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Certain Integral Formulae Associated with the Product of Generalized Hypergeometric Series and Several Elementary Functions Derived from Formulas for the Beta Function

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Abstract: The literature has an astonishingly large number of integral formulae involving a range of special functions. In this paper, by using three Beta function formulae, we aim to establish three integral formulas whose integrands are products of the generalized hypergeometric series ${}_{p+1}F_p$ and the integrands of the three Beta function formulae. Among the many particular instances for our formulae, several are stated clearly. Moreover, an intriguing inequality that emerges throughout the proving procedure is shown. It is worth noting that the three integral formulae shown here may be expanded further by using a variety of more generalized special functions than ${}_{p+1}F_p$. Symmetry occurs naturally in the Beta and ${}_{p+1}F_p$ functions, which are two of the most important functions discussed in this study.

Keywords: gamma function; beta function; generalized hypergeometric functions ${}_pF_q$; summation formulas for ${}_pF_q$

MSC: 33B15; 33C05; 33C20; 33C60; 33C70

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1. Introduction and Preliminaries

The generalized hypergeometric series ${}_pF_q$ is defined by (see [1], p. 73; see also [2,3]):

$$\begin{aligned}{}_pF_q\left[\begin{matrix}\mu_1,\,\ldots,\,\mu_p\,;\\ \nu_1,\,\ldots,\,\nu_q\,;\end{matrix}\,z\right] &= \sum_{n=0}^{\infty}\frac{(\mu_1)_n\cdots(\mu_p)_n}{(\nu_1)_n\cdots(\nu_q)_n}\frac{z^n}{n!}\\ &= {}_pF_q(\mu_1,\,\ldots,\,\mu_p;\,\nu_1,\,\ldots,\,\nu_q;\,z),\end{aligned} \tag{1}$$

where $(\xi)_\tau$ denotes the Pochhammer symbol or the shifted factorial, since

$$(1)_n = n! \quad (n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\};\ \mathbb{N} := \{1,2,3,\cdots\}),$$

which is defined (for $\xi,\,\tau \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by

$$(\xi)_\tau := \frac{\Gamma(\xi+\tau)}{\Gamma(\xi)} = \begin{cases}1 & (\tau = 0;\ \xi \in \mathbb{C}\setminus\{0\})\\ \xi(\xi+1)\cdots(\xi+n-1) & (\tau = n \in \mathbb{N};\ \xi \in \mathbb{C}),\end{cases} \tag{2}$$

it being traditionally considered that $(0)_0 := 1$ and $\mathbb{C}$ the set of complex numbers. Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume (for simplicity) that the variable z, the numerator parameters $\mu_1, \ldots, \mu_p$, and the denominator parameters $\nu_1, \ldots, \nu_q$ take on complex values, provided that no zeros appear in the denominator of (1), that is, that

$$(\nu_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ j = 1, \ldots, q),$$

where $\mathbb{Z}_0^-$ denotes the set of nonpositive integers. Thus, if a numerator parameter is a negative integer or zero, the ${}_pF_q$ series terminates in view of the known identity (see, for example [4], p. 5):

$$(-\eta)_j = \begin{cases} \dfrac{(-1)^j\, \eta!}{(\eta - j)!} & (0 \leqq j \leqq \eta;\ j,\, \eta \in \mathbb{N}_0), \\ 0 & (j > \eta). \end{cases} \tag{3}$$

For details on the convergence criteria for ${}_pF_q$ in (1), see (for example) [4], pp. 64 and 72 (see also [1–3]). The celebrated Gauss's summation theorem (see, e.g., [4], p. 64)

$${}_2F_1(\kappa,\, \lambda;\, \mu;\, 1) = \frac{\Gamma(\mu)\, \Gamma(\mu - \kappa - \lambda)}{\Gamma(\mu - \kappa)\, \Gamma(\mu - \kappa)} \quad (\Re(\mu - \kappa - \lambda) > 0;\, \mu \in \mathbb{C} \setminus \mathbb{Z}_0^-) \tag{4}$$

and the numerous subsequent summation formulas for ${}_pF_q$ (see, for example [1–16]) play critical roles in theories of special functions and have a wide range of applications in diverse fields such as number theory, combinatorics, and geometric analytic function theory.

The classical beta function (see, e.g., [4], p. 8)

$$B(\mu,\, \nu) = \begin{cases} \displaystyle\int_0^1 t^{\mu-1}(1-t)^{\nu-1}\, \mathrm{d}t & (\min\{\Re(\mu),\, \Re(\nu)\} > 0) \\ \dfrac{\Gamma(\mu)\, \Gamma(\nu)}{\Gamma\mu + \nu)} & (\mu,\, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{cases} \tag{5}$$

has numerous other integral forms (see, e.g., [8], Section 1.5). It is easy to find

$$\int_0^1 t^{\mu-1}(1-t^\eta)^{\nu-1}\, \mathrm{d}t = \frac{1}{\eta}\, B\left(\frac{\mu}{\eta},\, \nu\right) \quad (\min\{\Re(\mu),\, \Re(\nu)\} > 0,\, \eta > 0), \tag{6}$$

which is recorded in [8] (p. 10, Equation (17)).

Numerous integral formulae incorporating a variety of special functions have been published in the literature (see, for example [8,11,17,18]). By using three Beta function formulae (7)–(9), we want to offer three integral formulas whose integrands are products of the generalized hypergeometric series ${}_{p+1}F_p$ and their associated integrands. Several of our formulae' various special instances are fully shown. Furthermore, an intriguing inequality is formed throughout the proving procedure. Clearly, the three integral formulae described here may be extended further by using a variety of more generalized special functions (listed in Section 5) than ${}_{p+1}F_p$.

It is noted in passing that symmetry issues may arise overtly or indirectly in any discipline or aspect of human existence. It is self-evident that symmetry occurs in the Beta and ${}_pF_q$ functions, two of the most significant functions considered in this paper. Explicitly,

$$B(\mu, \nu) = B(\nu, \mu),$$

and, for example,

$${}_pF_q(\mu_1,\, \ldots,\, \mu_p;\, \nu_1,\, \ldots,\, \nu_q;\, z) = {}_pF_q(\mu_p,\, \ldots,\, \mu_1;\, \nu_q,\, \ldots,\, \nu_1;\, z),$$

where every reordering of the numerator parameters produces the same function, and every reordering of the denominator parameters provide the same function.

2. Beta Function Formulae

The Beta function, which encompasses a large variety of special functions, is often reported in the literature. There are 24 integral formulas expressed in terms of the Beta function in [8] (Sections 1.5 and 1.6), including (5) and (6). Similarly, [18] has over 100 integral formulae for the Beta function. The following two formulae (7) and (8) may be dropped from the two monographs cited above.

$$\int_0^1 x^{\mu-1}(1-x)^{2\nu-1}\left(1-\frac{x}{3}\right)^{2\mu-1}\left(1-\frac{x}{4}\right)^{\nu-1}\mathrm{d}x = \left(\frac{2}{3}\right)^{2\mu} B(\mu,\,\nu) \\ (\min\{\Re(\mu),\,\Re(\nu)\} > 0) \tag{7}$$

(see [19]);

$$\int_0^1\int_0^1 y^{\mu}(1-x)^{\mu-1}(1-y)^{\nu-1}(1-xy)^{1-\mu-\nu}\mathrm{d}x\mathrm{d}y = B(\mu,\,\nu) \\ (\min\{\Re(\mu),\,\Re(\nu)\} > 0) \tag{8}$$

(see, e.g., [20], p. 145, Problem 6);

$$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}(x-c)^{-\mu-\nu}\mathrm{d}x = (b-a)^{\mu+\nu-1}\,(b-c)^{-\mu}(a-c)^{-\nu}\,B(\mu,\,\nu) \\ (\min\{\Re(\mu),\,\Re(\nu)\} > 0,\ c < a < b) \tag{9}$$

(see [18], p. 315, Entry 3.199);

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}[ax+b(1-x)+c]^{-\mu-\nu}\mathrm{d}x = (a+c)^{-\mu}\,(b+c)^{-\nu}\,B(\mu,\,\nu) \\ (a \geq 0,\, b \geq 0,\, c > 0,\, \min\{\Re(\mu),\,\Re(\nu)\} > 0), \tag{10}$$

(see [18], p. 315, Entry 3.198);

$$\int_0^{\frac{\pi}{2}} e^{\mathbf{i}(\mu+\nu)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\nu-1}\mathrm{d}\theta = e^{\frac{\mathbf{i}\pi\mu}{2}}\,B(\mu,\,\nu) \\ \left(\mathbf{i} = \sqrt{-1},\, \min\{\Re(\mu),\,\Re(\nu)\} > 0\right) \tag{11}$$

(see [21]);

Equating the real and imaginary parts of both sides of (11) yields

$$\int_0^{\frac{\pi}{2}} \cos[(\mu+\nu)\theta](\sin\theta)^{\mu-1}(\cos\theta)^{\nu-1}\mathrm{d}\theta = \cos\left(\frac{1}{2}\pi\mu\right) B(\mu,\,\nu) \\ (\min\{\Re(\mu),\,\Re(\nu)\} > 0), \tag{12}$$

and

$$\int_0^{\frac{\pi}{2}} \sin[(\mu+\nu)\theta](\sin\theta)^{\mu-1}(\cos\theta)^{\nu-1}\mathrm{d}\theta = \sin\left(\frac{1}{2}\pi\mu\right) B(\mu,\,\nu) \\ (\Re(\mu) > -1,\, \Re(\nu) > 0) \tag{13}$$

(see [21]);

$$\int_u^\infty (x+v)^{-\mu}(x-u)^{\nu-1}\mathrm{d}x = (u+v)^{\nu-\mu}\,B(\mu-\nu,\nu) \\ \left(\left|\arg\left(\frac{u}{v}\right)\right| < \pi,\ \Re(\mu) > \Re(\nu) > 0\right) \tag{14}$$

(see [18], p. 314, Entry 3.196-2).

3. Three Integral Formulas Associated with ${}_{p+1}F_p$

Here and elsewhere, conventionally, let $\Delta(m;\alpha)$ denote the array of m parameters

$$\frac{\alpha}{m}, \frac{\alpha+1}{m}, \ldots, \frac{\alpha+m-1}{m}, \tag{15}$$

where $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$. In addition, let ${}^{+}\Delta(m;\alpha)$ denote the sum of the m parameters in the array in (15).

In the following lemma, we begin by introducing an intriguing inequality.

Lemma 1. *Let* $\mathtt{k},\ \mathtt{s} \in \mathbb{N}$. *Then,*

$$\left(\frac{2}{3}\right)^{2\mathtt{k}} \frac{\mathtt{k}^{\mathtt{k}}\mathtt{s}^{\mathtt{s}}}{(\mathtt{k}+\mathtt{s})^{\mathtt{k}+\mathtt{s}}} \leq \frac{1}{9}; \tag{16}$$

$$\frac{\mathtt{k}^{\mathtt{k}}\mathtt{s}^{\mathtt{s}}}{(\mathtt{k}+\mathtt{s})^{\mathtt{k}+\mathtt{s}}} \leq \frac{1}{4}. \tag{17}$$

More generally,

$$d^{2x} \frac{x^x\, y^y}{(x+y)^{x+y}} \leq \left(\frac{d}{2}\right)^2 \quad (0 < d < 1;\ x \geq 1,\ y \geq 1). \tag{18}$$

Proof. Let us denote $f(\mathtt{k},\mathtt{s})$ by the left-member of the inequality (16). We show that

$$g(x) := \left(\frac{x}{1+x}\right)^x \tag{19}$$

is strictly decreasing on $[1,\infty)$. Indeed, logarithmic differentiation affords

$$g'(x) = \left(\frac{x}{1+x}\right)^x \left\{\log\left(1 - \frac{1}{1+x}\right) + \frac{1}{1+x}\right\}.$$

Recall the Maclaurin series expansion

$$\log(1-t) = -\sum_{k=1}^{\infty} \frac{t^k}{k} \quad (|t| < 1).$$

We find

$$g'(x) = -\left(\frac{x}{1+x}\right)^x \sum_{k=2}^{\infty} \frac{1}{k(1+x)^k} < 0$$

for all $x \geq 1$. Hence $g(x)$ is strictly decreasing on $[1,\infty)$.

Direct computation gives $f(1,1) = \frac{1}{9}$ and $f(1,2) < \frac{1}{9}$. Since $g(x)$ in (19) is strictly decreasing on $[1,\infty)$, we have that, for $\mathtt{s} \geq 3$,

$$f(1,\mathtt{s}) = \frac{4}{9}\frac{\mathtt{s}^{\mathtt{s}}}{(1+\mathtt{s})^{1+\mathtt{s}}} = \frac{4}{9}\cdot\frac{1}{1+\mathtt{s}}\cdot\frac{\mathtt{s}^{\mathtt{s}}}{(1+\mathtt{s})^{\mathtt{s}}} < \frac{4}{9}\cdot\frac{1}{4}\cdot\frac{3^3}{4^3} < \frac{1}{9}.$$

Since $(\mathtt{k}+\mathtt{s})^{\mathtt{k}+\mathtt{s}} = (\mathtt{k}+\mathtt{s})^{\mathtt{k}}(\mathtt{k}+\mathtt{s})^{\mathtt{s}} \geq \mathtt{k}^{\mathtt{k}}\mathtt{s}^{\mathtt{s}}$, we find $\frac{\mathtt{k}^{\mathtt{k}}\mathtt{s}^{\mathtt{s}}}{(\mathtt{k}+\mathtt{s})^{\mathtt{k}+\mathtt{s}}} \leq 1$, and, therefore,

$$f(\mathtt{k},\mathtt{s}) \leq \left(\frac{2}{3}\right)^{2\mathtt{k}} \quad (\mathtt{k},\ \mathtt{s} \in \mathbb{N}).$$

Now, for $\mathtt{k} \geq 3$, we find

$$f(\mathtt{k},\mathtt{s}) \leq \left(\frac{2}{3}\right)^{2\mathtt{k}} \leq \left(\frac{2}{3}\right)^{6} = \frac{4}{9}\cdot\frac{4}{9}\cdot\frac{4}{9} < \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4}{9} = \frac{1}{9}.$$

Similarly, the inequality (17) can be proved. Indeed, to prove (18), it may be enough to show that

$$f(x,y) := \frac{x^x\, y^y}{(x+y)^{x+y}} \quad (x \geq 1,\, y \geq 1)$$

is a decreasing function on the variable $x \in [0,\infty)$ when y is fixed in $[0,\infty)$. This is true since

$$\frac{\partial f(x,y)}{\partial x} = \log x - \log(x+y) < 0 \quad (x \geq 1,\, y \geq 1).$$

Since $f(x,y)$ is symmetric with respect to the variables x and y, $f(x,y)$ is also decreasing on the variable $y \in [0,\infty)$ when x is fixed in $[0,\infty)$. Note that d^{2x} is decreasing on $[0,\infty)$ when $0 < d < 1$. Hence we find that $f(x,y) \leq f(1,1)$ $(x \geq 1,\, y \geq 1)$. $\square$

Theorem 1. *Let $p \in \mathbb{N}_0$, $\min\{\Re(\alpha),\, \Re(\beta)\} > 0$, $a_j \in \mathbb{C}$ $(j = 1, \ldots, p+1)$, and $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \ldots, p)$. Moreover, let $\Bbbk,\, \mathbb{s} \in \mathbb{N}$, and $\mu \in \mathbb{C}$ be such that $|\mu| < \left(\frac{3}{2}\right)^{2\Bbbk} \frac{(\Bbbk+\mathbb{s})^{\Bbbk+\mathbb{s}}}{\Bbbk^{\Bbbk} \mathbb{s}^{\mathbb{s}}}$. Then*

$$\begin{aligned}
&\mathscr{I}_1^{(p)}(\alpha, \beta; \Bbbk, \mathbb{s}; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \mu) \\
&:= \int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} \\
&\qquad \times {}_{p+1}F_p\left[\begin{matrix} a_1, \ldots, a_{p+1}\,; \\ b_1, \ldots, b_p\,; \end{matrix}\; \mu\, x^{\Bbbk}(1-x)^{2\mathbb{s}}\left(1-\frac{x}{3}\right)^{2\Bbbk}\left(1-\frac{x}{4}\right)^{\mathbb{s}}\right] \mathrm{d}x \\
&= \left(\frac{2}{3}\right)^{2\alpha} \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
&\quad \times {}_{p+1+\Bbbk+\mathbb{s}}F_{p+\Bbbk+\mathbb{s}}\left[\begin{matrix} a_1, \ldots, a_{p+1},\, \Delta(\Bbbk;\alpha),\, \Delta(\mathbb{s};\beta)\,; \\ b_1, \ldots, b_p,\, \Delta(\Bbbk+\mathbb{s};\alpha+\beta)\,; \end{matrix}\; \mu\left(\frac{2}{3}\right)^{2\Bbbk} \frac{\Bbbk^{\Bbbk}\mathbb{s}^{\mathbb{s}}}{(\Bbbk+\mathbb{s})^{\Bbbk+\mathbb{s}}}\right].
\end{aligned} \tag{20}$$

Furthermore, the integral $\mathscr{I}_1^{(p)}(\alpha, \beta; \Bbbk, \mathbb{s}; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \mu)$ converges absolutely for $|\mu| = \left(\frac{3}{2}\right)^{2\Bbbk} \frac{(\Bbbk+\mathbb{s})^{\Bbbk+\mathbb{s}}}{\Bbbk^{\Bbbk}\mathbb{s}^{\mathbb{s}}}$ if

$$\Re\left({}^{+}\Delta(\Bbbk+\mathbb{s};\alpha+\beta) + \sum_{j=1}^{p} b_j - {}^{+}\Delta(\Bbbk;\alpha) - {}^{+}\Delta(\mathbb{s};\beta) - \sum_{j=1}^{p+1} a_j\right) > 0, \tag{21}$$

converges conditionally for $|\mu| = \left(\frac{3}{2}\right)^{2\Bbbk} \frac{(\Bbbk+\mathbb{s})^{\Bbbk+\mathbb{s}}}{\Bbbk^{\Bbbk}\mathbb{s}^{\mathbb{s}}}$ $\left(\mu \neq \left(\frac{3}{2}\right)^{2\Bbbk} \frac{(\Bbbk+\mathbb{s})^{\Bbbk+\mathbb{s}}}{\Bbbk^{\Bbbk}\mathbb{s}^{\mathbb{s}}}\right)$ if

$$-1 < \Re\left({}^{+}\Delta(\Bbbk+\mathbb{s};\alpha+\beta) + \sum_{j=1}^{p} b_j - {}^{+}\Delta(\Bbbk;\alpha) - {}^{+}\Delta(\mathbb{s};\beta) - \sum_{j=1}^{p+1} a_j\right) \leq 0. \tag{22}$$

Proof. Now we prove Theorem 1. In view of Lemma 1, for the convergence of the expression ${}_{p+1+\mathrm{k}+\mathrm{s}}F_{p+\mathrm{k}+\mathrm{s}}$ in (20),

$$|\mu|\left(\frac{2}{3}\right)^{2\mathrm{k}} \frac{\mathrm{k}^{\mathrm{k}}\mathrm{s}^{\mathrm{s}}}{(\mathrm{k}+\mathrm{s})^{\mathrm{k}+\mathrm{s}}} \leq 1$$

implies that $|\mu| \leq 9$. Therefore, for the convergence of the ${}_{p+1}F_p$ in the integrand in (20), it suffices to show that, when $0 \leq x \leq 1$,

$$0 \leq f(x) := 9x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right) \leq 1.$$

In fact, we find

$$f'(x) = \frac{3}{2}(1-x)(2-x)(3-x)\left(x-2+\sqrt{3}\right)\left(x-2-\sqrt{3}\right),$$

which may depict that $f(x)$ has a local maximum value only at $x = 2 - \sqrt{3}$ on the interval $[0, 1]$. Furthermore, $f\left(2 - \sqrt{3}\right) = 1$ is the maximum value on the interval $[0, 1]$. We, therefore, have $0 < f(x) \leq 1$ when $0 < x < 1$.

Starting with this observation for $f(x)$ and the identity (7), and use the series definition in (1) to expand ${}_{p+1}F_p$ in the integrand in (20). It is easily found that the resultant series in the integrand converges uniformly under the given restrictions and, therefore, the termwise-integration gives

$$
\begin{aligned}
&\mathscr{I}_1^{(p)}(\alpha, \beta; \mathtt{k}, \mathtt{s}; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \mu) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p+1})_n}{n!\,(b_1)_n \cdots (b_p)_n}\, \mu^n \\
&\quad \times \int_0^1 x^{\alpha+n-1}(1-x)^{2\beta+2n-1}\left(1 - \frac{x}{3}\right)^{2\alpha+2n-1}\left(1 - \frac{x}{4}\right)^{\beta+n-1} \mathrm{d}x \\
&= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p+1})_n}{n!\,(b_1)_n \cdots (b_p)_n}\, \mu^n \left(\frac{2}{3}\right)^{2\alpha+2\mathtt{k}n} B(\alpha + \mathtt{k}n,\, \beta + \mathtt{s}n).
\end{aligned}
$$

Now employing (2) and (5) gives

$$
\begin{aligned}
&\mathscr{I}_1^{(p)}(\alpha, \beta; \mathtt{k}, \mathtt{s}; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \mu) \\
&= \left(\frac{2}{3}\right)^{2\alpha} \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p+1})_n}{n!\,(b_1)_n \cdots (b_p)_n}\, \mu^n \left(\frac{2}{3}\right)^{2\mathtt{k}n} \frac{(\alpha)_{\mathtt{k}n}\,(\beta)_{\mathtt{s}n}}{(\alpha+\beta)_{(\mathtt{k}+\mathtt{s})n}},
\end{aligned}
\tag{23}
$$

which, upon using the multiplication formula for the Pochhammer symbol (see, e.g., [4], p. 6, Equation (30))

$$
(\eta)_{\mathtt{m}n} = \mathtt{m}^{\mathtt{m}n} \prod_{r=1}^{\mathtt{m}} \left(\frac{\eta + r - 1}{\mathtt{m}}\right)_n \qquad (\eta \in \mathbb{C};\ \mathtt{m} \in \mathbb{N};\ n \in \mathbb{N}_0),
\tag{24}
$$

leads to the last expression in (20).

It is also noted that the integral formula in Theorem 1 is seen to hold true for $\mu = \pm 9$ by appealing to the Abel-type argument (see, e.g., [22], p. 243, 7.32 Theorem).

The convergence conditions (21) and (22) follow easily from the well known theory of ${}_pF_q$ (see, e.g., [4], p. 72). $\square$

Theorem 2. *Let $p \in \mathbb{N}_0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$, $a_j \in \mathbb{C}$ $(j = 1, \ldots, p+1)$, and $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \ldots, p)$. Also let k, $s \in \mathbb{N}$, and $\xi \in \mathbb{C}$ be such that $|\xi| < \frac{(k+s)^{k+s}}{k^k s^s}$. Then*

$$
\begin{aligned}
&\mathscr{I}_2^{(p)}(\alpha, \beta; k, s; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \xi) \\
&:= \int_0^1 \int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}\,(1-xy)^{1-\alpha-\beta} \\
&\quad \times {}_{p+1}F_p\left[\begin{matrix} a_1, \ldots, a_{p+1}\,; \\ b_1, \ldots, b_p\,; \end{matrix}\ \xi\, y^k(1-x)^k(1-y)^s(1-xy)^{-k-s}\right] \mathrm{d}x\mathrm{d}y \\
&= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, {}_{p+k+s+1}F_{p+k+s}\left[\begin{matrix} a_1, \ldots, a_{p+1},\, \Delta(k; \alpha),\, \Delta(s; \beta); \\ b_1, \ldots, b_p,\, \Delta(k+s; \alpha+\beta); \end{matrix}\ \xi\, \frac{k^k s^s}{(k+s)^{k+s}}\right].
\end{aligned}
\tag{25}
$$

Furthermore, the integral $\mathscr{I}_2^p(\alpha, \beta; k, s; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \xi)$ converges absolutely for $|\xi| = \frac{(k+s)^{k+s}}{k^k s^s}$ under the condition (21), and converges conditionally for $|\xi| = \frac{(k+s)^{k+s}}{k^k s^s}$ $\left(\xi \neq \frac{(k+s)^{k+s}}{k^k s^s}\right)$ under the condition (22).

Proof. Consider the inequality (17). It suffices to show the case $\mathtt{k} = \mathtt{s} = 1$. Let

$$
g(x, y) := 4y(1-x)(1-y)(1-xy)^{-2}.
$$

When $0 < x,\, y < 1$, obviously $g(x,y) > 0$. Under $0 < x,\, y < 1$, we want to show that $g(x,y) \leq 1$ if and only if $4y(1-x)(1-y) \leq (1-xy)^2$ if and only if

$$x^2y^2 - 4xy^2 + 4y^2 + 2xy - 4y + 1 = y^2(x-2)^2 + 2y(x-2) + 1 = \big((x-2)y+1\big)^2 \geq 0.$$

Using this observation and the identity (8), similarly as in the proof of Theorem 1, the proof can be complete. The details are omitted. □

Theorem 3. *Let $p \in \mathbb{N}_0$, $\min\{\Re(\alpha),\, \Re(\beta)\} > 0$, $a_j \in \mathbb{C}$ $(j = 1, \ldots, p+1)$, and $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \ldots, p)$. In addition let $\Bbbk,\, \mathtt{s} \in \mathbb{N}$, and $c < a < b$ and $\mu \in \mathbb{C}$ be such that $|\mu| < \frac{(b-c)^{\Bbbk}(a-c)^{\mathtt{s}}}{(b-a)^{\Bbbk+\mathtt{s}}}\frac{(\Bbbk+\mathtt{s})^{\Bbbk+\mathtt{s}}}{\Bbbk^{\Bbbk}\mathtt{s}^{\mathtt{s}}}$. Then*

$$\begin{aligned}
&\mathscr{I}_3^{(p)}\left(\alpha,\beta;c,a,b;\Bbbk,\mathtt{s};[a_1,\ldots,a_{p+1};b_1,\ldots,b_p];\mu\right)\\
&:= \int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(x-c)^{-\alpha-\beta}\\
&\qquad \times {}_{p+1}F_p\left[\begin{matrix} a_1,\ldots,a_{p+1}\,; \\ b_1,\ldots,b_p\,; \end{matrix}\ \mu\,(x-a)^{\Bbbk}(b-x)^{\mathtt{s}}(x-c)^{-\Bbbk-\mathtt{s}}\right]\mathrm{d}x\\
&= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,\frac{(b-a)^{\alpha+\beta-1}}{(b-c)^{\alpha}(a-c)^{\beta}}\\
&\quad \times {}_{p+\Bbbk+\mathtt{s}+1}F_{p+\Bbbk+\mathtt{s}}\left[\begin{matrix} a_1,\ldots,a_{p+1},\,\Delta(\Bbbk;\alpha),\,\Delta(\mathtt{s};\beta)\,; \\ b_1,\ldots,b_p,\,\Delta(\Bbbk+\mathtt{s};\alpha+\beta)\,; \end{matrix}\ \frac{\mu(b-a)^{\Bbbk+\mathtt{s}}}{(b-c)^{\Bbbk}(a-c)^{\mathtt{s}}}\,\frac{\Bbbk^{\Bbbk}\mathtt{s}^{\mathtt{s}}}{(\Bbbk+\mathtt{s})^{\Bbbk+\mathtt{s}}}\right].
\end{aligned} \tag{26}$$

Under further restrictions:

$$\textit{either}\quad c \geq 0 \quad \textit{or} \quad (c < 0 \ \textit{and}\ a+b < 0),$$

the integral $\mathscr{I}_3^{(p)}\left(\alpha,\beta;c,a,b;\Bbbk,\mathtt{s};[a_1,\ldots,a_{p+1};b_1,\ldots,b_p];\mu\right)$ converges absolutely for $|\mu| = \frac{(b-c)^{\Bbbk}(a-c)^{\mathtt{s}}}{(b-a)^{\Bbbk+\mathtt{s}}}\frac{(\Bbbk+\mathtt{s})^{\Bbbk+\mathtt{s}}}{\Bbbk^{\Bbbk}\mathtt{s}^{\mathtt{s}}}$ under the condition (21), and converges conditionally for $|\mu| = \frac{(b-c)^{\Bbbk}(a-c)^{\mathtt{s}}}{(b-a)^{\Bbbk+\mathtt{s}}}\frac{(\Bbbk+\mathtt{s})^{\Bbbk+\mathtt{s}}}{\Bbbk^{\Bbbk}\mathtt{s}^{\mathtt{s}}}\left(\mu \neq \frac{(b-c)^{\Bbbk}(a-c)^{\mathtt{s}}}{(b-a)^{\Bbbk+\mathtt{s}}}\frac{(\Bbbk+\mathtt{s})^{\Bbbk+\mathtt{s}}}{\Bbbk^{\Bbbk}\mathtt{s}^{\mathtt{s}}}\right)$ under the condition (22).

Proof. Consider the inequality (17). It suffices to show the case $\Bbbk = \mathtt{s} = 1$.

We observe the followings: Assume $c < a < b$ and $a < x < b$. We want to show that

$$\frac{4(b-c)(a-c)}{(b-a)^2}\,(x-a)(b-x)(x-c)^{-2} \leq 1$$

if and only if

$$4(b-c)(a-c)\,(x-a)(b-x) \leq (b-a)^2\,(x-c)^2$$

if and only if

$$\begin{aligned}
h(x) :&= (b-a)^2\,(x-c)^2 - 4(b-c)(a-c)\,(x-a)(b-x)\\
&= \Big\{(b-a)^2 + 4(b-c)(a-c)\Big\}x^2 - 2\Big\{c(b-a)^2 + 2(a+b)(b-c)(a-c)\Big\}x\\
&\quad + c^2(b-a)^2 + 4ab(b-c)(a-c) \geq 0.
\end{aligned}$$

Note that

(i) Since $(b-a)^2 + 4(b-c)(a-c) > 0$, the graph of the quadratic equation $y = h(x)$ has the form of a parabola which opens up.

(ii)

$$0 < h(a) = (b-a)^2(a-c)^2 < (b-a)^2(b-c)^2 = h(b).$$

(iii)

$$h'(x) = 2\Big\{(b-a)^2 + 4(b-c)(a-c)\Big\}x - 2\Big\{c(b-a)^2 + 2(a+b)(b-c)(a-c)\Big\}.$$

(iv) We find that, on the whole real x-axis, $h(x)$ has the minimum value at x with $h'(x) = 0$, i.e.,

$$x = \frac{c(b-a)^2 + 2(a+b)(b-c)(a-c)}{(b-a)^2 + 4(b-c)(a-c)},$$

which is the symmetric axis for the graph of $y = h(x)$.

(v) Since $c < a < b$, it is easy to see that, say,

$$x_0 = \frac{c(b-a)^2 + 2(a+b)(b-c)(a-c)}{(b-a)^2 + 4(b-c)(a-c)} < b.$$

Moreover, $a < x_0$ if and only if $c < \frac{a+b}{2}$. We observe that there is no case $a > x_0$ since $c < a < \frac{a+b}{2}$.

(vi) The minimum value of $h(x)$ on the whole real x-axis is

$$h(x_0) = \frac{8c(a+b)(b-a)^2(b-c)(a-c)}{(b-a)^2 + 4(b-c)(a-c)}.$$

(vii) From (vi), we see that $h(x_0) \geq 0$ if and only if either $c \geq 0$ or ($c < 0$ and $a + b \leq 0$). In this case, $h(x) \geq 0$ for all x on the whole real x-axis.

(viii) From (vi), we observe that $h(x_0) < 0$ if and only if $c < 0$ and $a + b > 0$. We also see that $a < x_0$ if and only if $c < \frac{a+b}{2}$. Therefore, we conclude that if $h(x_0) < 0$, then $a < x_0$, and hence $a < x_0 < b$.

(vix) From (viii), if $h(x_0) < 0$, then $a < x_0 < b$. In view of (ii), we can find two distinct zeros τ_1, τ_2 of $h(x)$ such that

$$h(x) = \Big\{(b-a)^2 + 4(b-c)(a-c)\Big\}(x-\tau_1)(x-\tau_2) \quad (a < \tau_1 < \tau_2 < b).$$

In view of (ii), we therefore observe that $h(x) \geq 0$ on *either* the interval $[a, \tau_1]$ *or* the interval $[\tau_2, b]$, and $h(x) < 0$ on the interval (τ_1, τ_2).
Since the integration in (26) is acting on the interval $[a, b]$ and $h(x) \geq 0$ on $[a, b]$, this case should be dropped when

$$\mu = \frac{4(b-c)(a-c)}{(b-a)^2}. \tag{27}$$

(x) Hence, we find from (vii) and (vix) that the restrictions (33) should be satisfied when μ is the case (27).

Using this observation and the identity (9), similarly to in the proof of Theorem 1, the proof can be complete. The details are omitted. □

4. Special Cases

By setting $\mathtt{k} = \mathtt{s} = 1$ in Theorems 1–3, we may obtain the following three relatively simple integral formulae.

Corollary 1. *Let $p \in \mathbb{N}_0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$, $a_j \in \mathbb{C}$ $(j = 1, \ldots, p+1)$, and $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \ldots, p)$. Furthermore, let $\nu \in \mathbb{C}$ be such that $|\nu| < 9$. Then,*

$$\begin{aligned} &\mathscr{I}_1^{(p)}(\alpha, \beta; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \nu) \\ &\quad := \int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\Big(1-\frac{x}{3}\Big)^{2\alpha-1}\Big(1-\frac{x}{4}\Big)^{\beta-1} \\ &\qquad \times {}_{p+1}F_p\left[\begin{matrix} a_1, \ldots, a_{p+1}\,; \\ b_1, \ldots, b_p\,; \end{matrix}\ \nu\, x(1-x)^2\Big(1-\frac{x}{3}\Big)^2\Big(1-\frac{x}{4}\Big)\right]\mathrm{d}x \\ &\quad = \Big(\frac{2}{3}\Big)^{2\alpha}\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,{}_{p+3}F_{p+2}\left[\begin{matrix} \alpha,\, \beta,\, a_1, \ldots, a_{p+1}\,; \\ \frac{\alpha+\beta}{2},\, \frac{\alpha+\beta+1}{2},\, b_1, \ldots, b_p\,; \end{matrix}\ \frac{\nu}{9}\right]. \end{aligned} \tag{28}$$

Furthermore, the integral $\mathscr{I}_1^{p}(\alpha, \beta; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \nu)$ converges absolutely for $|\nu| = 9$ if

$$\Re\Big(\sum_{j=1}^{p} b_j - \sum_{j=1}^{p+1} a_j + \frac{1}{2}\Big) > 0, \tag{29}$$

converges conditionally for $|\nu| = 9$ $(\nu \neq 9)$ if

$$-1 < \Re\Big(\sum_{j=1}^{p} b_j - \sum_{j=1}^{p+1} a_j + \frac{1}{2}\Big) \leq 0. \tag{30}$$

Corollary 2. *Let $p \in \mathbb{N}_0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$, $a_j \in \mathbb{C}$ $(j = 1, \ldots, p+1)$, and $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \ldots, p)$. In addition, let $\xi \in \mathbb{C}$ be such that $|\xi| < 4$. Then*

$$\begin{aligned} &\mathscr{I}_2^{(p)}(\alpha, \beta; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \xi) \\ &\quad := \int_0^1\int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}\,(1-xy)^{1-\alpha-\beta} \\ &\qquad \times {}_{p+1}F_p\left[\begin{matrix} a_1, \ldots, a_{p+1}\,; \\ b_1, \ldots, b_p\,; \end{matrix}\ \xi\, y(1-x)(1-y)(1-xy)^{-2}\right]\mathrm{d}x\mathrm{d}y \\ &\quad = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,{}_{p+3}F_{p+2}\left[\begin{matrix} \alpha,\, \beta,\, a_1, \ldots, a_{p+1}\,; \\ \frac{\alpha+\beta}{2},\, \frac{\alpha+\beta+1}{2},\, b_1, \ldots, b_p\,; \end{matrix}\ \frac{\xi}{4}\right]. \end{aligned} \tag{31}$$

Furthermore, the integral $\mathscr{I}_2^{p}(\alpha, \beta; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \xi)$ converges absolutely for $|\xi| = 4$ under the same condition in (29), and converges conditionally for $|\xi| = 4$ $(\xi \neq 4)$ under the same condition in (30).

Corollary 3. *Let* $p \in \mathbb{N}_0$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$, $a_j \in \mathbb{C}$ $(j = 1, \ldots, p+1)$, *and* $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \ldots, p)$. *Moreover, let* $c < a < b$ *and* $\mu \in \mathbb{C}$ *be such that* $|\mu| < \frac{4(b-c)(a-c)}{(b-a)^2}$. *Then*

$$\begin{aligned}
&\mathscr{I}_3^{(p)}(\alpha, \beta; c, a, b; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \mu) \\
&:= \int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(x-c)^{-\alpha-\beta} \\
&\qquad \times {}_{p+1}F_p\left[\begin{matrix} a_1, \ldots, a_{p+1}\,; \\ b_1, \ldots, b_p\,; \end{matrix}\ \mu\,(x-a)(b-x)(x-c)^{-2}\right] \mathrm{d}x \\
&= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,\frac{(b-a)^{\alpha+\beta-1}}{(b-c)^{\alpha}(a-c)^{\beta}} \\
&\qquad \times {}_{p+3}F_{p+2}\left[\begin{matrix} \alpha, \beta, a_1, \ldots, a_{p+1}\,; \\ \frac{\alpha+\beta}{2}, \frac{\alpha+\beta+1}{2}, b_1, \ldots, b_p\,; \end{matrix}\ \frac{\mu(b-a)^2}{4(b-c)(a-c)}\right].
\end{aligned} \tag{32}$$

Under further restrictions:

$$\text{either} \quad c \geq 0 \quad \text{or} \quad (c < 0 \ \text{and} \ a+b < 0), \tag{33}$$

the integral $\mathscr{I}_3^{p}(\alpha, \beta; c, a, b; [a_1, \ldots, a_{p+1}; b_1, \ldots, b_p]; \mu)$ *converges absolutely for* $|\mu| = \frac{4(b-c)(a-c)}{(b-a)^2}$ *under the same condition in* (29), *and converges conditionally for* $|\mu| = \frac{4(b-c)(a-c)}{(b-a)^2}$ $\left(\mu \neq \frac{4(b-c)(a-c)}{(b-a)^2}\right)$ *under the same condition in* (30).

For the sake of this section and the next, the following standard notation may be used to denote a product of many Gamma functions:

$$\frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p)}{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_q)} = \Gamma\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right] \quad (p, q \in \mathbb{N}_0). \tag{34}$$

Among the many special instances of integral formulae discussed in the preceding section, we have chosen to illustrate just a few of the special cases in Corollary 1.

Example 1.

$$\begin{aligned}
&\int_0^1 x^{a-1}(1-x)^{5-6a}\left(1-\frac{x}{3}\right)^{2a-1}\left(1-\frac{x}{4}\right)^{1-3a} \\
&\qquad \times \left(1 - \frac{9(2-\sqrt{3})}{4}\,x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right)^{a-1} \mathrm{d}x \\
&= \frac{2}{\sqrt{\pi}\,3^{\frac{a}{2}}}\,\Gamma\left[\begin{matrix} \frac{4}{3}, a, 2-3a, \frac{3}{2}-a \\ 2-2a, \frac{4}{3}-a \end{matrix}\right] \quad \left(0 < \Re(a) < \frac{2}{3}\right),
\end{aligned} \tag{35}$$

where [11] (p. 495, Entry 7.3.9-25) is used;

Example 2.

$$\begin{aligned}&\int_0^1 x^{a-1}(1-x)^{2b-1}\left(1-\frac{x}{3}\right)^{2a-1}\left(1-\frac{x}{4}\right)^{b-1}\\&\quad\times {}_2F_1\left[\begin{matrix}\dfrac{a+b}{2},\ \dfrac{a+b+1}{2}\,;\\ c\,;\end{matrix}\ 9\,x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]\mathrm{d}x\\&=\left(\frac{2}{3}\right)^{2a}\Gamma\left[\begin{matrix}a,\ b,\ c,\ c-a-b\\ a+b,\ c-a,\ c-b\end{matrix}\right]\\&\qquad\left(\min\{\Re(a),\ \Re(b)\}>0,\ \Re(c-a-b)>0;\ c\in\mathbb{C}\setminus\mathbb{Z}_0^-\right),\end{aligned}\tag{36}$$

where (4) is employed;

Example 3.

$$\begin{aligned}&\int_0^1 x^{a-1}(1-x)^{2b-1}\left(1-\frac{x}{3}\right)^{2a-1}\left(1-\frac{x}{4}\right)^{b-1}\\&\quad\times {}_2F_1\left[\begin{matrix}\dfrac{a+b}{2},\ \dfrac{a+b+1}{2}\,;\\ 1+a-b\,;\end{matrix}\ -9\,x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]\mathrm{d}x\\&=\left(\frac{2}{3}\right)^{2a}\Gamma\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ b,\ 1+a-b\\ 1+a,\ a+b,\ 1+\frac{1}{2}a-b\end{matrix}\right]\\&\qquad\left(\Re(a)>0,\ 0<\Re(b)<1;\ 1+a-b\in\mathbb{C}\setminus\mathbb{Z}_0^-\right),\end{aligned}\tag{37}$$

where Kummer's theorem for ${}_2F_1(-1)$ is used (see, e.g., [1], p. 42, Theorem 26);

Example 4.

$$\begin{aligned}&\int_0^1 x^{a-1}(1-x)^{2b-1}\left(1-\frac{x}{3}\right)^{2a-1}\left(1-\frac{x}{4}\right)^{b-1}\\&\quad\times {}_2F_1\left[\begin{matrix}\dfrac{a+b}{2},\ \dfrac{a+b+1}{2}\,;\\ \dfrac{a+b+1-m}{2}\,;\end{matrix}\ \frac{9}{2}\,x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]\mathrm{d}x\\&=\frac{2^{2a+b-1}}{3^{2a}}\Gamma\left[\begin{matrix}a,\ \frac{a+b+1-m}{2}\\ a+b\end{matrix}\right]\sum_{k=0}^{m}\binom{m}{k}\Gamma\left[\begin{matrix}\frac{b+k}{2}\\ \frac{1+a+k-m}{2}\end{matrix}\right]\\&\qquad\left(\min\{\Re(a),\ \Re(b)\}>0,\ m\in\mathbb{N}\right),\end{aligned}\tag{38}$$

where [11] (p. 491, Entry 7.3.7-2) is used;

Example 5.

$$\begin{aligned}&\int_0^1 x^{3a-3}(1-x)^{2a-1}\left(1-\frac{x}{3}\right)^{6a-5}\left(1-\frac{x}{4}\right)^{a-1}\\&\quad\times {}_2F_1\left[\begin{matrix}b,\ 2a-\dfrac{1}{2}\,;\\ 3a-b-1\,;\end{matrix}\ 9\,x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]\mathrm{d}x\\&=\left(\frac{2}{3}\right)^{6a-4}\Gamma\left[\begin{matrix}a,\ \frac{3}{2}a,\ 2a-1,\ 3a-2,\ 3a-b-1,\ \frac{1}{2}a-b\\ \frac{1}{2}a,\ 3a-1,\ 4a-2,\ 2a-b-1,\ \frac{3}{2}a-b\end{matrix}\right]\\&\qquad\left(\Re(a)>\frac{2}{3},\ \Re(a-2b)>0\right),\end{aligned}\tag{39}$$

where Dixon's theorem for ${}_3F_2(1)$ is employed (see, e.g., [3], p. 13);

Example 6.

$$\begin{aligned}&\int_0^1 x^{a-1}(1-x)^{a+1}\left(1-\frac{x}{3}\right)^{2a-1}\left(1-\frac{x}{4}\right)^{\frac{1}{2}a+\frac{1}{2}}\\&\times {}_3F_2\left[\begin{matrix}\frac{1}{2}+\frac{1}{4}a,\, 1+\frac{3}{4}a,\, b;\\ \frac{1}{2}a,\, 1+a-b;\end{matrix}\; 9\,x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]\mathrm{d}x\\&=\left(\frac{2}{3}\right)^{2a}\Gamma\left[\begin{matrix}a,\, 1+\frac{1}{2}a,\, \frac{1}{2}+\frac{1}{4}a,\, \frac{1}{2}+\frac{3}{4}a,\, 1+a-b,\, \frac{1}{4}a-b\\ \frac{1}{4}a,\, 1+\frac{3}{2}a,\, 1+a,\, \frac{1}{2}+\frac{1}{2}a-b,\, \frac{1}{2}+\frac{3}{4}a-b\end{matrix}\right]\\&\qquad(\Re(a)>0,\ \Re(a-4b)>0),\end{aligned}\tag{40}$$

where a summation formula for ${}_4F_3(1)$ is used (see, e.g., [12], p. 245, Entry (III.22)).

5. Concluding Remarks

As in previous section, by choosing to employ such a remarkably large number of summation formulas for ${}_{p+1}F_p$ with various arguments (see, e.g., [11]), more particular integral formulas for those in Corollaries 1–3 can be provided. For example,

$$\begin{aligned}&\int_0^1\int_0^1 y^a(1-x)^{a-1}(1-y)^{b-1}\,(1-xy)^{1-a-b}\\&\times\left(1-2\,y(1-x)(1-y)(1-xy)^{-2}\right)^{-\frac{a+b+1}{2}}\mathrm{d}x\mathrm{d}y\\&=\sqrt{\pi}\left\{\Gamma\left[\begin{matrix}a,\, b,\, \frac{a+b}{2}\\ a+b,\, \frac{b}{2},\, \frac{a+1}{2}\end{matrix}\right]+\Gamma\left[\begin{matrix}a,\, b,\, \frac{a+b}{2}\\ a+b,\, \frac{a}{2},\, \frac{b+1}{2}\end{matrix}\right]\right\}\\&\qquad(\min\{\Re(a),\, \Re(b)\}),\end{aligned}\tag{41}$$

where [11] (p. 491, Entry 7.3.7-4) is employed.

A variety of elementary functions and classical functions such as Legendre functions of the first and second kinds, Jacobi polynomials, and the incomplete Beta function (see, e.g., [1,23]) are expressed in terms of ${}_2F_1$. In this connection, for example, from (28), we obtain

$$\begin{aligned}&\int_0^1 x^{a-1}(1-x)^{2b-1}\left(1-\frac{x}{3}\right)^{2a-1}\left(1-\frac{x}{4}\right)^{b-1}\\&\times P_n^{(\alpha,\beta)}\left(18\,x(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)-1\right)\mathrm{d}x\\&=\frac{(-1)^n(1+\beta)_n}{n!}\left(\frac{2}{3}\right)^{2a}\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,{}_4F_3\left[\begin{matrix}a,\, b,\, -n,\, 1+\alpha+\beta+n;\\ \frac{a+b}{2},\, \frac{a+b+1}{2},\, 1+\beta;\end{matrix}\; 1\right],\end{aligned}\tag{42}$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials (see, e.g., [1], p. 254), $n \in \mathbb{N}_0$ and $\min\{\Re(a),\, \Re(b)\} > 0$.

By using the numerous other Beta function formulas including the ones in Section 2, many different integral formulas of the similar type presented in Section 3 may also be established.

Further diverse generalizations of the integral formulas in Section 3 may be established by replacing the integrand factor ${}_{p+1}F_p$ with more generalized functions such as the ${}_pF_q$ (see, e.g., [1], p. 104), the Fox–Wright function ${}_p\Psi_q$ (see, e.g., [24], p. 21), MacRobert's E-function (see, e.g., [8], pp. 203–206; for the similar kind integral formulas presented here, see [21]), the Meijer's G-function (see, e.g., [8], pp. 206–222), the H-function (see, e.g., [25]; see also [23], pp. 49–51), the I-function [26], the $\overline{H}$-function (see, e.g., [27,28]), the Aleph ($\aleph$)-function (see [29,30]).

Question: Is it feasible to extend the work in [31] in the analogous way that the integral formulae established in this article have been generalized?

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