then, all of these, W, Q_J $(\emptyset \subsetneq J \subsetneq I)$, are real and distinct from each other.

Proof. For the situation where all critical points W, Q_J $(\emptyset \subsetneq J \subsetneq I)$ are real and distinct from each other, we need the condition that each solution of (12) is real and is not a double point. This condition is equivalent to: (a) The discriminants $\mathfrak{D}_p$ $(1 \leq p \leq n)$ of the Equation (13) are greater than 0; and (b) The left-hand side of (13) at $\tau = \frac{n+1-p}{n+1}$ does not vanish. We discuss (b) first. The condition for (b) is written as

$$\left(\frac{n+1-p}{n+1}\right)^2 - \frac{2p}{n+1}\left(\frac{n+1-p}{n+1}\right) + \frac{n+1-p}{n+1}\left(p-1-2p\frac{r^2}{\rho^2}\right) \neq 0,$$

which is equivalent to

$$p\frac{n+1-p}{n+1}\left(\frac{n-2}{2(n+1)} - \frac{r^2}{\rho^2}\right) \neq 0, \quad \text{i.e.,} \quad \frac{r^2}{\rho^2} \neq \frac{n-2}{2(n+1)}. \tag{16}$$

Next, we consider the condition (a). Since the discriminant $\mathfrak{D}_p$ of (13) is rewritten as

$$\mathfrak{D}_p = 2p(n+1-p)(n+1)\left(\frac{r^2}{\rho^2} - \frac{n+2}{2(n+1)} + \frac{n+1}{2p(n+1-p)}\right),$$

if n is odd, then we need

$$\begin{aligned} &\min\left\{\frac{r^2}{\rho^2} - \frac{n+2}{2(n+1)} + \frac{n+1}{2p(n+1-p)} \,\middle|\, p = 1,2,\ldots,n\right\} \\ &= \frac{r^2}{\rho^2} - \frac{n+2}{2(n+1)} + \frac{n+1}{2p(n+1-p)}\bigg|_{p=\frac{n+1}{2}} = \frac{r^2}{\rho^2} - \frac{n-2}{2(n+1)} > 0, \end{aligned} \tag{17}$$

while if n is even, then we need

$$\begin{aligned} &\min\left\{\frac{r^2}{\rho^2} - \frac{n+2}{2(n+1)} + \frac{n+1}{2p(n+1-p)} \,\middle|\, p = 1,2,\ldots,n\right\} \\ &= \frac{r^2}{\rho^2} - \frac{n+2}{2(n+1)} + \frac{n+1}{2p(n+1-p)}\bigg|_{p=\frac{n}{2}} = \frac{r^2}{\rho^2} - \frac{n^3-4n-4}{2n(n+1)(n+2)} > 0. \end{aligned} \tag{18}$$

Therefore, (16)–(18) imply that conditions (a) and (b) are satisfied if (15) holds for r. This completes the proof. □

Before we prove Conjecture 2 under the condition (8), we show the following identities:

Lemma 2. *Suppose that J is fixed as $p = |J|$ $(1 \leq p \leq n)$. Then we have*

$$f_j(W) = \frac{n\rho^2}{2(n+1)} - r^2 \quad (1 \leq j \leq n+1). \tag{19}$$

If $j \in J$, $k \in J^c$, then $f_j(Q_J)$, $f_k(Q_J)$, $f_j(Q_{J^c})$ and $f_k(Q_{J^c})$ are expressed as

$$f_j(Q_J) = \frac{\rho^2}{n+1-p}\tau(Q_J), \quad f_k(Q_J) = \frac{\rho^2}{p}(1-\tau(Q_J)), \tag{20}$$

$$f_j(Q_{J^c}) = \frac{\rho^2}{n+1-p}\tau(Q_{J^c}), \quad f_k(Q_{J^c}) = \frac{\rho^2}{p}(1-\tau(Q_{J^c})), \tag{21}$$

where

$$\tau(Q_J) = \frac{p-\sqrt{\mathfrak{D}_p}}{n+1}, \quad \tau(Q_{J^c}) = \frac{p+\sqrt{\mathfrak{D}_p}}{n+1}, \tag{22}$$

$$1-\tau(Q_J) = \frac{n+1-p+\sqrt{\mathfrak{D}_p}}{n+1}, \quad 1-\tau(Q_{J^c}) = \frac{n+1-p-\sqrt{\mathfrak{D}_p}}{n+1}. \tag{23}$$

Moreover, if $j \in J$, then we have

$$f_j(Q_J)f_j(Q_{J^c}) = \frac{\rho^2((p-1)\rho^2 - 2pr^2)}{(n+1)(n+1-p)}. \tag{24}$$

Proof. From Lemma 1 and (12) we obtain (19)–(21), where $\tau_1 = \tau(Q_J)$ and $\tau_2 = \tau(Q_{J^c})$ are the solutions of the quadratic equation (13) satisfying $\tau_1 < \tau_2$. We simply have (22) and (23) by the explicit forms of τ_1 and τ_2. We also have

$$f_j(Q_J)f_j(Q_{J^c}) = \frac{\rho^4}{(n+1-p)^2}\tau(Q_J)\tau(Q_{J^c}) = \frac{\rho^4}{(n+1-p)^2}\frac{n+1-p}{n+1}\Big(p-1-2p\frac{r^2}{\rho^2}\Big),$$

which coincides with (24). □

Using Lemma 2 we see that Conjecture 2 holds true under the condition (8).

Theorem 1. *Under the condition (8) the norms $\mathcal{N}(f_j)$ and $\mathcal{N}(\sum_{j=1}^{n+1} f_j^{-1})$ are expressed as*

$$\mathcal{N}(f_j) = \frac{B(0\star 1)}{2(n+1)^{2^n}} \prod_{p=2}^{n+1} \left(\frac{B(0\star 12\dots p)}{B(023\dots p)}\right)^{\binom{n}{p-1}} \quad (1 \le j \le n+1), \tag{25}$$

$$\mathcal{N}\Big(\sum_{j=1}^{n+1}\frac{1}{f_j}\Big) = 2(n+1)^{2^{n+1}-1} \prod_{p=1}^{n+1} \left(\frac{B(012\dots p)}{B(0\star 12\dots p)}\right)^{\binom{n+1}{p}}. \tag{26}$$

Proof. We prove (25) first. Without loss of generality we may assume $j=1$ for the proof of (25). By definition $\mathcal{N}(f_1)$ is expressed as

$$\begin{aligned}\mathcal{N}(f_1) &= f_1(W)\prod_{\emptyset \subsetneq J \subsetneq I} f_1(Q_J) = f_1(W) \prod_{\substack{\emptyset \subsetneq J \subsetneq I \\ 1 \in J}} f_1(Q_J) \prod_{\substack{\emptyset \subsetneq J \subsetneq I \\ 1 \notin J}} f_1(Q_J) \\ &= f_1(W) \prod_{\substack{\emptyset \subsetneq J \subsetneq I \\ 1 \in J}} f_1(Q_J)f_1(Q_{J^c}) = f_1(W)\prod_{p=1}^{n} \prod_{\substack{\emptyset \subsetneq J \subsetneq I \\ 1 \in J, |J|=p}} f_1(Q_J)f_1(Q_{J^c}). \end{aligned} \tag{27}$$

Applying (19) and (24) in Lemma 2 to (27), $\mathcal{N}(f_1)$ is calculated as

$$\begin{aligned}
\mathcal{N}(f_1) &= \left(\frac{n\rho^2}{2(n+1)} - r^2\right) \prod_{p=1}^{n} \prod_{\substack{\emptyset \subsetneq J \subsetneq I \\ 1\in J,\, |J|=p}} \frac{\rho^2((p-1)\rho^2 - 2pr^2)}{(n+1)(n+1-p)} \\
&= \frac{n\rho^2 - 2(n+1)r^2}{2(n+1)} \prod_{p=1}^{n} \left(\frac{\rho^2((p-1)\rho^2 - 2pr^2)}{(n+1)(n+1-p)}\right)^{\binom{n}{p-1}} \\
&= -\frac{2(n+1)r^2 - n\rho^2}{2(n+1)} \prod_{p=1}^{n} \left(\frac{-1}{n+1}\frac{\rho^2(2pr^2-(p-1)\rho^2)}{n+1-p}\right)^{\binom{n}{p-1}} \\
&= \frac{2r^2}{2}\left(\frac{-1}{n+1}\right)^{1+\sum_{p=1}^{n}\binom{n}{p-1}} \prod_{p=2}^{n+1} \left(\frac{\rho^2(2pr^2-(p-1)\rho^2)}{p-1}\right)^{\binom{n}{p-1}} \\
&= \frac{2r^2}{2(n+1)^{2^n}} \prod_{p=2}^{n+1} \left(\frac{\rho^2(2pr^2-(p-1)\rho^2)}{p-1}\right)^{\binom{n}{p-1}}.
\end{aligned} \tag{28}$$

On the other hand, by definition, we obtain

$$B(0\star 1) = 2r^2, \quad B(0\star 12\ldots p) = (-1)^{p-1}\rho^{2(p-1)}\{2pr^2 - (p-1)\rho^2\} \tag{29}$$

and

$$B(012\ldots p) = (-1)^p \rho^{2(p-1)} p, \tag{30}$$

so that

$$B(023\ldots p) = B(012\ldots p-1) = (-1)^{p-1}\rho^{2(p-2)}(p-1). \tag{31}$$

Therefore, (29) and (31) imply that (28) coincides with (25).

Next, we prove (26). By definition $\mathcal{N}(\sum_{j=1}^{n+1} f_j^{-1})$ is expressed as

$$\begin{aligned}
\mathcal{N}\Big(\sum_{j=1}^{n+1}\frac{1}{f_j}\Big) &= \Big(\sum_{j=1}^{n+1}\frac{1}{f_j(W)}\Big) \prod_{\emptyset\subsetneq J\subsetneq I}\Big(\sum_{j=1}^{n+1}\frac{1}{f_j(Q_J)}\Big) = \Big(\sum_{j=1}^{n+1}\frac{1}{f_j(W)}\Big) \prod_{p=1}^{n}\prod_{\substack{\emptyset\subsetneq J\subsetneq I\\ |J|=p}}\Big(\sum_{j=1}^{n+1}\frac{1}{f_j(Q_J)}\Big) \\
&= \Big(\sum_{j=1}^{n+1}\frac{1}{f_j(W)}\Big) \prod_{p=1}^{n}\prod_{\substack{\emptyset\subsetneq J\subsetneq I\\ |J|=p}}\Big(\sum_{j\in J}\frac{1}{f_j(Q_J)} + \sum_{k\in J^c}\frac{1}{f_k(Q_J)}\Big).
\end{aligned} \tag{32}$$

Applying (19)–(21) in Lemma 2 to (32), we have

$$\begin{aligned}
\mathcal{N}\Big(\sum_{j=1}^{n+1}\frac{1}{f_j}\Big) &= \Big(\sum_{j=1}^{n+1}\frac{1}{f_j(W)}\Big) \prod_{p=1}^{n}\prod_{\substack{\emptyset\subsetneq J\subsetneq I\\ |J|=p}}\Big(\sum_{j\in J}\frac{n+1-p}{\rho^2\tau(Q_J)} + \sum_{k\in J^c}\frac{p}{\rho^2(1-\tau(Q_J))}\Big) \\
&= \frac{2(n+1)^2}{n\rho^2 - 2(n+1)r^2} \prod_{p=1}^{n}\prod_{\substack{\emptyset\subsetneq J\subsetneq I\\ |J|=p}}\Big(\frac{p(n+1-p)}{\rho^2\tau(Q_J)} + \frac{(n+1-p)p}{\rho^2(1-\tau(Q_J))}\Big) \\
&= \frac{2(n+1)^2}{n\rho^2 - 2(n+1)r^2} \prod_{p=1}^{n}\prod_{\substack{\emptyset\subsetneq J\subsetneq I\\ |J|=p}}\frac{p(n+1-p)}{\rho^2\tau(Q_J)(1-\tau(Q_J))}.
\end{aligned} \tag{33}$$

Using (22), (23) and $\mathfrak{D}_p = \mathfrak{D}_{n+1-p}$ for (33) we obtain

$$\begin{aligned}
\mathcal{N}(\sum_{j=1}^{n+1}\frac{1}{f_j}) &= \frac{2(n+1)^2}{n\rho^2-2(n+1)r^2}\prod_{p=1}^{n}\left(\frac{(n+1)^2p(n+1-p)}{\rho^2(p-\sqrt{\mathfrak{D}_p})(n+1-p+\sqrt{\mathfrak{D}_{n+1-p}})}\right)^{\binom{n+1}{p}} \\
&= \frac{2(n+1)^2}{n\rho^2-2(n+1)r^2}\prod_{p=1}^{n}\left(\frac{(n+1)^2p(n+1-p)}{\rho^2(p-\sqrt{\mathfrak{D}_p})(p+\sqrt{\mathfrak{D}_p})}\right)^{\binom{n+1}{p}} \\
&= \frac{2(n+1)^2}{n\rho^2-2(n+1)r^2}\prod_{p=1}^{n}\left(\frac{(n+1)^2p(n+1-p)}{\rho^2(p^2-\mathfrak{D}_p)}\right)^{\binom{n+1}{p}}.
\end{aligned} \tag{34}$$

Since we have $\rho^2(p^2-\mathfrak{D}_p) = (n+1)(n+1-p)(\rho^2(p-1)-2pr^2)$ from (14), (34) is written as

$$\begin{aligned}
\mathcal{N}(\sum_{j=1}^{n+1}\frac{1}{f_j}) &= \frac{2(n+1)^2}{n\rho^2-2(n+1)r^2}\prod_{p=1}^{n}\left(\frac{(n+1)^2p(n+1-p)}{(n+1)(n+1-p)(\rho^2(p-1)-2pr^2)}\right)^{\binom{n+1}{p}} \\
&= \frac{2(n+1)^2}{n\rho^2-2(n+1)r^2}\prod_{p=1}^{n}\left(\frac{(n+1)p}{\rho^2(p-1)-2pr^2}\right)^{\binom{n+1}{p}} \\
&= 2(n+1)^{\sum_{p=1}^{n+1}\binom{n+1}{p}}\prod_{p=1}^{n+1}\left(\frac{p}{\rho^2(p-1)-2pr^2}\right)^{\binom{n+1}{p}} \\
&= 2(n+1)^{2^{n+1}-1}\prod_{p=1}^{n+1}\left(\frac{p}{\rho^2(p-1)-2pr^2}\right)^{\binom{n+1}{p}}.
\end{aligned} \tag{35}$$

Hence, (29) and (30) imply that (35) coincides with (26). □

3. Special Coordinates (Three-Dimensional Case)

In this section, for a general three-dimensional case, we define the special coordinate system (denoted by t_1, t_2, t_3) attached to the fundamental three-dimensional simplex $\triangle O_1O_2O_3O_4$.

Each plane $\mathfrak{p}_{jkl}$ containing the three vertices O_j, O_k, O_l is described by the equation

$$\mathfrak{p}_{jkl} : L_{jkl} = 0,$$

where the linear functions L_{jkl} on X are given by

$$\begin{aligned}
L_{123}(x) &:= -\det(x+\alpha_1, x+\alpha_2, x+\alpha_3), \\
L_{124}(x) &:= \det(x+\alpha_1, x+\alpha_2, x+\alpha_4), \\
L_{134}(x) &:= -\det(x+\alpha_1, x+\alpha_3, x+\alpha_4), \\
L_{234}(x) &:= \det(x+\alpha_2, x+\alpha_3, x+\alpha_4)
\end{aligned} \tag{36}$$

for $\alpha_j = (\alpha_{j1}, \alpha_{j2}, \alpha_{j3}) \in \mathbb{R}^3$. Under our setting (1) the functions L_{jkl} are explicitly expanded as

$$\begin{aligned}
L_{123} &= \alpha_{13}\alpha_{22}(x_1+\alpha_{31}) - \alpha_{13}(\alpha_{21}-\alpha_{31})x_2 + \{-(\alpha_{11}-\alpha_{31})\alpha_{22} + \alpha_{12}(\alpha_{21}-\alpha_{31})\}x_3, \\
L_{124} &= -\alpha_{13}\alpha_{22}x_1 + \alpha_{21}\alpha_{13}x_2 + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_3, \\
L_{134} &= -\alpha_{13}\alpha_{31}x_2 + \alpha_{31}\alpha_{12}x_3, \\
L_{234} &= -\alpha_{31}\alpha_{22}x_3,
\end{aligned}$$

so that

$$L_{123} + L_{124} + L_{134} + L_{234} = \alpha_{31}\alpha_{22}\alpha_{13}. \tag{37}$$

Hence, the simplex $\triangle O_1O_2O_3O_4$ can be defined by $L_{jkl} \geq 0$ $(1 \leq j \leq k \leq 4)$. Remark that

$$\alpha_{31} = \sqrt{\frac{B(034)}{2}} > 0,\ \alpha_{31}\alpha_{22} = \sqrt{-\frac{B(0234)}{4}} > 0,\ \alpha_{31}\alpha_{22}\alpha_{13} = \sqrt{\frac{B(01234)}{8}} > 0. \tag{38}$$

Definition 2. *Two rational functions φ_1, φ_2 on X are said to be congruent with respect to $\mathcal{C}$ and is denoted by*

$$\varphi_1 \equiv \varphi_2 \mod Ann(\mathcal{C})$$

if φ_1, φ_2 have definite values at every point of $\mathcal{C}$ and

$$\varphi_1(x) = \varphi_2(x)$$

at each critical point x in $\mathcal{C}$ ($Ann(\mathcal{C})$ means the annihilator of $\mathcal{C}$).

Lemma 3. *Let g_1, g_2, g_3 be polynomials in x of degree 3 specified by*

$$g_1 := L_{123}f_4 - L_{234}f_1, \quad g_2 := L_{123}f_4 - L_{134}f_2, \quad g_3 := L_{123}f_4 - L_{124}f_3. \tag{39}$$

Then, we have

$$g_j = -f_jf_4M_j, \tag{40}$$

where the functions M_j are given by

$$M_j = \pm\begin{vmatrix} x_1+\alpha_{k1} & x_1+\alpha_{l1} & G_1 \\ x_2+\alpha_{k2} & x_2+\alpha_{l2} & G_2 \\ x_3+\alpha_{k3} & x_3+\alpha_{l3} & G_3 \end{vmatrix}.$$

Here, $\{j,k,l\}$ is a permutation of $\{1,2,3\}$ and $\pm$ denotes its sign. Moreover, we have the congruences

$$\frac{f_4}{f_1} \equiv \frac{L_{234}}{L_{123}}, \quad \frac{f_4}{f_2} \equiv \frac{L_{134}}{L_{123}}, \quad \frac{f_4}{f_3} \equiv \frac{L_{124}}{L_{123}} \mod Ann(\mathcal{C}). \tag{41}$$

Proof. Without loss of generality we prove (40) for $j = 1$. Since G_ν $(1 \leq \nu \leq 3)$ are given as $G_\nu = \sum_{j=1}^4 (x_\nu + \alpha_{j\nu})/f_j$, using (36) M_1 is written as

$$\begin{aligned} M_1 &= \sum_{j=1}^{4} \det(x+\alpha_2, x+\alpha_3, x+\alpha_j)/f_j \\ &= \det(x+\alpha_2, x+\alpha_3, x+\alpha_1)/f_1 + \det(x+\alpha_2, x+\alpha_3, x+\alpha_4)/f_4 \\ &= -\frac{L_{123}}{f_1} + \frac{L_{234}}{f_4}, \end{aligned}$$

which is equivalent to (40) for $j = 1$. From (40), we obtain

$$\frac{f_4}{f_1} = \frac{L_{234}}{L_{123}} - \frac{f_4}{L_{123}}M_1, \quad \frac{f_4}{f_2} = \frac{L_{134}}{L_{123}} - \frac{f_4}{L_{123}}M_2, \quad \frac{f_4}{f_3} = \frac{L_{124}}{L_{123}} - \frac{f_4}{L_{123}}M_3.$$

This implies (41). □

According to Lemma 3, we can characterize the set of critical points $\mathcal{C}$ in X by the polynomials g_j as follows.

Lemma 4. *Under the condition $(\mathcal{H}_0)$, the system*

$$G_1 = G_2 = G_3 = 0$$

on X is equivalent to the system

$$g_1 = g_2 = g_3 = 0$$

on X.

Lemma 5. *The following identity holds as function of $x = (x_1, x_2, x_3)$:*

$$dg_1 \wedge dg_2 \wedge dg_3 \equiv -L_{123}^2 f_1 f_2 f_3 f_4^3 dG_1 \wedge dG_2 \wedge dG_3 \mod Ann(\mathcal{C}). \tag{42}$$

Proof. By taking the derivatives of both sides of (40) in Lemma 3 one obtains

$$\frac{\partial g_j}{\partial x_\nu} \equiv -f_j f_4 \Big(a_{jj}^* \frac{\partial G_j}{\partial x_\nu} + a_{kj}^* \frac{\partial G_k}{\partial x_\nu} + a_{lj}^* \frac{\partial G_l}{\partial x_\nu} \Big) \mod Ann(\mathcal{C})$$

for the triple $\{j, k, l\}$, which is an even permutation of $\{1, 2, 3\}$, where a_{pq}^* denotes the cofactor of the (p, q)-component of the 3×3 matrix $A = (x + \alpha_1, x + \alpha_2, x + \alpha_3)$. Thus, we have

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_3}{\partial x_1} & \frac{\partial g_3}{\partial x_2} & \frac{\partial g_3}{\partial x_3} \end{pmatrix} \equiv -f_4 \begin{pmatrix} f_1 a_{11}^* & f_1 a_{21}^* & f_1 a_{31}^* \\ f_2 a_{12}^* & f_2 a_{22}^* & f_2 a_{32}^* \\ f_3 a_{13}^* & f_3 a_{23}^* & f_3 a_{33}^* \end{pmatrix} \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} & \frac{\partial G_1}{\partial x_3} \\ \frac{\partial G_2}{\partial x_1} & \frac{\partial G_2}{\partial x_2} & \frac{\partial G_2}{\partial x_3} \\ \frac{\partial G_3}{\partial x_1} & \frac{\partial G_3}{\partial x_2} & \frac{\partial G_3}{\partial x_3} \end{pmatrix} \mod Ann(\mathcal{C}).$$

Hence, we obtain

$$\frac{\partial(g_1, g_2, g_3)}{\partial(x_1, x_2, x_3)} \equiv -f_1 f_2 f_3 f_4^3 (\det A^*) \frac{\partial(G_1, G_2, G_3)}{\partial(x_1, x_2, x_3)} \mod Ann(\mathcal{C}), \tag{43}$$

where A^* is the cofactor matrix of A given by

$$A^* = \begin{pmatrix} a_{11}^* & a_{21}^* & a_{31}^* \\ a_{12}^* & a_{22}^* & a_{32}^* \\ a_{13}^* & a_{23}^* & a_{33}^* \end{pmatrix}, \quad A = \begin{pmatrix} x_1 + \alpha_{11} & x_1 + \alpha_{21} & x_1 + \alpha_{31} \\ x_2 + \alpha_{12} & x_2 + \alpha_{22} & x_2 + \alpha_{32} \\ x_3 + \alpha_{13} & x_2 + \alpha_{23} & x_3 + \alpha_{33} \end{pmatrix}.$$

Since A^* satisfies $A^* A = (\det A) I$, where I is the identity matrix, (36) implies

$$\det A^* = (\det A)^2 = (-L_{123})^2 = L_{123}^2.$$

Therefore, we see that (43) is equivalent to (42). □

We now introduce special coordinates $t = (t_1, t_2, t_3)$ instead of $x = (x_1, x_2, x_3)$, given by

$$t_1 := \frac{L_{234}}{L_{123}}, \quad t_2 := \frac{L_{134}}{L_{123}}, \quad t_3 := \frac{L_{124}}{L_{123}} \quad \text{and} \quad t_\infty := \frac{\alpha_{31}\alpha_{22}\alpha_{13}}{L_{123}}. \tag{44}$$

The identity (37) implies

$$t_\infty = 1 + t_1 + t_2 + t_3.$$

Conversely, for given $t = (t_1, t_2, t_3)$ solving the system (44) with respect to $x = (x_1, x_2, x_3)$, we obtain

$$x_1 = -\frac{\alpha_{11} t_1 + \alpha_{21} t_2 + \alpha_{31} t_3}{t_\infty}, \quad x_2 = -\frac{\alpha_{12} t_1 + \alpha_{22} t_2}{t_\infty}, \quad x_3 = -\frac{\alpha_{13} t_1}{t_\infty}. \tag{45}$$

Therefore, (45) defines the projective map $\iota : \{t \in \mathbb{C}^3 \mid 1 + t_1 + t_2 + t_3 \neq 0\} \to \mathbb{C}^3$, i.e.,

$$t \mapsto x = \iota(t) := -\frac{t_1}{t_\infty}\alpha_1 - \frac{t_2}{t_\infty}\alpha_2 - \frac{t_3}{t_\infty}\alpha_3 \in \mathbb{C}^3.$$

By definition, we notice that

Lemma 6.
$$dx_1 \wedge dx_2 \wedge dx_3 = \frac{\alpha_{31}\alpha_{22}\alpha_{13}}{t_\infty^4}\, dt_1 \wedge dt_2 \wedge dt_3. \tag{46}$$

Proof. Differentiating both sides of $t_\infty x_j = -(\alpha_{1j}t_1 + \alpha_{2j}t_2 + \alpha_{3j}t_3)$ with respect to t_k, we have $t_\infty \partial x_j/\partial t_k = -(x_j + \alpha_{kj})$, so that

$$\frac{\partial(x_1, x_2, x_3)}{\partial(t_1, t_2, t_3)} = \frac{-1}{t_\infty^3} \det\left(x_j + \alpha_{kj}\right)_{j,k=1,2,3} = \frac{L_{123}}{t_\infty^3} = \frac{\alpha_{31}\alpha_{22}\alpha_{13}}{t_\infty^4},$$

which is equivalent to (46). □

Under the condition $(\mathcal{H}_0)$, for the sets $\mathcal{C}$ and $X \subset \mathbb{C}^3$, we put $\tilde{\mathcal{C}} = \iota^{-1}\mathcal{C}$ and $\tilde{X} := \iota^{-1}X$, respectively.

Definition 3. *Two rational functions φ_1, φ_2 on $\tilde{X}$ are said to be congruent with respect to $\tilde{\mathcal{C}}$ and written by*
$$\varphi_1 \equiv \varphi_2 \mod Ann(\tilde{\mathcal{C}})$$
if their restriction to $\tilde{\mathcal{C}}$ are equal, i.e., if φ_1, φ_2 have definite values at every point of $\tilde{\mathcal{C}}$ and $\varphi_1(t) = \varphi_2(t)$ at each critical point t in $\tilde{\mathcal{C}}$ ($Ann(\tilde{\mathcal{C}})$ means the annihilator of $\tilde{\mathcal{C}}$).

Remark 5. *From* (41) *of Lemma* 3, (44) *implies the congruences*

$$t_1 \equiv \frac{f_4}{f_1}, \quad t_2 \equiv \frac{f_4}{f_2}, \quad t_3 \equiv \frac{f_4}{f_3} \mod Ann(\tilde{\mathcal{C}}). \tag{47}$$

Through the projective map $\iota : \tilde{X} \to X$ we can characterize the set of critical points $\tilde{\mathcal{C}}$ in $\tilde{X}$ as follows:

Lemma 7. *Under the condition $(\mathcal{H}_0)$, the system*

$$g_1 = g_2 = g_3 = 0$$

on X is equivalent to the system
$$\tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3 = 0$$
on $\tilde{X}$, where $\tilde{g}_j$ are polynomials in t of degree 3 given by

$$\tilde{g}_j(t) := \frac{t_\infty^3}{\alpha_{31}\alpha_{22}\alpha_{13}} g_j(x) = \frac{t_\infty^3}{\alpha_{31}\alpha_{22}\alpha_{13}} g_j(\iota(t)). \tag{48}$$

Moreover we have

$$dg_1 \wedge dg_2 \wedge dg_3 \equiv \frac{\alpha_{31}^3\alpha_{22}^3\alpha_{13}^3}{t_\infty^9}\, d\tilde{g}_1 \wedge d\tilde{g}_2 \wedge d\tilde{g}_3 \mod Ann(\tilde{\mathcal{C}}). \tag{49}$$

Proof. By definition the equivalence between $g_1 = g_2 = g_3 = 0$ and $\tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3 = 0$ is obvious. The identity (49) is straightforward from (48). Here, we just confirm that $\tilde{g}_j(t)$ are polynomials in t of degree 3. By the definition (39) of g_j, we have

$$\tilde{g}_1(t) = \frac{t_\infty^3}{\alpha_{31}\alpha_{22}\alpha_{13}}(L_{123}f_4 - L_{234}f_1) = \frac{t_\infty^3 L_{123}}{\alpha_{31}\alpha_{22}\alpha_{13}}(f_4 - \frac{L_{234}}{L_{123}}f_1) = t_\infty^2(f_4 - t_1 f_1).$$

In the same way, we have the expression

$$\tilde{g}_j(t) = t_\infty^2(f_4 - t_j f_j) \quad (j = 1, 2, 3). \tag{50}$$

For $1 \le j \le 4$, we have

$$t_\infty^2 f_j = t_\infty^2 \Big((x,x) + 2(\alpha_j, x) + \alpha_{j0} \Big) = (t_\infty x, t_\infty x) + 2t_\infty(\alpha_j, t_\infty x) + \alpha_{j0} t_\infty^2 \tag{51}$$
$$= |\sum_{k=1}^{3} t_k \alpha_k|^2 - 2(1 + \sum_{k=1}^{3} t_k)(\alpha_j, \sum_{l=1}^{3} t_l \alpha_l) + \alpha_{j0}(1 + \sum_{k=1}^{3} t_k)^2$$
$$= \sum_{k=1}^{3} |\alpha_k|^2 t_k^2 + 2 \sum_{1\le k<l\le 3} (\alpha_k, \alpha_l) t_k t_l - 2(1 + \sum_{k=1}^{3} t_k) \sum_{l=1}^{3} (\alpha_j, \alpha_l) t_l + \alpha_{j0}(1 + \sum_{k=1}^{3} t_k)^2,$$

which are polynomials in t of degree 2. From (50), we see that $\tilde{g}_j$ are polynomials in t of degree 3. □

Before we show the explicit forms of the polynomials $\tilde{g}_j$, we prove two lemmas.

Lemma 8. *The following identities hold:*

$$t_\infty(f_j - f_4) = \sum_{k=1}^{3} B\begin{pmatrix} 0 & \star & k \\ 0 & j & 4 \end{pmatrix} t_k + B\begin{pmatrix} 0 & \star & 4 \\ 0 & j & 4 \end{pmatrix} \quad (1 \le j \le 3). \tag{52}$$

Proof. By definition, we have

$$B\begin{pmatrix} 0 & \star & k \\ 0 & j & 4 \end{pmatrix} = (\rho_{kj}^2 - r_j^2) - (\rho_{k4}^2 - r_4^2)$$
$$= |\alpha_k - \alpha_j|^2 - (|\alpha_j|^2 - \alpha_{j0}) - |\alpha_k - \alpha_4|^2 + (|\alpha_4|^2 - \alpha_{40})$$
$$= 2(\alpha_k, \alpha_4 - \alpha_j) + \alpha_{j0} - \alpha_{40}. \tag{53}$$

On the other hand, by the definition of f_j we have

$$t_\infty(f_j - f_4) = t_\infty\{2(x, \alpha_j) - 2(x, \alpha_4) + \alpha_{j0} - \alpha_{40}\}$$
$$= 2(-t_\infty x, \alpha_4 - \alpha_j) + t_\infty(\alpha_{j0} - \alpha_{40}).$$

Applying the relations $-t_\infty x = t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3$ and $t_\infty = 1 + t_1 + t_2 + t_3$ to the above identity, we have

$$t_\infty(f_j - f_4) = \sum_{k=1}^{3} \{2(\alpha_k, \alpha_4 - \alpha_j) + (\alpha_{j0} - \alpha_{40})\} t_k + (\alpha_{j0} - \alpha_{40}). \tag{54}$$

Under $\alpha_4 = 0$, i.e., $\rho_{j4}^2 = |\alpha_j|^2$, (53) implies $B\begin{pmatrix} 0 & \star & 4 \\ 0 & j & 4 \end{pmatrix} = \alpha_{j0} - \alpha_{40}$. Therefore, we obtain (52) from (53) and (54). □

Lemma 9.

$$t_\infty^2 f_4 = \sum_{j=1}^{3} (\rho_{j4}^2 - r_4^2) t_j^2 + \sum_{1\le j<k\le 3} \Big\{ B\begin{pmatrix} 0 & j & 4 \\ 0 & k & 4 \end{pmatrix} - 2r_4^2 \Big\} t_j t_k - 2r_4^2(t_1 + t_2 + t_3) - r_4^2. \tag{55}$$

Proof. From (51), we have

$$t_\infty^2 f_4 = \sum_{k=1}^{3} |\alpha_k|^2 t_k^2 + 2 \sum_{1\le k<l\le 3} (\alpha_k, \alpha_l) t_k t_l + \alpha_{40}(1 + t_1 + t_2 + t_3)^2$$
$$= \sum_{j=1}^{3} \rho_{j4}^2 t_j^2 + \sum_{1\le j<k\le 3} B\begin{pmatrix} 0 & j & 4 \\ 0 & k & 4 \end{pmatrix} t_j t_k - r_4^2(1 + t_1 + t_2 + t_3)^2,$$

which coincides with (55). □

Proposition 2. *The polynomials $\tilde{g}_j$ in t of degree 3 given in (48) are expressed as*

$$\begin{aligned}\tilde{g}_j &= r_j^2 t_j^3 + (r_j^2 - \rho_{jk}^2) t_k^2 t_j + (r_j^2 - \rho_{jl}^2) t_l^2 t_j + 2r_j^2 t_j^2 (t_k + t_l) - \left\{ B\begin{pmatrix} 0 & k & j \\ 0 & l & j \end{pmatrix} - 2r_j^2 \right\} t_j t_k t_l \\ &+ (\rho_{j4}^2 - r_4^2 + 2r_j^2) t_j^2 + (\rho_{k4}^2 - r_4^2) t_k^2 + (\rho_{l4}^2 - r_4^2) t_l^2 \\ &- 2B\begin{pmatrix} 0 & \star & k \\ 0 & j & 4 \end{pmatrix} t_j t_k - 2B\begin{pmatrix} 0 & \star & l \\ 0 & j & 4 \end{pmatrix} t_j t_l + \left\{ B\begin{pmatrix} 0 & k & 4 \\ 0 & l & 4 \end{pmatrix} - 2r_4^2 \right\} t_k t_l \\ &+ (r_j^2 - 2r_4^2 - \rho_{j4}^2) t_j - 2r_4^2 (t_k + t_l) - r_4^2.\end{aligned}$$

Proof. From (50), for $1 \le j \le 3$, we have

$$\begin{aligned}\tilde{g}_j &= (1 - t_j) t_\infty^2 f_4 - t_j t_\infty^2 (f_j - f_4) \\ &= (1 - t_j) t_\infty^2 f_4 - t_j (1 + t_1 + t_2 + t_3) t_\infty (f_j - f_4).\end{aligned}$$

Since the explicit forms of $t_\infty^2 f_4$ and $t_\infty (f_j - f_4)$ have already been given in (55) and (52), respectively, we eventually obtain the result of Proposition 2. In particular, the identity

$$B\begin{pmatrix} 0 & k & j \\ 0 & l & j \end{pmatrix} - 2r_j^2 = B\begin{pmatrix} 0 & k & 4 \\ 0 & j & 4 \end{pmatrix} - 2r_4^2 + B\begin{pmatrix} 0 & \star & k \\ 0 & j & 4 \end{pmatrix} + B\begin{pmatrix} 0 & \star & l \\ 0 & j & 4 \end{pmatrix}$$

was applied to the coefficient of $t_j t_k t_l$. □

Lemma 10. *For $1 \le j < k \le 3$ let $\tilde{g}_{jk}$ be functions specified by*

$$\tilde{g}_{jk} := \frac{(1 - t_k)\tilde{g}_j - (1 - t_j)\tilde{g}_k}{t_\infty}. \tag{56}$$

Then, $\tilde{g}_{jk}$ are polynomials in t of degree 3, which are explicitly written as follows:

$$\begin{aligned}\tilde{g}_{jk} &= -t_j^2 t_k B\begin{pmatrix} 0 & \star & j \\ 0 & k & j \end{pmatrix} + t_j t_k^2 B\begin{pmatrix} 0 & \star & k \\ 0 & j & k \end{pmatrix} + t_j t_k t_l B\begin{pmatrix} 0 & l & \star \\ 0 & k & j \end{pmatrix} \\ &+ t_j^2 B\begin{pmatrix} 0 & \star & j \\ 0 & 4 & j \end{pmatrix} - t_k^2 B\begin{pmatrix} 0 & \star & k \\ 0 & 4 & k \end{pmatrix} + t_j \left\{ -B\begin{pmatrix} 0 & \star & 4 \\ 0 & j & 4 \end{pmatrix} - B\begin{pmatrix} 0 & \star & l \\ 0 & j & 4 \end{pmatrix} t_l \right\} \\ &+ t_k \left\{ B\begin{pmatrix} 0 & \star & 4 \\ 0 & k & 4 \end{pmatrix} + B\begin{pmatrix} 0 & \star & l \\ 0 & k & 4 \end{pmatrix} t_l \right\}\end{aligned}$$

such that $\{j, k, l\}$ is the uniquely determined permutation of $\{1, 2, 3\}$.

Proof. By the definition (50) of $\tilde{g}_j$, (56) implies

$$\tilde{g}_{jk} = -t_j (1 - t_k) t_\infty (f_j - f_4) + t_k (1 - t_j) t_\infty (f_k - f_4).$$

Since the explicit form of $t_\infty (f_j - f_4)$ has already been given in (52) in Lemma 8, we obtain the result of Lemma 10. □

Remark 6. *As a consequence of Lemma 10, we have*

$$(1 - t_1)\tilde{g}_{23} - (1 - t_2)\tilde{g}_{13} + (1 - t_3)\tilde{g}_{12} = 0,$$

so that we immediately have

$$d\tilde{g}_{12} \wedge d\tilde{g}_{13} \wedge d\tilde{g}_{23} \equiv 0 \mod Ann(\tilde{C}).$$

The following is a key lemma to characterize the set $\tilde{\mathcal{C}}$ of critical points.

Lemma 11. *Suppose $t_1 \neq 1$. Under the condition $(\mathcal{H}_0)$ the system*

$$\tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3 = 0$$

in $\tilde{X}$ is equivalent to the system

$$\tilde{g}_1 = \tilde{g}_{12} = \tilde{g}_{13} = 0 \tag{57}$$

in $\tilde{X}$. Moreover we have

$$d\tilde{g}_1 \wedge d\tilde{g}_2 \wedge d\tilde{g}_3 \equiv \frac{t_\infty^2}{(1-t_1)^2} d\tilde{g}_1 \wedge d\tilde{g}_{12} \wedge d\tilde{g}_{13} \mod Ann(\tilde{\mathcal{C}}). \tag{58}$$

Proof. By definition the equivalence between $\tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3 = 0$ and $\tilde{g}_1 = \tilde{g}_{12} = \tilde{g}_{13} = 0$ is obvious under $t_1 \neq 1$. From (56) we have

$$t_\infty d\tilde{g}_{1j} \equiv (1-t_j)d\tilde{g}_1 - (1-t_1)d\tilde{g}_j \mod Ann(\tilde{\mathcal{C}}),$$

which implies (58). □

Then the following congruence identity holds true:

Lemma 12. *Regarding $G_j = G_j(x)$ as functions on $\tilde{X}$ through the map ι, i.e., $G_j = G_j(\iota(t))$, we have*

$$dG_1 \wedge dG_2 \wedge dG_3 \equiv -\frac{\alpha_{31}\alpha_{22}\alpha_{13}}{f_1 f_2 f_3 f_4^3} \frac{d\tilde{g}_1 \wedge d\tilde{g}_2 \wedge d\tilde{g}_3}{t_\infty^7} \mod Ann(\tilde{\mathcal{C}}). \tag{59}$$

If $t \in \tilde{\mathcal{C}}$ satisfies $t_1 \neq 1$, then

$$dG_1 \wedge dG_2 \wedge dG_3 \equiv -\frac{\alpha_{31}\alpha_{22}\alpha_{13}}{f_1 f_2 f_3 f_4^3} \frac{d\tilde{g}_1 \wedge d\tilde{g}_{12} \wedge d\tilde{g}_{13}}{t_\infty^5 (1-t_1)^2} \mod Ann(\tilde{\mathcal{C}}). \tag{60}$$

Proof. From Lemma 5 and (49) in Lemma 7 we have (59) using the definition (44) of t_∞. Furthermore, from (58) we see that (59) implies (60). □

Lemma 13. *For an arbitrary critical point $t \in \tilde{\mathcal{C}}$, the Hessian at $x = \iota(t)$ is expressed as*

$$\frac{1}{2^3}\mathrm{Hess}(F)\Big|_{x=\iota(t)} = -\frac{1}{f_1 f_2 f_3 f_4^3\, t_\infty^3} \frac{\partial(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)}{\partial(t_1, t_2, t_3)}. \tag{61}$$

In particular, if $t \in \tilde{\mathcal{C}}$ satisfies $t_1 \neq 1$, then

$$\frac{1}{2^3}\mathrm{Hess}(F)\Big|_{x=\iota(t)} = -\frac{1}{f_1 f_2 f_3 f_4^3\, t_\infty (1-t_1)^2} \frac{\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_1, t_2, t_3)}. \tag{62}$$

Proof. From (4), we have

$$\frac{1}{2^3}\mathrm{Hess}(F) = \frac{\partial(G_1, G_2, G_3)}{\partial(x_1, x_2, x_3)}.$$

According to (46) in Lemma 6 and (59), for $t \in \tilde{\mathcal{C}}$ we have

$$\begin{aligned} \frac{\partial(G_1, G_2, G_3)}{\partial(x_1, x_2, x_3)}\Big|_{x=\iota(t)} &= \frac{\partial(G_1, G_2, G_3)}{\partial(t_1, t_2, t_3)} \Big/ \frac{\partial(x_1, x_2, x_3)}{\partial(t_1, t_2, t_3)} \\ &= -\frac{\alpha_{31}\alpha_{22}\alpha_{13}}{f_1 f_2 f_3 f_4^3 t_\infty^7} \left(\frac{\alpha_{31}\alpha_{22}\alpha_{13}}{t_\infty^4}\right)^{-1} \frac{\partial(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)}{\partial(t_1, t_2, t_3)}, \end{aligned}$$

which coincides with (61). On the other hand, if $t_1 \neq 1$, then from (60) we obtain

$$\left.\frac{\partial(G_1, G_2, G_3)}{\partial(x_1, x_2, x_3)}\right|_{x=\iota(t)} = \frac{\partial(G_1, G_2, G_3)}{\partial(t_1, t_2, t_3)} \Big/ \frac{\partial(x_1, x_2, x_3)}{\partial(t_1, t_2, t_3)}$$
$$= -\frac{\alpha_{31}\alpha_{22}\alpha_{13}}{f_1 f_2 f_3 f_4^3 t_\infty^5 (1-t_1)^2} \left(\frac{\alpha_{31}\alpha_{22}\alpha_{13}}{t_\infty^4}\right)^{-1} \frac{\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_1, t_2, t_3)},$$

which coincides with the right-hand side of (62). □

We denote by $\mathbf{c}_j$ $(1 \leq j \leq \kappa)$ all the critical points in X, i.e., $\mathcal{C} = \{\mathbf{c}_j \in X \,|\, 1 \leq j \leq \kappa\}$. For a rational function $\varphi(x)$ on X we denote by $\mathcal{N}(\varphi)$ the product of the critical values at all points in $\mathcal{C}$, i.e., $\mathcal{N}(\varphi) := \prod_{j=1}^{\kappa} \varphi(\mathbf{c}_j)$, which is called the *norm* of φ on X. We also denote by $\tilde{\mathbf{c}}_j$ the critical points in $\tilde{X}$ specified by $\tilde{\mathbf{c}}_j = \iota^{-1}(\mathbf{c}_j)$, i.e., $\tilde{\mathcal{C}} = \iota^{-1}\mathcal{C} = \{\tilde{\mathbf{c}}_j \in \tilde{X} \,|\, 1 \leq j \leq \kappa\}$, where $\iota : \tilde{X} \to X$ is the projective map given by (45). For a rational function $\varphi(t)$ on $\tilde{X}$ we also denote by $\mathcal{N}(\varphi)$ the product of the critical values at all points in $\tilde{\mathcal{C}}$, i.e.,

$$\mathcal{N}(\varphi) := \prod_{j=1}^{\kappa} \varphi(\tilde{\mathbf{c}}_j), \tag{63}$$

which is called the *norm* of φ on $\tilde{X}$.

Our aim is to study the norm $\mathcal{N}(\mathrm{Hess}(F))$ on X, and the following proposition gives the formula for $\mathcal{N}(\mathrm{Hess}(F))$ on X to be written by norms of several functions on $\tilde{X}$.

Proposition 3. *If an arbitrary $t \in \tilde{\mathcal{C}}$ satisfies $t_1 \neq 1$, then*

$$\mathcal{N}(\mathrm{Hess}(F)/2^3)$$
$$= -\frac{1}{\mathcal{N}(f_1)\mathcal{N}(f_2)\mathcal{N}(f_3)\{\mathcal{N}(f_4)\}^3\mathcal{N}(t_\infty)\{\mathcal{N}(1-t_1)\}^2} \mathcal{N}\left(\frac{\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_1, t_2, t_3)}\right). \tag{64}$$

Proof. From (62) in Lemma 13 we have

$$\prod_{j=1}^{\kappa} \left.\frac{\mathrm{Hess}(F)}{2^3}\right|_{x=\mathbf{c}_j} = \prod_{j=1}^{\kappa} \left.\left\{\frac{-1}{f_1 f_2 f_3 f_4^3 t_\infty (1-t_1)^2} \frac{\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_1, t_2, t_3)}\right\}\right|_{t=\tilde{\mathbf{c}}_j},$$

which coincides with the right-hand side of (64). □

In order to calculate the part $\mathcal{N}\big(\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})/\partial(t_1, t_2, t_3)\big)$ in the right-hand side of (64) in Proposition 3 we will use the following lemma later.

Lemma 14. *Suppose that there exists rational curve $\omega : \mathbb{C} \to \tilde{X}$ in $\tilde{X}$ given by*

$$\omega : t_1 \longmapsto (t_1, t_2 = \omega_2(t_1), t_3 = \omega_3(t_1)) \in \tilde{X}$$

satisfies the equations

$$\tilde{g}_{12}(t_1, \omega_2(t_1), \omega_3(t_1)) = 0 \quad \text{and} \quad \tilde{g}_{13}(t_1, \omega_2(t_1), \omega_3(t_1)) = 0. \tag{65}$$

Suppose also that the curve ω interpolates some critical point in $\tilde{\mathcal{C}}$, i.e., there exists $\tau \in \mathbb{C}$ such that

$$\tilde{g}_1(\tau, \omega_2(\tau), \omega_3(\tau)) = 0.$$

Let ψ be function on $\mathbb{C}$ specified by

$$\psi(t_1) := \tilde{g}_1(t_1, \omega_2(t_1), \omega_3(t_1)). \tag{66}$$

The following identity as a function on the curve ω in $\tilde{X}$ (i.e., as a function of t_1) holds.

$$\frac{\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_1, t_2, t_3)} = \psi'(t_1) \times \frac{\partial(\tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_2, t_3)}, \tag{67}$$

where $\psi'(t_1)$ denotes the derivative $d\psi/dt_1$. In particular, for the point $\tau \in \mathbb{C}$ such that $T = (\tau, \omega_2(\tau), \omega_3(\tau)) \in \tilde{C}$ it follows that

$$\left.\frac{\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_1, t_2, t_3)}\right|_{t=T} = \psi'(\tau) \times \left.\frac{\partial(\tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_2, t_3)}\right|_{t=T}. \tag{68}$$

Proof. Applying chain rule to (65) and (66), we have

$$\begin{pmatrix} -\psi' + \frac{\partial \tilde{g}_1}{\partial t_1} & \frac{\partial \tilde{g}_1}{\partial t_2} & \frac{\partial \tilde{g}_1}{\partial t_3} \\ \frac{\partial \tilde{g}_{12}}{\partial t_1} & \frac{\partial \tilde{g}_{12}}{\partial t_2} & \frac{\partial \tilde{g}_{12}}{\partial t_3} \\ \frac{\partial \tilde{g}_{13}}{\partial t_1} & \frac{\partial \tilde{g}_{13}}{\partial t_2} & \frac{\partial \tilde{g}_{13}}{\partial t_3} \end{pmatrix} \begin{pmatrix} 1 \\ \omega_2' \\ \omega_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so that

$$0 = \begin{vmatrix} -\psi' + \frac{\partial \tilde{g}_1}{\partial t_1} & \frac{\partial \tilde{g}_1}{\partial t_2} & \frac{\partial \tilde{g}_1}{\partial t_3} \\ \frac{\partial \tilde{g}_{12}}{\partial t_1} & \frac{\partial \tilde{g}_{12}}{\partial t_2} & \frac{\partial \tilde{g}_{12}}{\partial t_3} \\ \frac{\partial \tilde{g}_{13}}{\partial t_1} & \frac{\partial \tilde{g}_{13}}{\partial t_2} & \frac{\partial \tilde{g}_{13}}{\partial t_3} \end{vmatrix} = \begin{vmatrix} \frac{\partial \tilde{g}_1}{\partial t_1} & \frac{\partial \tilde{g}_1}{\partial t_2} & \frac{\partial \tilde{g}_1}{\partial t_3} \\ \frac{\partial \tilde{g}_{12}}{\partial t_1} & \frac{\partial \tilde{g}_{12}}{\partial t_2} & \frac{\partial \tilde{g}_{12}}{\partial t_3} \\ \frac{\partial \tilde{g}_{13}}{\partial t_1} & \frac{\partial \tilde{g}_{13}}{\partial t_2} & \frac{\partial \tilde{g}_{13}}{\partial t_3} \end{vmatrix} - \begin{vmatrix} \psi' & \frac{\partial \tilde{g}_1}{\partial t_2} & \frac{\partial \tilde{g}_1}{\partial t_3} \\ 0 & \frac{\partial \tilde{g}_1}{\partial t_2} & \frac{\partial \tilde{g}_1}{\partial t_3} \\ 0 & \frac{\partial \tilde{g}_1}{\partial t_2} & \frac{\partial \tilde{g}_1}{\partial t_3} \end{vmatrix},$$

which is equivalent to (67), and (68) is a special case of (67) when $t_1 = \tau$. □

In the next section we consider a special symmetric case when $\rho_{12}^2 = \rho_{23}^2 = \rho_{13}^2$ and $\rho_{14}^2 = \rho_{24}^2 = \rho_{34}^2$, r_j^2 being the same. We shall present $\psi(t_1)$, $\frac{\partial(\tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_2, t_3)}$ and the norms of f_j explicitly by taking the basic parameter t_1.

We shall also consider the cases when we take the basic parameter t_∞ and s in place of t_1, where

$$t_\infty = 1 + t_1 + t_2 + t_3, \qquad s := \frac{\rho_{13}^2 - \rho_{14}^2}{\rho_{34}^2} t_1 + \frac{\rho_{23}^2 - \rho_{24}^2}{\rho_{34}^2} t_2 - t_3. \tag{69}$$

From (52) of Lemma 8, for t_∞ and s, we see that

$$f_3 - f_4 = \rho_{34}^2 \frac{s+1}{t_\infty} + \frac{(r_4^2 - r_1^2)t_1 + (r_4^2 - r_2^2)t_2 + (r_4^2 - r_3^2)(t_3 + 1)}{t_\infty}, \tag{70}$$

and we also see from (47) that

$$\sum_{j=1}^{4} \frac{1}{f_j} \equiv \frac{t_\infty}{f_4} \mod Ann(\tilde{C}). \tag{71}$$

The relations (70) and (71) will be used in Section 5 to evaluate the norms of $f_3 - f_4$ and $\sum_{j=1}^{4} f_j^{-1}$.

4. A Special Symmetric Case of $\triangle O_1O_2O_3O_4$

In this section, we restrict ourselves to a special symmetric case when $\triangle O_1O_2O_3O_4$ is a pyramid with axis of symmetry whose base triangle $\triangle O_1O_2O_3$ is regular and all spheres have the same radius, i.e., throughout this section we assume

$$(\mathcal{H}_1): \qquad \rho_{12}^2 = \rho_{13}^2 = \rho_{23}^2, \quad \rho_{14}^2 = \rho_{24}^2 = \rho_{34}^2, \quad r_j^2 = r^2 \quad (1 \le j \le 4).$$

We first see the fundamental invariants, i.e., the explicit forms of the Cayley–Menger determinants.

Lemma 15.

$$
\begin{aligned}
&B(012) = B(013) = B(023) = 2\rho_{12}^2 > 0, \quad B(014) = B(024) = B(034) = 2\rho_{14}^2 > 0,\\
&B(0123) = -3\rho_{12}^4 < 0, \quad B(0124) = \rho_{12}^2(\rho_{12}^2 - 4\rho_{14}^2) < 0,\\
&B(01234) = -2\rho_{12}^4(\rho_{12}^2 - 3\rho_{14}^2) > 0.
\end{aligned}
$$

Lemma 16.

$$
\begin{aligned}
&B(0\star j) = 2r^2 \quad (1 \le j \le 4), \quad B(0\star jk) = \rho_{12}^2(\rho_{12}^2 - 4r^2) \quad (1 \le j < k \le 3),\\
&B(0\star j4) = \rho_{14}^2(\rho_{14}^2 - 4r^2) \quad (1 \le j \le 3),\\
&B(0\star 123) = 2\rho_{12}^4(3r^2 - \rho_{12}^2), \quad B(0\star 124) = 2\rho_{12}^2(4r^2\rho_{14}^2 - r^2\rho_{12}^2 - \rho_{14}^4),\\
&B(0\star 1234) = \rho_{12}^4(3\rho_{14}^4 + 4r^2\rho_{12}^2 - 12r^2\rho_{14}^2) = \frac{3}{2}\rho_{12}^2 B(124) - 2r^2 B(01234).
\end{aligned}
$$

Remark 7. *$B(0\star jk) < 0$, $B(0\star jkl) > 0$, $B(0\star 1234) < 0$ for sufficiently large $r \gg 0$.*

Lemma 17. *Under the condition $(\mathcal{H}_1)$ the polynomials $\tilde{g}_j$ $(1 \le j \le 3)$ and $\tilde{g}_{jk}$ $(1 \le j < k \le 3)$ defined in (48) and (56) are written as*

$$
\begin{aligned}
\tilde{g}_j(t) = {}& r^2t_j^3 + (r^2 - \rho_{12}^2)(t_k^2 + t_l^2)t_j + 2r^2t_j^2(t_k + t_l) - (\rho_{12}^2 - 2r^2)t_jt_kt_l\\
&+ (\rho_{14}^2 + r^2)(t_j^2 + t_k^2 + t_l^2) - 2(\rho_{12}^2 - \rho_{14}^2)t_j(t_k + t_l)\\
&+ (2\rho_{14}^2 - \rho_{12}^2 - 2r^2)t_kt_l - (\rho_{14}^2 + r^2)t_j - 2r^2(t_k + t_l) - r^2,
\end{aligned} \tag{72}
$$

$$
\tilde{g}_{jk}(t) = (t_k - t_j)\hat{g}_{jk}(t), \quad \hat{g}_{jk}(t) := \rho_{12}^2t_jt_k + (\rho_{12}^2 - \rho_{14}^2)t_l - \rho_{14}^2(t_j + t_k - 1), \tag{73}
$$

respectively, where $\{j,k,l\}$ is a permutation of $\{1,2,3\}$.

Proof. Under the condition $(\mathcal{H}_1)$, the following symbols become as

$$
\begin{aligned}
B\begin{pmatrix} 0 & k & j\\ 0 & l & j\end{pmatrix} &= \rho_{kj}^2 + \rho_{lj}^2 - \rho_{kj}^2 = \rho_{12}^2,\\
B\begin{pmatrix} 0 & k & 4\\ 0 & j & 4\end{pmatrix} &= \rho_{k4}^2 + \rho_{l4}^2 - \rho_{kj}^2 = 2\rho_{14}^2 - \rho_{12}^2,\\
B\begin{pmatrix} 0 & \star & k\\ 0 & j & 4\end{pmatrix} &= \rho_{kj}^2 - r_j^2 - \rho_{k4}^2 + r_4^2 = \rho_{12}^2 - \rho_{14}^2 = \Delta_0,
\end{aligned}
$$

where $\{j,k,l\}$ is a permutation of $\{1,2,3\}$. Applying them to Proposition 2 and Lemma 10 in Section 3, we obtain (72) and (73). □

Notice that $\tilde{g}_{jk}$ are independent of r^2 under the condition $(\mathcal{H}_1)$. For the succeeding arguments we write $\tilde{g}_1$ as polynomial in t_2 and t_3 explicitly as follows.

$$
\begin{aligned}
&\tilde{g}_1(t_1,t_2,t_3)\\
&= \left[(r^2 - \rho_{12}^2)t_1 + \rho_{14}^2 - r^2\right](t_2^2 + t_3^2) + \left[(2r^2 - \rho_{12}^2)t_1 + 2\rho_{14}^2 - \rho_{12}^2 - 2r^2\right]t_2t_3\\
&\quad + 2\left[r^2t_1^2 - (\rho_{12}^2 - \rho_{14}^2)t_1 - r^2\right](t_2 + t_3) + (t_1 - 1)(r^2t_1^2 + (2r^2 + \rho_{14}^2)t_1 + r^2).
\end{aligned} \tag{74}
$$

By definition, we have the identity

$$
\hat{g}_{12}(t) - \hat{g}_{13}(t) = \rho_{12}^2(t_1 - 1)(t_2 - t_3). \tag{75}
$$

We state a property of the set $\tilde{\mathcal{C}} = \{t \in \tilde{X} \mid \tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3 = 0\}$ as follows.

Lemma 18. *There exists no point $t = (t_1, t_2, t_3)$ in $\tilde{\mathcal{C}} \subset \tilde{X}$ such that $t_1 \neq t_2, t_2 \neq t_3, t_1 \neq t_3$.*

Proof. Assume that $t = (t_1, t_2, t_3) \in \tilde{\mathcal{C}}$ satisfies $t_1 \neq t_2, t_2 \neq t_3, t_1 \neq t_3$. If $t_1 \neq 1$, then from Lemma 11 we see that $t \in \tilde{\mathcal{C}}$ satisfies the system $\tilde{g}_1 = \tilde{g}_{12} = \tilde{g}_{13} = 0$. Seeing (73) and $t_1 \neq t_2$, $t_1 \neq t_3$ we have $0 = \hat{g}_{12} - \hat{g}_{13} = \rho_{12}^2(t_1 - 1)(t_2 - t_3)$, which contradicts the assumption. If $t_1 = 1$, then we have $t_2 \neq t_1 = 1$, so that again from Lemma 11 we see that $t \in \tilde{\mathcal{C}}$ satisfies the system $\tilde{g}_2 = \tilde{g}_{12} = \tilde{g}_{23} = 0$. In the same way as above we have a contradiction again. □

As a consequence of Lemma 18, the set $\tilde{\mathcal{C}}$ is partitioned into the following:

$$\tilde{\mathcal{C}}_1 : t_1 = t_2 = t_3, \quad \tilde{\mathcal{C}}_2 : t_1 = t_2 \neq t_3, \quad \tilde{\mathcal{C}}_3 : t_1 = t_3 \neq t_2, \quad \tilde{\mathcal{C}}_4 : t_2 = t_3 \neq t_1. \tag{76}$$

Denote $\mathcal{C}_j = \iota\tilde{\mathcal{C}}_j \subset X$, such that $\mathcal{C}$ is the disjoint union of $\mathcal{C}_j$ and that $\tilde{\mathcal{C}}$ is the disjoint union of $\tilde{\mathcal{C}}_j$:

$$\mathcal{C} = \bigsqcup_{j=1}^{4} \mathcal{C}_j, \quad \tilde{\mathcal{C}} = \bigsqcup_{j=1}^{4} \tilde{\mathcal{C}}_j.$$

Remark 8. *The number of the critical points is $2^{3+1} - 1 = 15$, i.e., $|\tilde{\mathcal{C}}| = 15$. As we will see below, it is confirmed that $|\tilde{\mathcal{C}}_1| = 3$ and $|\tilde{\mathcal{C}}_2| = |\tilde{\mathcal{C}}_3| = |\tilde{\mathcal{C}}_4| = 4$.*

For the set $\tilde{\mathcal{C}}_1$, we immediately have the following:

Lemma 19. *For $(1,1,1) \in \tilde{X}$ the polynomials $\tilde{g}_j$ $(j = 1,2,3)$ are evaluated as*

$$\tilde{g}_1(1,1,1) = \tilde{g}_2(1,1,1) = \tilde{g}_3(1,1,1) = -8\Delta_0,$$

where $\Delta_0 := \rho_{12}^2 - \rho_{14}^2$. In other words, the following equivalence holds:

$$\Delta_0 = 0 \iff (1,1,1) \in \tilde{\mathcal{C}}_1.$$

Our approach to study the structure of $\tilde{\mathcal{C}}$ depends on whether we impose the condition $\Delta_0 = 0$ or not. In the following section, we first consider the case $\Delta_0 \neq 0$, while we devote Section 6 to the case $\Delta_0 = 0$, i.e., the case where $\triangle O_1O_2O_3O_4$ is the regular tetrahedron.

5. Critical Points Under $(\mathcal{H}_0)$, $(\mathcal{H}_1)$ and $\rho_{12} \neq \rho_{14}$

Throughout this section, in addition to the imposed conditions $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$, we suppose further

$$\Delta_0 := \rho_{12}^2 - \rho_{14}^2 \neq 0. \tag{77}$$

In this setting, the special parameter s introduced in (69) is given by

$$s := \delta(t_1 + t_2) - t_3, \quad \text{where} \quad \delta = \frac{\rho_{12}^2 - \rho_{14}^2}{\rho_{14}^2}, \tag{78}$$

and (70) is reduced to

$$f_3 - f_4 = \rho_{14}^2(s+1)/t_\infty, \tag{79}$$

where $t_\infty = 1 + t_1 + t_2 + t_3$. Thus, $\mathcal{N}(s+1)$ will be used indirectly for calculation of $\mathcal{N}(f_3 - f_4)$ later.

Lemma 20. *Under the conditions* $(\mathcal{H}_0)$, $(\mathcal{H}_1)$ *and* $\Delta_0 \neq 0$, *there exists no point* $t = (t_1, t_2, t_3)$ *in* $\tilde{C} \subset \tilde{X}$ *such that* $t_1 = 1$ *if and only if* $\Delta_1 \neq 0$, *where*

$$\Delta_1 := 4r^2 - 3\rho_{12}^2 - \rho_{14}^2. \quad (80)$$

Moreover, $\Delta_1 = 0$ *if and only if* $\{(t_1, t_2, t_3) \in \tilde{C} \,|\, t_1 = 1\} = \{(1, 1, -1), (1, -1, 1)\}$.

Proof. If $(1, t_2, t_3) \in \tilde{C}$ then $(1, t_2, t_3)$ satisfies the condition (76) and the equation

$$\tilde{g}_1(1, t_2, t_3) = -\Delta_0(t_2 + t_3)(t_2 + t_3 + 2) = 0. \quad (81)$$

Then, under the assumption $\Delta_0 \neq 0$, we need six possibilities for $(1, t_2, t_3) \in \tilde{C}$, i.e.,

$$(1, t_2, t_3) = (1, 1, -3),\ (1, 1, -1),\ (1, -3, 1),\ (1, -1, 1),\ (1, -1, -1),\ (1, 0, 0).$$

Under the condition $(\mathcal{H}_0)$, for these points we have

$$\begin{aligned}
&\tilde{g}_1(1,1,-3) = \tilde{g}_2(1,1,-3) = 0, \quad \tilde{g}_3(1,1,-3) = 4(5\rho_{12}^2 + \rho_{14}^2) > 0,\\
&\tilde{g}_1(1,1,-1) = \tilde{g}_2(1,1,-1) = 0, \quad \tilde{g}_3(1,1,-1) = -2\Delta_1,\\
&\tilde{g}_1(1,-3,1) = \tilde{g}_3(1,-3,1) = 0, \quad \tilde{g}_2(1,-3,1) = 4(5\rho_{12}^2 + \rho_{14}^2) > 0,\\
&\tilde{g}_1(1,-1,1) = \tilde{g}_3(1,-1,1) = 0, \quad \tilde{g}_2(1,-1,1) = -2\Delta_1,\\
&\tilde{g}_1(1,-1,-1) = 0, \quad \tilde{g}_2(1,-1,-1) = \tilde{g}_3(1,-1,-1) = 2(\rho_{12}^2 + \rho_{14}^2) > 0,\\
&\tilde{g}_1(1,0,0) = 0, \quad \tilde{g}_2(1,0,0) = \tilde{g}_3(1,0,0) = \rho_{14}^2 - 4r^2 = \frac{2B(0 \star 14)}{B(014)} \neq 0,
\end{aligned}$$

so that we see

$$(1,1,-3),\ (1,-3,1),\ (1,-1,-1),\ (1,0,0) \notin \tilde{C}, \quad (82)$$

and therefore obtain

$$\begin{aligned}
\Delta_1 \neq 0 \quad &\Longleftrightarrow \quad \{(t_1,t_2,t_3) \in \tilde{C} \,|\, t_1 = 1\} = \emptyset,\\
\Delta_1 = 0 \quad &\Longleftrightarrow \quad \{(t_1,t_2,t_3) \in \tilde{C} \,|\, t_1 = 1\} = \{(1,1,-1),(1,-1,1)\}.
\end{aligned}$$

This completes the proof. □

Lemma 11 states the equivalence between the systems $\tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3 = 0$ and $\tilde{g}_1 = \tilde{g}_{12} = \tilde{g}_{13} = 0$ under $t_1 \neq 1$. If $\Delta_1 \neq 0$, then we can omit the condition $t_1 \neq 1$ for this equivalence, because Lemma 20 says that each point $t \in \tilde{C} = \{t \in \tilde{X} \,|\, \tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3 = 0\}$ satisfies $t_1 \neq 1$. Namely, $\tilde{C}$ coincides with $\{t \in \tilde{X} \,|\, \tilde{g}_1 = \tilde{g}_{12} = \tilde{g}_{13} = 0\}$ if $\Delta_1 \neq 0$. On the other hand, if $\Delta_1 = 0$, then Lemma 20 implies that $\tilde{C}$ is expressed as

$$\tilde{C} = \{(1,1,-1),(1,-1,1)\} \cup \{t \in \tilde{X} \,|\, \tilde{g}_1 = \tilde{g}_{12} = \tilde{g}_{13} = 0, t_1 \neq 1\}.$$

However we eventually realize that this distinction is unnecessary whether $\Delta_1 = 0$ or not (see explanation in Remark 10 after Lemma 21). Hereafter, we analyze the set $\tilde{C}$ regarded as that of solutions of the system $\tilde{g}_1 = \tilde{g}_{12} = \tilde{g}_{13} = 0$ without constraint $t_1 \neq 1$, i.e.,

$$\tilde{C} = \{t \in \tilde{X} \,|\, \tilde{g}_1 = \tilde{g}_{12} = \tilde{g}_{13} = 0\}.$$

The aim of succeeding four subsections is to evaluate the norms $\mathcal{N}(t_j), \mathcal{N}(1 - t_j), \mathcal{N}(t_\infty)$ and $\mathcal{N}(s+1)$ for each $\tilde{C}_j$ $(j = 1,2,3,4)$ given in (76). We denote $\mathcal{N}_j(\varphi)$ the partial product of $\varphi(t)$ $(t \in \tilde{C}_j)$, i.e.,

$$\mathcal{N}_j(\varphi) := \prod_{t \in \tilde{C}_j} \varphi(t).$$

5.1. The Set $\tilde{\mathcal{C}}_1 : t_1 = t_2 = t_3$

In this subsection, we assume that $t_1 = t_2 = t_3$ for the critical points. When $t_2 = t_1$ and $t_3 = t_1$, from (73) $\tilde{g}_{12} = \tilde{g}_{13} = 0$ is automatically satisfied. Then, the solutions of the equation $\tilde{g}_1(t_1, t_1, t_1) = 0$ correspond to the critical points in $\tilde{\mathcal{C}}_1$. We define the characteristic polynomial $\psi_1(t_1)$ of the set $\tilde{\mathcal{C}}_1$ by

$$\psi_1(t_1) := \tilde{g}_1(t_1, t_1, t_1) = 3(3r^2 - \rho_{12}^2)t_1^3 + (9\rho_{14}^2 - 5\rho_{12}^2 - 3r^2)t_1^2 - (5r^2 + \rho_{14}^2)t_1 - r^2, \quad (83)$$

which gives $\tilde{\mathcal{C}}_1 = \{(t_1, t_1, t_1) \in \tilde{X} \,|\, \psi_1(t_1) = 0\}$. We denote the roots of the equation $\psi_1(t_1) = 0$ by $\zeta_1, \zeta_2, \zeta_3$, then the points corresponding to ζ_j give the set of critical points $\tilde{\mathcal{C}}_1$ in the straight line $t_1 = t_2 = t_3$. Let $\overline{\psi}_1$ be monic polynomial in t_1 specified by

$$\overline{\psi}_1(t_1) := \prod_{j=1}^{3}(t_1 - \zeta_j) = \frac{\psi_1(t_1)}{h_1}, \quad (84)$$

where $h_1 = 3(3r^2 - \rho_{12}^2)$ is the coefficient of highest degree of ψ_1. Then we obtain the following.

Proposition 4.

$$\begin{aligned}
\mathcal{N}_1(t_j) &= \frac{r^2}{3(3r^2 - \rho_{12}^2)} = \frac{2r^2\rho_{12}^4}{3B(0\star 123)} \quad (j = 1, 2, 3), \\
\mathcal{N}_1(1 - t_j) &= -\frac{8\Delta_0}{3(3r^2 - \rho_{12}^2)} = -\frac{16\Delta_0\rho_{12}^4}{3B(0\star 123)} \quad (j = 1, 2, 3), \\
\mathcal{N}_1(t_\infty) &= 4\,\frac{\rho_{12}^2 - 3\rho_{14}^2}{3r^2 - \rho_{12}^2} = -4\,\frac{B(01234)}{B(0\star 123)}, \\
\mathcal{N}_1(s+1) &= \frac{2\Delta_0(\rho_{12}^2 - 3\rho_{14}^2)(3\rho_{14}^2 + 4r^2\rho_{12}^2 - 12r^2\rho_{14}^2)}{3\rho_{14}^6(3r^2 - \rho_{12}^2)} = \frac{2\Delta_0 B(01234)B(0\star 1234)}{\rho_{14}^6 B(0123)B(0\star 123)}.
\end{aligned}$$

Proof. Since $t_1 = t_2 = t_3$, for $1 \le j \le 3$ we have $\mathcal{N}_1(t_j) = \mathcal{N}_1(t_1)$ and $\mathcal{N}_1(1 - t_j) = \mathcal{N}_1(1 - t_1)$, which are evaluated as special values of $\psi_1(t)$, as follows:

$$\begin{aligned}
\mathcal{N}_1(t_1) &= \zeta_1\zeta_2\zeta_3 = -\overline{\psi}_1(0) = -\frac{\psi_1(0)}{h_1} = \frac{r^2}{3(3r^2 - \rho_{12}^2)}, \\
\mathcal{N}_1(1 - t_1) &= \prod_{j=1}^{3}(1 - \zeta_j) = \overline{\psi}_1(1) = \frac{\psi_1(1)}{h_1} = -\frac{8\Delta_0}{3(3r^2 - \rho_{12}^2)}.
\end{aligned}$$

In general, for arbitrary γ_1, γ_2 we can calculate the norm of $\gamma_1 t_1 - \gamma_2$ by

$$\mathcal{N}_1(\gamma_1 t_1 - \gamma_2) = \prod_{j=1}^{3}(\gamma_1\zeta_j - \gamma_2) = -\gamma_1^3 \prod_{j=1}^{3}\left(\frac{\gamma_2}{\gamma_1} - \zeta_j\right) = -\frac{\gamma_1^3}{h_1}\psi_1\left(\frac{\gamma_2}{\gamma_1}\right).$$

We can evaluate $\psi_1(\gamma_2/\gamma_1)$ by a direct calculation from (83). Using this formula, we obtain

$$\begin{aligned}
\mathcal{N}_1(t_\infty) &= \mathcal{N}_1(3t_1 + 1) = -\frac{27}{h_1}\psi_1\left(-\frac{1}{3}\right) = 4\,\frac{\rho_{12}^2 - 3\rho_{14}^2}{3r^2 - \rho_{12}^2}, \\
\mathcal{N}_1(s+1) &= \mathcal{N}_1((2\delta - 1)t_1 + 1) = -\frac{(2\delta - 1)^3}{h_1}\psi_1\left(\frac{-1}{2\delta - 1}\right) \\
&= \frac{(3\rho_{14}^2 - 2\rho_{12}^2)^3}{h_1\rho_{14}^6}\psi_1\left(\frac{\rho_{14}^2}{3\rho_{14}^2 - 2\rho_{12}^2}\right),
\end{aligned}$$

which coincides with the result for $\mathcal{N}_1(s+1)$ in Proposition 4. □

5.2. The Set $\tilde{\mathcal{C}}_2 : t_1 = t_2 \neq t_3$

We assume that $t_1 = t_2 \neq t_3$ for the critical points. From (73) $t_1 = t_2$ implies that $\tilde{g}_{12} = 0$ is automatically satisfied. When $t_1 \neq t_3$, according to (73) it is necessary for $\tilde{g}_{13} = 0$ that

$$\hat{g}_{13}(t_1, t_1, t_3) = \rho_{12}^2 t_1 t_3 + (\rho_{12}^2 - \rho_{14}^2) t_1 - \rho_{14}^2 (t_1 + t_3 - 1) = 0$$

is satisfied. Solving this equation, with respect to t_3, we have

$$t_3 = \omega_3(t_1) := \frac{(2\rho_{14}^2 - \rho_{12}^2) t_1 - \rho_{14}^2}{\rho_{12}^2 t_1 - \rho_{14}^2}. \tag{85}$$

Hence, for the basic parameter t_1 the rational curve $(t_1, t_1, \omega_3(t_1)) \in \tilde{X}$ interpolates the set of all critical points in $\tilde{\mathcal{C}}_2$.

Lemma 21. *Let ψ_2 be function specified by $\psi_2(t_1) := \tilde{g}_1(t_1, t_1, \omega_3(t_1))$. Then ψ_2 has the factor $t_1 - 1$, namely it is written as*

$$\psi_2(t_1) = \frac{(t_1 - 1)\hat{\psi}_2(t_1)}{(\rho_{12}^2 t_1 - \rho_{14}^2)^2}, \tag{86}$$

where $\hat{\psi}_2(t_1)$ is a polynomial in t_1 of degree 4. Moreover the explicit form of $\hat{\psi}_2(t_1)$ is

$$\begin{aligned} \hat{\psi}_2(t_1) = {} & \rho_{12}^4 (4r^2 - \rho_{12}^2) t_1^4 + 2\rho_{12}^4 (2\rho_{14}^2 - \rho_{12}^2) t_1^3 + \rho_{12}^2 \rho_{14}^2 (-8r^2 + \rho_{12}^2 - 3\rho_{14}^2) t_1^2 \\ & + 2\rho_{12}^2 \rho_{14}^4 t_1 + \rho_{14}^4 (4r^2 - \rho_{14}^2). \end{aligned} \tag{87}$$

Proof. Since $\omega_3(t_1)$ is a ratio of two polynomials in t_1 of degree 1 as (85), and $\tilde{g}_1(t_1, t_1, t_3)$ is a polynomial in t_3 of degree 2 and in t_1 of degree 3, $\psi_2(t_1) = \tilde{g}_1(t_1, t_1, \omega_3(t_1))$ can be written as $\psi_2(t_1) = (\text{polynomial in } t_1 \text{ of degree } 5)/(\rho_{12}^2 t_1 - \rho_{14}^2)^2$. In particular, from (81) we have $\psi_2(1) = \tilde{g}_1(1, 1, \omega_3(1)) = \tilde{g}_1(1, 1, -1) = 0$, so that $\psi_2(t_1)$ is divisible by $t_1 - 1$. Therefore, we obtain the expression (86). The explicit form (87) is obtained by direct calculation. □

Remark 9. *From Lemma 20, we see that $(1, 1, \omega_3(1)) = (1, 1, -1) \notin \tilde{\mathcal{C}}$ if $\Delta_1 \neq 0$. This means that the root $t_1 = 1$ of the equation $\psi_2(t_1) = 0$ does not correspond to any point in $\tilde{\mathcal{C}}_2$.*

Since $\hat{\psi}_2(t_1)$ is evaluated at $t_1 = \rho_{14}^2/\rho_{12}^2$ as $\hat{\psi}_2(\rho_{14}^2/\rho_{12}^2) = 4r^2 \rho_{14}^4 \Delta_0^2 / \rho_{12}^4 \neq 0$, we have the expression

$$\tilde{\mathcal{C}}_2 = \{(t_1, t_1, \omega_3(t_1)) \in \tilde{X} \mid \hat{\psi}_2(t_1) = 0\},$$

where $\omega_3(t_1)$ is given by (85), and we call $\hat{\psi}_2(t_1)$ the *characteristic polynomial* of $\tilde{\mathcal{C}}_2$.

Remark 10. *When $\Delta_1 = 0$, i.e., $r^2 = (3\rho_{12}^2 + \rho_{14}^2)/4$, the characteristic polynomial $\hat{\psi}_2(t_1)$ is expressed as $\hat{\psi}_2(t_1) = \rho_{12}^2 r^2 (t_1 - 1)\left(\rho_{12}^2 (2\rho_{12}^2 + \rho_{14}^2) t_1^3 + 5\rho_{12}^2 \rho_{14}^2 t_1^2 - 5\rho_{14}^4 t_1 - 3\rho_{14}^4\right)$, which has the factor $(t_1 - 1)$, so that $t_1 = 1$ is the double root of $\psi_2(t_1) = 0$. The polynomial $\hat{\psi}_2(t_1)$ was originally defined from the system $\tilde{g}_1 = \tilde{g}_{12} = \tilde{g}_{13} = 0$ for $t_1 \neq 1$ under $\Delta_1 \neq 0$. In this sense $t_1 = 1$ is meaningless as a solution of $\hat{\psi}_2(t_1) = 0$, which corresponds to a point in $\tilde{\mathcal{C}}_2$. However, the point $(1, 1, \omega_3(1)) = (1, 1, -1)$ formally corresponding to $t_1 = 1$ is indeed an element of $\in \tilde{\mathcal{C}}$ when $\Delta_1 = 0$ (the fact $(1, 1, -1) \in \tilde{\mathcal{C}}_2$ if $\Delta_1 = 0$ was also confirmed in Lemma 20). This makes sense even when $t_1 = 1$, and eventually the imposed condition for t_1 or Δ_1 can be removed.*

We denote the roots of the equation $\hat{\psi}_2(t_1) = 0$ by $\zeta_1, \zeta_2, \zeta_3, \zeta_4$, then the points corresponding to ζ_j give the set of critical points $\tilde{\mathcal{C}}_2$. Let $\overline{\psi}_2$ be monic polynomial in t_1 specified by

$$\overline{\psi}_2(t_1) := \prod_{j=1}^{4} (t_1 - \zeta_j) = \frac{\hat{\psi}_2(t_1)}{h_2},$$

where $h_2 := \rho_{12}^4(4r^2 - \rho_{12}^2)$ is the coefficient of highest degree of $\hat{\psi}_2$. Then we obtain the following:

Proposition 5.

$$\mathcal{N}_2(t_1) = \mathcal{N}_2(t_2) = \frac{\rho_{14}^4(4r^2 - \rho_{14}^2)}{\rho_{12}^4(4r^2 - \rho_{12}^2)} = -\frac{\rho_{14}^2 B(0 \star 14)}{\rho_{12}^2 B(0 \star 12)},$$

$$\mathcal{N}_2(1 - t_1) = \mathcal{N}_2(1 - t_2) = \frac{\Delta_0^2 \Delta_1}{\rho_{12}^4(4r^2 - \rho_{12}^2)} = -\frac{\Delta_0^2 \Delta_1}{\rho_{12}^2 B(0 \star 12)}, \tag{88}$$

$$\mathcal{N}_2(\rho_{12}^2 t_1 - \rho_{14}^2) = \frac{4r^2 \rho_{14}^4 \Delta_0^2}{4r^2 - \rho_{12}^2} = -\frac{4r^2 \rho_{12}^2 \rho_{14}^4 \Delta_0^2}{B(0 \star 12)}, \tag{89}$$

$$\mathcal{N}_2((2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2) = \frac{4\rho_{14}^4 \Delta_0^2 (\rho_{12}^2 - 4\rho_{14}^2)(\rho_{14}^4 + r^2\rho_{12}^2 - 4r^2\rho_{14}^2)}{\rho_{12}^4(4r^2 - \rho_{12}^2)} = \frac{2\rho_{14}^4 \Delta_0^2 B(0124) B(0 \star 124)}{\rho_{12}^6 B(0 \star 12)},$$

$$\mathcal{N}_2(\rho_{12}^2 t_1^2 - \rho_{14}^2) = -\frac{4\rho_{14}^6 \Delta_0^2 (\rho_{12}^2 - 4\rho_{14}^2)}{(4r^2 - \rho_{12}^2)^2} = -\frac{4\rho_{12}^2 \rho_{14}^6 \Delta_0^2 B(0124)}{\{B(0 \star 12)\}^2}, \tag{90}$$

where Δ_1 *is given in* (80).

Proof. Since $t_1 = t_2$, we have $\mathcal{N}_2(t_1) = \mathcal{N}_2(t_2)$ and $\mathcal{N}_2(1 - t_1) = \mathcal{N}_2(1 - t_2)$. Indeed one can apply the formula

$$\mathcal{N}_2(\gamma_1 t_1 - \gamma_2) = \prod_{j=1}^{4}(\gamma_1 \zeta_j - \gamma_2) = \gamma_1^4 \prod_{j=1}^{4}\left(\frac{\gamma_2}{\gamma_1} - \zeta_j\right) = \gamma_1^4 \overline{\psi}_2\left(\frac{\gamma_2}{\gamma_1}\right) = \frac{\gamma_1^4}{h_2}\hat{\psi}_2\left(\frac{\gamma_2}{\gamma_1}\right) \tag{91}$$

to every case except (90). $\hat{\psi}_2(\gamma_2/\gamma_1)$ can be evaluated by a direct calculation from (87).

For an arbitrary quadratic polynomial $c_2 t_1^2 + c_1 t_1 + c_0 = c_2(t_1 - \alpha)(t_1 - \beta)$, there exist polynomials $P(t_1)$ and $q_1 t_1 + q_0$ such that

$$\hat{\psi}_2(t_1) = (c_2 t_1^2 + c_1 t_1 + c_0)P(t_1) + q_1 t_1 + q_0.$$

Then the norm of $c_2 t_1^2 + c_1 t_1 + c_0$ is calculated by reciprocity law as

$$\begin{aligned}
\mathcal{N}_2(c_2 t_1^2 + c_1 t_1 + c_0) &= c_2^4 \prod_{j=1}^{4}(\zeta_j - \alpha)(\zeta_j - \beta) = c_2^4 \overline{\psi}_2(\alpha)\overline{\psi}_2(\beta) \\
&= \frac{c_2^4}{h_2^2}\hat{\psi}_2(\alpha)\hat{\psi}_2(\beta) = \frac{c_2^4}{h_2^2}(q_1\alpha + q_0)(q_1\beta + q_0) = \frac{c_2^4}{h_2^2}(q_1^2\alpha\beta + q_0 q_1(\alpha + \beta) + q_0^2) \\
&= \frac{c_2^4}{h_2^2}\left(q_1^2\frac{c_0}{c_2} - q_0 q_1 \frac{c_1}{c_2} + q_0^2\right) = \frac{c_2^3}{h_2^2}(q_1^2 c_0 - q_0 q_1 c_1 + q_0^2 c_2).
\end{aligned} \tag{92}$$

For $c_2 = \rho_{12}^2$, $c_1 = 0$, $c_0 = -\rho_{14}^2$, by Euclidean division we have

$$\hat{\psi}_2(t_1) = (\rho_{12}^2 t_1^2 - \rho_{14}^2)P(t_1) + q_1 t_1 + q_0,$$

where

$$P(t_1) = \rho_{12}^2(4r^2 - \rho_{12}^2)t_1^2 + 2\rho_{12}^2(2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2(4r^2 - \rho_{14}^2)$$

and $q_1 = 2\rho_{12}^2\rho_{14}^2(3\rho_{14}^2 - \rho_{12}^2)$, $q_0 = -4\rho_{14}^6$. Then, using (92) we obtain

$$\mathcal{N}_2(\rho_{12}^2 t_1^2 - \rho_{14}^2) = \frac{\rho_{12}^6}{\rho_{12}^8(4r^2 - \rho_{12}^2)^2}\{-4\rho_{12}^4\rho_{14}^6(3\rho_{14}^2 - \rho_{12}^2)^2 + 16\rho_{14}^{12}\rho_{12}^2\},$$

which is factorized simply and coincides with (90). □

Corollary 1.

$$\mathcal{N}_2(t_3) = \frac{(\rho_{12}^2 - 4\rho_{14}^2)(\rho_{14}^4 + r^2\rho_{12}^2 - 4r^2\rho_{14}^2)}{r^2\rho_{12}^4} = -\frac{B(0124)B(0\star 124)}{2r^2\rho_{12}^8},$$
$$\mathcal{N}_2(1-t_3) = \frac{4\Delta_0^2(4r^2 - \rho_{14}^2)}{r^2\rho_{12}^4} = -4\frac{\Delta_0^2 B(0\star 14)}{r^2\rho_{12}^4\rho_{14}^2},$$
$$\mathcal{N}_2(s+1) = 4\frac{\Delta_0^2(4r^2 - \rho_{14}^2)^2}{r^2\rho_{14}^4(4r^2 - \rho_{12}^2)} = -4\frac{\Delta_0^2\rho_{12}^2\{B(0\star 14)\}^2}{r^2\rho_{14}^8 B(0\star 12)},$$
$$\mathcal{N}_2(t_\infty) = -16\frac{\rho_{14}^2(\rho_{12}^2 - 4\rho_{14}^2)}{r^2(4r^2 - \rho_{12}^2)} = 16\,\frac{\rho_{14}^2 B(0124)}{r^2 B(0\star 12)}.$$

Proof. Since parameters t_3, $1-t_3$, $s+1$ and t_∞ are written as

$$t_3 = \frac{(2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2}{\rho_{12}^2 t_1 - \rho_{14}^2}, \quad 1 - t_3 = \frac{2\Delta_0\, t_1}{\rho_{12}^2 t_1 - \rho_{14}^2},$$
$$s + 1 = 2\delta t_1 - t_3 + 1 = \frac{2\rho_{12}^2\Delta_0\, t_1^2}{\rho_{14}^2(\rho_{12}^2 t_1 - \rho_{14}^2)}, \quad t_\infty = 1 + 2t_1 + t_3 = \frac{2(\rho_{12}^2 t_1^2 - \rho_{14}^2)}{\rho_{12}^2 t_1 - \rho_{14}^2},$$

we obtain

$$\mathcal{N}_2(t_3) = \frac{\mathcal{N}_2((2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2)}{\mathcal{N}_2(\rho_{12}^2 t_1 - \rho_{14}^2)}, \quad \mathcal{N}_2(1-t_3) = \frac{2^4\Delta_0^4\mathcal{N}_2(t_1)}{\mathcal{N}_2(\rho_{12}^2 t_1 - \rho_{14}^2)},$$
$$\mathcal{N}_2(s+1) = \frac{2^4\rho_{12}^8\Delta_0^4\mathcal{N}_2(t_1)^2}{\rho_{14}^8\mathcal{N}_2(\rho_{12}^2 t_1 - \rho_{14}^2)}, \quad \mathcal{N}_2(t_\infty) = \frac{2^4\mathcal{N}_2(\rho_{12}^2 t_1^2 - \rho_{14}^2)}{\mathcal{N}_2(\rho_{12}^2 t_1 - \rho_{14}^2)},$$

respectively. They are all combinations of factors evaluated in Proposition 5. We therefore obtain the results. □

5.3. The Set $\tilde{C}_3 : t_1 = t_3 \neq t_2$

The case $\tilde{C}_3 : t_1 = t_3 \neq t_2$ for the admissible parameter t_1 is evaluated from that of $\tilde{C}_2 : t_1 = t_2 \neq t_3$ in previous subsection by the use of the transposition σ_{23} of the coordinates t_2 and t_3. In fact, one may take as in (86) and (87), i.e.,

$$\psi_3(t_1) := \psi_2(t_1), \quad \hat{\psi}_3(t_1) := \hat{\psi}_2(t_1), \tag{93}$$

and for the basic parameter t_1, the rational curve $(t_1, \omega_2(t_1), t_1) \in \tilde{X}$ interpolates the set of all critical points in $\tilde{C}_3$, where

$$t_2 = \omega_2(t_1) = \frac{(2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2}{\rho_{12}^2 t_1 - \rho_{14}^2}, \quad t_3 = t_1, \tag{94}$$

so that we have the expression

$$\tilde{C}_3 = \{(t_1, \omega_2(t_1), t_1) \in \tilde{X} \mid \hat{\psi}_2(t_1) = 0\}.$$

Then the same assertion as the preceding proposition holds true.

Proposition 6. *The points in $\tilde{C}_3$ consist of the four points corresponding to the solutions ζ_j to the equation $\hat{\psi}_3(t_1) = 0$ with $t_2 = \omega_2(t_1), t_3 = t_1$, and we have*

$$\begin{aligned}
&\mathcal{N}_3(t_1) = \mathcal{N}_2(t_1), \quad \mathcal{N}_3(1-t_1) = \mathcal{N}_2(1-t_1),\\
&\mathcal{N}_3(\rho_{12}^2 t_1 - \rho_{14}^2) = \mathcal{N}_2(\rho_{12}^2 t_1 - \rho_{14}^2),\\
&\mathcal{N}_3((2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2) = \mathcal{N}_2((2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2),\\
&\mathcal{N}_3(t_\infty) = \mathcal{N}_2(t_\infty)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{N}_3(t_2) = \mathcal{N}_2(t_3), \quad \mathcal{N}_3(1-t_2) = \mathcal{N}_2(1-t_3),\\
&\mathcal{N}_3(t_3) = \mathcal{N}_2(t_2), \quad \mathcal{N}_3(1-t_3) = \mathcal{N}_2(1-t_2).
\end{aligned} \tag{95}$$

These are explicitly given in Proposition 5 and Corollary 1.

Proof. Indeed t_1, t_∞ leave invariant under the transposition σ_{23}. Therefore, $\mathcal{N}_j(t_1), \mathcal{N}_j(1-t_1), \mathcal{N}_j(\rho_{12}^2 t_1 - \rho_{14}^2), \mathcal{N}_j\{(2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2\}$ and $\mathcal{N}_j(t_\infty)$ are all invariant under the transposition σ_{23}. The symmetry with respect to σ_{23} also implies (95). □

Proposition 7. *For the special parameter s, we have*

$$\mathcal{N}_3(s+1) = \frac{\Delta_0^2 \Delta_1 (\rho_{12}^2 - 4\rho_{14}^2)(\rho_{14}^4 + r^2\rho_{12}^2 - 4r^2\rho_{14}^2)}{r^2 \rho_{14}^8 (4r^2 - \rho_{12}^2)} = \frac{\Delta_0^2 \Delta_1 B(0124) B(0 \star 124)}{2r^2 \rho_{14}^8 \rho_{12}^2 B(0 \star 12)}. \tag{96}$$

Proof. From (94) the special parameter $s+1$ is calculated as

$$\begin{aligned}
s+1 = \delta(t_1+t_2) - t_3 + 1 &= \frac{\rho_{12}^2 - \rho_{14}^2}{\rho_{14}^2}\left(t_1 + \frac{(2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2}{\rho_{12}^2 t_1 - \rho_{14}^2}\right) - t_1 + 1\\
&= \frac{\rho_{12}^2}{\rho_{14}^2}(1-t_1)\frac{(2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2}{\rho_{12}^2 t_1 - \rho_{14}^2} = \frac{\rho_{12}^2}{\rho_{14}^2}(1-t_1)t_2,
\end{aligned}$$

so that, using (95) we have

$$\mathcal{N}_3(s+1) = \frac{\rho_{12}^8}{\rho_{14}^8}\mathcal{N}_3(1-t_1)\mathcal{N}_3(t_2) = \frac{\rho_{12}^8}{\rho_{14}^8}\mathcal{N}_2(1-t_1)\mathcal{N}_2(t_3).$$

Since $\mathcal{N}_2(1-t_1)$ and $\mathcal{N}_2(t_3)$ are given in Proposition 5 and Corollary 1, respectively, we obtain

$$\mathcal{N}_3(s+1) = \frac{\rho_{12}^8}{\rho_{14}^8} \times \frac{\Delta_0^2 \Delta_1}{\rho_{12}^4 (4r^2 - \rho_{12}^2)} \times \frac{(\rho_{12}^2 - 4\rho_{14}^2)(\rho_{14}^4 + r^2\rho_{12}^2 - 4r^2\rho_{14}^2)}{r^2 \rho_{12}^4},$$

which coincides with (96). □

5.4. The Set $\tilde{C}_4 : t_2 = t_3 \neq t_1$

We assume that $t_2 = t_3 \neq t_1$ for the critical points. Since $t_1 \neq t_2$ and $t_2 = t_3$, from (73) it is necessary for $\tilde{g}_{12} = \tilde{g}_{13} = 0$ that

$$\hat{g}_{12}(t_1, t_2, t_2) = \hat{g}_{13}(t_1, t_2, t_2) = \rho_{12}^2 t_1 t_2 + (\rho_{12}^2 - \rho_{14}^2)t_2 - \rho_{14}^2(t_1 + t_2 - 1) = 0.$$

is satisfied. Solving this equation with respect to t_2 we have

$$t_2 = \omega_2(t_1) := \frac{V(t_1)}{U(t_1)},$$

where

$$U(t_1) := \rho_{12}^2 t_1 + \rho_{12}^2 - 2\rho_{14}^2, \quad V(t_1) := \rho_{14}^2(t_1 - 1). \tag{97}$$

We may take the interpolating curve $(t_1, \omega_2(t_1), \omega_3(t_1)) \in \tilde{X}$ of the set $\tilde{\mathcal{C}}_4$ satisfying

$$\tilde{g}_1(t_1, \omega_2(t_1), \omega_3(t_1)) = 0,$$

where

$$t_2 = \omega_2(t_1) = \frac{V}{U}, \quad t_3 = \omega_3(t_1) := \frac{V}{U}.$$

Furthermore

Lemma 22. *Let ψ_4 be function specified by $\psi_4(t_1) := \tilde{g}_1(t_1, \frac{V}{U}, \frac{V}{U})$. Then ψ_4 has the factor $t_1 - 1$, namely it is written as*

$$\psi_4(t_1) = \frac{(t_1 - 1)\hat{\psi}_4(t_1)}{U^2}, \tag{98}$$

where $\hat{\psi}_4(t_1)$ is a polynomial in t_1 of degree 4. Moreover the explicit form of $\hat{\psi}_4(t_1)$ is

$$\begin{aligned}\hat{\psi}_4(t_1) = &\left(r^2t_1^2 + (\rho_{14}^2 + 2r^2)t_1 + r^2\right)U^2 + 4\rho_{14}^2\left(r^2t_1^2 - (\rho_{12}^2 - \rho_{14}^2)t_1 - r^2\right)U \\ &+ \rho_{14}^4(t_1 - 1)\left((4r^2 - 3\rho_{12}^2)t_1 + 4\rho_{14}^2 - \rho_{12}^2 - 4r^2\right)\end{aligned} \tag{99}$$

$$\begin{aligned}&= \rho_{12}^4 r^2 t_1^4 + \rho_{12}^4(\rho_{14}^2 + 4r^2)t_1^3 + \rho_{12}^2(6\rho_{12}^2 r^2 - 8\rho_{14}^2 r^2 - 2\rho_{12}^2\rho_{14}^2 - 3\rho_{14}^4)t_1^2 \\ &\quad + \rho_{12}^2(4\rho_{12}^2 r^2 - 16\rho_{14}^2 r^2 - 3\rho_{12}^2\rho_{14}^2 + 10\rho_{14}^4)t_1 \\ &\quad + (\rho_{12}^2 - 4\rho_{14}^2)(\rho_{14}^4 + \rho_{12}^2 r^2 - 4\rho_{14}^2 r^2).\end{aligned} \tag{100}$$

Proof. By the definition (74) of $\tilde{g}_1$, we have

$$\begin{aligned}U^2\psi_4(t_1) &= U^2\tilde{g}_1(t_1, \frac{V}{U}, \frac{V}{U}) \\ &= \left[2\left((r^2 - \rho_{12}^2)t_1 + \rho_{14}^2 - r^2\right) + \left((2r^2 - \rho_{12}^2)t_1 + 2\rho_{14}^2 - \rho_{12}^2 - 2r^2\right)\right]V^2 \\ &\quad + 4\left[r^2t_1^2 - (\rho_{12}^2 - \rho_{14}^2)t_1 - r^2\right]UV + (t_1 - 1)\left[r^2t_1^2 + (\rho_{14}^2 + 2r^2)t_1 + r^2\right]U^2, \\ &= (t_1 - 1)\left[\rho_{14}^4(t_1 - 1)\left((4r^2 - 3\rho_{12}^2)t_1 + 4\rho_{14}^2 - \rho_{12}^2 - 4r^2\right)\right. \\ &\qquad \left. + 4\rho_{14}^2\left(r^2t_1^2 - (\rho_{12}^2 - \rho_{14}^2)t_1 - r^2\right)U + \left(r^2t_1^2 + (\rho_{14}^2 + 2r^2)t_1 + r^2\right)U^2\right],\end{aligned}$$

which is a polynomial in t_1 of degree 5. Thus, we obtain

$$U^2\psi_4(t_1) = (t_1 - 1)\hat{\psi}_4(t_1),$$

where $\hat{\psi}_4(t_1)$ is a polynomial in t_1 of degree 4 explicitly given by (99). We therefore obtain (98). The explicit form (100) is obtained by direct calculation from (99). □

Remark 11. *From (82), we see that $(1, \omega_2(1), \omega_3(1)) = (1, 0, 0) \notin \tilde{\mathcal{C}}$. This means that the root $t_1 = 1$ of the equation $\psi_4(t_1) = 0$ does not correspond to any point in $\tilde{\mathcal{C}}_4$.*

Since $\hat{\psi}_4(t_1)$ is evaluated at $t_1 = (-\rho_{12}^2 + 2\rho_{14}^2)/\rho_{12}^2$ as

$$\hat{\psi}_4\left(\frac{-\rho_{12}^2 + 2\rho_{14}^2}{\rho_{12}^2}\right) = \frac{4\rho_{14}^4(4r^2 - \rho_{12}^2)\Delta_0^2}{\rho_{12}^4} = -\frac{4\rho_{14}^4 B(0 \star 12)}{\rho_{12}^6}\Delta_0^2 \neq 0, \tag{101}$$

we have the expression

$$\tilde{\mathcal{C}}_4 = \{(t_1, \frac{V}{U}, \frac{V}{U}) \in \tilde{X} \,|\, \hat{\psi}_4(t_1) = 0\},$$

where $U(t_1), V(t_1)$ are given by (97), and we call $\hat{\psi}_4(t_1)$ the *characteristic polynomial* of $\tilde{\mathcal{C}}_4$. We denote the roots of the equation $\hat{\psi}_4(t_1) = 0$ by $\zeta_1, \zeta_2, \zeta_3, \zeta_4$, then the points corresponding to ζ_j give the set of critical points $\tilde{\mathcal{C}}_4$. Let $\overline{\psi}_4$ be monic polynomial in t_1 specified by

$$\overline{\psi}_4(t_1) := \prod_{j=1}^{4}(t_1 - \zeta_j) = \frac{\hat{\psi}_4(t_1)}{h_4},$$

where $h_4 := \rho_{12}^4 r^2$ is the coefficient of highest degree of $\hat{\psi}_4$. Due to Lemma 22 we obtain the following.

Lemma 23. *$\hat{\psi}_4(t_1)$ is a polynomial in t_1 of degree 4 with the leading term*

$$\hat{\psi}_4(t_1) \approx h_4 t_1^4 \qquad (|t_1| \to \infty),$$

and the leading coefficient is given by $h_4 = \rho_{12}^4 r^2$. Furthermore we have

$$\begin{aligned} \hat{\psi}_4(0) &= (\rho_{12}^2 - 4\rho_{14}^2)(\rho_{14}^4 + \rho_{12}^2 r^2 - 4\rho_{14}^2 r^2), \\ \hat{\psi}_4(1) &= 4(4r^2 - \rho_{14}^2)\Delta_0^2, \\ \hat{\psi}_4(-1) &= 4\rho_{14}^4 \Delta_1, \end{aligned}$$

and $\hat{\psi}_4((-\rho_{12}^2 + 2\rho_{14}^2)/\rho_{12}^2)$ is provided as (101).

Proof. The results are calculated directly using (99) or (100). □

From the symmetry between $\tilde{\mathcal{C}}_4$ and $\tilde{\mathcal{C}}_2$, we immediately have the following:

Proposition 8.

$$\begin{aligned} &\mathcal{N}_4(t_1) = \mathcal{N}_2(t_3), \quad \mathcal{N}_4(1 - t_1) = \mathcal{N}_2(1 - t_3) \\ &\mathcal{N}_4(t_2) = \mathcal{N}_4(t_3) = \mathcal{N}_2(t_1), \quad \mathcal{N}_4(1 - t_2) = \mathcal{N}_4(1 - t_3) = \mathcal{N}_2(1 - t_1) \\ &\mathcal{N}_4(t_\infty) = \mathcal{N}_2(t_\infty). \end{aligned}$$

These are explicitly given in Proposition 5 and Corollary 1.

Since $s = \delta(t_1 + t_2) - t_3$, from the symmetry between $\tilde{\mathcal{C}}_4$ and $\tilde{\mathcal{C}}_3$, we also immediately have the following:

Proposition 9. $\mathcal{N}_4(s + 1) = \mathcal{N}_3(s + 1)$. *The explicit form is given in Proposition 7.*

Remark 12. *As a consequence of Lemma 23, we can explain another way to have the explicit forms of $\mathcal{N}_4(t_j)$ $(j = 1, 2, 3, \infty)$, $\mathcal{N}_4(1 - t_j)$ $(j = 1, 2, 3)$ and $\mathcal{N}_4(s + 1)$ using special values of $\hat{\psi}_4(t_1)$ as follows. The basic idea is to use the following formula for arbitrary γ_1, γ_2:*

$$\mathcal{N}_4(\gamma_1 t_1 - \gamma_2) = \frac{\gamma_1^4}{h_4} \hat{\psi}_4\Big(\frac{\gamma_2}{\gamma_1}\Big),$$

which is explained in (91). Then, using Lemma 23, we obtain

$$\begin{aligned}
\mathcal{N}_4(t_1) &= \frac{\hat{\psi}_4(0)}{h_4}, \quad \mathcal{N}_4(1-t_1) = \frac{\hat{\psi}_4(1)}{h_4}, \quad \mathcal{N}_4(t_1+1) = \frac{\hat{\psi}_4(-1)}{h_4}, \\
\mathcal{N}_4(U) &= \mathcal{N}_4(\rho_{12}^2 t_1 + \rho_{12}^2 - 2\rho_{14}^2) = \frac{\rho_{12}^8}{h_4}\hat{\psi}_4\Big(\frac{-\rho_{12}^2 + 2\rho_{14}^2}{\rho_{12}^2}\Big) \\
&= \frac{4\rho_{12}^4\rho_{14}^4(4r^2 - \rho_{12}^2)\Delta_0^2}{h_4}, \\
\mathcal{N}_4(V) &= \mathcal{N}_4(\rho_{14}^2(t_1 - 1)) = \frac{\rho_{14}^8}{h_4}\hat{\psi}_4(1).
\end{aligned} \tag{102}$$

For our setting $t_2 = t_3 = V/U$*, we have* $1 - t_2 = 1 - t_3 = \Delta_0(t_1+1)/U$*, so that*

$$\mathcal{N}_4(t_2) = \mathcal{N}_4(t_3) = \frac{\mathcal{N}_4(V)}{\mathcal{N}_4(U)}, \quad \mathcal{N}_4(1-t_2) = \mathcal{N}_4(1-t_3) = \Delta_0^4\frac{\mathcal{N}_4(t_1+1)}{\mathcal{N}_4(U)}$$

are simply calculated. Since $s+1$ *is written as*

$$s + 1 = \delta(t_1 + t_2) - t_3 + 1 = \delta\Big(t_1 + \frac{V}{U}\Big) - \frac{V}{U} + 1 = \frac{\rho_{12}^2\Delta_0}{\rho_{14}^2}\frac{t_1(t_1+1)}{U},$$

we have

$$\mathcal{N}_4(s+1) = \frac{\rho_{12}^8\Delta_0^4}{\rho_{14}^8}\frac{\mathcal{N}_4(t_1)\mathcal{N}_4(t_1+1)}{\mathcal{N}_4(U)},$$

which is also simply calculated. Lastly, we evaluate $\mathcal{N}_4(t_\infty)$*. The parameter* t_∞ *is written as*

$$t_\infty = 1 + t_1 + t_2 + t_3 = 1 + t_1 + 2\frac{V}{U} = \frac{\rho_{12}^2 t_1^2 + 2\rho_{12}^2 t_1 + \rho_{12}^2 - 4\rho_{14}^2}{U},$$

so that we have

$$\mathcal{N}_4(t_\infty) = \frac{\mathcal{N}_4(\rho_{12}^2 t_1^2 + 2\rho_{12}^2 t_1 + \rho_{12}^2 - 4\rho_{14}^2)}{\mathcal{N}_4(U)}.$$

To evaluate the above numerator we use another method. By Euclidean division, we have

$$\hat{\psi}_4(t_1) = (c_2 t_1^2 + c_1 t_1 + c_0)P(t_1) + q_1 t_1 + q_0,$$

where, for setting $c_2 t_1^2 + c_1 t_1 + c_0 = \rho_{12}^2 t_1^2 + 2\rho_{12}^2 t_1 + \rho_{12}^2 - 4\rho_{14}^2$*, there exist*

$$P(t_1) = \rho_{12}^2 r^2 t_1^2 + \rho_{12}^2(\rho_{14}^2 + 2r^2)t_1 + \rho_{12}^2 r^2 - 4\rho_{14}^2 r^2 - 3\rho_{14}^4 - 4\rho_{12}^2\rho_{14}^2$$

and $q_1 = 4\rho_{12}^2\rho_{14}^2(\rho_{12}^2 + 5\rho_{14}^2)$*,* $q_0 = 4\rho_{14}^2(\rho_{12}^2 - 4\rho_{14}^2)(\rho_{12}^2 + \rho_{14}^2)$*. Using (92) we finally obtain*

$$\mathcal{N}_4(\rho_{12}^2 t_1^2 + 2\rho_{12}^2 t_1 + \rho_{12}^2 - 4\rho_{14}^2) = \frac{c_2^3}{h_4^2}(q_1^2 c_0 - q_0 q_1 c_1 + q_0^2 c_2) = \frac{64\rho_{14}^6(4\rho_{14}^2 - \rho_{12}^2)\Delta_0^2}{r^4}.$$

5.5. Conclusions of This Section

In this subsection, we give a proof of Conjecture 2 under the conditions $(\mathcal{H}_0)$, $(\mathcal{H}_1)$ and $\Delta_0 \neq 0$. As we saw in (63), for a rational function φ on $\tilde{X}$, the norm of φ is defined by the product of the values over the set of all critical points $\tilde{\mathcal{C}}$, i.e.,

$$\mathcal{N}(\varphi) := \prod_{Q \in \tilde{\mathcal{C}}} \varphi(Q) = \prod_{j=1}^{4} \mathcal{N}_j(\varphi).$$

Summing up Propositions 4–9 and Corollary 1, we have

Theorem 2.

$$
\begin{aligned}
\mathcal{N}(t_j) &= -\frac{\rho_{14}^8(\rho_{12}^2-4\rho_{14}^2)(\rho_{14}^2-4r^2)^2(\rho_{14}^4+\rho_{12}^2r^2-4\rho_{14}^2r^2)}{3\rho_{12}^{12}(\rho_{12}^2-3r^2)(\rho_{12}^2-4r^2)^2} \\
&= \frac{B(0124)B(0\star 124)\{B(0\star 14)B(014)\}^2}{B(0123)B(0\star 123)\{B(0\star 12)B(012)\}^2} \quad (j=1,2,3),
\end{aligned} \tag{103}
$$

$$
\begin{aligned}
\mathcal{N}(1-t_j) &= -\frac{32}{3}\frac{\Delta_0^7\Delta_1^2(\rho_{14}^2-4r^2)}{r^2\rho_{12}^{12}(\rho_{12}^2-3r^2)(\rho_{12}^2-4r^2)^2} \\
&= \frac{2^{10}}{3}\frac{\Delta_0^7\Delta_1^2 B(0\star 14)}{B(0\star 1)B(014)B(0\star 123)\{B(0\star 12)B(012)\}^2} \quad (j=1,2,3),
\end{aligned} \tag{104}
$$

$$
\mathcal{N}(t_\infty) = -\frac{4^7\,\rho_{14}^6(\rho_{12}^2-3\rho_{14}^2)(\rho_{12}^2-4\rho_{14}^2)^3}{r^6(\rho_{12}^2-3r^2)(\rho_{12}^2-4r^2)^3} = -\frac{4^7\,B(01234)\{B(014)B(0124)\}^3}{B(0\star 123)\{B(0\star 1)B(0\star 12)\}^3}, \tag{105}
$$

$$
\begin{aligned}
\mathcal{N}(s+1) &= \frac{8}{3}\frac{\Delta_0^7\Delta_1^2(\rho_{12}^2-3\rho_{14}^2)(\rho_{12}^2-4\rho_{14}^2)^2(\rho_{14}^2-4r^2)^2}{r^6\rho_{14}^{26}(\rho_{12}^2-3r^2)(\rho_{12}^2-4r^2)^3} \\
&\quad\times(3\rho_{14}^4+4\rho_{12}^2r^2-12\rho_{14}^2r^2)(\rho_{14}^4+\rho_{12}^2r^2-4\rho_{14}^2r^2)^2 \\
&= \frac{2^7}{3}\frac{\Delta_0^7\Delta_1^2}{\rho_{14}^{30}}\frac{B(01234)B(0\star 1234)\{B(0\star 14)B(0\star 124)B(0124)\}^2}{B(0\star 123)\{B(0\star 1)B(0\star 12)B(012)\}^3}.
\end{aligned} \tag{106}
$$

Corollary 2.

$$
\begin{aligned}
\mathcal{N}(f_j) &= \frac{r^2\rho_{12}^{12}(\rho_{12}^2-3r^2)(\rho_{12}^2-4r^2)^2(\rho_{14}^2-4r^2)}{2^{16}\rho_{14}^2(\rho_{12}^2-4\rho_{14}^2)} \\
&\quad\times(3\rho_{14}^4+4\rho_{12}^2r^2-12\rho_{14}^2r^2)(\rho_{14}^4+\rho_{12}^2r^2-4\rho_{14}^2r^2)^2 \\
&= -\frac{B(0\star 1)}{2\cdot 4^8}\,\frac{B(0\star 1234)B(0\star 123)B(0\star 14)\{B(0\star 124)B(0\star 12)\}^2}{B(0124)B(012)\{B(014)\}^2} \\
&\quad (j=1,2,3),
\end{aligned} \tag{107}
$$

$$
\begin{aligned}
\mathcal{N}(f_4) &= -\frac{r^2\rho_{14}^6}{3\cdot 4^8}(\rho_{14}^2-4r^2)^3(3\rho_{14}^4+4\rho_{12}^2r^2-12\rho_{14}^2r^2)(\rho_{14}^4+\rho_{12}^2r^2-4\rho_{14}^2r^2)^3 \\
&= -\frac{B(0\star 1)}{2\cdot 4^8}\,\frac{B(0\star 1234)\{B(0\star 124)B(0\star 14)\}^3}{B(0123)\{B(012)\}^3},
\end{aligned} \tag{108}
$$

$$
\begin{aligned}
\mathcal{N}(f_j-f_4) &= -\frac{\Delta_0^7\Delta_1^2(\rho_{14}^2-4r^2)^2(3\rho_{14}^4+4\rho_{12}^2r^2-12\rho_{14}^2r^2)(\rho_{14}^4+\rho_{12}^2r^2-4\rho_{14}^2r^2)^2}{3\cdot 2^{11}\rho_{14}^2(\rho_{12}^2-4\rho_{14}^2)} \\
&= -\frac{\Delta_0^7\Delta_1^2}{3\cdot 2^7}\frac{B(0\star 1234)\{B(0\star 14)B(0\star 124)\}^2}{B(0124)\{B(012)B(014)\}^3} \quad (j=1,2,3),
\end{aligned} \tag{109}
$$

$$
\mathcal{N}(f_j-f_k) = 0 \quad (1\le j,k\le 3), \tag{110}
$$

$$
\begin{aligned}
\mathcal{N}\Big(\sum_{j=1}^{4}\frac{1}{f_j}\Big) &= \frac{3\cdot 4^{15}\,(\rho_{12}^2-3\rho_{14}^2)(\rho_{12}^2-4\rho_{14}^2)^3}{(\rho_{12}^2-3r^2)(\rho_{12}^2-4r^2)^3(\rho_{14}^2-4r^2)^3} \\
&\quad\times\frac{1}{(3\rho_{14}^4+4\rho_{12}^2r^2-12\rho_{14}^2r^2)(\rho_{14}^4+\rho_{12}^2r^2-4\rho_{14}^2r^2)^3} \\
&= \frac{2\cdot 4^{15}\,B(01234)B(0123)\{B(0124)B(012)B(014)\}^3}{\{B(0\star 1)\}^4B(0\star 1234)B(0\star 123)\{B(0\star 124)B(0\star 12)B(0\star 14)\}^3},
\end{aligned} \tag{111}
$$

$$
\mathcal{N}(L_{123}) = \frac{\{\frac{B(01234)}{8}\}^{\frac{15}{2}}}{\mathcal{N}(t_\infty)}, \quad \mathcal{N}(L_{jk4}) = \mathcal{N}(t_l)\mathcal{N}(L_{123}), \tag{112}
$$

where $\{j,k,l\}$ *denotes a permutation of* $\{1,2,3\}$.

Proof. From (47) we have $f_j - f_4 \equiv f_4(1-t_j)/t_j \mod Ann(\tilde{\mathcal{C}})$ $(j = 1,2,3)$, so that we have $\mathcal{N}(f_j - f_4) = \mathcal{N}(f_4)\mathcal{N}(1-t_j)/\mathcal{N}(t_j)$ for $j = 1,2,3$. Using (103) and (104) in Theorem 2, we see that $\mathcal{N}(f_1 - f_4) = \mathcal{N}(f_2 - f_4) = \mathcal{N}(f_3 - f_4)$. On the other hand, from (79) we have $\mathcal{N}(f_3 - f_4) = \rho_{14}^{30}\mathcal{N}(s+1)/\mathcal{N}(t_\infty)$, which coincides with the right-hand side of (109) by using (105) and (106) in Theorem 2. We therefore obtain (109). From (47) we also have $\mathcal{N}(f_j) = \mathcal{N}(f_4)/\mathcal{N}(t_j)$ for $j = 1,2,3$, so that we have $\mathcal{N}(f_1) = \mathcal{N}(f_2) = \mathcal{N}(f_3)$. On the other hand, using (47) again we have $f_3 - f_4 \equiv (1-t_3)f_3 \mod Ann(\tilde{\mathcal{C}})$, so that we obtain $\mathcal{N}(f_3) = \mathcal{N}(f_3 - f_4)/\mathcal{N}(1-t_3)$, which is evaluated as the right-hand side of (107) by using (104) in Theorem 2 and (109). We therefore obtain (107). Moreover, from (47) we also obtain $\mathcal{N}(f_4) = \mathcal{N}(t_1)\mathcal{N}(f_1)$, which is evaluated as (108) by using (103) in Theorem 2 and (107). From (76), we have $t_j - t_k \equiv 0 \mod Ann(\tilde{\mathcal{C}})$ for $j,k \in \{1,2,3\}$, so that we have $\mathcal{N}(t_j - t_k) = 0$, which implies

$$\mathcal{N}(f_j - f_k) = \mathcal{N}(f_j f_k)\mathcal{N}(f_k^{-1} - f_j^{-1}) = \frac{\mathcal{N}(f_j)\mathcal{N}(f_k)}{\mathcal{N}(f_4)}\mathcal{N}(t_k - t_j) = 0 \text{ for } j,k \in \{1,2,3\}.$$

We therefore obtain (110). From (71) we obtain $\mathcal{N}(\sum_{j=1}^4 f_j^{-1}) = \mathcal{N}(t_\infty)/\mathcal{N}(f_4)$, which coincides with (111) by using (105) in Theorem 2 and (108). Lastly (112) follows from the definition (44) of L_{jkl} and (38). □

As we mentioned as Remark 2 of Conjecture 2 in the introduction, we have the following:

Theorem 3. *Under the conditions* $(\mathcal{H}_0)$, $(\mathcal{H}_1)$, *and* $\Delta_0 \neq 0$,

$$f_1,\ f_2,\ f_3,\ f_4,\ \sum_{j=1}^4 \frac{1}{f_j},\ L_{123},\ L_{124},\ L_{134},\ L_{234}$$

are all units.

Proof. From the product expressions for $\mathcal{N}(f_j)$, $\mathcal{N}(\sum_{j=1}^4 f_j^{-1})$ and $\mathcal{N}(L_{jkl})$ in Corollary 2 we see that there appears no factor of their numerators which vanishes. □

6. Regular Tetrahedron Case ($\rho_{12} = \rho_{14}$)

In this section, we impose the conditions $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$ with $\Delta_0 = \rho_{12}^2 - \rho_{14}^2 = 0$, which means $\triangle O_1O_2O_3O_4$ is a regular tetrahedron and all spheres S_j have the same radius, i.e., $\rho_{jk}^2 = \rho^2$ $(1 \le j < k \le 4)$ and $r_j^2 = r^2$ $(1 \le j \le 4)$. Under this setting, we present the explicit formulae for $\mathcal{N}(f_j)$, $\mathcal{N}(\sum_{j=1}^4 f_j^{-1})$ and $\mathcal{N}(\mathrm{Hess}(F))$ using the admissible parameters t_1, t_2, t_3, and show that Conjectures 1 and 2 stated in the introduction hold true.

The polynomials $\tilde{g}_j$ $(1 \le j \le 3)$ and $\tilde{g}_{jk}$ $(1 \le j < k \le 3)$ defined in (72) and (73) are simplified as

$$\begin{aligned}\tilde{g}_j(t) = (t_j - 1)\Big[&r^2(t_j+1)^2 + \rho^2 t_j + 2r^2(t_j+1)(t_k+t_l) \\ &+ (r^2-\rho^2)(t_k^2+t_l^2) + (2r^2-\rho^2)t_k t_l\Big],\end{aligned} \tag{113}$$

$$\tilde{g}_{jk}(t) = (t_k - t_j)\hat{g}_{jk}(t), \quad \hat{g}_{jk}(t) := \rho^2(t_j-1)(t_k-1), \tag{114}$$

respectively, where $\{j,k,l\}$ is a permutation of $\{1,2,3\}$. Let $\tilde{\mathcal{C}} = \iota^{-1}\mathcal{C}$ be the set of critical points characterized by $\tilde{\mathcal{C}} = \{t \in \tilde{X} \,|\, \tilde{g}_1(t) = \tilde{g}_2(t) = \tilde{g}_3(t) = 0\}$. By Lemma 11 if $t_1 \neq 1$ for $t \in \tilde{\mathcal{C}}$, then the system $\tilde{g}_1(t) = \tilde{g}_2(t) = \tilde{g}_3(t) = 0$ is equivalent to

$$\tilde{g}_1(t) = \tilde{g}_{12}(t) = \tilde{g}_{13}(t) = 0.$$

We may use the same notation for the points corresponding to these points in $\tilde{X}$. As a result, the set of 15 critical points are tabulated as W, Q_j, Q_{jk}, Q_{jkl}. One can also classify these points by the property (76). The set $\tilde{\mathcal{C}}$ is partitioned into four parts, i.e., $\tilde{\mathcal{C}} = \bigsqcup_{j=1}^{4} \tilde{\mathcal{C}}_j$.

6.1. The Set $\tilde{\mathcal{C}}_1 : t_1 = t_2 = t_3$

The point $(t_1, t_2, t_3) = (1, 1, 1) \in \tilde{X}$ satisfies the system $\tilde{g}_1(t) = \tilde{g}_2(t) = \tilde{g}_3(t) = 0$, and this point corresponds to the point W. Since $\tilde{g}_1(t_1, t_1, t_1) = (t_1 - 1)\left[3(3r^2 - \rho^2)t_1^2 + (6r^2 + \rho^2)t_1 + r^2\right]$, if $t_1 \neq 1$, then the two solutions of the quadratic equation

$$3(3r^2 - \rho^2)t_1^2 + (6r^2 + \rho^2)t_1 + r^2 = 0,$$

correspond to the points Q_{123}, Q_4. We obtain $\mathcal{C}_1 = \{W, Q_{123}, Q_4\}$.

6.2. The Set $\tilde{\mathcal{C}}_2 : t_1 = t_2 \neq t_3$

If $t_3 = 1$ for $t \in \tilde{\mathcal{C}}_2$, then $t_1 \neq 1$. Thus, $t \in \tilde{\mathcal{C}}_2$ satisfies $\tilde{g}_{12}(t_1, t_1, 1) = \tilde{g}_{13}(t_1, t_1, 1) = 0$ automatically. Since $\tilde{g}_1(t_1, t_1, 1) = (t_1 - 1)[(4r^2 - \rho^2)t_1^2 + 8r^2 t_1 + (4r^2 - \rho^2)]$, the two solutions of the quadratic equation

$$(4r^2 - \rho^2)t_1^2 + 8r^2 t_1 + (4r^2 - \rho^2) = 0 \quad (115)$$

correspond to Q_{12}, Q_{34}. On the other hand, if $t_3 \neq 1$ for $t \in \tilde{\mathcal{C}}_2$, then $t \in \tilde{\mathcal{C}}_2$ satisfies the system

$$\tilde{g}_3(t_1, t_1, t_3) = \tilde{g}_{31}(t_1, t_1, t_3) = \tilde{g}_{32}(t_1, t_1, t_3) = 0.$$

Since $\tilde{g}_{31}(t_1, t_1, t_3) = \tilde{g}_{32}(t_1, t_1, t_3) = \rho^2(t_1 - 1)(t_3 - 1)(t_3 - t_1)$, we need $t_1 = 1$ for $t \in \tilde{\mathcal{C}}_2$. Then we also need $\tilde{g}_3(1, 1, t_3) = (t_3 - 1)\left[r^2 t_3^2 + (6r^2 + \rho^2)t_3 + 3(3r^2 - \rho^2)\right] = 0$. Thus, the two solutions of the quadratic equation

$$r^2 t_3^2 + (6r^2 + \rho^2)t_3 + 3(3r^2 - \rho^2) = 0$$

correspond to Q_3, Q_{124}. We obtain $\mathcal{C}_2 = \{Q_{12}, Q_{34}, Q_3, Q_{124}\}$.

6.3. The Set $\tilde{\mathcal{C}}_3 : t_1 = t_3 \neq t_2$

This occurs from $\tilde{\mathcal{C}}_2$ by exchange of t_2, t_3. The cases $t_2 = 1$ or $t_2 \neq 1$ correspond to Q_{13}, Q_{24} or Q_2, Q_{134}, respectively. We obtain $\mathcal{C}_3 = \{Q_{13}, Q_{24}, Q_2, Q_{134}\}$.

6.4. The Set $\tilde{\mathcal{C}}_4 : t_2 = t_3 \neq t_1$

If $t_1 = 1$ for $t \in \tilde{\mathcal{C}}_4$, then $t_2 \neq 1$. Thus, $t \in \tilde{\mathcal{C}}_4$ satisfies $\tilde{g}_{21}(1, t_2, t_2) = \tilde{g}_{23}(1, t_2, t_2) = 0$ automatically. Since $\tilde{g}_2(1, t_2, t_2) = (t_2 - 1)[(4r^2 - \rho^2)t_2^2 + 8r^2 t_2 + (4r^2 - \rho^2)]$, the two solutions of the quadratic equation

$$(4r^2 - \rho^2)t_2^2 + 8r^2 t_2 + (4r^2 - \rho^2) = 0$$

correspond to Q_{14}, Q_{23}. On the other hand, if $t_1 \neq 1$ for $t \in \tilde{\mathcal{C}}_4$, then $t \in \tilde{\mathcal{C}}_4$ satisfies the system

$$\tilde{g}_1(t_1, t_2, t_2) = \tilde{g}_{12}(t_1, t_2, t_2) = \tilde{g}_{13}(t_1, t_2, t_2) = 0.$$

Since $\tilde{g}_{12}(t_1, t_2, t_2) = \tilde{g}_{13}(t_1, t_2, t_2) = \rho^2(t_1 - 1)(t_2 - 1)(t_2 - t_1)$, we need $t_2 = 1$ for $t \in \tilde{\mathcal{C}}_4$. Then we also need $\tilde{g}_1(t_1, 1, 1) = (t_1 - 1)\left[r^2 t_1^2 + (6r^2 + \rho^2)t_1 + 3(3r^2 - \rho^2)\right] = 0$. Thus, the two solutions of the quadratic equation

$$r^2 t_1^2 + (6r^2 + \rho^2)t_1 + 3(3r^2 - \rho^2) = 0 \quad (116)$$

correspond to Q_1, Q_{234}. We obtain $\mathcal{C}_4 = \{Q_{14}, Q_{23}, Q_1, Q_{234}\}$.

6.5. Conclusions of This Section

We have the following two lemmas by a direct calculation:

Lemma 24.

$$
\begin{aligned}
f_j(W) &= \frac{3\rho^2 - 8r^2}{8} \quad \text{for } j \in \{1,2,3,4\}, \\
f_j(Q_j) &= \frac{\rho(\rho - \sqrt{24r^2+\rho^2})}{12} \quad \text{for } j \in \{1,2,3,4\}, \\
f_k(Q_j) &= \frac{\rho(3\rho + \sqrt{24r^2+\rho^2})}{4} \quad \text{for } k \neq j, \\
f_j(Q_{jkl}) &= \frac{\rho(3\rho - \sqrt{24r^2+\rho^2})}{4} \quad \text{for } \{j,k,l\} \subset \{1,2,3,4\}, \\
f_m(Q_{jkl}) &= \frac{\rho(\rho + \sqrt{24r^2+\rho^2})}{12} \quad \text{for } m \notin \{j,k,l\} \subset \{1,2,3,4\}, \\
f_j(Q_{jk}) &= \frac{\rho(\rho - \sqrt{8r^2-\rho^2})}{4} \quad \text{for } \{j,k\} \subset \{1,2,3,4\}, \\
f_l(Q_{jk}) &= \frac{\rho(\rho + \sqrt{8r^2-\rho^2})}{4} \quad \text{for } l \notin \{j,k\}.
\end{aligned}
$$

As a consequence

Corollary 3.

$$
\begin{aligned}
\mathcal{N}(f_j) = \mathcal{N}(f_1) &= -\frac{1}{3\cdot 4^8}\, r^2\rho^{14}(3\rho^2-8r^2)(\rho^2-3r^2)^3(\rho^2-4r^2)^3 \\
&= -\frac{2}{4^9}\,\frac{B(0\star 1)\{B(0\star 12)B(0\star 123)\}^3 B(0\star 1234)}{\{B(012)\}^3 B(0123)} \quad (1\le j\le 4),
\end{aligned} \tag{117}
$$

$$
\begin{aligned}
\mathcal{N}\Big(\sum_{j=1}^{4}\frac{1}{f_j}\Big) &= \frac{2\cdot 3^4\cdot 4^{15}}{r^8(3\rho^2-8r^2)(\rho^2-3r^2)^4(\rho^2-4r^2)^6} \\
&= 2\cdot 4^{15}\,\frac{B(01234)\{B(0123)\}^4\{B(012)\}^6}{B(0\star 1234)\{B(0\star 123)\}^4\{B(0\star 12)\}^6\{B(0\star 1)\}^4}.
\end{aligned} \tag{118}
$$

Proof. The above formulae are obtained by definition and from Lemma 24 in view of the following identities: $B(0\star j) = 2r^2$, $B(0\star jk) = \rho^2(\rho^2-4r^2)$, $B(0\star jkl) = 2\rho^4(3r^2-\rho^2)$, $B(0\star 1234) = \rho^6(3\rho^2-8r^2)$, $B(0jk) = 2\rho^2$, $B(0jkl) = -3\rho^4$, $B(01234) = 4\rho^6$. □

Lemma 25.

$$
\operatorname{Hess}(F)\big|_{x=W} = 4^9\,\frac{(\rho^2-8r^2)^3}{(3\rho^2-8r^2)^6}, \tag{119}
$$

$$
\operatorname{Hess}(F)\big|_{x=Q_j} \times \operatorname{Hess}(F)\big|_{x=Q_{klm}} = 4^7\,\frac{(\rho^2+24r^2)(\rho^2-8r^2)^3}{r^6\rho^4(\rho^2-3r^2)^5}, \tag{120}
$$

$$
\operatorname{Hess}(F)\big|_{x=Q_{jk}} \times \operatorname{Hess}(F)\big|_{x=Q_{lm}} = 4^{13}\,\frac{(\rho^2-8r^2)^4}{\rho^4(\rho^2-4r^2)^8} \tag{121}
$$

for $\{j,k,l,m\}$ *a permutation of* $\{1,2,3,4\}$.

Proof. We prove (119) first. By the definition (113) of $\tilde{g}_j$ we have

$$
\frac{\partial(\tilde{g}_1,\tilde{g}_2,\tilde{g}_3)}{\partial(t_1,t_2,t_3)}\bigg|_{t_1=t_2=t_3=1} = \begin{vmatrix} 2(8r^2-\rho^2) & & \\ & 2(8r^2-\rho^2) & \\ & & 2(8r^2-\rho^2) \end{vmatrix} = 8(8r^2-\rho^2)^3. \tag{122}
$$

By definition, we also have

$$t_\infty\big|_{t_1=t_2=t_3=1} = (1+t_1+t_2+t_3)\big|_{t_1=t_2=t_3=1} = 4. \tag{123}$$

From Lemma 24, we obtain

$$f_1 f_2 f_3 f_4^3\big|_{x=W} = (3\rho^2 - 8r^2)^6/8^6. \tag{124}$$

Applying (122)–(124) to the formula (61), we therefore obtain (119).

Next, we show (120). Without loss of generality, we prove the case $\mathrm{Hess}(F)\big|_{x=Q_1} \times \mathrm{Hess}(F)\big|_{x=Q_{234}}$ only. We denote by τ_1, τ_2 the solutions of (116). Then Q_1 and Q_{234} are written as $Q_1 = \iota(\tau_1, 1, 1)$ and $Q_{234} = \iota(\tau_2, 1, 1)$, respectively. From (62) of Lemma 13 we have

$$\begin{aligned}&\mathrm{Hess}(F)\big|_{x=Q_1} \times \mathrm{Hess}(F)\big|_{x=Q_{234}} \\ &= 2^6 \prod_{j=1}^{2} \left(\frac{1}{f_1 f_2 f_3 f_4^3\, t_\infty (1-t_1)^2} \frac{\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_1, t_2, t_3)} \right)\Bigg|_{\substack{t_1=\tau_j \\ t_2=1 \\ t_3=1}}.\end{aligned} \tag{125}$$

We now calculate the right-hand side of (125) precisely. From Lemma 24 we have

$$\begin{aligned}&(f_1 f_2 f_3 f_4^3)\big|_{x=Q_1} \times (f_1 f_2 f_3 f_4^3)\big|_{x=Q_{234}} \\ &= -\frac{\rho^2 r^2}{6}\left(\frac{\rho^2(\rho^2-3r^2)}{2}\right)^5 = -\frac{r^2\rho^{12}(\rho^2-3r^2)^5}{2^6 3}.\end{aligned} \tag{126}$$

From (68) in Lemma 14 we see that

$$\frac{\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_1, t_2, t_3)}\Bigg|_{\substack{t_2=1 \\ t_3=1}} = \frac{d\psi}{dt_1} \frac{\partial(\tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_2, t_3)}\Bigg|_{\substack{t_2=1, \\ t_3=1}}$$

where $\psi(t_1) = \tilde{g}_1(t_1, 1, 1) = (t_1 - 1)\left[r^2 t_1^2 + (6r^2+\rho^2)t_1 + 3(3r^2-\rho^2)\right]$. This implies that for the solution τ of (116), we have

$$\begin{aligned}\frac{d\psi}{dt_1}\Big|_{t_1=\tau} &= (\tau-1)(2r^2\tau + (6r^2+\rho^2)) = 2r^2\tau^2 + (4r^2+\rho^2)\tau - (6r^2+\rho^2) \\ &= 2[-(6r^2+\rho^2)\tau - 3(3r^2-\rho^2)] + (4r^2+\rho^2)\tau - (6r^2+\rho^2) \\ &= -(8r^2+\rho^2)\tau - (24r^2 - 5\rho^2),\end{aligned}$$

so that we obtain

$$\begin{aligned}\frac{d\psi}{dt_1}\Big|_{t_1=\tau_1} \times \frac{d\psi}{dt_1}\Big|_{t_1=\tau_2} &= (8r^2+\rho^2)^2\tau_1\tau_2 + (8r^2+\rho^2)(24r^2-5\rho^2)(\tau_1+\tau_2) + (24r^2-5\rho^2)^2 \\ &= (8r^2+\rho^2)^2\frac{3(3r^2-\rho^2)}{r^2} - (8r^2+\rho^2)(24r^2-5\rho^2)\frac{6r^2+\rho^2}{r^2} + (24r^2-5\rho^2)^2 \\ &= 2\rho^2(\rho^2-8r^2)(\rho^2+24r^2)/r^2.\end{aligned} \tag{127}$$

Since we can calculate

$$\frac{\partial(\tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_2, t_3)}\Bigg|_{\substack{t_2=1 \\ t_3=1}} = \begin{vmatrix} -\rho^2(t_1-1)^2 & 0 \\ 0 & -\rho^2(t_1-1)^2 \end{vmatrix} = \rho^4(t_1-1)^4,$$

we have

$$\frac{\partial(\tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_2, t_3)}\Bigg|_{\substack{t_1=\tau_1 \\ t_2=1 \\ t_3=1}} \times \frac{\partial(\tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_2, t_3)}\Bigg|_{\substack{t_1=\tau_2 \\ t_2=1 \\ t_3=1}} = \rho^8(\tau_1-1)^4(\tau_2-1)^4. \tag{128}$$

Moreover, by definition we have

$$t_\infty\big|_{\substack{t_2=1\\t_3=1}} = (1+t_1+t_2+t_3)\big|_{\substack{t_2=1\\t_3=1}} = t_1+3. \tag{129}$$

Applying (126)–(129) to the Equation (125) we therefore obtain

$$\mathrm{Hess}(F)\big|_{x=Q_1} \times \mathrm{Hess}(F)\big|_{x=Q_{234}} = -2^{13}\frac{3(\rho^2-8r^2)(\rho^2+24r^2)}{r^4\rho^2(\rho^2-3r^2)^5}\frac{(\tau_1-1)^2(\tau_2-1)^2}{(\tau_1+3)(\tau_2+3)},$$

which coincides with (120) by calculating $(\tau_1-1)(\tau_2-1)$ and $(\tau_1+3)(\tau_2+3)$ as follows:

$$(\tau_1-1)(\tau_2-1) = \tau_1\tau_2 - (\tau_1+\tau_2) + 1 = \frac{3(3r^2-\rho^2)}{r^2} + \frac{6r^2+\rho^2}{r^2} + 1 = \frac{2(8r^2-\rho^2)}{r^2},$$

$$(\tau_1+3)(\tau_2+3) = \tau_1\tau_2 + 3(\tau_1+\tau_2) + 9 = \frac{3(3r^2-\rho^2)}{r^2} - 3\frac{6r^2+\rho^2}{r^2} + 9 = -\frac{6\rho^2}{r^2}.$$

Finally, we show (121). Without loss of generality, we prove the case $\mathrm{Hess}(F)\big|_{x=Q_{12}} \times \mathrm{Hess}(F)\big|_{x=Q_{34}}$ only. We denote by σ_1, σ_2 the solutions of (115). Then Q_{12} and Q_{34} are written as $Q_{12} = \iota(\sigma_1, \sigma_1, 1)$ and $Q_{34} = \iota(\sigma_2, \sigma_2, 1)$, respectively. From (62) of Lemma 13, we have

$$\begin{aligned}&\mathrm{Hess}(F)\big|_{x=Q_{12}} \times \mathrm{Hess}(F)\big|_{x=Q_{34}}\\ &= 2^6\prod_{j=1}^{2}\left(\frac{1}{f_1f_2f_3f_4^3\, t_\infty(1-t_1)^2}\frac{\partial(\tilde{g}_1,\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_1,t_2,t_3)}\right)\Bigg|_{\substack{t_1=\sigma_j\\t_2=\sigma_j\\t_3=1}}.\end{aligned} \tag{130}$$

We now calculate the right-hand side of (130) precisely. From Lemma 24, we have

$$(f_1f_2f_3f_4^3)\big|_{x=Q_{12}} \times (f_1f_2f_3f_4^3)\big|_{x=Q_{34}} = \left(\frac{\rho^2(\rho^2-4r^2)}{8}\right)^6 = \frac{\rho^{12}(\rho^2-4r^2)^6}{2^{18}}. \tag{131}$$

From (68) in Lemma 14, we see that

$$\frac{\partial(\tilde{g}_1,\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_1,t_2,t_3)}\bigg|_{\substack{t_2=t_1\\t_3=1}} = \frac{d\psi}{dt_1}\frac{\partial(\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_2,t_3)}\bigg|_{\substack{t_2=t_1\\t_3=1}},$$

where $\psi(t_1) = \tilde{g}_1(t_1,t_1,1) = (t_1-1)\left[(4r^2-\rho^2)t_1^2 + 8r^2t_1 + (4r^2-\rho^2)\right]$. This implies that for the solution σ of (115), we have

$$\begin{aligned}\frac{d\psi}{dt_1}\bigg|_{t_1=\sigma} &= (\sigma-1)(2(4r^2-\rho^2)\sigma + 8r^2) = 2[(4r^2-\rho^2)\sigma^2 + \rho^2\sigma - 4\rho^2]\\ &= 2[-(8r^2\sigma + 4r^2 - \rho^2) + \rho^2\sigma - 4\rho^2] = 2(\rho^2-8r^2)(\sigma+1),\end{aligned}$$

so that we obtain

$$\frac{d\psi}{dt_1}\bigg|_{t_1=\sigma_1} \times \frac{d\psi}{dt_1}\bigg|_{t_1=\sigma_2} = 2^2(\rho^2-8r^2)^2(\sigma_1+1)(\sigma_2+1). \tag{132}$$

Since we can calculate

$$\frac{\partial(\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_2,t_3)}\bigg|_{\substack{t_2=t_1\\t_3=1}} = \begin{vmatrix}\rho^2(t_1-1)^2 & 0\\ 0 & \rho^2(t_1-1)^2\end{vmatrix} = \rho^4(t_1-1)^4,$$

we have

$$\frac{\partial(\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_2,t_3)}\bigg|_{\substack{t_1=\sigma_1\\t_2=\sigma_1\\t_3=1}} \times \frac{\partial(\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_2,t_3)}\bigg|_{\substack{t_1=\sigma_2\\t_2=\sigma_2\\t_3=1}} = \rho^8(\sigma_1-1)^4(\sigma_2-1)^4. \tag{133}$$

Moreover, by definition, we have

$$t_\infty\big|_{\substack{t_2=t_1\\t_3=1}} = (1+t_1+t_2+t_3)\big|_{\substack{t_2=t_1\\t_3=1}} = 2(t_1+1),$$

so that

$$t_\infty\big|_{\substack{t_1=\sigma_1\\t_2=\sigma_1\\t_3=1}} \times t_\infty\big|_{\substack{t_1=\sigma_2\\t_2=\sigma_2\\t_3=1}} = 4(\sigma_1+1)(\sigma_2+1). \tag{134}$$

Applying (131)–(134) to the Equation (130) we therefore obtain

$$\mathrm{Hess}(F)\big|_{x=Q_{12}} \times \mathrm{Hess}(F)\big|_{x=Q_{34}} = 2^{24}\frac{(\rho^2-8r^2)^2}{\rho^4(\rho^2-4r^2)^6}(\sigma_1-1)^2(\sigma_2-1)^2,$$

which coincides with (121) by calculating $(\sigma_1-1)(\sigma_2-1)$ as follows:

$$(\sigma_1-1)(\sigma_2-1) = \sigma_1\sigma_2-(\sigma_1+\sigma_2)+1 = 1+\frac{8r^2}{4r^2-\rho^2}+1 = \frac{2(8r^2-\rho^2)}{4r^2-\rho^2}.$$

This completes the proof. □

From Lemma 25, we obtain:

Proposition 10.

$$\mathcal{N}(\mathrm{Hess}(F)) = 4^{76}\frac{(\rho^2+24r^2)^4(\rho^2-8r^2)^{27}}{r^{24}\rho^{28}(3\rho^2-8r^2)^6(\rho^2-3r^2)^{20}(\rho^2-4r^2)^{24}}. \tag{135}$$

If $\mathcal{N}(\mathrm{Hess}\,F) = 0$ then $8r^2 = \rho^2$ and vice versa. This case occurs when all critical points Q_{jk} $(j<k)$ and Q_{jkl} $(j<k<l)$ coincides with W the center of gravity.

Remark 13. *If $r^2 > \frac{1}{8}\rho^2$, then all 15 critical points $\mathbf{c}_\nu$ are real and distinct from each other. If $r^2 > \frac{3}{8}\rho^2$, then $D_1\cap D_2\cap D_3\cap D_4 \neq \emptyset$. If $\frac{3}{8}\rho^2 > r^2 > \frac{1}{3}\rho^2$, then $D_j\cap D_k\cap D_l\cap \overline{D}_m^c \neq \emptyset$. If $\frac{1}{3}\rho^2 > r^2 > \frac{1}{4}\rho^2$, then $D_j\cap D_k\cap \overline{D}_l^c\cap \overline{D}_m^c \neq \emptyset$.*

Let $\{j,k,l,m\}$ be an arbitrary permutation of $\{1,2,3,4\}$. Then $\mathrm{grad}\,\mathfrak{Re}F$ preserves every affine plane $\mathfrak{p}_{jk,lm}$ and the lines $\mathfrak{l}(W_j,W_{klm}), \mathfrak{l}(W_{jk},W_{lm})$ are trajectories of $\mathrm{grad}\,\mathfrak{Re}F$.

If $\frac{1}{4}\rho^2 > r^2 > \frac{1}{8}\rho^2$, then $D_j\cap D_k = \emptyset$. The four points Q_j lie one by one in the inside of each $\mathfrak{Re}S_j$. The remaining 11 points lie in the common part of the inside of the pyramid $\Delta O_1O_2O_3O_4$ and the outside of all $\overline{D}_k$. The values of $\mathfrak{Re}F$ at $\mathfrak{Re}S_j, Q_{jk}, Q_{jkl}, W$ satisfy the ordering

$$\mathfrak{Re}F\big|_{\mathfrak{Re}S_j} = -\infty < \mathfrak{Re}F\big|_{Q_{jk}} < \mathfrak{Re}F\big|_{Q_{jkl}} < \mathfrak{Re}F\big|_{W}.$$

There exist the unique trajectories (separatrices) of the real vector field $\mathrm{grad}\,\mathfrak{Re}F$ starting from some point of $\mathfrak{Re}S_j$ and tending to Q_j, starting from Q_j and tending to Q_{jk}, starting from Q_{jkl} and tending to W respectively.

We assume that $\rho = 2$. Take the axis y_1 and the ordinate y_2 to be the lines $\mathfrak{l}(W_{jk},W_{lm})$ and $\mathfrak{l}(O_j,O_k)$ such that W_{jk} is the origin in the plane $\mathfrak{p}_{jk,lm}$. The restriction of f_j to the plane is represented by

$$f_j = y_1^2+(y_2+1)^2-r^2,\ f_k = y_1^2+(y_2-1)^2-r^2,\ f_l = f_m = (y_1-\sqrt{2})^2+y_2^2+1-r^2,$$

and the vector field $\mathrm{grad}\,\mathfrak{Re}F$ on $\mathfrak{p}_{jk.lm}$ is defined by the differential equation

$$\frac{dy_2}{dy_1} = \frac{v_2}{v_1},$$

where

$$v_1 = \frac{2y_1}{f_j} + \frac{2y_1}{f_k} + \frac{4(y_1 - \sqrt{2})}{f_l}, \quad v_2 = \frac{2(y_2+1)}{f_j} + \frac{2(y_2-1)}{f_k} + \frac{4y_2}{f_l}.$$

Then every trajectory in $\mathfrak{p}_{jk,lm}$ *tending to the infinity has an asymptotic expansion*

$$y_2 \approx C_{-1}(y_1 - \frac{1}{\sqrt{2}}) + \frac{C_2}{y_1^2} + \frac{C_3}{y_1^3} + \cdots \quad (|y_1| \to \infty)$$

or

$$y_1 \approx C'_{-1} y_2 + \frac{1}{\sqrt{2}} + \frac{C'_2}{y_2^2} + \frac{C'_3}{y_2^3} + \cdots \quad (|y_2| \to \infty).$$

where C_{-1} *or* C'_{-1} *denotes an arbitrary real constant and the remaining* C_ν, C'_ν $(\nu \geq 2)$ *are uniquely determined in a successive way. The phase portrait of* $\mathrm{grad}\,\mathfrak{Re}F$ *in* $\mathfrak{p}_{jk,lm}$ *is given as in the Figure 1.*

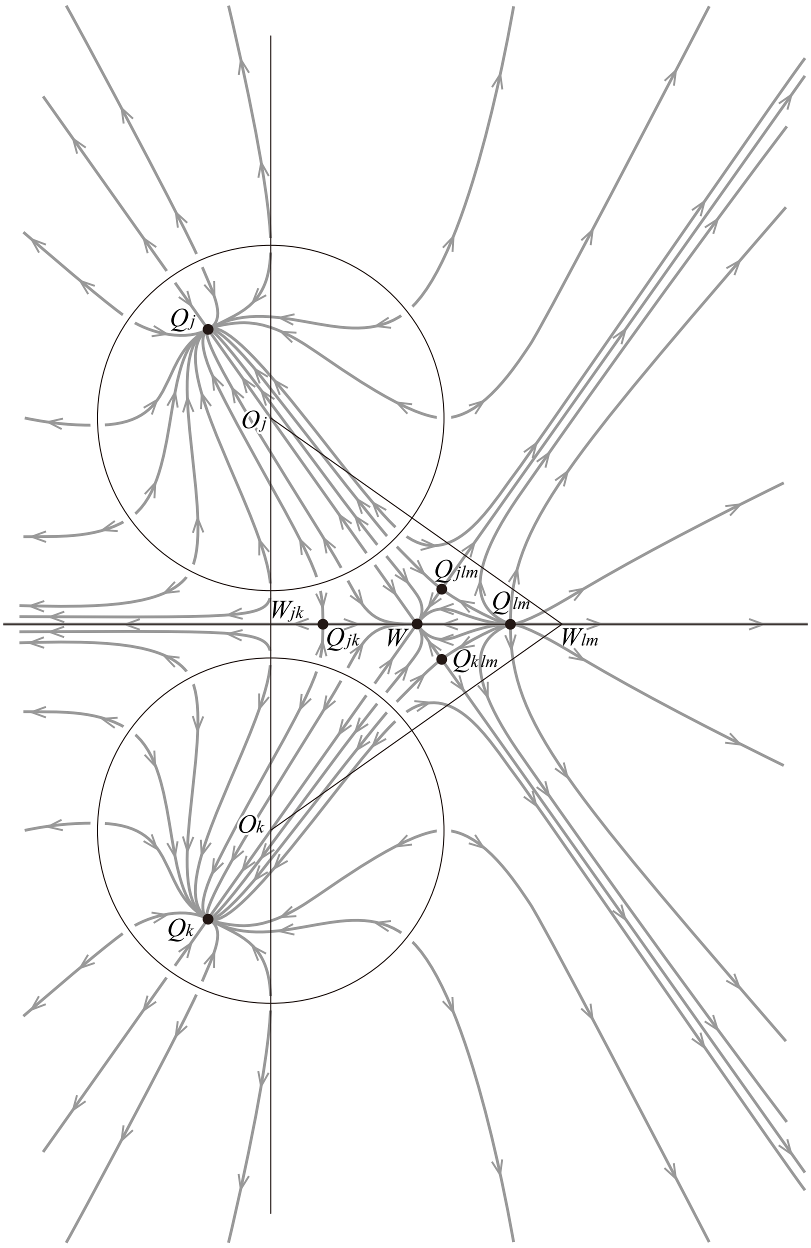

Figure 1. Phase portrait of $\mathrm{grad}\,\mathfrak{Re}F$ in $\mathfrak{p}_{jk,lm}$.

The real vector field grad $\mathfrak{Re}F$ *preserves the two dimensional real plane* $\mathfrak{p}_{jk,lm}$*, which contains the critical points* Q_j*,* Q_k*,* Q_{jk}*,* Q_{lm}*,* Q_{jlm} *and* Q_{klm}*. The three straight lines* $\overline{Q_jW}$*,* $\overline{Q_kW}$ *and* $\overline{W_{jk}W_{lm}}$ *are trajectories themselves. Every trajectory starts from each point of the circles* $\mathfrak{Re}S_j \cap \mathfrak{p}_{jk,lm}$ *and* $\mathfrak{Re}S_k \cap \mathfrak{p}_{jk,lm}$ *in a perpendicular manner to the circles, or from the point* Q_{lm} *(unstable node). The points* Q_{jk}*,* Q_{jlm} *and* Q_{klm} *are saddle points. Every trajectory finally tends to one of the points* Q_j*,* Q_k *and* W *(stable nodes), or to the infinity.*

7. Product of Hessians

In this section, we evaluate the norm of the Hessian of F under the same constraints as Section 5, i.e., we still impose the conditions $(\mathcal{H}_0)$, $(\mathcal{H}_1)$ and $\Delta_0 \neq 0$. From (64) of Proposition 3 the Hessian of F satisfies that

$$\mathcal{N}\left(\frac{1}{2^3}\mathrm{Hess}(F)\right) = -\frac{1}{\mathcal{N}(f_1)\mathcal{N}(f_2)\mathcal{N}(f_3)\mathcal{N}(f_4)^3\mathcal{N}(t_\infty)\mathcal{N}(1-t_1)^2}\mathcal{N}\left(\frac{\partial(\tilde{g}_1,\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_1,t_2,t_3)}\right).$$

Since we have already evaluated $\mathcal{N}(t_j)$ $(j=1,2,3,\infty)$, $\mathcal{N}(1-t_j)$ $(j=1,2,3)$ and $\mathcal{N}(f_j)$ $(j=1,2,3,4)$ in Section 5, our aim in this section is to study the remaining part

$$\mathcal{N}\left(\frac{\partial(\tilde{g}_1,\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_1,t_2,t_3)}\right).$$

Since the set $\tilde{\mathcal{C}}$ is separated into four parts, which are specified in (76), i.e., $\tilde{\mathcal{C}} = \bigsqcup_{j=1}^{4}\tilde{\mathcal{C}}_j$, we have

$$\mathcal{N}\left(\frac{\partial(\tilde{g}_1,\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_1,t_2,t_3)}\right) = \prod_{j=1}^{4}\mathcal{N}_j\left(\frac{\partial(\tilde{g}_1,\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_1,t_2,t_3)}\right).$$

Furthermore, from Lemma 14, it follows that

$$\mathcal{N}_j\left(\frac{\partial(\tilde{g}_1,\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_1,t_2,t_3)}\right) = \mathcal{N}_j\left(\frac{d\psi_j}{dt_1}\right)\mathcal{N}_j\left(\frac{\partial(\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_2,t_3)}\right), \tag{136}$$

where $\psi_j(t_1) = \tilde{g}_1(\omega(t_1))$ are the characteristic functions defined by the interpolation curve $\omega : \mathbb{C} \to \tilde{X}$ associated with $\tilde{\mathcal{C}}_j$, respectively (see Section 5 for further detail on the functions $\psi_j(t_1)$). In the sequel, we shall abbreviate

$$Z = \frac{\partial(\tilde{g}_1,\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_1,t_2,t_3)}, \quad Z_0 = \frac{\partial(\tilde{g}_{12},\tilde{g}_{13})}{\partial(t_2,t_3)}$$

respectively.

Definition 4. *Let* $\hat{\psi}_j(t_1)$ $(1 \le j \le 4)$ *be the characteristic polynomials of* $\tilde{\mathcal{C}}_j$ *defined in Section 4 (see (83), (87), (93) and (100) for explicit forms of* $\hat{\psi}_1(t_1) = \psi_1(t_1)$*,* $\hat{\psi}_2(t_1)$*,* $\hat{\psi}_3(t_1)$ *and* $\hat{\psi}_4(t_1)$*, respectively). For the polynomial* $\hat{\psi}_j(t_1)$ *of degree* m*, let* $\{\zeta_k \,|\, 1 \le k \le m\}$ *be the set of roots of the equation* $\hat{\psi}_j(t_1) = 0$*.*

$$\overline{\psi}_j(t_1) := \prod_{k=1}^{m}(t_1 - \zeta_k) = \frac{\hat{\psi}_j(t_1)}{h_j},$$

where h_j *are the coefficients of the leading terms of* $\hat{\psi}_j(t_1)$*. We define the discriminant of the polynomial* $\overline{\psi}_j(t_1)$ *associated with each* $\tilde{\mathcal{C}}_j$ *as follows:*

$$\mathrm{Discri}_j := \prod_{1 \le k < l \le m}(\zeta_k - \zeta_l)^2. \tag{137}$$

By definition, we can immediately confirm that

$$\mathrm{Discri}_j = (-1)^{\binom{m}{2}}\mathcal{N}_j(\overline{\psi}'_j(t_1)) = (-1)^{\binom{m}{2}}h_j^{-m}\mathcal{N}_j(\hat{\psi}'_j(t_1)), \tag{138}$$

i.e., $\mathrm{Discri}_1 = -h_1^{-3}\mathcal{N}_1(\psi_1'(t_1))$ and $\mathrm{Discri}_j = h_j^{-4}\mathcal{N}_j(\hat{\psi}_j'(t_1))$ $(j = 2,3,4)$.

7.1. $\mathcal{N}_1(Z)$

In this subsection, we consider $\mathcal{N}_1(Z)$ for the set $\tilde{\mathcal{C}}_1$. As we saw in Section 5.1, an arbitrary critical point $t = (t_1, t_2, t_3) \in \tilde{\mathcal{C}}_1$ is characterized as a point on the interpolation curve $\omega : \mathbb{C} \to \tilde{X}$ defined by $\omega(t_1) = (t_1, \omega_2(t_1), \omega_3(t_1)) \in \tilde{X}$, where

$$t_2 = \omega_2(t_1) := t_1, \quad t_3 = \omega_3(t_1) := t_1,$$

and $t = \omega(t_1) \in \tilde{X}$ satisfies the equation $\tilde{g}_1(\omega(t_1)) = 0$. Since $\tilde{g}_{12}$ and $\tilde{g}_{13}$ are expressed as (73), $\tilde{g}_{12} = \tilde{g}_{13} = 0$ is automatically satisfied. The characteristic function ψ_1 relative to the parameter t_1 is defined by

$$\psi_1(t_1) = \tilde{g}_1(t_1, t_1, t_1) = a_0 t_1^3 + a_2 t_1^2 + a_3 t_1 + a_3 = h_1 \overline{\psi}_1(t_1),$$

where

$$h_1 = a_0 = 3(3r^2 - \rho_{12}^2),\ a_2 = -3r^2 + 9\rho_{14}^2 - 5\rho_{12}^2,\ a_3 = -(5r^2 + \rho_{14}^2),\ a_4 = -r^2. \quad (139)$$

Lemma 26. *Let $\hat{g}_{12}$ and $\hat{g}_{13}$ be polynomials in t given by (73). Then we have*

$$Z_0 \equiv \hat{g}_{12}\,\hat{g}_{13} \mod Ann(\tilde{\mathcal{C}}_1), \quad (140)$$

namely

$$\mathcal{N}_1(Z_0) = \mathcal{N}_1(\hat{g}_{12})\mathcal{N}_1(\hat{g}_{13}). \quad (141)$$

Proof. Since $\tilde{g}_{12} = (t_2 - t_1)\hat{g}_{12}$ and $\tilde{g}_{13} = (t_3 - t_1)\hat{g}_{13}$, the point on the interpolation line $t_1 = t_2 = t_3$ satisfies

$$\left.\frac{\partial(\tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_2, t_3)}\right|_{\substack{t_2=t_1\\ t_3=t_1}} = \det\begin{pmatrix} \hat{g}_{12} + (t_2 - t_1)\frac{\partial \hat{g}_{12}}{\partial t_2} & (t_2 - t_1)\frac{\partial \hat{g}_{12}}{\partial t_3} \\ (t_3 - t_1)\frac{\partial \hat{g}_{13}}{\partial t_2} & \hat{g}_{13} + (t_3 - t_1)\frac{\partial \hat{g}_{13}}{\partial t_3} \end{pmatrix}\Bigg|_{\substack{t_2=t_1\\ t_3=t_1}}$$
$$= \hat{g}_{12}(t_1, t_1, t_1)\hat{g}_{13}(t_1, t_1, t_1),$$

which implies (140). □

By definition, the polynomial $\hat{g}_{12}(t_1, t_1, t_1)$ coincides with $\hat{g}_{13}(t_1, t_1, t_1)$, and they are written as

$$\hat{g}_{12}(t_1, t_1, t_1) = \hat{g}_{13}(t_1, t_1, t_1) = \rho_{12}^2 t_1^2 + (\rho_{12}^2 - 3\rho_{14}^2)t_1 + \rho_{14}^2.$$

Lemma 27.

$$\mathcal{N}_1(\hat{g}_{12}(t_1, t_1, t_1)) = \mathcal{N}_1(\hat{g}_{13}(t_1, t_1, t_1)) = \frac{2\Delta_0\Delta_2}{h_1^2} = \frac{2\Delta_0\Delta_2}{9(3r^2 - \rho_{12}^2)^2}, \quad (142)$$

where Δ_2 denotes

$$\Delta_2 := 4r^4(\rho_{12}^2 - 9\rho_{14}^2)^2 + r^2(\rho_{12}^2 - 9\rho_{14}^2)(\rho_{12}^4 - 2\rho_{12}^2\rho_{14}^2 + 9\rho_{14}^4) + 4\rho_{12}^4\rho_{14}^4. \quad (143)$$

Proof. For an arbitrary quadratic polynomial $c_2t_1^2 + c_1t_1 + c_0 = c_2(t_1 - \alpha)(t_1 - \beta)$, there exist unique polynomials $P(t_1)$ and $q_1t_1 + q_0$, such that $\hat{\psi}_1(t_1) = (c_2t_1^2 + c_1t_1 + c_0)P(t_1) + q_1t_1 + q_0$. Then the norm of $c_2t_1^2 + c_1t_1 + c_0$ is calculated by reciprocity law as

$$
\begin{aligned}
\mathcal{N}_1(c_2t_1^2 + c_1t_1 + c_0) &= c_2^3 \prod_{j=1}^{3} (\zeta_j - \alpha)(\zeta_j - \beta) = c_2^3 \overline{\psi}_1(\alpha)\overline{\psi}_1(\beta) \\
&= \frac{c_2^3}{h_1^2} \hat{\psi}_1(\alpha)\hat{\psi}_1(\beta) = \frac{c_2^3}{h_1^2}(q_1\alpha + q_0)(q_1\beta + q_0) = \frac{c_2^3}{h_1^2}(q_1^2\alpha\beta + q_0q_1(\alpha + \beta) + q_0^2) \\
&= \frac{c_2^3}{h_1^2}(q_1^2\frac{c_0}{c_2} - q_0q_1\frac{c_1}{c_2} + q_0^2) = \frac{c_2^2}{h_1^2}(q_1^2c_0 - q_0q_1c_1 + q_0^2c_2).
\end{aligned} \tag{144}
$$

By Euclidean division, we have

$$
\hat{\psi}_1(t_1) = (c_2t_1^2 + c_1t_1 + c_0)P(t_1) + q_1t_1 + q_0,
$$

where, for setting $\hat{g}_{12}(t_1, t_1, t_1) = c_2t_1^2 + c_1t_1 + c_0 = \rho_{12}^2 t_1^2 + (\rho_{12}^2 - 3\rho_{14}^2)t_1 + \rho_{14}^2$, there exist

$$
\begin{aligned}
P(t_1) &= 3\rho_{12}^{-2}(3r^2 - \rho_{12}^2)t_1 + \rho_{12}^{-4}(27\rho_{14}^2 r^2 - 12\rho_{12}^2 r^2 - 2\rho_{12}^4), \\
q_1 &= \rho_{12}^{-4}(2\rho_{12}^6 - 4\rho_{14}^2\rho_{12}^4 + 7\rho_{12}^4 r^2 - 72\rho_{12}^2\rho_{14}^2 r^2 + 81\rho_{14}^4 r^2), \\
q_0 &= \rho_{12}^{-4}(2\rho_{12}^4\rho_{14}^2 - \rho_{12}^4 r^2 + 12\rho_{12}^2\rho_{14}^2 r^2 - 27\rho_{14}^4 r^2).
\end{aligned}
$$

Using (144), we obtain

$$
\mathcal{N}_1(\hat{g}_{12}(t_1, t_1, t_1)) = c_2^2(q_1^2c_0 - q_0q_1c_1 + q_0^2c_2)h_1^{-2} = 2\Delta_0\Delta_2 h_1^{-2},
$$

which coincides with (142). □

Lemma 28. *The explicit form of* Discri$_1$ *is given by*

$$
\mathrm{Discri}_1 = -\frac{(\rho_{12}^2 - 3\rho_{14}^2)\Delta_3}{h_1^4}, \tag{145}
$$

where Δ_3 *denotes*

$$
\begin{aligned}
\Delta_3 := 3072r^6 &- 64(13\rho_{12}^2 - 3\rho_{14}^2)r^4 + 4(125\rho_{12}^4 - 430\rho_{12}^2\rho_{14}^2 + 309\rho_{14}^4)r^2 \\
&- (25\rho_{12}^2 - 27\rho_{14}^2)\rho_{14}^4.
\end{aligned} \tag{146}
$$

The explicit form of $\mathcal{N}_1(\psi_1')$ *is also expressed as*

$$
\mathcal{N}_1(\psi_1') = -h_1^3\,\mathrm{Discri}_1 = \frac{(\rho_{12}^2 - 3\rho_{14}^2)\Delta_3}{h_1} = \frac{(\rho_{12}^2 - 3\rho_{14}^2)\Delta_3}{3(3r^2 - \rho_{12}^2)}. \tag{147}
$$

Proof. The resultant of ψ_1 and ψ_1' gives the discriminant of $\psi_1(t_1)$, i.e.,

$$
R(\psi_1, \psi_1') := \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \\ & a_0 & a_1 & a_2 & a_3 \\ 3a_0 & 2a_1 & a_2 & & \\ & 3a_0 & 2a_1 & a_2 & \\ & & 3a_0 & 2a_1 & a_2 \end{vmatrix} = -a_0^5 \prod_{1 \le j < k \le 3} (\zeta_j - \zeta_k)^2 = -h_1^5\,\mathrm{Discri}_1,
$$

where a_0, a_1,a_2,a_3 are given in (139). From direct calculation of the above determinant, we obtain

$$
R(\psi_1, \psi_1') = 3(3r^2 - \rho_{12}^2)(\rho_{12}^2 - 3\rho_{14}^2)\Delta_3 = (\rho_{12}^2 - 3\rho_{14}^2)h_1\Delta_3,
$$

which implies (145). Using (138) we obtain (147) from (145). □

Due to (145) we immediately have the following:

Corollary 4. *There exists no double point in $\tilde{C}_1$ if and only if $\Delta_3 \neq 0$.*

We conclude the following from (136) and Lemmas 26–28.

Proposition 11.

$$\mathcal{N}_1(Z_0) = \frac{4\Delta_0^2\Delta_2^2}{h_1^4} = \frac{4\Delta_0^2\Delta_2^2}{3^4(3r^2 - \rho_{12}^2)^4}, \tag{148}$$

$$\mathcal{N}_1(Z) = -\frac{4\Delta_0^2\Delta_2^2\mathrm{Discri}_1}{h_1} = \frac{4(\rho_{12}^2 - 3\rho_{14}^2)\Delta_0^2\Delta_2^2\Delta_3}{h_1^5}. \tag{149}$$

Proof. Applying (142) to (141) we have (148). Using (148) and (147), (136) implies (149). □

7.2. $\mathcal{N}_2(Z)$

As we saw in Section 5.2, an arbitrary critical point $t = (t_1, t_2, t_3) \in \tilde{C}_2$ is characterized as a point on the interpolation curve $\omega : \mathbb{C} \to \tilde{X}$ defined by $\omega(t_1) = (t_1, \omega_2(t_1), \omega_3(t_1)) \in \tilde{X}$, where

$$t_2 = \omega_2(t_1) := t_1, \quad t_3 = \omega_3(t_1) := \frac{(2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2}{\rho_{12}^2 t_1 - \rho_{14}^2}, \tag{150}$$

and $t = \omega(t_1) \in \tilde{X}$ satisfies the equation $\tilde{g}_1(\omega(t_1)) = 0$. Since $\tilde{g}_{12}$ are expressed as (73), $\tilde{g}_{12} = 0$ is automatically satisfied when $t_2 = t_1$. The relation $t_3 = \omega_3(t_1)$ in (150) is determined by solving the equation $\hat{g}_{13}(t_1, t_1, t_3) = 0$, where

$$\hat{g}_{13}(t_1, t_1, t_3) = \rho_{12}^2 t_1 t_3 + \Delta_0 t_1 - \rho_{14}^2(t_1 + t_3 - 1).$$

In this setting,

$$\hat{g}_{12}(t_1, t_1, \omega_3(t_1)) = \rho_{12}^2 t_1^2 + \Delta_0 t_3 - \rho_{14}^2(2t_1 - 1).$$

From (75), $\hat{g}_{12}$ is also expressed as

$$\hat{g}_{12}(t_1, t_1, \omega_3(t_1)) = \hat{g}_{12}(t_1, t_1, \omega_3(t_1)) - \hat{g}_{13}(t_1, t_1, \omega_3(t_1)) = \rho_{12}^2(t_1 - 1)(t_1 - t_3).$$

The characteristic function ψ_2 relative to t_1 is defined by $\psi_2(t_1) = \tilde{g}_1(t_1, t_1, \omega_3(t_1))$, and from Lemma 21 $\psi_2(t_1)$ is expressed as

$$\psi_2(t_1) = \frac{(t_1 - 1)\hat{\psi}_2(t_1)}{(\rho_{12}^2 t_1 - \rho_{14}^2)^2}. \tag{151}$$

Here, $\hat{\psi}_2(t_1)$ is a polynomial in t_1 of degree 4 given by

$$\hat{\psi}_2(t_1) = a_0 t_1^4 + a_2 t_1^3 + a_3 t_2 + a_3 t_1 + a_4 = h_2 \overline{\psi}_2(t_1),$$

where

$$\begin{aligned} &h_2 = a_0 = \rho_{12}^4(4r^2 - \rho_{12}^2), \quad a_1 = 2\rho_{12}^4(2\rho_{14}^2 - \rho_{12}^2), \\ &a_2 = \rho_{12}^2\rho_{14}^2(-8r^2 + \rho_{12}^2 - 3\rho_{14}^2), \quad a_3 = 2\rho_{12}^2\rho_{14}^4, \quad a_4 = \rho_{14}^4(4r^2 - \rho_{14}^2). \end{aligned} \tag{152}$$

Lemma 29.

$$Z_0 \equiv \rho_{12}^2(\rho_{12}^2 t_1 - \rho_{14}^2)(1 - t_1)(t_1 - t_3)^2 \mod Ann(\tilde{C}_2), \tag{153}$$

namely

$$\mathcal{N}_2(Z_0) = \rho_{12}^8 \mathcal{N}_2(\rho_{12}^2 t_1 - \rho_{14}^2)\mathcal{N}_2(1 - t_1)\{\mathcal{N}_2(t_1 - t_3)\}^2. \tag{154}$$

Proof. Since $\tilde{g}_{12} = (t_2 - t_1)\hat{g}_{12}$ and $\tilde{g}_{13} = (t_3 - t_1)\hat{g}_{13}$, the point on the interpolation curve $t_2 = t_1, t_3 = \omega_3(t_1)$ satisfies

$$\frac{\partial(\tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_2, t_3)}\bigg|_{\substack{t_2=t_1\\ t_3=\omega_3(t_1)}} = \det\begin{pmatrix} \hat{g}_{12} + (t_2 - t_1)\frac{\partial \hat{g}_{12}}{\partial t_2} & (t_2 - t_1)\frac{\partial \hat{g}_{12}}{\partial t_3} \\ (t_3 - t_1)\frac{\partial \hat{g}_{13}}{\partial t_2} & \hat{g}_{13} + (t_3 - t_1)\frac{\partial \hat{g}_{13}}{\partial t_3}\end{pmatrix}\bigg|_{\substack{t_2=t_1\\ t_3=\omega_3(t_1)}}$$
$$= \hat{g}_{12}(t_1, t_1, t_3)\left[(t_3 - t_1)\frac{\partial \hat{g}_{13}}{\partial t_3}\right] = (\rho_{12}^2 t_1 - \rho_{14}^2)\rho_{12}^2(1 - t_1)(t_1 - t_3)^2,$$

which implies (153). □

Lemma 30.
$$\mathcal{N}_2(t_1 - t_3) = \mathcal{N}_2(t_2 - t_3) = \frac{\Delta_2}{r^2 h_2} = \frac{\Delta_2}{r^2\rho_{12}^4(4r^2 - \rho_{12}^2)}, \tag{155}$$
where Δ_2 is given by (143).

Proof. Since $t_1 - t_3$ on the curve ω is written as

$$t_1 - t_3 = t_1 - \omega_3(t_1) = t_1 - \frac{(2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2}{\rho_{12}^2 t_1 - \rho_{14}^2} = \frac{\hat{g}_{12}(t_1, t_1, t_1)}{\rho_{12}^2 t_1 - \rho_{14}^2},$$

where $\hat{g}_{12}(t_1, t_1, t_1) = \rho_{12}^2 t_1^2 + (\rho_{12}^2 - 3\rho_{14}^2)t_1 + \rho_{14}^2$, we have

$$\mathcal{N}_2(t_1 - t_3) = \frac{\mathcal{N}_2(\hat{g}_{12}(t_1, t_1, t_1))}{\mathcal{N}_2(\rho_{12}^2 t_1 - \rho_{14}^2)}. \tag{156}$$

By Euclidean division, we have

$$\hat{\psi}_2(t_1) = (c_2 t_1^2 + c_1 t_1 + c_0)P(t_1) + q_1 t_1 + q_0,$$

where, for setting $\hat{g}_{12}(t_1, t_1, t_1) = c_2 t_1^2 + c_1 t_1 + c_0 = \rho_{12}^2 t_1^2 + (\rho_{12}^2 - 3\rho_{14}^2)t_1 + \rho_{14}^2$, there exist

$$\begin{aligned} P(t_1) &= \rho_{12}^2(4r^2 - \rho_{12}^2)t_1^2 + (12\rho_{14}^2 r^2 - 4\rho_{12}^2 r^2 + \rho_{12}^2\rho_{14}^2 - \rho_{12}^4)t_1 \\ &\quad + \rho_{12}^{-2}(36\rho_{14}^4 r^2 - 36\rho_{12}^2\rho_{14}^2 r^2 + 4\rho_{12}^4 r^2 - 2\rho_{12}^4\rho_{14}^2 - \rho_{12}^6), \\ q_1 &= -\rho_{12}^{-2}\Delta_0(108\rho_{14}^4 r^2 - 408\rho_{12}^2\rho_{12}^2 r^2 + 4\rho_{12}^4 r^2 - 5\rho_{12}^2\rho_{14}^2 + \rho_{12}^2), \\ q_0 &= \rho_{12}^{-2}\Delta_0\rho_{14}^2(36\rho_{14}^2 r^2 - 4\rho_{12}^2 r^2 + \rho_{12}^2\rho_{14}^2 - \rho_{12}^4). \end{aligned}$$

Using (92), we obtain

$$\mathcal{N}_2(\hat{g}_{12}(t_1, t_1, t_1)) = c_2^3(q_1^2 c_0 - q_0 q_1 c_1 + q_0^2 c_2)h_2^{-2} = 4\rho_{12}^4\rho_{14}^4\Delta_0^2\Delta_2 h_2^{-2}.$$

Since $\mathcal{N}_2(\rho_{12}^2 t_1 - \rho_{14}^2) = 4r^2\rho_{12}^4\rho_{14}^4\Delta_0^2 h_2^{-1}$ is presented in Proposition 5, using (156), we therefore obtain $\mathcal{N}_2(t_1 - t_3) = \Delta_2 r^{-2} h_2^{-1}$, which coincides with (155). □

Corollary 5.
$$\mathcal{N}_2(t_1 - t_3) = \frac{9(3r^2 - \rho_{12}^2)^2}{2r^2\rho_{12}^4\Delta_0(4r^2 - \rho_{12}^2)}\mathcal{N}_1(\hat{g}_{12}).$$

Proof. See (142) in Lemma 27. □

Proposition 12. $\Delta_2 = 0$ *if and only if* $\tilde{C}_j \cap \tilde{C}_k \neq \emptyset$ $(1 \leq j < k \leq 4)$.

Proof. By definition, the curve $\omega(t_1) = (t_1, t_1, \omega_3(t_1)) \in \tilde{X}$ defined by (150) cross the line $t_1 = t_2 = t_3$, when $\omega_3(t_1) = t_1$, i.e., the parameter t_1 satisfies

$$\hat{g}_{13}(t_1, t_1, t_1) = \rho_{12}^2 t_1^2 + (\rho_{12}^2 - 3\rho_{14}^2)t_1 + \rho_{14}^2 = 0, \tag{157}$$

which is equivalent to $\hat{g}_{12}(t_1, t_1, t_1) = 0$, since $\hat{g}_{12}(t_1, t_1, t_1) = \hat{g}_{13}(t_1, t_1, t_1)$. This means that $\tilde{C}_2 \cap \tilde{C}_1 \neq \emptyset$ if and only if there exists a solution $t_1 = \zeta$ of the equation $\hat{\psi}_2(t_1) = 0$ satisfying (157), also means that $\tilde{C}_2 \cap \tilde{C}_1 \neq \emptyset$ if and only if there exists a solution $t_1 = \zeta$ of $\psi_1(t_1) = 0$ satisfying (157). Hence, by definition, $\mathcal{N}_2(t_1 - t_3) = 0$ holds if and only if there exists a solution $t_1 = \zeta$ of the equation $\hat{\psi}_2(t_1) = 0$, such that $\zeta - \omega_3(\zeta) = 0$, which is equivalent to $\tilde{C}_1 \cap \tilde{C}_2 \neq \emptyset$. The evaluation (155) in Lemma 30 implies the following:

$$\Delta_2 = 0 \quad \Longleftrightarrow \quad \mathcal{N}_2(t_1 - t_3) = 0 \quad \Longleftrightarrow \quad \tilde{C}_1 \cap \tilde{C}_2 \neq \emptyset. \tag{158}$$

Due to the symmetry of $\tilde{C}_2$, $\tilde{C}_3$ and $\tilde{C}_4$,

$$\tilde{C}_1 \cap \tilde{C}_2 \neq \emptyset, \quad \tilde{C}_1 \cap \tilde{C}_3 \neq \emptyset \quad \text{and} \quad \tilde{C}_1 \cap \tilde{C}_4 \neq \emptyset$$

occur at the same time. Hence, we see that $\Delta_2 = 0$ if and only if $\tilde{C}_1 \cap \tilde{C}_j \neq \emptyset$ $(j = 2, 3, 4)$. Moreover, since $\mathcal{N}_2(t_2 - t_3) = \mathcal{N}_2(t_1 - t_3)$, (158) implies that $\Delta_2 = 0$ if and only if $\mathcal{N}_2(t_2 - t_3) = 0$, which means $\tilde{C}_2 \cap \tilde{C}_3 \neq \emptyset$. Due to the symmetry of $\tilde{C}_2$, $\tilde{C}_3$, and $\tilde{C}_4$,

$$\tilde{C}_2 \cap \tilde{C}_3 \neq \emptyset, \quad \tilde{C}_2 \cap \tilde{C}_4 \neq \emptyset \quad \text{and} \quad \tilde{C}_3 \cap \tilde{C}_4 \neq \emptyset$$

occur at the same time. As a result, we therefore obtain Proposition 12. □

Lemma 31. *The explicit form of* Discri$_2$ *is given by*

$$\mathrm{Discri}_2 = \frac{16\rho_{12}^{12}\rho_{14}^8\Delta_0^2\Delta_4}{h_2^6} = \frac{16\rho_{14}^8\Delta_0^2\Delta_4}{\rho_{12}^{12}(4r^2 - \rho_{12}^2)^6}, \tag{159}$$

where Δ_4 *is the polynomial in* r^2, ρ_{12}^2, ρ_{14}^2 *of degree* 12 *given by*

$$\begin{aligned}\Delta_4 := {} & 16384\rho_{14}^2(4\rho_{14}^2 - \rho_{12}^2)r^8 + 2048\rho_{14}^2(3\rho_{12}^4 - 12\rho_{12}^2\rho_{14}^2 + \rho_{14}^4)r^6 \\ & - 48(9\rho_{12}^8 - 56\rho_{12}^6\rho_{14}^2 + 122\rho_{12}^4\rho_{14}^4 - 177\rho_{12}^2\rho_{14}^6 + 69\rho_{14}^8)r^4 \\ & + 4\Delta_0\rho_{14}^2(35\rho_{12}^6 - 141\rho_{12}^4\rho_{14}^2 + 9\rho_{12}^2\rho_{14}^4 + 81\rho_{14}^6)r^2 \\ & + \rho_{12}^2\rho_{14}^4(9\rho_{12}^6 - 49\rho_{12}^4\rho_{14}^2 + 63\rho_{12}^2\rho_{14}^4 - 27\rho_{14}^6).\end{aligned} \tag{160}$$

The explicit form of $\mathcal{N}_2(\hat{\psi}_2')$ *is also expressed as*

$$\mathcal{N}_2(\hat{\psi}_2') = h_2^4\,\mathrm{Discri}_2 = \frac{16\rho_{12}^{12}\rho_{14}^8\Delta_0^2\Delta_4}{h_2^2} = \frac{16\rho_{12}^4\rho_{14}^8\Delta_0^2\Delta_4}{(4r^2 - \rho_{12}^2)^2}. \tag{161}$$

Proof. The resultant of $\hat{\psi}_2$ and $\hat{\psi}_2'$ gives the discriminant of $\hat{\psi}_2(t_1)$, i.e.,

$$R(\hat{\psi}_2, \hat{\psi}_2') := \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & & \\ & a_0 & a_1 & a_2 & a_3 & a_4 & \\ & & a_0 & a_1 & a_2 & a_3 & a_4 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & & & \\ & 4a_0 & 3a_1 & 2a_2 & a_3 & & \\ & & 4a_0 & 3a_1 & 2a_2 & a_3 & \\ & & & 4a_0 & 3a_1 & 2a_2 & a_3 \end{vmatrix} = a_0^7 \prod_{j<k}(\zeta_j - \zeta_k)^2 = h_2^7\,\mathrm{Discri}_2,$$

where a_0, a_1, a_2, a_3, a_4 are given in (152). From direct calculation of the above determinant, we obtain

$$R(\hat{\psi}_2, \hat{\psi}_2') = 16\rho_{12}^{16}\rho_{14}^{8}(4r^2 - \rho_{12}^2)\Delta_0^2\Delta_4 = 16\rho_{12}^{12}\rho_{14}^{8}h_2\Delta_0^2\Delta_4,$$

which implies (159). Using (138) we therefore obtain (161) from (159). □

Due to (159) we immediately have the following:

Corollary 6. *There exists no double point in $\tilde{C}_2$ if and only if $\Delta_4 \neq 0$.*

We conclude the following from Lemmas 30 and 31:

Proposition 13.

$$\mathcal{N}_2(Z_0) = \frac{4\rho_{14}^4\Delta_0^4\Delta_1\Delta_2^2}{r^2\rho_{12}^4(4r^2 - \rho_{12}^2)^4}, \tag{162}$$

$$\mathcal{N}_2(Z) = \frac{\rho_{12}^4 h_2}{4r^6\rho_{14}^4}\Delta_0^2\Delta_1^2\Delta_2^2\,\mathrm{Discri}_2 = 4\frac{\rho_{12}^{16}\rho_{14}^4}{r^6 h_2^5}\Delta_0^4\Delta_1^2\Delta_2^2\Delta_4. \tag{163}$$

Proof. Applying (88), (89) in Proposition 5 and (155) in Lemma 30 to (154) we have (162). Differentiating both sides of (151) with respect to t_1 we have

$$\psi_2'(t_1) \equiv \frac{(t_1 - 1)\hat{\psi}_2'(t_1)}{(\rho_{12}^2 t_1 - \rho_{14}^2)^2} \quad \mathrm{mod}\ Ann(\tilde{C}_2),$$

so that

$$\mathcal{N}_2(\psi_2') = \frac{\mathcal{N}_2(t_1 - 1)}{\{\mathcal{N}_2(\rho_{12}^2 t_1 - \rho_{14}^2)\}^2}\mathcal{N}_2(\hat{\psi}_2'). \tag{164}$$

Using (164) and (154) in Lemma 29, (136) implies

$$\mathcal{N}_2(Z) = \mathcal{N}_2(Z_0)\mathcal{N}_2(\psi_2') = \frac{\rho_{12}^8\{\mathcal{N}_2(1 - t_1)\,\mathcal{N}_2(t_1 - t_3)\}^2}{\mathcal{N}_2(\rho_{12}^2 t_1 - \rho_{14}^2)}\mathcal{N}_2(\hat{\psi}_2'). \tag{165}$$

According to Proposition 5, Lemma 30 and (161) in Lemma 31, the right-hand side of (165) coincides with (163). □

7.3. $\mathcal{N}_3(Z)$

An arbitrary critical point $t = (t_1, t_2, t_3) \in \tilde{C}_3$ is characterized as a point on the interpolation curve $\omega : \mathbb{C} \to \tilde{X}$ defined by $\omega(t_1) = (t_1, \omega_2(t_1), \omega_3(t_1)) \in \tilde{X}$, where

$$t_2 = \omega_2(t_1) := \frac{(2\rho_{14}^2 - \rho_{12}^2)t_1 - \rho_{14}^2}{\rho_{12}^2 t_1 - \rho_{14}^2}, \quad t_3 = \omega_3(t_1) := t_1,$$

and $t = \omega(t_1) \in \tilde{X}$ satisfies the equation $\tilde{g}_1(\omega(t_1)) = 0$. This situation is represented by the transposition σ_{23} of the coordinates t_2 and t_3 from that of $\tilde{C}_2$. Thus, the characteristic function of $\tilde{C}_3$ is the same as $\tilde{C}_2$, i.e., $\psi_3(t_1) = \tilde{g}_1(t_1, \omega_2(t_1), \omega_3(t_1)) = \psi_2(t_1)$. Hence, our conclusion is:

Proposition 14. $\mathcal{N}_3(Z_0) = \mathcal{N}_2(Z_0)$, $\mathcal{N}_3(Z) = \mathcal{N}_2(Z)$. *The explicit forms are given in Proposition 13.*

7.4. $\mathcal{N}_4(Z)$

As we saw in Section 5.4, an arbitrary critical point $t = (t_1, t_2, t_3) \in \tilde{C}_4$ is characterized as a point on the interpolation curve $\omega : \mathbb{C} \to \tilde{X}$ defined by $\omega(t_1) = (t_1, \omega_2(t_1), \omega_3(t_1)) \in \tilde{X}$, where

$$t_2 = \omega_2(t_1) := \frac{V(t_1)}{U(t_1)}, \quad t_3 = \omega_3(t_1) := \frac{V(t_1)}{U(t_1)}, \tag{166}$$

$$U(t_1) := \rho_{12}^2 t_1 + \rho_{12}^2 - 2\rho_{14}^2, \quad V(t_1) := \rho_{14}^2(t_1 - 1),$$

and $t = \omega(t_1) \in \tilde{X}$ satisfies the equation $\tilde{g}_1(\omega(t_1)) = 0$. The relation $t_2 = t_3 = V(t_1)/U(t_1)$ in (166) is determined by solving the equation $\hat{g}_{12}(t_1, t_2, t_2) = 0$, where

$$\hat{g}_{12}(t_1, t_2, t_2) = \hat{g}_{13}(t_1, t_2, t_2) = \rho_{12}^2 t_1 t_2 + \Delta_0 t_2 - \rho_{14}^2(t_1 + t_2 - 1) = t_2 U(t_1) - V(t_1).$$

The characteristic function ψ_4 relative to t_1 is defined by $\psi_4(t_1) = \tilde{g}_1(t_1, \omega_2(t_1), \omega_3(t_1))$, and from Lemma 22 $\psi_4(t_1)$ is expressed as

$$\psi_4(t_1) = \frac{(t_1 - 1)\hat{\psi}_4(t_1)}{U^2}. \tag{167}$$

Here $\hat{\psi}_4(t_1)$ is a polynomial in t_1 of degree 4 given by

$$\hat{\psi}_4(t_1) = a_0 t_1^4 + a_2 t_1^3 + a_3 t_2 + a_3 t_1 + a_4 = h_4 \overline{\psi}_4(t_1),$$

where

$$\begin{aligned} h_4 = a_0 &= \rho_{12}^4 r^2, \quad a_1 = \rho_{12}^4(\rho_{14}^2 + 4r^2), \\ a_2 &= \rho_{12}^2(6\rho_{12}^2 r^2 - 8\rho_{14}^2 r^2 - 2\rho_{12}^2\rho_{14}^2 - 3\rho_{14}^4), \\ a_3 &= \rho_{12}^2(4\rho_{12}^2 r^2 - 16\rho_{14}^2 r^2 - 3\rho_{12}^2\rho_{14}^2 + 10\rho_{14}^4), \\ a_4 &= (\rho_{12}^2 - 4\rho_{14}^2)(\rho_{14}^4 + \rho_{12}^2 r^2 - 4\rho_{14}^2 r^2). \end{aligned} \tag{168}$$

Lemma 32.

$$Z_0 \equiv \rho_{12}^2(t_1 - 1)(t_1 - t_2)^2 U(t_1) \equiv \frac{\rho_{12}^2}{\rho_{14}^2}(t_1 - t_2)^2 U(t_1) V(t_1) \mod Ann(\tilde{C}_4), \tag{169}$$

namely

$$\mathcal{N}_4(Z_0) = \rho_{12}^8 \mathcal{N}_4(1 - t_1)\{\mathcal{N}_4(t_1 - t_2)\}^2 \mathcal{N}_4(U) = \frac{\rho_{12}^8}{\rho_{14}^8}\{\mathcal{N}_4(t_1 - t_2)\}^2 \mathcal{N}_4(U)\mathcal{N}_4(V). \tag{170}$$

Proof. Since $\tilde{g}_{12} = (t_2 - t_1)\hat{g}_{12}$ and $\tilde{g}_{13} = \tilde{g}_{12}$, the point on the interpolation curve $t_2 = t_3 = V(t_1)/U(t_1)$ satisfies

$$\begin{aligned} \frac{\partial(\tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_2, t_3)}\bigg|_{\substack{t_2 = V/U \\ t_3 = V/U}} &= \det\begin{pmatrix} \hat{g}_{12} + (t_2 - t_1)\frac{\partial \hat{g}_{12}}{\partial t_2} & (t_2 - t_1)\frac{\partial \hat{g}_{12}}{\partial t_3} \\ (t_3 - t_1)\frac{\partial \hat{g}_{13}}{\partial t_2} & \hat{g}_{13} + (t_3 - t_1)\frac{\partial \hat{g}_{13}}{\partial t_3} \end{pmatrix}\bigg|_{\substack{t_2 = V/U \\ t_3 = V/U}} \\ &= \left\{(t_2 - t_1)\frac{\partial \hat{g}_{12}}{\partial t_2}\right\}^2 - \left\{(t_2 - t_1)\frac{\partial \hat{g}_{13}}{\partial t_2}\right\}^2 = (t_2 - t_1)^2\{(\rho_{12}^2 t_1 - \rho_{14}^2)^2 - \Delta_0^2\} \\ &= (t_2 - t_1)^2 \rho_{12}^2 (t_1 - 1)(\rho_{12}^2 t_1 + \rho_{12}^2 - 2\rho_{14}^2), \end{aligned}$$

which implies (169). □

From the symmetry between $\tilde{C}_4$ and $\tilde{C}_2$, we immediately have $\mathcal{N}_4(t_1 - t_2) = \mathcal{N}_2(t_3 - t_1)$, which has already been evaluated as (155) in Lemma 30. Hence, we have the following:

Lemma 33.

$$\mathcal{N}_4(t_1 - t_2) = \mathcal{N}_4(t_1 - t_3) = \frac{\Delta_2}{r^2\rho_{12}^4(4r^2 - \rho_{12}^2)}. \tag{171}$$

Lemma 34. *The explicit form of* Discri$_4$ *is given by*

$$\text{Discri}_4 = \frac{16\rho_{12}^{12}\rho_{14}^8\Delta_0^2\Delta_4}{h_4^6} = \frac{16\rho_{14}^8\Delta_0^2\Delta_4}{r^{12}\rho_{12}^{12}}, \tag{172}$$

where Δ_4 is the polynomial in r^2, ρ_{12}^2, ρ_{14}^2 of degree 12 given by (160). $\mathcal{N}_4(\hat{\psi}_4')$ is also given as

$$\mathcal{N}_4(\hat{\psi}_4') = h_4^4\,\text{Discri}_4 = \frac{16\rho_{12}^{12}\rho_{14}^8\Delta_0^2\Delta_4}{h_4^2} = \frac{16\rho_{12}^4\rho_{14}^8\Delta_0^2\Delta_4}{r^4}. \tag{173}$$

Proof. The resultant of $\hat{\psi}_4$ and $\hat{\psi}_4'$ gives the discriminant of $\hat{\psi}_4(t_1)$, i.e.,

$$R(\hat{\psi}_4, \hat{\psi}_4') := \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & & \\ & a_0 & a_1 & a_2 & a_3 & a_4 & \\ & & a_0 & a_1 & a_2 & a_3 & a_4 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & & & \\ & 4a_0 & 3a_1 & 2a_2 & a_3 & & \\ & & 4a_0 & 3a_1 & 2a_2 & a_3 & \\ & & & 4a_0 & 3a_1 & 2a_2 & a_3 \end{vmatrix} = a_0^7\prod_{j<k}(\zeta_j - \zeta_k)^2 = h_4^7\,\text{Discri}_4,$$

where a_0, a_1, a_2, a_3, a_4 are given in (168). From direct calculation of the above determinant, we obtain

$$R(\hat{\psi}_4, \hat{\psi}_4') = 16\rho_{12}^{16}\rho_{14}^8 r^2\Delta_0^2\Delta_4 = 16\rho_{12}^{12}\rho_{14}^8 h_4\Delta_0^2\Delta_4,$$

which implies (172). Using (138) we therefore obtain (173) from (172). □

Due to (172) we immediately have the following:

Corollary 7. *There exists no double point in $\tilde{\mathcal{C}}_4$ if and only if $\Delta_4 \neq 0$.*

We conclude the following from Lemmas 33 and 34:

Proposition 15.

$$\mathcal{N}_4(Z_0) = 16\frac{\rho_{14}^4(4r^2 - \rho_{14}^2)\Delta_0^4\Delta_2^2}{r^8\rho_{12}^4(4r^2 - \rho_{12}^2)}, \tag{174}$$

$$\mathcal{N}_4(Z) = 4\frac{\rho_{12}^4(4r^2 - \rho_{14}^2)^2}{\rho_{14}^4(4r^2 - \rho_{12}^2)^3}\Delta_0^2\Delta_2^2\,h_4\text{Discri}_4 = 64\frac{\rho_{12}^{16}\rho_{14}^4(4r^2 - \rho_{14}^2)^2}{(4r^2 - \rho_{12}^2)^3}\frac{\Delta_0^4\Delta_2^2\Delta_4}{h_4^5}. \tag{175}$$

Proof. Applying (102) and (171) to (170) in Lemma 32 we have (174). Differentiating both sides of (167), with respect to t_1, we have

$$\psi_4'(t_1) \equiv \frac{(t_1 - 1)\hat{\psi}_4'(t_1)}{U^2} \mod Ann(\tilde{\mathcal{C}}_4),$$

so that

$$\mathcal{N}_4(\psi_4') = \frac{\mathcal{N}_4(t_1 - 1)}{\{\mathcal{N}_4(U)\}^2}\mathcal{N}_4(\hat{\psi}_4'). \tag{176}$$

Using (136), (176) and (170) in Lemma 32, (173) in Lemma 34 and Proposition 5, we obtain

$$\begin{aligned}\mathcal{N}_4(Z) = \mathcal{N}_4(Z_0)\mathcal{N}_4(\psi_4') &= \frac{\rho_{12}^8\,\mathcal{N}_4(V)}{\rho_{14}^8\,\mathcal{N}_4(U)}\mathcal{N}_4(1-t_1)\{\mathcal{N}_4(t_1-t_2)\}^2\mathcal{N}_4(\hat{\psi}_4') \\ &= \frac{\rho_{12}^8}{\rho_{14}^8}\mathcal{N}_4(t_2)\mathcal{N}_4(1-t_1)\{\mathcal{N}_4(t_1-t_2)\}^2\mathcal{N}_4(\hat{\psi}_4') \\ &= \frac{\rho_{12}^8}{\rho_{14}^8}\mathcal{N}_2(t_1)\mathcal{N}_2(1-t_3)\{\mathcal{N}_2(t_1-t_3)\}^2\,h_4^4\,\mathrm{Discri}_4 \\ &= \frac{\rho_{12}^8}{\rho_{14}^8}\times\frac{\rho_{14}^4(4r^2-\rho_{14}^2)}{\rho_{12}^4(4r^2-\rho_{12}^2)}\times\frac{4\Delta_0^2(4r^2-\rho_{14}^2)}{r^2\rho_{12}^4}\times\left[\frac{\Delta_2}{r^2\rho_{12}^4(4r^2-\rho_{12}^2)}\right]^2 h_4^4\,\mathrm{Discri}_4,\end{aligned}$$

which coincides with (175). □

7.5. Conclusions of This Section

In this subsection, we give a proof of Conjecture 1 under the conditions $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$. And we try to prove Conjecture 1 without the constraint $(\mathcal{H}_1)$ in Appendix A.

Theorem 4. *Under the conditions $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$ the norm of the Hessian of F relative to $\mathcal{C}$ is expressed as*

$$\begin{aligned}\mathcal{N}(\mathrm{Hess}(F)) = 2^{129}&\frac{\Delta_2^8\Delta_3\Delta_4^3}{r^{24}\rho_{12}^{24}\rho_{14}^6(\rho_{12}^2-4r^2)^{12}(\rho_{14}^2-4r^2)^{12}(3r^2-\rho_{12}^2)^5} \\ &\times\frac{1}{(\rho_{14}^4+\rho_{12}^2r^2-4\rho_{14}^2r^2)^{15}(3\rho_{14}^4+4\rho_{12}^2r^2-12\rho_{14}^2r^2)^6},\end{aligned}\tag{177}$$

where Δ_2, Δ_3 and Δ_4 are the polynomials in r^2, ρ_{12}^2, ρ_{14}^2 given by (143), (146), and (160), respectively. The right-hand side is written in terms of the Cayley–Menger determinants as

$$\frac{2^{161}\,\rho_{12}^{74}\rho_{14}^{18}}{\{B(0\star 1)\}^{12}}\frac{\Delta_2^8\Delta_3\Delta_4^3}{\{B(0\star 12)B(0\star 14)\}^{12}\{B(0\star 123)\}^5\{B(0\star 124)\}^{15}\{B(0\star 1234)\}^6}.$$

Proof. From Propositions 11, 13, 14, and 15, we obtain

$$\mathcal{N}(Z) = \prod_{j=1}^4\mathcal{N}_j(Z) = \frac{2^{12}}{3^5}\frac{\rho_{14}^{12}(\rho_{12}^2-3\rho_{14}^2)(4r^2-\rho_{14}^2)^2\Delta_0^{14}\Delta_1^4\Delta_2^8\Delta_3\Delta_4^3}{r^{22}\rho_{12}^{12}(3r^2-\rho_{12}^2)^5(4r^2-\rho_{12}^2)^{13}}.\tag{178}$$

Since we already had

$$\mathcal{N}(\mathrm{Hess}(F)) = 2^{45}\frac{-\mathcal{N}(Z)}{\mathcal{N}(f_1)\mathcal{N}(f_2)\mathcal{N}(f_3)\{\mathcal{N}(f_4)\}^3\mathcal{N}(t_\infty)\{\mathcal{N}(1-t_1)\}^2}\tag{179}$$

by Proposition 3, using (178) and the results for $\mathcal{N}(f_j)$, $\mathcal{N}(t_\infty)$ and $\mathcal{N}(1-t_1)$ stated in Theorem 2 or Corollary 2, we therefore see that (179) coincides with (177). □

Remark 14. *While Theorem 4 was proved under $\Delta_0 \neq 0$ in the above proof, the formula (177) is also valid for $\Delta_0 = 0$. When $\Delta_0 = 0$, i.e., $\rho_{12}^2 = \rho_{14}^2$ the invariants Δ_2, Δ_3 and Δ_4 degenerate to*

$$\Delta_2 = 4\rho^4(\rho^2-8r^2)^2,\ \Delta_3 = 2(\rho^2-8r^2)^2(\rho^2+24r^2) \text{ and } \Delta_4 = -4\rho^4(\rho^2-8r^2)^3(\rho^2+24r^2),$$

respectively, so that we can confirm that the right-hand side of (177) degenerates to (135) in Proposition 10, which is the result proved independently under $\Delta_0 = 0$.

Remark 15. *The factors Δ_0 and Δ_1 do not appear in the expression (177) of $\mathcal{N}(\mathrm{Hess}(F))$, while $\mathcal{N}(Z)$ in (178) is divisible by $\Delta_0^{14}\Delta_1^4$. Since $\mathcal{N}(1-t_1)$ given in Theorem 2 is also divisible by*

$\Delta_0^7\Delta_1^2$, the factor $\Delta_0^{14}\Delta_1^4$ in the numerator and that in denominator of (179) are cancelled. For the right-hand side of the formula (177) as a meromorphic function of r^2, we see that the point $r^2 = (3\rho_{12}^2 + \rho_{14}^2)/4$ for $\Delta_1 = 0$ is a removable singularity.

Corollary 8. *Under the conditions $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$*

$$\mathcal{N}(\mathrm{Hess}(F)) \neq 0$$

if and only if every critical point in $\mathcal{C}$ is different from each other.

Proof. By (177) in Theorem 4 we have $\mathcal{N}(\mathrm{Hess}(F)) \neq 0$ if and only if $\Delta_2^8\Delta_3\Delta_4^3 \neq 0$. According to Proposition 12, Corollaries 4, 6 and 7, we see that $\Delta_2\Delta_3\Delta_4 \neq 0$ if and only if every critical point in $\vec{\mathcal{C}} = \sqcup_{j=1}^4 \vec{\mathcal{C}}_j$ is different from each other. □

8. Conclusions

We discussed the norm of the Hessian of the level function F at critical points $\mathcal{C}$ involved in asymptotic behaviors of hypergeometric integrals associated with a symmetric arrangement of three-dimensional spheres. We also provided two conjectures (Conjectures 1 and 2) relevant to this topic. We provide a proof in a special symmetric case where $\triangle O_1O_2O_3O_4$ is a pyramid with the axis of symmetry, whose base triangle $\triangle O_1O_2O_3$ is regular and all spheres have the same radius.

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Appendix A. Further Reduction and an Associated Characteristic Function

As we saw in Section 7.5 (Theorem 4), we calculated $\mathcal{N}(\mathrm{Hess}(F))$ explicitly under the conditions $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$, and we consequently confirmed that Conjecture 1 holds true under the conditions $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$. However, we want to prove Conjecture 1 without the constraint $(\mathcal{H}_1)$, if possible. For that purpose, we show a way to compute the part

$$\frac{\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_1, t_2, t_3)}$$

in the expression (62) of $\mathrm{Hess}(F)$ under a more general setting.

Appendix A.1. Step 1

We fix the admissible parameter t_1. Then $\tilde{g}_1$ given in Proposition 2 is polynomial in t_2, t_3 whose coefficients are explicitly written using the parameter t_1 as follows:

$$\tilde{g}_1 = \beta_{1,22}t_2^2 + 2\beta_{1,23}t_2t_3 + \beta_{1,33}t_3^2 + 2\beta_{1,2}t_2 + 2\beta_{1,3}t_3 + \beta_{1,\emptyset}, \quad \text{(A1)}$$

where the coefficients $\beta_{1,*}$ are polynomials in t_1 of, at most, second degree:

$$\begin{aligned}
\beta_{1,22} &= (r_1^2 - \rho_{12}^2)t_1 + \rho_{24}^2 - r_4^2,\\
\beta_{1,33} &= (r_1^2 - \rho_{13}^2)t_1 + \rho_{34}^2 - r_4^2,\\
2\beta_{1,23} &= \left\{2r_1^2 - B\begin{pmatrix} 0 & 2 & 1 \\ 0 & 3 & 1 \end{pmatrix}\right\}t_1 - 2r_4^2 + B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix},\\
2\beta_{1,2} &= 2\left\{r_1^2 t_1^2 + B\begin{pmatrix} 0 & \star & 2 \\ 0 & 4 & 1 \end{pmatrix}t_1 - r_4^2\right\},\\
2\beta_{1,3} &= 2\left\{r_1^2 t_1^2 + B\begin{pmatrix} 0 & \star & 3 \\ 0 & 4 & 1 \end{pmatrix}t_1 - r_4^2\right\},\\
\beta_{1,\emptyset} &= r_1^2 t_1^3 + (\rho_{14}^2 - r_4^2 + 2r_1^2)t_1^2 + (-\rho_{14}^2 + r_1^2 - 2r_4^2)t_1 - r_4^2.
\end{aligned}$$

Moreover, $\tilde{g}_{12}$ and $\tilde{g}_{13}$ given in Lemma 10 are also polynomials in t_2, t_3, whose coefficients are explicitly written using the parameter t_1 as follows:

$$\tilde{g}_{1j} = \beta_{1j,jj}t_j^2 + 2\beta_{1j,jk}t_j t_k + \beta_{1j,kk}t_k^2 + 2\beta_{1j,j}t_j + 2\beta_{1j,k}t_k + \beta_{1j,\emptyset} \quad (j = 2,3), \tag{A2}$$

where $\{j,k\}$ is a permutation of $\{2,3\}$ and the coefficients $\beta_{1j,*}$ are polynomials in t_1 of at most second degree:

$$\begin{aligned}
\beta_{1j,jj} &= t_1 B\begin{pmatrix} 0 & \star & j \\ 0 & 1 & j \end{pmatrix} - B\begin{pmatrix} 0 & \star & j \\ 0 & 4 & j \end{pmatrix}, \quad \beta_{1j,kk} = 0,\\
2\beta_{1j,jk} &= t_1 B\begin{pmatrix} 0 & k & \star \\ 0 & j & 1 \end{pmatrix} + B\begin{pmatrix} 0 & \star & k \\ 0 & j & 4 \end{pmatrix},\\
2\beta_{1j,j} &= -t_1^2 B\begin{pmatrix} 0 & \star & 1 \\ 0 & j & 1 \end{pmatrix} + B\begin{pmatrix} 0 & \star & 4 \\ 0 & j & 4 \end{pmatrix},\\
2\beta_{1j,k} &= -t_1 B\begin{pmatrix} 0 & \star & k \\ 0 & 1 & 4 \end{pmatrix},\\
\beta_{1j,\emptyset} &= t_1^2 B\begin{pmatrix} 0 & \star & 1 \\ 0 & 4 & 1 \end{pmatrix} - t_1 B\begin{pmatrix} 0 & \star & 4 \\ 0 & 1 & 4 \end{pmatrix} \quad (j = 2,3).
\end{aligned}$$

Appendix A.2. Step 2

We modify $\tilde{g}_1$ as

$$\tilde{g}_1' := \tilde{g}_1 - \frac{\beta_{1,22}}{\beta_{12,22}}\tilde{g}_{12} - \frac{\beta_{1,33}}{\beta_{13,33}}\tilde{g}_{13},$$

which can be represented as

$$\tilde{g}_1' = 2\beta_{1,23}' t_2 t_3 + 2\beta_{1,2}' t_2 + 2\beta_{1,3}' t_3 + \beta_{1,\emptyset}',$$

where

$$\begin{aligned}
2\beta_{1,23}' &= 2\beta_{1,23} - \frac{\beta_{1,22}}{\beta_{12,22}}2\beta_{12,23} - \frac{\beta_{1,33}}{\beta_{13,33}}2\beta_{13,23},\\
2\beta_{1,2}' &= 2\beta_{1,2} - \frac{\beta_{1,22}}{\beta_{12,22}}2\beta_{12,2} - \frac{\beta_{1,33}}{\beta_{13,33}}2\beta_{13,2},\\
2\beta_{1,3}' &= 2\beta_{1,3} - \frac{\beta_{1,33}}{\beta_{13,33}}2\beta_{13,3} - \frac{\beta_{1,22}}{\beta_{12,22}}2\beta_{12,3},\\
\beta_{1,\emptyset}' &= \beta_{1,\emptyset} - \frac{\beta_{1,22}}{\beta_{12,22}}\beta_{12,\emptyset} - \frac{\beta_{1,33}}{\beta_{13,33}}\beta_{13,\emptyset}.
\end{aligned}$$

We also modify $\tilde{g}_{1j}$ as

$$\tilde{g}'_{12} := \tilde{g}_{12} - \frac{\beta_{12,23}}{\beta'_{1,23}}\tilde{g}'_1 = \beta'_{12,22}t_2^2 + 2\beta'_{12,2}t_2 + 2\beta'_{12,3}t_3 + \beta'_{12,\emptyset},$$
$$\tilde{g}'_{13} := \tilde{g}_{13} - \frac{\beta_{13,23}}{\beta'_{1,23}}\tilde{g}'_1 = \beta'_{13,33}t_3^2 + 2\beta'_{13,2}t_2 + 2\beta'_{13,3}t_3 + \beta'_{13,\emptyset},$$

where

$$\beta'_{12,22} = \beta_{12,22}, \quad \beta'_{13,33} = \beta_{13,33},$$
$$2\beta'_{12,2} = -\frac{\beta_{12,23}}{\beta'_{1,23}}2\beta'_{1,2} + 2\beta_{12,2}, \quad 2\beta'_{13,3} = -\frac{\beta_{13,23}}{\beta'_{1,23}}2\beta'_{1,3} + 2\beta_{13,3},$$
$$2\beta'_{12,3} = -\frac{\beta_{12,23}}{\beta'_{1,23}}2\beta'_{1,3} + 2\beta_{12,3}, \quad 2\beta'_{13,2} = -\frac{\beta_{13,23}}{\beta'_{1,23}}2\beta'_{1,2} + 2\beta_{13,2},$$
$$\beta'_{12,\emptyset} = -\frac{\beta_{12,23}}{\beta'_{1,23}}\beta'_{1,\emptyset} + \beta_{12,\emptyset}, \quad \beta'_{13,\emptyset} = -\frac{\beta_{13,23}}{\beta'_{1,23}}\beta'_{1,\emptyset} + \beta_{13,\emptyset}.$$

Remark that $2\beta'_{1j,jk} = 0$ for $j = 2,3$ and $j \neq k$. According to Lemma 11, we may conclude

Lemma A1. *Suppose that $\beta_{12,22}\beta_{13,33} \neq 0$. Then under the condition $(\mathcal{H}_0)$ the system (57) holds if and only if*

$$\tilde{g}'_1 = \tilde{g}'_{12} = \tilde{g}'_{13} = 0. \tag{A3}$$

In this case, the identity

$$d\tilde{g}'_1 \wedge d\tilde{g}'_{12} \wedge d\tilde{g}'_{13} \equiv d\tilde{g}_1 \wedge d\tilde{g}_{12} \wedge d\tilde{g}_{13} \mod Ann(\tilde{\mathcal{C}})$$

holds, i.e.,

$$\frac{\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_1, t_2, t_3)} \equiv \frac{\partial(\tilde{g}'_1, \tilde{g}'_{12}, \tilde{g}'_{13})}{\partial(t_1, t_2, t_3)} \mod Ann(\tilde{\mathcal{C}}).$$

Appendix A.3. Step 3

In this subsection, we want to express

$$\frac{\partial(\tilde{g}'_1, \tilde{g}'_{12}, \tilde{g}'_{13})}{\partial(t_1, t_2, t_3)}$$

explicitly in terms of the resultant of $\tilde{g}'_1$, $\tilde{g}'_{12}$ and $\tilde{g}'_{13}$. We assume that the monomials in t_2, t_3, of at most fourth degree, are arranged in the following order:

$$t_2^4 \succ t_2^3t_3 \succ t_2^2t_3^2 \succ t_2t_3^3 \succ t_3^4 \succ t_2^3 \succ t_3^3 \succ t_2^2t_3 \succ t_2t_3^2 \succ t_2^2 \succ t_3^2 \succ t_2t_3 \succ t_2 \succ t_3 \succ 1. \tag{A4}$$

Three fundamental linear relations among $\tilde{g}'_1, \tilde{g}'_{12}, \tilde{g}'_{13}$ over the coefficients of quadratic polynomials are given as follows:

$$(\beta'_{12,22}t_2^2 + 2\beta'_{12,2}t_2 + 2\beta'_{12,3}t_3 + \beta'_{12,\emptyset})\tilde{g}'_{13} - (\beta'_{13,33}t_3^2 + 2\beta'_{13,2}t_2 + 2\beta'_{13,3}t_3 + \beta'_{13,\emptyset})\tilde{g}'_{12} = 0,$$
$$(\beta'_{12,22}t_2^2 + 2\beta'_{12,2}t_2 + 2\beta'_{12,3}t_3 + \beta'_{12,\emptyset})\tilde{g}'_1 - (2\beta'_{1,23}t_2t_3 + 2\beta'_{1,2}t_2 + 2\beta'_{1,3}t_3 + \beta'_{1,\emptyset})\tilde{g}'_{12} = 0,$$
$$(\beta'_{13,33}t_3^2 + 2\beta'_{13,2}t_2 + 2\beta'_{13,3}t_3 + \beta'_{13,\emptyset})\tilde{g}'_1 - (2\beta'_{1,23}t_2t_3 + 2\beta'_{1,2}t_2 + 2\beta'_{1,3}t_3 + \beta'_{1,\emptyset})\tilde{g}'_{13} = 0.$$

As a result $t_2^2\tilde{g}'_{13}$ is a linear combination of

$$t_3^2\tilde{g}'_{12}, \quad t_2\tilde{g}'_{12}, \quad t_3\tilde{g}'_{12}, \quad \tilde{g}'_{12}, \quad t_2\tilde{g}'_{13}, \quad t_3\tilde{g}'_{13}, \quad \tilde{g}'_{13},$$

and $t_2^2\tilde{g}_1'$ is a linear combination of

$$t_2\tilde{g}_1', \quad t_3\tilde{g}_1', \quad \tilde{g}_1', \quad t_2t_3\tilde{g}_{12}', \quad t_2\tilde{g}_{12}', \quad t_3\tilde{g}_{12}', \quad \tilde{g}_{12}'.$$

Moreover, $t_3^2\tilde{g}_1'$ is also a linear combination of

$$t_2\tilde{g}_1', \quad t_3\tilde{g}_1', \quad \tilde{g}_1', \quad t_2t_3\tilde{g}_{13}', \quad t_2\tilde{g}_{13}', \quad t_3\tilde{g}_{13}', \quad \tilde{g}_{13}'.$$

One can see:

Lemma A2. *The system of (ordered) 15 polynomials (denoted by $\mathcal{X}$)*

$$\begin{aligned} &\mathbf{x}_1 = t_2^2\tilde{g}_{12}' \succ \mathbf{x}_2 = t_2t_3\tilde{g}_{12}' \succ \mathbf{x}_3 = t_3^2\tilde{g}_{12}' \succ \mathbf{x}_4 = t_2t_3\tilde{g}_{13}' \succ \mathbf{x}_5 = t_3^2\tilde{g}_{13}' \\ &\succ \mathbf{x}_6 = t_2\tilde{g}_{12}' \succ \mathbf{x}_7 = t_3\tilde{g}_{13}' \succ \mathbf{x}_8 = t_3\tilde{g}_{12}' \succ \mathbf{x}_9 = t_2\tilde{g}_{13}', \succ \mathbf{x}_{10} = \tilde{g}_{12}' \\ &\succ \mathbf{x}_{11} = \tilde{g}_{13}' \succ \mathbf{x}_{12} = \tilde{g}_1' \succ \mathbf{x}_{13} = t_2\tilde{g}_1' \succ \mathbf{x}_{14} = t_3\tilde{g}_1' \succ \mathbf{x}_{15} = t_2t_3\tilde{g}_1' \end{aligned}$$

are linearly independent and span the linear space of polynomials at most fourth degree.

Definition A1. *The Macaulay's diagram $\mathcal{X}$ 15 × 15 associated with the ordered basis $\mathbf{x}_j$ with respect to the monomials in the ordering (A4) is defined by the following equation:*

$$\mathcal{X}: \ {}^T(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{15}) = \mathcal{X}\ {}^T(t_2^4, t_2^3t_3, \ldots, t_2, t_3, 1),$$

where $\mathcal{X}$ is the 15 × 15 matrix given by

$$\mathcal{X} = \begin{pmatrix}
\beta'_{12,22} & & & & & 2\beta'_{12,2} & & 2\beta'_{12,3} & & \beta_{12,\emptyset} & & & & & \\
& \beta'_{12,22} & & & & & & 2\beta'_{12,2} & 2\beta'_{12,3} & & & \beta'_{12,\emptyset} & & & \\
& & \beta'_{12,22} & & & & 2\beta'_{12,3} & & 2\beta'_{12,2} & & \beta'_{12,\emptyset} & & & & \\
& & & \beta'_{13,33} & & & & 2\beta'_{13,2} & 2\beta'_{13,3} & & & \beta'_{13,\emptyset} & & & \\
& & & & \beta'_{13,33} & & 2\beta'_{13,3} & & 2\beta'_{13,2} & & \beta'_{13,\emptyset} & & & & \\
& & & & & \beta'_{12,22} & & & & 2\beta'_{12,2} & & 2\beta'_{12,3} & \beta'_{12,\emptyset} & & \\
& & & & & & \beta'_{13,33} & & & & 2\beta'_{13,3} & 2\beta'_{13,2} & & \beta'_{13,\emptyset} & \\
& & & & & & & \beta'_{12,22} & & & 2\beta'_{12,3} & 2\beta'_{12,2} & & \beta'_{12,\emptyset} & \\
& & & & & & & & \beta'_{13,33} & 2\beta'_{13,2} & & 2\beta'_{13,3} & \beta'_{13,\emptyset} & & \\
& & & & & & & & & \beta'_{12,22} & & & 2\beta'_{12,2} & 2\beta'_{12,3} & \beta'_{12,\emptyset} \\
& & & & & & & & & & \beta'_{13,33} & & 2\beta'_{13,2} & 2\beta'_{13,3} & \beta'_{13,\emptyset} \\
& & & & & & & & & & & 2\beta'_{1,23} & 2\beta'_{1,2} & 2\beta'_{1,3} & \beta'_{1,\emptyset} \\
& & & & & & & 2\beta'_{1,23} & & 2\beta'_{1,2} & & 2\beta'_{1,3} & \beta'_{1,\emptyset} & & \\
& & & & & & & & 2\beta'_{1,23} & & 2\beta'_{1,3} & 2\beta'_{1,2} & & \beta'_{1,\emptyset} & \\
& & 2\beta'_{1,23} & & & & & 2\beta'_{1,2} & 2\beta'_{1,3} & & & \beta'_{1,\emptyset} & & &
\end{pmatrix}.$$

The resultant of $\tilde{g}_{12}', \tilde{g}_{13}', \tilde{g}_1'$, which we denote by $R(\tilde{g}_{12}', \tilde{g}_{13}', \tilde{g}_1')$ is related to $\det\mathcal{X}$ as follows:

$$\det\mathcal{X} = (\beta'_{12,22})^2(\beta'_{13,33})\, R(\tilde{g}_{12}', \tilde{g}_{13}', \tilde{g}_1'), \tag{A5}$$

where

$$R(\tilde{g}_{12}', \tilde{g}_{13}', \tilde{g}_1') = (\beta'_{12,22}\beta'_{13,33})^3(\beta'_{1.\emptyset})^2\{\beta'_{12,22}\beta'_{13,33}(\beta'_{1,\emptyset})^2 - 4(\beta'_{1,23})^2\beta'_{12,\emptyset}\beta'_{13,\emptyset}\} + \cdots.$$

See [15] and W. Gröbner : Moderne Algebraische Geometrie [16] (pp. 70–71). Using (A1) and (A2), we first define

$$\tilde{g}''_{12} := t_2\tilde{g}'_1 - \frac{2\beta'_{1,23}}{\beta'_{12,22}}t_3\tilde{g}'_{12} = \zeta_{11}t_2^2 + \zeta_{12}t_3^2 + \zeta_{13}t_2t_3 + \zeta_{14}t_2 + \zeta_{15}t_3 + \zeta_{16},$$
$$\tilde{g}''_{13} := t_3\tilde{g}'_1 - \frac{2\beta'_{1,23}}{\beta'_{13,33}}t_2\tilde{g}'_{13} = \zeta_{21}t_2^2 + \zeta_{22}t_3^2 + \zeta_{23}t_2t_3 + \zeta_{24}t_2 + \zeta_{25}t_3 + \zeta_{26},$$

where

$$\zeta_{11} = 2\beta'_{1,2}, \quad \zeta_{12} = -\frac{4\beta'_{12,3}\beta'_{1,23}}{\beta'_{12,22}}, \quad \zeta_{13} = 2\beta'_{1,3} - \frac{4\beta'_{12,2}\beta'_{1,23}}{\beta'_{12,22}},$$
$$\zeta_{14} = \beta'_{1,\emptyset}, \quad \zeta_{15} = -\frac{2\beta'_{12,\emptyset}\beta'_{1,23}}{\beta'_{12,22}}, \quad \zeta_{16} = 0,$$
$$\zeta_{21} - \frac{4\beta'_{13,2}\beta'_{1,23}}{\beta'_{13,33}}, \quad \zeta_{22} = 2\beta'_{1,3}, \quad \zeta_{23} = 2\beta'_{1,2} - \frac{4\beta'_{13,3}\beta'_{1,23}}{\beta'_{13,33}},$$
$$\zeta_{24} = \beta'_{1,\emptyset}, \quad \zeta_{25} = -\frac{2\beta'_{13,\emptyset}\beta'_{1,23}}{\beta'_{13,33}}, \quad \zeta_{26} = 0.$$

Then we have

$${}^T(\tilde{g}''_{12}, \tilde{g}''_{13}, \tilde{g}'_1, \tilde{g}'_{12}, \tilde{g}'_{13}) = \Xi\,{}^T(t_2^2, t_3^2, t_2t_3, t_2, t_3, 1),$$

where the 5×6 matrix Ξ is given by

$$\Xi := \begin{pmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} & \zeta_{14} & \zeta_{15} & \zeta_{16} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} & \zeta_{24} & \zeta_{25} & \zeta_{26} \\ 0 & 0 & \zeta_{33} & \zeta_{34} & \zeta_{35} & \zeta_{36} \\ \zeta_{41} & 0 & 0 & \zeta_{44} & \zeta_{45} & \zeta_{46} \\ 0 & \zeta_{52} & 0 & \zeta_{54} & \zeta_{55} & \zeta_{56} \end{pmatrix}. \tag{A6}$$

Here, the entries ζ_{ij} of the matrix Ξ can be expressed as

$$\zeta_{33} = 2\beta'_{1,23}, \quad \zeta_{34} = 2\beta'_{1,2}, \quad \zeta_{35} = 2\beta'_{1,3}, \quad \zeta_{36} = \beta'_{1,\emptyset},$$
$$\zeta_{41} = \beta'_{12,22}, \quad \zeta_{44} = 2\beta'_{12,2}, \quad \zeta_{45} = 2\beta'_{12,3}, \quad \zeta_{46} = \beta'_{12,\emptyset},$$
$$\zeta_{52} = \beta'_{13,33}, \quad \zeta_{54} = 2\beta'_{13,2}, \quad \zeta_{55} = 2\beta'_{13,3}, \quad \zeta_{56} = \beta'_{13,\emptyset}.$$

Define further

$$\tilde{g}'''_{12} := \tilde{g}''_{12} - \frac{\zeta_{13}}{\zeta_{33}}\tilde{g}'_1 - \frac{\zeta_{11}}{\zeta_{41}}\tilde{g}'_{12} - \frac{\zeta_{12}}{\zeta_{52}}\tilde{g}'_{13} = \zeta'_{14}t_2 + \zeta'_{15}t_3 + \zeta'_{16}, \tag{A7}$$
$$\tilde{g}'''_{13} := \tilde{g}''_{13} - \frac{\zeta_{23}}{\zeta_{33}}\tilde{g}'_1 - \frac{\zeta_{21}}{\zeta_{41}}\tilde{g}'_{12} - \frac{\zeta_{22}}{\zeta_{52}}\tilde{g}'_{13} = \zeta'_{24}t_2 + \zeta'_{25}t_3 + \zeta'_{26}, \tag{A8}$$

where ζ'_{jk} can be expressed more concretely

$$\zeta'_{14} = \beta'_{1,\emptyset} - 2\frac{\beta'_{1,2}\beta'_{1,3}}{\beta'_{1,23}} + 8\frac{\beta'_{1,23}\beta'_{12,3}\beta'_{13,2}}{\beta'_{12,22}\beta'_{13,33}}, \tag{A9}$$
$$\zeta'_{15} = -\frac{4\beta'_{1,2}\beta'_{12,3} + 2\beta'_{1,23}\beta'_{12,\emptyset}}{\beta'_{12,22}} + 8\frac{\beta'_{1,23}\beta'_{12,3}\beta'_{13,3}}{\beta'_{12,22}\beta'_{13,33}} - 2\frac{\beta'_{1,3}(\beta'_{1,3}\beta'_{12,22} - 2\beta'_{1,23}\beta'_{12,2})}{\beta'_{1,23}\beta'_{12,22}},$$
$$\zeta'_{16} = -\frac{\beta'_{1,3}\beta'_{1,\emptyset}}{\beta'_{1,23}} + 2\frac{-\beta'_{1,2}\beta'_{12,\emptyset} + \beta'_{1,\emptyset}\beta'_{12,2}}{\beta'_{12,22}} + 4\frac{\beta'_{1,23}\beta'_{12,3}\beta'_{13,\emptyset}}{\beta'_{12,22}\beta'_{13,33}},$$

and likewise $\xi'_{24} = \sigma_{23}(\xi'_{15})$, $\xi'_{25} = \sigma_{23}(\xi'_{14}) = \xi'_{14}$, $\xi'_{26} = \sigma_{23}(\xi'_{16})$, where σ_{23} denotes the transposition between the subscript $\{2,3\}$. The polynomials $\tilde{g}'''_{12}$ and $\tilde{g}'''_{13}$ are linear in t_2, t_3. Using the matrix $\Xi' := (\xi'_{jk})_{1\le j\le 2, 1\le k\le 3}$ we have

$$\begin{pmatrix} \tilde{g}'''_{12} \\ \tilde{g}'''_{13} \end{pmatrix} = \Xi' \begin{pmatrix} t_2 \\ t_3 \\ 1 \end{pmatrix}.$$

Lemma A3. *Under the condition*

$$(\mathcal{H}_2): \quad \beta'_{12,22} \neq 0, \quad \beta'_{13,33} \neq 0, \quad \xi'_{14} \neq 0,$$

the system (A3) *are equivalent to*

$$\tilde{g}'_1 = \tilde{g}'''_{12} = \tilde{g}'''_{13} = 0. \tag{A10}$$

Proof. It is obvious from (A7) and (A8) that (A3) implies (A10). Conversely suppose that (A10) holds true. Then from (A7) and (A8) we have

$$0 = \Big(\frac{2\beta'_{1,23}}{\beta'_{12,22}} t_3 + \frac{\xi_{11}}{\xi_{41}}\Big)\tilde{g}'_{12} + \frac{\xi_{12}}{\xi_{52}}\tilde{g}'_{13}, \quad 0 = \frac{\xi_{21}}{\xi_{41}}\tilde{g}'_{12} + \Big(\frac{2\beta'_{1,23}}{\beta'_{13,33}} t_2 + \frac{\xi_{22}}{\xi_{52}}\Big)\tilde{g}'_{13}.$$

Since t_2, t_3 satisfies $\tilde{g}'_1 = 0$, the determinant

$$\begin{vmatrix} \frac{2\beta'_{1,23}}{\beta'_{12,22}} t_3 + \frac{\xi_{11}}{\xi_{41}} & \frac{\xi_{12}}{\xi_{52}} \\ \frac{\xi_{21}}{\xi_{41}} & \frac{2\beta'_{1,23}}{\beta'_{13,33}} t_2 + \frac{\xi_{22}}{\xi_{52}} \end{vmatrix} \equiv -\frac{2\beta'_{1,23}\xi'_{14}}{\beta'_{12,22}\beta'_{13,33}} \tag{A11}$$

does not vanish by hypothesis. Hence, we obtain $\tilde{g}'_{12} = \tilde{g}'_{13} = 0$. □

One can also express $t_3\tilde{g}'''_{12}$ and $t_3\tilde{g}'''_{13}$ as linear combination of the basis $\mathcal{X}$:

Lemma A4.

$$\begin{aligned} t_3\tilde{g}'''_{12} &= t_3(\xi'_{14}t_2 + \xi'_{15}t_3 + \xi'_{16}) \\ &= t_2t_3\tilde{g}'_1 - \frac{2\beta'_{1,23}}{\beta'_{12,22}} t_3^2\tilde{g}'_{12} - \frac{\xi_{13}}{\xi_{33}} t_3\tilde{g}'_1 - \frac{\xi_{11}}{\xi_{41}} t_3\tilde{g}'_{12} - \frac{\xi_{12}}{\xi_{52}} t_3\tilde{g}'_{13}, \\ t_3\tilde{g}'''_{13} &= t_3(\xi'_{24}t_2 + \xi'_{25}t_3 + \xi'_{26}) \\ &= t_3^2\tilde{g}'_1 - \frac{2\beta'_{1,23}}{\beta'_{13,33}} t_2t_3\tilde{g}'_{13} - \frac{\xi_{23}}{\xi_{33}} t_3\tilde{g}'_1 - \frac{\xi_{21}}{\xi_{41}} t_3\tilde{g}'_{12} - \frac{\xi_{22}}{\xi_{52}} t_3\tilde{g}'_{13} \\ &= -\Big\{\frac{2\beta'_{13,2}}{\beta'_{13,33}} t_2 + \Big(\frac{2\beta'_{13,3}}{\beta'_{13,33}} + \frac{\xi_{23}}{\xi_{33}}\Big) t_3 + \frac{\beta'_{13,\emptyset}}{\beta'_{13,33}}\Big\}\tilde{g}'_1 - \frac{\xi_{21}}{\xi_{41}} t_3\tilde{g}'_{12} \\ &\quad + \frac{2\beta'_{1,2}t_2 + (2\beta'_{1,3} - \xi_{22})t_3 + \beta'_{1,\emptyset}}{\beta'_{13,33}}\tilde{g}'_{13}. \end{aligned}$$

Definition A2. Macaulay's diagram Y 5×5 corresponding to $\tilde{g}'_1, \tilde{g}'''_{12}, \tilde{g}'''_{13}, t_3\tilde{g}'''_{12}, t_3\tilde{g}'''_{13}$, is defined by

$$\begin{pmatrix} \tilde{g}'_1 \\ \tilde{g}'''_{12} \\ \tilde{g}'''_{13} \\ t_3\tilde{g}'''_{12} \\ t_3\tilde{g}'''_{13} \end{pmatrix} = Y \begin{pmatrix} t_3^2 \\ t_2t_3 \\ t_2 \\ t_3 \\ 1 \end{pmatrix}, \quad \text{where} \quad Y = \begin{pmatrix} 0 & \xi_{33} & \xi_{34} & \xi_{35} & \xi_{36} \\ 0 & 0 & \xi'_{14} & \xi'_{15} & \xi'_{16} \\ 0 & 0 & \xi'_{24} & \xi'_{25} & \xi'_{26} \\ \xi'_{15} & \xi'_{14} & 0 & \xi'_{16} & 0 \\ \xi'_{25} & \xi'_{24} & 0 & \xi'_{26} & 0 \end{pmatrix}.$$

Let U_1 be function specified by

$$U_1 := \frac{\partial(\tilde{g}'''_{12}, \tilde{g}'''_{13})}{\partial(t_2, t_3)} = \begin{vmatrix} \xi'_{14} & \xi'_{15} \\ \xi'_{24} & \xi'_{25} \end{vmatrix} = \frac{\Xi\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}}{\xi_{33}\xi_{41}\xi_{52}},$$

where the minor determinants of order l for the matrix $\Xi = (\xi_{jk})$ given in (A6) are defined as

$$\Xi\begin{pmatrix} j_1 & j_2 & \cdots & j_l \\ k_1 & k_2 & \cdots & k_l \end{pmatrix} := \det\left(\xi_{j_p k_q}\right)_{p,q=1,2,\ldots,l}$$

for $1 \le j_1 < j_2 < \cdots < j_l \le 5$ and $1 \le k_1 < k_2 < \cdots < k_l \le 6$.

Lemma A5. *Suppose that U_1 never vanishes at any point of $\tilde{C}$. Then the equations*

$$\tilde{g}'''_{12} = \tilde{g}'''_{13} = 0$$

concerning t_2, t_3 can be uniquely solved by

$$(\omega_2, \omega_3): \quad t_2 = \frac{U_2}{U_1}, \quad t_3 = \frac{U_3}{U_1},$$

which defines a rational curve interpolating $\tilde{C}$, where

$$U_2 = \begin{vmatrix} \xi'_{15} & \xi'_{16} \\ \xi'_{25} & \xi'_{26} \end{vmatrix} = \frac{\Xi\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 6 \end{pmatrix}}{\xi_{33}\xi_{41}\xi_{52}},$$

$$U_3 = -\begin{vmatrix} \xi'_{14} & \xi'_{16} \\ \xi'_{24} & \xi'_{26} \end{vmatrix} = -\frac{\Xi\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 6 \end{pmatrix}}{\xi_{33}\xi_{41}\xi_{52}}.$$

The associated characteristic function $\psi = \psi(t_1)$ given by

$$\psi := \tilde{g}'_1(t_1, \frac{U_2}{U_1}, \frac{U_3}{U_1})U_1^2$$

equals

$$\psi = -\det Y = \xi_{33}U_2U_3 + \xi_{34}U_2U_1 + \xi_{35}U_3U_1 + \xi_{36}U_1^2. \tag{A12}$$

Furthermore Lemma A3 shows that if U_1 is finite and $U_1 \neq 0$ at all points of $\tilde{C}$ then (A3) holds if and only if $\psi(t_1) = 0$.

Lemma A6. *The system of (ordered) polynomials (denoted by $\mathcal{Y}$) is obtained from $\mathcal{X}$ after exchanging $\{\mathbf{x}_3, \mathbf{x}_{11}, \mathbf{x}_{13}, \mathbf{x}_{14}\}$ for $\{\mathbf{y}_{12}, \mathbf{y}_{13}, \mathbf{y}_{14}, \mathbf{y}_{15}\}$*

$$\mathcal{Y} := (\mathcal{X} - \{\mathbf{x}_3, \mathbf{x}_{11}, \mathbf{x}_{13}, \mathbf{x}_{14}\}) \cup \{\mathbf{y}_{12}, \mathbf{y}_{13}, \mathbf{y}_{14}, \mathbf{y}_{15}\} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{15}\},$$

where

$$\mathbf{y}_{12} := \tilde{g}'''_{12}, \quad \mathbf{y}_{13} := \tilde{g}'''_{13}, \quad \mathbf{y}_{14} := t_3\tilde{g}'''_{12}, \quad \mathbf{y}_{15} := t_3\tilde{g}'''_{13}$$

and $\mathbf{y}_j$ *are connected with* $\mathbf{x}_k$ *by the matrix* $\mathcal{T}$*:*

$$\mathcal{T}: \begin{cases} \mathbf{y}_j = \mathbf{x}_j \quad (1 \le j \le 2), \\ \mathbf{y}_j = \mathbf{x}_{j+1} \quad (3 \le j \le 9), \\ \mathbf{y}_{10} = \mathbf{x}_{15}, \\ \mathbf{y}_{11} = \mathbf{x}_{12} \\ \mathbf{y}_{12} \equiv c_2\mathbf{x}_{11} + \mathbf{x}_{13} \mod (\mathbf{x}_8, \mathbf{x}_{10}, \mathbf{x}_{12}), \\ \mathbf{y}_{13} \equiv c_3\mathbf{x}_{11} + \mathbf{x}_{14} \mod (\mathbf{x}_9, \mathbf{x}_{10}, \mathbf{x}_{12}), \\ \mathbf{y}_{14} \equiv c_1\mathbf{x}_3 + \mathbf{x}_{15} \mod (\mathbf{x}_7, \mathbf{x}_8, \mathbf{x}_{14}), \\ \mathbf{y}_{15} \equiv c_4\mathbf{x}_{11} + c_5\mathbf{x}_{13} + c_6\mathbf{x}_{14} \mod (\mathbf{x}_7, \mathbf{x}_8, \mathbf{x}_9, \mathbf{x}_{12}) \end{cases}$$

such that

$$c_1 = -\frac{2\beta'_{1,23}}{\beta'_{12,22}}, \ c_2 = -\frac{\xi_{12}}{\xi_{52}}, \ c_3 = -\frac{\xi_{22}}{\xi_{52}}, \ c_4 = \frac{\beta'_{1,\emptyset}}{\beta'_{13,33}}, \ c_5 = -\frac{2\beta'_{13,2}}{\beta'_{13,33}}, \ c_6 = \frac{2\beta'_{13,3}}{\beta'_{13,33}} + \frac{\xi_{23}}{\xi_{33}},$$

and

$$\det \mathcal{T} = c_1(c_4 - c_2c_5 - c_3c_6) = -\frac{2\beta'_{1,23}\xi'_{14}}{\beta'_{12,22}\beta'_{13,33}}. \tag{A13}$$

In other words,

$$\mathcal{Y} = \mathcal{T}\mathcal{X}$$

and hence

$$\det \mathcal{Y} = \det \mathcal{T} \det \mathcal{X}. \tag{A14}$$

The ordered system $(\mathbf{y}_j)_{1 \le j \le 15}$ *are linearly independent and span the linear space of polynomials at most fourth degree.*

On the other hand

Lemma A7. *Macaulay's diagram associated with the system* $\mathcal{Y}$

$$\mathcal{Y} := {}^T(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{15})$$

is given as follows (each sum is expressed in the ordering (A4)):

$$\begin{aligned}
\mathbf{y}_1 &= t_2^2\tilde{g}'_{12} = \beta'_{12.22}t_2^4 + \cdots (\textit{lower order terms}), \\
\mathbf{y}_2 &= t_2t_3\tilde{g}'_{12} = \beta'_{12,22}t_2^3t_3 + \cdots, \\
\mathbf{y}_3 &= t_2t_3\tilde{g}'_{13} = \beta'_{13,33}t_2t_3^3 + \cdots, \\
\mathbf{y}_4 &= t_3^2\tilde{g}'_{13} = \beta'_{13,33}t_3^4 + \cdots, \\
\mathbf{y}_5 &= t_2\tilde{g}'_{12} = \beta'_{12,22}t_2^3 + \cdots, \\
\mathbf{y}_6 &= t_3\tilde{g}'_{13} = \beta'_{13,33}t_3^3 + \cdots, \\
\mathbf{y}_7 &= t_3\tilde{g}'_{12} = \beta'_{12,22}t_2^2t_3 + \cdots, \\
\mathbf{y}_8 &= t_2\tilde{g}'_{13} = \beta'_{13,33}t_2t_3^2 + \cdots, \\
\mathbf{y}_9 &= \tilde{g}'_{12} = \beta'_{12,22}t_2^2 + \cdots, \\
\mathbf{y}_{10} &= t_2t_3\tilde{g}'_1 = 2\beta'_{1,23}t_2^2t_3^2 + 2\beta'_{1,2}t_2^2t_3 + 2\beta'_{1,3}t_2t_3^2 + \beta'_{1,\emptyset}t_2t_3, \\
\mathbf{y}_{11} &= \tilde{g}'_1 = 2\beta'_{1,23}t_2t_3 + 2\beta'_{1,2}t_2 + 2\beta'_{1,3}t_3 + \beta'_{1,\emptyset}, \\
\mathbf{y}_{12} &= \tilde{g}'''_{12} = \xi'_{14}t_2 + \xi'_{15}t_3 + \xi'_{16}, \\
\mathbf{y}_{13} &= \tilde{g}'''_{13} = \xi'_{24}t_2 + \xi'_{25}t_3 + \xi'_{26}, \\
\mathbf{y}_{14} &= t_3\tilde{g}'''_{12} = \xi'_{15}t_3^2 + \xi'_{14}t_2t_3 + \xi'_{16}t_3, \\
\mathbf{y}_{15} &= t_3\tilde{g}'''_{13} = \xi'_{25}t_3^2 + \xi'_{24}t_2t_3 + \xi'_{26}t_3.
\end{aligned}$$

so that

$$\det \mathcal{Y} = 2\beta'_{1,23}(\beta'_{12,22})^5(\beta'_{13,33})^4 \det Y. \tag{A15}$$

Hence, (A5), (A13), (A14) and (A15) imply the following identity:

Proposition A1. *ψ is related with the resultant $R(\tilde{g}'_{12}, \tilde{g}'_{13}, \tilde{g}'_1)$ as follows:*

$$\psi = -\frac{\xi'_{14}}{(\beta'_{12,22}\beta'_{13,33})^4} R(\tilde{g}'_{12}, \tilde{g}'_{13}, \tilde{g}'_1),$$

where ξ'_{14} is given by (A9).

The determinant identity (A11) in the proof of Lemma A3 shows

Lemma A8.

$$d\tilde{g}'_1 \wedge d\tilde{g}'''_{12} \wedge d\tilde{g}'''_{13} \equiv -\frac{2\beta'_{1,23}\xi'_{14}}{\beta'_{12,22}\beta'_{13,33}} d\tilde{g}'_1 \wedge d\tilde{g}'_{12} \wedge d\tilde{g}'_{13} \mod Ann(\tilde{\mathcal{C}}). \tag{A16}$$

Proof. Indeed in view of (A7)–(A9)

$$\begin{aligned} &\text{The left-hand side of (A16)} \\ &\equiv \frac{1}{\beta'_{12,22}\beta'_{13,33}} \begin{vmatrix} 2\beta'_{1,23}t_3 + \xi_{11} & \xi_{12} \\ \xi_{21} & 2\beta'_{1,23}t_2 + \xi_{22} \end{vmatrix} d\tilde{g}'_1 \wedge d\tilde{g}'_{12} \wedge d\tilde{g}'_{13} \\ &\equiv -\frac{2\beta'_{1,23}\xi'_{14}}{\beta'_{12,22}\beta'_{13,33}} d\tilde{g}'_1 \wedge d\tilde{g}'_{12} \wedge d\tilde{g}'_{13} \mod Ann(\tilde{\mathcal{C}}) \end{aligned}$$

since $\tilde{g}'_1 \equiv 0$. □

We now assume that

$$(\mathcal{H}_3): \quad U_1 = \frac{\partial(\tilde{g}'''_{12}, \tilde{g}'''_{13})}{\partial(t_2, t_3)} \text{ does not have any zero or pole at } \tilde{\mathcal{C}}.$$

Then, we finally obtain the following fundamental equality:

Proposition A2. *Suppose that the conditions $(\mathcal{H}_0)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are satisfied. Then,*

$$t = (t_1, \frac{U_2}{U_1}, \frac{U_3}{U_1}) \in \tilde{\mathcal{C}} \quad \textit{if and only if} \quad \psi(t_1) = 0,$$

and

$$\frac{\partial(\tilde{g}_1, \tilde{g}_{12}, \tilde{g}_{13})}{\partial(t_1, t_2, t_3)} \equiv -\frac{\beta'_{12,22}\beta'_{13,33}}{2\beta'_{1,23}\xi'_{14}} \frac{\partial(\tilde{g}'_1, \tilde{g}'''_{12}, \tilde{g}'''_{13})}{\partial(t_1, t_2, t_3)} \mod Ann(\tilde{\mathcal{C}}), \tag{A17}$$

where

$$\frac{\partial(\tilde{g}'_1, \tilde{g}'''_{12}, \tilde{g}'''_{13})}{\partial(t_1, t_2, t_3)} \equiv \frac{1}{U_1}\frac{d\psi}{dt_1} \mod Ann(\tilde{\mathcal{C}}). \tag{A18}$$

Proof. Equation (A17) is a direct consequence of Lemmas A1 and A8. On the other hand, by a direct calculation, we have the identity

$$\frac{\partial(\tilde{g}'_1, \tilde{g}'''_{12}, \tilde{g}'''_{13})}{\partial(t_1, t_2, t_3)} \equiv \frac{1}{U_1^2}\frac{d\psi}{dt_1} \cdot \frac{\partial(\tilde{g}'''_{12}, \tilde{g}'''_{13})}{\partial(t_2, t_3)} \mod Ann(\tilde{\mathcal{C}}).$$

This means (A18) in view of (A12). □

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