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Some New James Type Geometric Constants in Banach Spaces

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Abstract: We will introduce four new geometric constants closely related to the James constant $J(X)$, which have symmetric structure, along with a discussion on the relationships among them and some other well-known geometric constants via several inequalities, together with the calculation of several values on some specific spaces. In addition, we will characterize geometric properties of $J_1(X)$, such as uniform non-squareness and uniformly normal structure.

Keywords: Banach spaces; geometric constants; normal structure

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1. Introduction and Preliminaries

Recently a number of geometric constants, which play a significant role in the theory of Banach space geometry, have been widely investigated. One of the well-known geometric constants is the James constant $J(X)$ proposed by Gao and Lau [1,2], which is defined as follows:

$$\begin{aligned} J(X) &= \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\} \\ &= \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in B_X\}. \end{aligned}$$

The relationship between it and other geometric constants, and its significant geometric properties such as normal structure in the context of fixed point property have been extensively discussed in [1–7].

It is noteworthy that a Banach space X has the fixed point property for nonexpansive mappings if each nonexpansive self-mapping of each non-empty bounded closed convex subset of X has a fixed point. When it comes to the non-empty weakly compact convex subset of X with normal structure, it is found that X has the weak fixed point property for nonexpansive mappings. It has been shown that further studies of uniform non-squareness are very useful in the description of a fixed point property. Particularly, García-Falset et al. obtained an important generalization of Browder–Göhde and Kirk theorems for the existence of fixed points of a nonexpansive mapping, i.e., every uniformly non-square Banach space has the fixed point property [8]. For more details of applications of the fixed point property, we recommand the references [9–12].

Recall that a classical constant $A_2(X)$ of a Banach space X, which is defined by

$$A_2(X) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} : x, y \in X, \|x\| = \|y\| = 1\right\},$$

has been intensively investigated by Baroni et al. [13]. By extension of the domain from $\|x\| = \|y\| = 1$ to $\|x\|^2 + \|y\|^2 = 2$, Takahashi and Kato have discussed a new constant $A(X)$ in relation with $A_2(X)$ and other geometric constants such as the James constant $J(X)$ and von Neumann–Jordan constant $C_{NJ}(X)$. From this germ of the idea emerged a sequence of the strengthened and improved relationships among the geometric constants. Furthermore, the characterization of uniform non-squareness has also been shown by means of the aforementioned discussion. For readers who are interested in pursuing more introduction and theoretical results of this constant, we recommend reference [14].

Motivated by the characterizations of the James constant $J(X)$ with its prominent properties due to its symmetric structure, we will introduce four new James type constants $J_1(X)$, $J_2(X)$, $J_3(X)$, and $J_4(X)$, which are also endowed with symmetric structure. In Section 2, we mainly focus on a new James type constant $J_1(X)$, which is derived from the original James constant $J(X)$ combined with the notion of metric. We then bring out several relationships between it and some other well-known geometric constants such as the James constant $J(X)$, which will be subsequently employed to explore the connection between it and its dual, together with the estimation of the upper and lower bounds of it. Moreover, a few connections between its value and some geometrical properties of the space, such as uniform non-squareness and uniformly normal structure, will be shown by inequalities. In Section 3, we will introduce another new James type constant $J_2(X)$ combined with the notion of isosceles orthogonality, whereby the relationship between it and the James constant $J(X)$ and the connection between it and its dual will be different. By considering the extension of its domain from the unit sphere to the whole Banach space, we define a new constant $J_3(X)$ and therefore illustrate the difference between $J_2(X)$ and $J_3(X)$ by giving an example on an inner product space in terms of their values. Inspired by the characterizations of the constant $A(X)$, we will discuss the last James type constant $J_4(X)$ in Section 4, and obtain several results mainly by conducting a comparison between it and James constant $J(X)$, which will show their similarities and differences on some specific Banach spaces.

Throughout the paper, we consider the real Banach space X with $\dim X \geq 2$ and the infinite-dimensional Banach space X, and use S_X and B_X to symbolize the unit sphere and closed unit ball of X, respectively.

We recall several geometric properties closely related to the geometric structure of Banach space as follows.

Definition 1. *A Banach space X is called uniformly non-square if there exists $\delta \in (0,1)$ such that for any $x, y \in S_X$, we have either $\frac{\|x+y\|}{2} \leq 1-\delta$ or $\frac{\|x-y\|}{2} \leq 1-\delta$.*

Definition 2 ([15])**.** *We define $\mathrm{diam}A = \sup\{\|x-y\| : x, y \in A\}$ to represent diameter of A and $r(A) = \inf\{\sup\{\|x-y\|\} : y \in A\}$ is called Chebyshev radius of A. A Banach space X has normal structure provided*

$$r(A) < \mathrm{diam}A$$

for every bounded closed convex subset A of X with $\mathrm{diam}A > 0$. A Banach space X is said to have uniform normal structure if

$$\inf\left\{\frac{\mathrm{diam}A}{r(A)}\right\} > 1,$$

with $\mathrm{diam}A > 0$.

In order to study the property of James type constants that appear in the paper, we also recall the following modulus of smoothness of X [16].

Definition 3. *Let X be a Banach space, then the modulus of smoothness $\rho_X(t)$ is defined by*

$$\rho_X(t) = \sup\left\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in S_X\right\}, t \geq 0.$$

The applications of aforementioned constants $J(X)$, $A_2(X)$, $A(X)$, and $\rho_X(t)$ can be presented as simply and plainly as possible in the following terms [5,13,14].

(i) For any Banach space,

$$\sqrt{2} \leq J(X) \leq A_2(X) \leq A(X) \leq 2.$$

(ii) For any Banach space,

$$\rho_X(1) \le 2\left(1 - \frac{1}{J(X)}\right),$$

and

$$\begin{aligned} A(X) &= \max_{0\le t\le 1} \frac{\sqrt{2}(1+\rho_X(t))}{\sqrt{1+t^2}} \\ &\le \sqrt{2}\sqrt{1+(A_2(X)-1)^2} \\ &\le \sqrt{2}\sqrt{1+4\left(1-\frac{1}{J(X)}\right)^2}. \\ &\le (1+\sqrt{J(X)-1})^2 \end{aligned}$$

(iii) If X is a Hilbert space, then $J(X) = \sqrt{2}$, resp. $A_2(X) = \sqrt{2}$, resp. $A(X) = \sqrt{2}$, resp. $\rho_X(t) = \sqrt{1+t^2} - 1$.

(iv) X is uniformly non-square if and only if one of the following conditions is true: (a) $J(X) < 2$, (b) $A_2(X) < 2$, (c) $A(X) < 2$, (d) $\rho_X(1) < 1$.

(v) Let X be a Banach space. Then

$$2J(X) - 2 \le J(X^*) \le \frac{1}{2}J(X) + 1,$$

resp.

$$A_2(X) = A_2(X^*),$$

resp.

$$\sqrt{\frac{A(X)^2}{2} - 1} + 1 \le A(X^*) \le \sqrt{2}\sqrt{1+(A(X)-1)^2},$$

resp.

$$\rho_X(1) = \rho_{X^*}(1).$$

The following lemma will be employed in the proofs of this paper.

Lemma 1. *$f(t) = \frac{t}{1+t}$ is continuously increasing on $(0, +\infty)$.*

2. James Type Constant Related to Metric

For the sake of revealing the origin of $J_1(X)$, we first bring up the following notion of metric. It is easy to see $d(x,y) = \|x-y\|$ is a metric on X, and then we can easily transform $J(X)$ into following term.

$$J(X) = \sup\{\min\{d(x,y), d(x,-y)\} : x, y \in S_X\}.$$

Let $\rho(x,y) = \frac{\|x-y\|}{1+\|x-y\|}$. Then $\rho(x,y)$ is a metric on X. Thus we consider the following symmetric James type constant:

$$J_1(X) = \sup\left\{\min\left\{\frac{\|x+y\|}{1+\|x+y\|}, \frac{\|x-y\|}{1+\|x-y\|}\right\} : \text{for all } x, y \in S_X\right\}.$$

Theorem 1. *Let X be a Banach space. Then*

$$\frac{1}{3}J(X) \le J_1(X) \le J(X).$$

Proof. Since $x, y \in S_X$,

$$\frac{\|x+y\|}{1+\|x+y\|} \geq \frac{\|x+y\|}{1+\|x\|+\|y\|} = \frac{\|x+y\|}{3}.$$

Similarly, we can deduce that

$$\frac{\|x-y\|}{1+\|x-y\|} \geq \frac{\|x-y\|}{1+\|x\|+\|y\|} = \frac{\|x-y\|}{3}.$$

Therefore,

$$\min\left\{\frac{\|x+y\|}{1+\|x+y\|}, \frac{\|x-y\|}{1+\|x-y\|}\right\} \geq \frac{1}{3}\min\{\|x+y\|, \|x-y\|\},$$

which implies that

$$J_1(X) \geq \frac{1}{3}J(X).$$

On the other hand, since

$$\frac{\|x+y\|}{1+\|x+y\|} \leq \|x+y\|,$$

and

$$\frac{\|x-y\|}{1+\|x-y\|} \leq \|x-y\|,$$

therefore,

$$\min\left\{\frac{\|x+y\|}{1+\|x+y\|}, \frac{\|x-y\|}{1+\|x-y\|}\right\} \leq \min\{\|x+y\|, \|x-y\|\},$$

i.e., $J_1(X) \leq J(X)$. □

Proposition 1. *Let X be a Banach space. Then*

$$J_1(X) \leq \frac{J(X)}{1+J(X)}.$$

Proof. For any $x, y \in S_X$, since

$$\frac{\|x+y\|}{1+\|x+y\|} = 1 - \frac{1}{1+\|x+y\|} \geq \min\left\{1 - \frac{1}{1+\|x+y\|}, 1 - \frac{1}{1+\|x-y\|}\right\},$$

and

$$\frac{\|x-y\|}{1+\|x-y\|} = 1 - \frac{1}{1+\|x-y\|} \geq \min\left\{1 - \frac{1}{1+\|x+y\|}, 1 - \frac{1}{1+\|x-y\|}\right\},$$

therefore,

$$1 - \frac{1}{1+\min\{\|x+y\|, \|x-y\|\}} \geq \min\left\{1 - \frac{1}{1+\|x+y\|}, 1 - \frac{1}{1+\|x-y\|}\right\},$$

which implies that

$$\begin{aligned}
\frac{J(X)}{1+J(X)} &= 1-\frac{1}{1+J(X)} \\
&= 1-\frac{1}{1+\sup\{\min\{\|x+y\|,\|x-y\|\}: x,y\in S_X\}} \\
&= 1-\inf\left\{\frac{1}{1+\min\{\|x+y\|,\|x-y\|\}}: x,y\in S_X\right\} \\
&= \sup\left\{1-\frac{1}{1+\min\{\|x+y\|,\|x-y\|\}}: x,y\in S_X\right\} \\
&\geq \sup\left\{\min\left\{1-\frac{1}{1+\|x+y\|}, 1-\frac{1}{1+\|x-y\|}\right\}: x,y\in S_X\right\} \\
&= J_1(X),
\end{aligned}$$

namely, $\frac{J(X)}{1+J(X)} \geq J_1(X)$. □

Example 1. *Let X be the space ℓ_p.*

By utilizing Proposition 1 and results from [17],

$$J(X) = \begin{cases} 2^{\frac{1}{p}} & \text{if } 1 \leq p \leq 2, \\ 2^{1-\frac{1}{p}} & \text{if } p \geq 2. \end{cases}$$

we can easily obtain

$$J_1(X) \leq \begin{cases} \frac{2^{\frac{1}{p}}}{1+2^{\frac{1}{p}}} & \text{if } 1 \leq p \leq 2, \\ \frac{2}{2+2^{\frac{1}{p}}} & \text{if } p \geq 2. \end{cases}$$

If $p \geq 2$, let $x = 2^{-\frac{1}{p}}(1,1,0,\ldots), y = 2^{-\frac{1}{p}}(1,-1,0,\ldots) \in S_{\ell_p}$. Then

$$\|x+y\| = 2^{1-\frac{1}{p}},\ \|x-y\| = 2^{1-\frac{1}{p}},$$

$$J_1(X) \geq \min\left\{\frac{\|x+y\|}{1+\|x+y\|}, \frac{\|x-y\|}{1+\|x-y\|}\right\} = \frac{2}{2+2^{\frac{1}{p}}}.$$

If $1 \leq p \leq 2$, let $x = (1,0,0,\ldots), y = (0,1,0,\ldots) \in S_{\ell_p}$. Then

$$\|x+y\| = 2^{\frac{1}{p}},\ \|x-y\| = 2^{\frac{1}{p}},$$

$$J_1(X) \geq \min\left\{\frac{\|x+y\|}{1+\|x+y\|}, \frac{\|x-y\|}{1+\|x-y\|}\right\} = \frac{2^{\frac{1}{p}}}{1+2^{\frac{1}{p}}}.$$

Therefore,

$$J_1(X) = \begin{cases} \frac{2^{\frac{1}{p}}}{1+2^{\frac{1}{p}}} & \text{if } 1 \leq p \leq 2, \\ \frac{2}{2+2^{\frac{1}{p}}} & \text{if } p \geq 2. \end{cases}$$

Proposition 2. *Let X be a non-trivial Banach space. Then*

$$2-\sqrt{2} \leq J_1(X) \leq \frac{2}{3}.$$

Proof. By utilizing result from James constant $J(X)$ [1],

$$\sqrt{2} \leq J(X) \leq 2.$$

By employing Lemma 1 and Proposition 1, we obtain

$$J_1(X) \leq \frac{2}{3}.$$

On the other hand, by utilizing Theorem 10 in [18], we can deduce that there exist $x_0, y_0 \in S_X$ such that

$$\|x_0 + y_0\| = \|x_0 - y_0\| = \sqrt{2},$$

which shows

$$J_1(X) \geq \frac{\sqrt{2}}{1+\sqrt{2}} = 2 - \sqrt{2}.$$

This completes the proof. □

Theorem 2. *Let X be Hilbert space. Then $J_1(X) = 2 - \sqrt{2}$.*

Proof. Assume that X is Hilbert space, then

$$\|x+y\|^2 + \|x-y\|^2 = 4.$$

For any $x, y \in S_X$, let $\|x+y\| \geq \|x-y\|$. By utilizing Lemma 1, we have

$$\frac{\|x+y\|}{1+\|x+y\|} \geq \frac{\|x-y\|}{1+\|x-y\|}.$$

In addition, since $\|x+y\| \geq \|x-y\|$, then $4 = \|x+y\|^2 + \|x-y\|^2 \geq 2\|x-y\|^2$, which implies that $\|x-y\| \leq \sqrt{2}$.

Thus

$$J_1(X) = \sup\left\{\frac{\|x-y\|}{1+\|x-y\|} : x, y \in S_X\right\} \leq \frac{\sqrt{2}}{1+\sqrt{2}} = 2 - \sqrt{2},$$

i.e., $J_1(X) = 2 - \sqrt{2}$. □

In order to reveal the relationship between $J_1(X)$ and the uniform smoothness, the aforementioned constant $\rho_X(t)$ will be employed in the following theorems and corollaries.

Theorem 3. *Let X be a Banach space. Then*

$$\rho_X(t) \leq \frac{[2 - J_1(X)]\max\{1, t\} - 1 + J_1(X)}{2 - 2J_1(X)}.$$

Proof. Let $x, y \in S_X$. Then

$$\|x+ty\| = \left\|\frac{1+t}{2} \cdot (x+y) + \frac{1-t}{2} \cdot (x-y)\right\| \leq \frac{1+t}{2}\|x+y\| + \frac{|1-t|}{2}\|x-y\|,$$

and

$$\|x-ty\| = \left\|\frac{1+t}{2} \cdot (x-y) + \frac{1-t}{2} \cdot (x+y)\right\| \leq \frac{1+t}{2}\|x-y\| + \frac{|1-t|}{2}\|x+y\|.$$

Thus

$$\begin{aligned}\|x+ty\|+\|x-ty\| &\leq \frac{1+t+|1-t|}{2}\cdot(\|x+y\|+\|x-y\|)\\ &= \max\{1,t\}(\|x+y\|+\|x-y\|)\\ &\leq \max\{1,t\}(2+\min\{\|x+y\|,\|x-y\|\}),\end{aligned}$$

which implies that

$$\frac{\|x+ty\|+\|x-ty\|}{\max\{1,t\}}-2 \leq \min\{\|x+y\|,\|x-y\|\}.$$

Note that

$$J_1(X) \geq \frac{\min\{\|x+y\|,\|x-y\|\}}{1+\min\{\|x+y\|,\|x-y\|\}} = 1-\frac{1}{1+\min\{\|x+y\|,\|x-y\|\}},$$

we have

$$\min\{\|x+y\|,\|x-y\|\} \leq \frac{J_1(X)}{1-J_1(X)}.$$

Therefore,

$$\frac{\|x+ty\|+\|x-ty\|}{\max\{1,t\}}-2 \leq \frac{J_1(X)}{1-J_1(X)},$$

i.e., $\frac{2\rho_X(t)+1}{\max\{1,t\}}-2 \leq \frac{J_1(X)}{1-J_1(X)}$.

Hence

$$\begin{aligned}2\rho_X(t) &\leq \max\{1,t\}\left(\frac{J_1(X)}{1-J_1(X)}+2\right)-1\\ &= \frac{[2-J_1(X)]\max\{1,t\}-1+J_1(X)}{1-J_1(X)}.\end{aligned}$$

This completes the proof. □

Corollary 1. *Let X be a Banach space. Then*

$$J_1(X) \leq \frac{\rho_X(1)+1}{2+\rho_X(1)}.$$

Proof. For any $x,y \in S_X$, we can deduce that

$$J(X) \leq \rho_X(1)+1.$$

By utilizing Lemma 1 and Proposition 1, we obtain

$$J_1(X) \leq \frac{\rho_X(1)+1}{2+\rho_X(1)}.$$

This completes the proof. □

Corollary 2. *Let X be a Banach space. Then*

$$\rho_X(1) \leq 2\left\{1-\frac{3}{J_1(X)}\right\}.$$

Proof. Since X is a Banach space, and by [6], we have

$$\rho_X(1) \leq 2\left\{1-\frac{1}{J(X)}\right\}.$$

By employing Theorem 1, we obtain

$$\rho_X(1) \leq 2\left\{1 - \frac{1}{J(X)}\right\} \leq 2\left\{1 - \frac{3}{J_1(X)}\right\}.$$

This completes the proof. □

Next, we will consider the dual space X^*, and manage to bring out the relationship between $J_1(X)$ and $J_1(X^*)$ by utilizing the aforementioned theorem.

Theorem 4. *Let X be a Banach space. Then*

$$\frac{1}{3}(2J_1(X) - 2) \leq J_1(X^*) \leq \frac{3}{2}J_1(X) + 1.$$

Proof. By [5], we have

$$2J(X) - 2 \leq J(X^*) \leq \frac{1}{2}J(X) + 1.$$

By utilizing Theorem 1, we have

$$\frac{1}{3}J(X) \leq J_1(X) \leq J(X).$$

By employing Lemma 1, we have

$$\begin{aligned} J_1(X^*) &\geq \frac{1}{3}J(X^*) \\ &\geq \frac{1}{3}(2J(X) - 2) \\ &\geq \frac{1}{3}(2J_1(X) - 2), \end{aligned}$$

and

$$\begin{aligned} J_1(X^*) &\leq J(X^*) \\ &\leq \frac{1}{2}J(X) + 1 \\ &\leq \frac{3}{2}J_1(X) + 1. \end{aligned}$$

Therefore,

$$\frac{1}{3}(2J_1(X) - 2) \leq J_1(X^*) \leq \frac{3}{2}J_1(X) + 1.$$

This completes the proof. □

Theorem 5. *Let X be a non-trivial Banach space. Then $J_1(X) < \frac{2}{3}$ if and only if X is uniformly non-square.*

Proof. According to the definition of uniformly non-square, there exists a $\delta \in (0, 1)$ such that for any $x, y \in S_X$, either $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$ or $\left\|\frac{x-y}{2}\right\| \leq 1 - \delta$. We first consider the case $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$. Then we have

$$\frac{\|x+y\|}{1 + \|x+y\|} = 1 - \frac{1}{1 + \|x+y\|} \leq 1 - \frac{1}{3 - 2\delta} < \frac{2}{3}.$$

In the case $\left\|\frac{x-y}{2}\right\| \leq 1-\delta$, by utilizing the same method above, we can obtain that

$$\frac{\|x-y\|}{1+\|x-y\|} \leq 1-\frac{1}{3-2\delta} < \frac{2}{3}.$$

Furthermore, if $J_1(X) < \frac{2}{3}$, by applying Theorem 1, $J(X) < 2$, then X is uniformly non-square. This completes the proof. □

Next, we consider the uniform normal structure of $J_1(X)$. The concept of normal structure plays an important role in Banach space geometry and fixed point theory. It was proved by Kirk [19] that every reflexive Banach space with normal structure has the fixed point property. We recall a lemma from Dhompongsa et al. [3] as follows.

Lemma 2. *Let X be a Banach space with $J(X) < \frac{1+\sqrt{5}}{2}$. Then X has uniformly normal structure.*

Theorem 6. *Let X be a Banach space. If $J_1(X) < \frac{\sqrt{5}+1}{6}$, then X has uniformly normal structure.*

Proof. Since $J_1(X) < \frac{\sqrt{5}+1}{6}$, by employing Theorem 1, we have

$$J(X) \leq 3J_1(X),$$

hence

$$J(X) < \frac{1+\sqrt{5}}{2}.$$

By utilizing Lemma 2, we obtain that X has uniformly normal structure. □

3. Several Inequalities Related to New Constant $J_2(X)$

In this section, we continue to discuss the James type constant $J_2(X)$, which is different from the aforesaid constant $J_1(X)$ when it comes to the conditions of vectors x and y. It is defined as follows.

$$J_2(X) = \sup\left\{\frac{\|x+y\|}{1+\|x+y\|} : x, y \in S_X, x \perp_I y\right\}.$$

Theorem 7. *If X is an inner product space, then*

$$J_2(X) = 2-\sqrt{2}.$$

Proof. For any $x, y \in S_X$ satisfying $x \perp_I y$, by parallelogram law, we have

$$2\|x+y\|^2 = \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 = 4,$$

thus

$$\|x+y\| = \|x-y\| = \sqrt{2}.$$

Therefore,

$$\frac{\|x+y\|}{1+\|x+y\|} = 2-\sqrt{2},$$

that is, $J_2(X) = 2-\sqrt{2}$. □

Theorem 8. *Let X be a Banach space. Then*

$$\frac{1}{3}J(X) \leq J_2(X) \leq \frac{1}{2}J(X).$$

Proof. Since $x, y \in S_X$, $x \perp_I y$,

$$\frac{\|x+y\|}{1+\|x+y\|} \geq \frac{\|x+y\|}{1+\|x\|+\|y\|} = \frac{\|x+y\|}{3}.$$

Therefore,

$$J_2(X) \geq \frac{1}{3}J(X).$$

However, since $x \perp_I y$, thus

$$2\|x+y\| = \|x+y\| + \|x-y\| \geq \|2x\| = 2,$$

that is, $\|x+y\| \geq 1$.

Hence

$$1 + \|x+y\| = 1 + \|x-y\| \geq 2,$$

and then

$$\frac{\|x+y\|}{1+\|x+y\|} \leq \frac{\|x+y\|}{2}.$$

Therefore,

$$J_2(X) \leq \frac{1}{2}J(X).$$

This completes the proof. □

For dual space X^*, we will bring out the relationship between $J_2(X)$ and $J_2(X^*)$ by utilizing the aforementioned theorem.

Theorem 9. *Let X be a Banach space. Then*

$$\frac{4}{3}J_2(X) - \frac{2}{3} \leq J_2(X^*) \leq \frac{3}{4}J_2(X) + \frac{1}{2}.$$

Proof. By [5], we have

$$2J(X) - 2 \leq J(X^*) \leq \frac{1}{2}J(X) + 1.$$

For any $x, y \in S_X$, $x \perp_I y$, by utilizing Theorem 8, we have

$$\begin{aligned} J_2(X^*) &\leq \frac{1}{2}J(X^*) \\ &\leq \frac{1}{4}J(X) + \frac{1}{2} \\ &\leq \frac{3}{4}J_2(X) + \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} J_2(X^*) &\geq \frac{1}{3}J(X^*) \\ &\geq \frac{1}{3}(2(J(X) - 1)) \\ &= \frac{2}{3}J(X) - \frac{2}{3} \\ &\geq \frac{4}{3}J_2(X^*) - \frac{2}{3}. \end{aligned}$$

This completes the proof. □

Note: We consider the following case:

$$J_3(X) = \sup\left\{ \frac{\|x+y\|}{1+\|x+y\|} \ : \ x, y \in X, x \perp_I y \right\}.$$

It is easy to prove that for any Banach space X, we have $J_3(X) = 1$.
In fact, for any $x \in S_X, y \in B_X$ satisfying $x \perp_I y$, by [20], we have

$$\|x+y\| \geq 2(\sqrt{2}-1)\|x\| = 2(\sqrt{2}-1).$$

Now take $nx \in nS_X$, $ny \in nB_X$, of course they satisfy $nS_X, nB_X \subset X$ and $nx \perp_I ny$, then we have

$$\|nx+ny\| \geq 2(\sqrt{2}-1)\|nx\| = 2(\sqrt{2}-1)n.$$

Therefore,

$$J_3(X) \geq \frac{\|nx+ny\|}{1+\|nx+ny\|} = \frac{2(\sqrt{2}-1)n}{1+2(\sqrt{2}-1)n}.$$

Since n can be arbitrarily large, then

$$J_3(X) \geq \lim_{n\to+\infty} \frac{\|nx+ny\|}{1+\|nx+ny\|} = 1,$$

and $J_3(X) \leq 1$.
Therefore,

$$J_3(X) = 1.$$

Example 2. *Let H be any inner product space. By utilizing Theorem 7, we have*

$$J_2(H) = 2-\sqrt{2}.$$

However, by employing the aforementioned Note, $J_3(X) = 1$ holds for any Banach space, hence

$$J_3(H) = 1.$$

Therefore,

$$J_3(H) \neq J_2(H).$$

4. James Type Constant $J_4(X)$

In this section, we will discuss the last James type constant $J_4(X)$ by utilizing several heuristic ideas from the investigation of the constant $A(X)$ proposed by Takahashi and Kato [14]. By considering the extension of the domain of the James constant from $\|x\| = \|y\| = 1$ to $\|x\|^2 + \|y\|^2 = 2$, we can define the symmetric constant $J_4(X)$ as follows.

$$J_4(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} \ : \ \|x\|^2 + \|y\|^2 = 2\}.$$

Obviously, for all Banach space X,

$$\sqrt{2} \leq J(X) \leq J_4(X) \leq A(X) \leq 2.$$

Theorem 10. *Let X be a Banach space. Then*

$$2A(X) - 2 \leq J_4(X) \leq A(X).$$

Proof. For any $x, y \in X$ satisfying $\|x\|^2 + \|y\|^2 = 2$, we have

$$\|x+y\|^2 \leq 2(\|x\|^2 + \|y\|^2) = 4,$$

and

$$\|x-y\|^2 \leq 2(\|x\|^2 + \|y\|^2) = 4,$$

hence

$$\|x+y\| \leq 2, \ \|x-y\| \leq 2.$$

Therefore,

$$\begin{aligned}\frac{\|x+y\| + \|x-y\|}{2} &\leq \frac{1}{2}(\min\{\|x+y\|, \|x-y\|\} + 2) \\ &= \frac{1}{2}\min\{\|x+y\|, \|x-y\|\} + 1,\end{aligned}$$

which implies that $A(X) \leq \frac{1}{2}J_4(X) + 1$.

To prove the right-hand side of the inequalities, we use the following fact that

$$\min\{\|x+y\|, \|x-y\|\} \leq \|x+y\|, \ \min\{\|x+y\|, \|x-y\|\} \leq \|x-y\|,$$

hence

$$\min\{\|x+y\|, \|x-y\|\} \leq \frac{\|x+y\| + \|x-y\|}{2},$$

which implies that

$$J_4(X) \leq A(X).$$

This completes the proof. □

Proposition 3. *Let X be a Banach space. Then*

$$J_4(X) = \sup\left\{\min\left\{\frac{\sqrt{2}\|x+ty\|}{\sqrt{1+t^2}}, \frac{\sqrt{2}\|x-ty\|}{\sqrt{1+t^2}}\right\} : \ x, y \in S_X, 0 \leq t \leq 1\right\}.$$

Proof. Let $\|u\|^2 + \|v\|^2 = 2$ and $\|u\| \geq \|v\| > 0$, $u, v \in X$. Then, since $1 \leq \|u\| < \sqrt{2}$, we have $\sqrt{1+t^2}\|u\| = \sqrt{2}$ with some $t \in (0,1]$. Now let $x = \frac{\sqrt{1+t^2}u}{\sqrt{2}}$ and $y = \frac{\sqrt{1+t^2}v}{\sqrt{2}t}$. Then $x, y \in S_X$ and we have

$$\|u+v\| = \frac{\sqrt{2}\|x+ty\|}{\sqrt{1+t^2}},$$

and

$$\|u-v\| = \frac{\sqrt{2}\|x-ty\|}{\sqrt{1+t^2}}.$$

Therefore,

$$J_4(X) \leq \sup\left\{\min\left\{\frac{\sqrt{2}\|x+ty\|}{\sqrt{1+t^2}}, \frac{\sqrt{2}\|x-ty\|}{\sqrt{1+t^2}}\right\} : \ x, y \in S_X, 0 \leq t \leq 1\right\}.$$

Conversely, let $x, y \in S_X$ and $t \in [0,1]$. Let $u = \frac{\sqrt{2}x}{\sqrt{1+t^2}}$ and $v = \frac{\sqrt{2}ty}{\sqrt{1+t^2}}$. Then $\|u\|^2 + \|v\|^2 = 2$, and then the opposite inequality holds. □

Proposition 4. *Let X be a Banach space. Then*

$$J_4(X) \leq \max_{0 \leq \tau \leq 1} \frac{\sqrt{2}(1+\rho_X(\tau))}{\sqrt{1+\tau^2}}.$$

Proof. Since

$$A(X) = \max_{0 \leq \tau \leq 1} \frac{\sqrt{2}(1+\rho_X(\tau))}{\sqrt{1+\tau^2}},$$

we have

$$J_4(X) \le A(X) = \max_{0\le\tau\le 1} \frac{\sqrt{2}(1+\rho_X(\tau))}{\sqrt{1+\tau^2}}.$$

This completes the proof. □

Theorem 11. *Let X be a Banach space. Then*

$$J(X) \le J_4(X) \le \sqrt{2}\sqrt{J(X)}.$$

Proof. For any $x, y \in X$ satisfying $\|x\| = \|y\| = 1$, of course we have$\|x\|^2 + \|y\|^2 = 2$. Then we get $J_4(X) \ge J(X)$ for Banach space X.

However, it is well-known that

$$\frac{\rho_X(\tau)}{\tau} \le \rho_X(1) \ \text{ for all } \tau \in (0,1].$$

By employing Proposition 4, we have

$$J_4(X) \le \frac{\sqrt{2}(1+\rho_X(\tau_0))}{\sqrt{1+\tau_0^2}} \le \frac{\sqrt{2}(1+\rho_X(1)\tau_0)}{\sqrt{1+\tau_0^2}} \le \sqrt{2}\sqrt{1+\rho_X(1)^2},$$

for some $\tau_0 \in (0,1]$. Since $\rho_X(1) \le \sqrt{J(X)-1}$ from [6], then we have

$$J_4(X) \le \sqrt{2}\sqrt{J(X)}.$$

This completes the proof. □

Corollary 3. *Let X be a Banach space. Then*

$$J_4(X) - J(X) \le 2^{\frac{3}{4}} - \sqrt{2},$$

and the equality holds only if $J_4(X) = 2^{\frac{3}{4}}$ and $J(X) = \sqrt{2}$.

Proof. Let $f(t) = \sqrt{2}\sqrt{t} - t$. Then $f(t)$ is strictly increasing on $t \in \left[0, \frac{1}{2}\right]$, and decreasing on $t \in \left(\frac{1}{2}, 2\right]$. Since $\sqrt{2} \le J(X) \le 2$, it follows from the aforementioned inequality that

$$J_4(X) - J(X) \le f(J(X)) \le f(\sqrt{2}) = 2^{\frac{3}{4}} - \sqrt{2}.$$

The latter assertion is easily deduced. □

Corollary 4. *Let X be a Banach space. Then*

$$1 + \left(\frac{J_4(X)^2}{2} - 1\right)^2 \le J_4(X^*) \le \sqrt{2}\sqrt{1+\sqrt{J_4(X)-1}}.$$

Proof. By [7], we have $1 + (J(X)-1)^2 \le J(X^*) \le 1 + \sqrt{J(X)-1}$. Then

$$\begin{aligned} J_4(X^*) &\le \sqrt{2}\sqrt{J(X^*)} \\ &\le \sqrt{2}\sqrt{1+\sqrt{J(X)-1}} \\ &\le \sqrt{2}\sqrt{1+\sqrt{J_4(X)-1}}, \end{aligned}$$

and

$$\begin{aligned} J_4(X^*) &\geq J(X^*) \\ &\geq 1 + (J(X) - 1)^2 \\ &\geq 1 + \left(\frac{J_4(X)^2}{2} - 1 \right)^2. \end{aligned}$$

Therefore we complete the proof. □

Example 3. *Let X be L_p, $1 \leq p < \infty$. Then $J_4(L_p) = 2^{\frac{1}{\min\{p,p'\}}}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

By [14], $J(L_p) = A(L_p) = 2^{\frac{1}{\min\{p,p'\}}}$, along with Theorems 10 and 11, we get

$$J(L_p) \leq J_4(L_p) \leq A(L_p),$$

which implies that

$$J_4(L_p) = 2^{\frac{1}{\min\{p,p'\}}}.$$

Example 4. *Let X be $\mathbb{R}^2$ endowed with $\ell_\infty - \ell_1$ norm*

$$\|x\| = \begin{cases} \|x\|_\infty & \text{if } x_1x_2 \geq 0, \\ \|x\|_1 & \text{if } x_1x_2 \leq 0. \end{cases}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$.

By [14], we know that $J_4(\ell_\infty - \ell_1) \leq A(\ell_\infty - \ell_1) = \frac{\sqrt{10}}{2}$.
Furthermore, let $t = \frac{1}{2}$, $x = (1,1)$, $y = \left(-\frac{1}{2}, \frac{1}{2}\right)$. Obviously, $x, y \in S_{\ell_\infty - \ell_1}$. Then

$$\|x + ty\| = \|x - ty\| = \frac{5}{4}.$$

Therefore,

$$\begin{aligned} J_4(\ell_\infty - \ell_1) &\geq \min\left\{ \frac{\sqrt{2}\|x+ty\|}{\sqrt{1+t^2}}, \frac{\sqrt{2}\|x+ty\|}{\sqrt{1+t^2}} \right\} \\ &= \frac{\sqrt{2} \cdot \frac{5}{4}}{\sqrt{1 + \left(\frac{1}{2}\right)^2}} \\ &= \frac{\sqrt{10}}{2}, \end{aligned}$$

which implies that $J_4(\ell_\infty - \ell_1) = \frac{\sqrt{10}}{2}$. We know that $J(\ell_\infty - \ell_1) = \frac{3}{2}$ from [5]. Then $J_4(\ell_\infty - \ell_1) > J(\ell_\infty - \ell_1)$.

Theorem 12. *Let X be a Banach space. Then X is a Hilbert space if and only if $J_4(X) = \sqrt{2}$.*

Proof. By [14], X is a Hilbert space if and only if $A(X) = \sqrt{2}$, and $J(X) = \sqrt{2}$ by [1]; we can easily obtain $J_4(X) = \sqrt{2}$ by the inequality $J(X) \leq J_4(X) \leq A(X)$. □

Theorem 13. *Let X be a Banach space. Then $J_4(X) < 2$ if and only if X is uniformly non-square.*

Proof. If X is uniformly non-square, by [14], we have $A(X) < 2$, then $J_4(X) \leq A(X) < 2$. Conversely, if $J_4(X) < 2$, then $J(X) < 2$, which implies that X is uniformly non-square. □

5. Conclusions

In this paper, we introduced a new James type constant $J_1(X)$, which combines with the notion of metric. It is of interest to characterize its relationships with a diversity of well-known geometric constants and investigate its geometric properties, such as uniform non-squareness and uniform normal structure. Moreover, we provide a study of its derived forms $J_2(X)$ and $J_3(X)$ with different conditions, thus making a comparison between them in terms of their values of the specific Banach space. Finally, we bring up the last James type constant $J_4(X)$ with the condition $\|x\|^2 + \|y\|^2 = 2$, which can be very intriguing, by conducting a contrast between it and James constant $J(X)$. However, there are still plenty of interesting problems that await discussion. How can all four James type constants be utilized to characterize more geometric properties? Henceforth, more results about James type constants will be presented in future research for the readers who are interested in the theory of geometric constants of Banach space.

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