symmetry

MDPI

*Article*

# New Applications of Fractional Integral for Introducing Subclasses of Analytic Functions

**Alina Alb Lupaş**

Department of Mathematics and Computer Science, University of Oradea, 1 Universitatii Street, 410087 Oradea, Romania; dalb@uoradea.ro or alblupas@gmail.com

**Abstract:** The fractional integral is prolific in giving rise to interesting outcomes when associated with different operators. For the study presented in this paper, the fractional integral is associated with the convolution product of multiplier transformation and the Ruscheweyh derivative. Using the operator obtained as a result of this association and inspired by previously published results obtained with similarly introduced operators, the class of analytic functions $\mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$ is defined and investigated concerning various characteristics such as distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to this class.

**Keywords:** analytic functions; univalent functions; radii of starlikeness and convexity; neighborhood property; multiplier transformation; Ruscheweyh derivative

check for updates

**Citation:** Alb Lupaş, A. New Applications of Fractional Integral for Introducing Subclasses of Analytic Functions. *Symmetry* **2022**, *14*, 419. https://doi.org/10.3390/sym14020419

Academic Editor: Wei-Shih Du

Received: 30 January 2022
Accepted: 18 February 2022
Published: 20 February 2022



## 1. Introduction

The fractional integral was recently investigated in relation with many different functions. Interesting results were obtained when applying the fractional integral to a confluent hypergeometric function [1], in connection with Sălăgean and Ruscheweyh operators [2], related to Bessel functions [3] or for the Mittag–Leffler Confluent Hypergeometric Function [4]. Applications of fractional calculus emerged in many studies related to convexity [5,6] and involving generalized fractional integral operators [7–9].

The applications of the fractional integral involved in the present study are related to a previously introduced operator obtained as a convolution product of a multiplier transformation and a Ruscheweyh derivative. In order to present the original results of the study, known notations and known definitions are used.

Consider $\mathcal{H}(U)$, the class of analytic function in $U = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disc of the complex plane, $\mathcal{H}(a, n)$, the subclass of $\mathcal{H}(U)$ of functions with the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots, z \in U\}$ with $\mathcal{A} = \mathcal{A}_1$.

The Hadamard product (or convolution) of analytic functions in the open unit disc $U$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, denoted by $f * g$, is defined as:

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

The operators used for the present study are the following.

**Definition 1** ([10]). *For $f \in \mathcal{A}$, $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \geq 0$, the multiplier transformation $I(m, \alpha, l) f(z)$ is defined by the following infinite series:*

$$I(m, \alpha, l) f(z) := z + \sum_{k=2}^{\infty} \left( \frac{1 + \alpha(k-1) + l}{1 + l} \right)^m a_k z^k.$$

**Remark 1.** *For $l = 0$, $\alpha \geq 0$, the operator $D_\alpha^m = I(m, \alpha, 0)$ was introduced and studied by Al-Oboudi [11], and was reduced to the Sălăgean differential operator $S^m = I(m, 1, 0)$ [12] for $\alpha = 1$.*

**Definition 2** ([13])**.** *For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the Ruscheweyh derivative $R^n$ is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\ldots \\ (n+1) R^{n+1} f(z) &= z(R^n f(z))' + n R^n f(z), \quad z \in U. \end{aligned}$$

**Remark 2.** *If $f \in \mathcal{A}$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then $R^n f(z) = z + \frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{\Gamma(n+k)}{\Gamma(k)} a_k z^k$ for $z \in U$.*

**Definition 3** ([14])**.** *Let $\alpha, l \geq 0$ and $n, m \in \mathbb{N}$. Denote by $IR_{\alpha,l}^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the multiplier transformation $I(m, \alpha, l)$ and the Ruscheweyh derivative $R^n$,*

$$IR_{\alpha,l}^{m,n} f(z) = (I(m, \alpha, l) * R^n) f(z),$$

*for any $z \in U$ and each of the nonnegative integers $m, n$.*

**Remark 3.** *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*
$IR_{\alpha,l}^{m,n} f(z) = z + \frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m \frac{\Gamma(n+k)}{\Gamma(k)} a_k^2 z^k$, $z \in U$.

**Definition 4** ([15,16])**.** *The fractional integral of order $\lambda$ ($\lambda > 0$) is defined for a function $f$ by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{1}$$

*where $f$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.*

**Definition 5.** *The fractional integral associated with the convolution product of a multiplier transformation and a Ruscheweyh derivative is defined by:*

$$D_z^{-\lambda} IR_{\alpha,l}^{m,n} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{IR_{\alpha,l}^{m,n} f(t)}{(z-t)^{1-\lambda}} dt = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{t}{(z-t)^{1-\lambda}} dt +$$

$$\frac{1}{\Gamma(\lambda)\Gamma(n+1)} \sum_{k=2}^{\infty} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m \frac{\Gamma(n+k)}{\Gamma(k)} a_k^2 \int_0^z \frac{t^k}{(z-t)^{1-\lambda}} dt,$$

*which has the following form, after a simple calculation:*

$$D_z^{-\lambda} IR_{\alpha,l}^{m,n} f(z) = \frac{1}{\Gamma(\lambda+2)} z^{\lambda+1} +$$

$$\frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m \frac{k\Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 z^{k+\lambda},$$

*for the function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$. We note that $D_z^{-\lambda} IR_{\alpha,l}^{m,n} f(z) \in \mathcal{A}(\lambda+1, 1)$.*

Inspired by the results seen in [17], a new subclass of analytic functions is defined using the operator given in Definition 5.

**Definition 6.** *For $\mu, \alpha, l \geq 0$, $\lambda, m, n \in \mathbb{N}$, $\gamma \in \mathbb{C} - \{0\}$ and $\frac{\lambda+\mu}{\lambda+\mu+|\gamma|\Gamma(\lambda+2)} < \beta \leq 1$, let $\mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$ be the subclass of $\mathcal{A}$ consisting of functions that satisfy the following inequality:*

$$\left| \frac{\lambda(1-\mu)\frac{D_z^{-\lambda} IR_{\alpha,l}^{m,n} f(z)}{z} + \mu\left(D_z^{-\lambda} IR_{\alpha,l}^{m,n} f(z)\right)'}{\lambda(1-\mu)\frac{D_z^{-\lambda} IR_{\alpha,l}^{m,n} f(z)}{z} + \mu\left(D_z^{-\lambda} IR_{\alpha,l}^{m,n} f(z)\right)' - \gamma} \right| < \beta \quad (2)$$

The study of the newly introduced subclass $\mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$ is presented in the next sections of the paper. Section 2 contains a new outcome of the coefficient-related studies and extreme points of the functions in the class $\mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$. In Section 3, distortion properties for the functions in class $\mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$ are given and properties of starlikeness and the convexity of this class are presented in Section 4.

## 2. Coefficient Bounds

In this section, coefficient bounds and extreme points for functions in $\mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$ are obtained.

**Theorem 1.** *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 < \frac{\beta|\gamma|\Gamma(n+1)}{\beta+1} - \frac{(\lambda+\mu)\Gamma(n+1)}{\Gamma(\lambda+2)}. \quad (3)$$

*The result is sharp for the function*

$$F(z) = z + \sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(k+\lambda+1)}{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}}\, z^k, \quad k \geq 2. \quad (4)$$

**Proof.** Assume that function $f \in \mathcal{A}$ and that Inequality (3) holds. Then we obtain:

$$\left| \frac{\lambda(1-\mu)\frac{D_z^{-\lambda} IR_{\alpha,l}^{m,n} f(z)}{z} + \mu\left(D_z^{-\lambda} IR_{\alpha,l}^{m,n} f(z)\right)'}{\lambda(1-\mu)\frac{D_z^{-\lambda} IR_{\alpha,l}^{m,n} f(z)}{z} + \mu\left(D_z^{-\lambda} IR_{\alpha,l}^{m,n} f(z)\right)' - \gamma} \right| =$$

$$\left| \frac{\frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda + \frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 z^{k+\lambda-1}}{\frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda + \frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 z^{k+\lambda-1} - \gamma} \right| =$$

$$\frac{\left| \frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda + \frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 z^{k+\lambda-1} \right|}{\left| \frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda + \frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 z^{k+\lambda-1} - \gamma \right|} <$$

$$\frac{\left| \frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda \right| + \left| \frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 z^{k+\lambda-1} \right|}{|\gamma| - \left| \frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda \right| - \left| \frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 z^{k+\lambda-1} \right|} =$$

$$\frac{\frac{\lambda+\mu}{\Gamma(\lambda+2)}\left|z^{\lambda}\right|+\frac{1}{\Gamma(n+1)}\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}{\Gamma(k+\lambda+1)}a_k^2\left|z^{k+\lambda-1}\right|}{|\gamma|-\frac{\lambda+\mu}{\Gamma(\lambda+2)}\left|z^{\lambda}\right|-\frac{1}{\Gamma(n+1)}\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}{\Gamma(k+\lambda+1)}a_k^2\left|z^{k+\lambda-1}\right|}<\beta,\ z\in U.$$

Choosing values of $z$ on the real axis and considering $z \to 1^-$, we have:

$$\frac{\lambda+\mu}{\Gamma(\lambda+2)}+\frac{1}{\Gamma(n+1)}\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}{\Gamma(k+\lambda+1)}a_k^2<$$

$$\beta|\gamma|-\beta\frac{\lambda+\mu}{\Gamma(\lambda+2)}-\beta\frac{1}{\Gamma(n+1)}\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}{\Gamma(k+\lambda+1)}a_k^2,$$

equivalently with

$$\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}{\Gamma(k+\lambda+1)}a_k^2<\frac{\beta|\gamma|\Gamma(n+1)}{\beta+1}-\frac{(\lambda+\mu)\Gamma(n+1)}{\Gamma(\lambda+2)},$$

obtaining that $f \in \mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$.

Conversely, suppose that $f \in \mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$, then we obtain the following inequality:

$$Re\left\{\left|\frac{\lambda(1-\mu)\frac{D_z^{-\lambda}IR_{\alpha,l}^{m,n}f(z)}{z}+\mu\left(D_z^{-\lambda}IR_{\alpha,l}^{m,n}f(z)\right)'}{\lambda(1-\mu)\frac{D_z^{-\lambda}IR_{\alpha,l}^{m,n}f(z)}{z}+\mu\left(D_z^{-\lambda}IR_{\alpha,l}^{m,n}f(z)\right)'-\gamma}\right|\right\}>-\beta$$

$$Re\left\{\frac{\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^{\lambda}+\frac{1}{\Gamma(n+1)}\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}{\Gamma(k+\lambda+1)}a_k^2z^{k+\lambda-1}}{\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^{\lambda}+\frac{1}{\Gamma(n+1)}\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}{\Gamma(k+\lambda+1)}a_k^2z^{k+\lambda-1}-\gamma}+\beta\right\}>0$$

$$Re\left\{\frac{(1+\beta)\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^{\lambda}+\frac{1+\beta}{\Gamma(n+1)}\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}{\Gamma(k+\lambda+1)}a_k^2z^{k+\lambda-1}-\beta\gamma}{\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^{\lambda}+\frac{1}{\Gamma(n+1)}\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}{\Gamma(k+\lambda+1)}a_k^2z^{k+\lambda-1}-\gamma}\right\}>0.$$

Taking account that $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to:

$$\frac{-(1+\beta)\frac{\lambda+\mu}{\Gamma(\lambda+2)}r^{\lambda}-\frac{1+\beta}{\Gamma(n+1)}\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}{\Gamma(k+\lambda+1)}a_k^2r^{k+\lambda-1}+\beta|\gamma|}{\frac{\lambda+\mu}{\Gamma(\lambda+2)}r^{\lambda}+\frac{1}{\Gamma(n+1)}\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}{\Gamma(k+\lambda+1)}a_k^2r^{k+\lambda-1}-\gamma}>0.$$

Letting $r \to 1^-$ and applying the mean value theorem, we have the desired inequality (3).

This completes the proof of Theorem 1. □

**Corollary 1.** *Function $f \in \mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$ implies:*

$$a_k \leq \sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(k+\lambda+1)}{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^{m}\Gamma(n+k)}},\quad k\geq 2, \tag{5}$$

*with equality only for functions defined by (4).*

**Theorem 2.** *Consider $f_1(z) = z$ and*

$$f_k(z) = z - \sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(k+\lambda+1)}{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}}z^k, \quad k \geq 2, \tag{6}$$

*for $\mu, \alpha, l \geq 0$, $\lambda, m, n \in \mathbb{N}$, $\gamma \in \mathbb{C} - \{0\}$ and $0 < \beta \leq 1$.*
*Then, $f \in \mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$ if and only if it can be written in the form:*

$$f(z) = \sum_{k=1}^{\infty} \omega_k f_k(z), \tag{7}$$

*where $\omega_k \geq 0$ and $\sum_{k=1}^{\infty} \omega_k = 1$.*

**Proof.** Assume $f$ can be written as in (7). Then:

$$f(z) = z - \sum_{k=2}^{\infty} \omega_k \sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(k+\lambda+1)}{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}}z^k.$$

Now,

$$\sum_{k=2}^{\infty} \sqrt{\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(k+\lambda+1)}}\omega_k$$

$$\sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(k+\lambda+1)}{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}} = \sum_{k=2}^{\infty} \omega_k = 1 - \omega_1 \leq 1.$$

Thus, $f \in \mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$.

Conversely, let $f \in \mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$. Then by using (5), setting

$$\omega_k = \sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(k+\lambda+1)}{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}}a_k, \quad k \geq 2$$

and $\omega_1 = 1 - \sum_{k=2}^{\infty} \omega_k$, we obtain $f(z) = \sum_{k=1}^{\infty} \omega_k f_k(z)$, completing the proof of Theorem 2. □

## 3. Distortion Bounds

In this section distortion bounds for the class $\mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$ are obtained.

**Theorem 3.** *For $f \in \mathcal{IR}(\mu, \lambda, \beta, \gamma, \alpha, l, m, n)$, inequality*

$$r - \sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(n+1)(\lambda+2\mu)\left(\frac{1+\alpha+l}{l+1}\right)^m}}r^2 \leq |f(z)| \tag{8}$$

$$\leq r + \sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(n+1)(\lambda+2\mu)\left(\frac{1+\alpha+l}{l+1}\right)^m}}r^2$$

*holds if the sequence* $\{\sigma_k(\mu,\lambda,\alpha,l,m,n)\}_{j=2}^{\infty}$ *is non-decreasing, and*

$$1-2\sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(n+1)(\lambda+2\mu l)\left(\frac{1+\alpha+l}{l+1}\right)^m}}r^2 \le |f'(z)| \tag{9}$$

$$\le 1+2\sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(n+1)(\lambda+2\mu)\left(\frac{1+\alpha+l}{l+1}\right)^m}}r^2$$

*holds if the sequence* $\{\frac{\sigma_k(\mu,\lambda,\alpha,l,m,n)}{k}\}_{k=2}^{\infty}$ *is non- decreasing, where*

$$\sigma_k(\mu,\lambda,\alpha,l,m,n)=\sqrt{\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m\Gamma(n+k)}{\Gamma(k+\lambda+1)}}.$$

*The bounds in (8) and (9) are sharp, for f given by*

$$f(z)=z+\sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(n+1)(\lambda+\mu)\left(\frac{1+\alpha+l}{l+1}\right)^m}}z^2, \quad z=\pm r. \tag{10}$$

**Proof.** Using Theorem 1, we obtain:

$$\sum_{k=2}^{\infty} a_k \le \sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(n+1)(\lambda+2\mu)\left(\frac{1+\alpha+l}{l+1}\right)^m}}. \tag{11}$$

We have

$$|z|-|z|^2\sum_{k=2}^{\infty} a_2 \le |f(z)| \le |z|+|z|^2\sum_{k=2}^{\infty} a_2.$$

Thus,

$$r-\sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(n+1)(\lambda+2\mu)\left(\frac{1+\alpha+l}{l+1}\right)^m}}r^2 \le |f(z)| \tag{12}$$

$$\le r+\sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(n+1)(\lambda+2\mu)\left(\frac{1+\alpha+l}{l+1}\right)^m}}r^2.$$

Hence (8) follows from (12). Furthermore,

$$\sum_{k=2}^{\infty} ka_k \le \sqrt{\frac{\left(\frac{\beta|\gamma|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(n+1)(\lambda+2\mu)\left(\frac{1+\alpha+l}{l+1}\right)^m}}.$$

Hence (9) follows from

$$1-r\sum_{k=2}^{\infty} ka_k \le |f'(z)| \le 1+r\sum_{k=2}^{\infty} ka_k.$$

□

## 4. Radius of Starlikeness and Convexity

In this section we give the radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$.

**Theorem 4.** *The function $f \in \mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$ is close-to-convex of the order $\delta$, $0 \le \delta < 1$ in the disc $|z| < r$, where:*

$$r := \inf_{k\ge 2} \sqrt{\frac{(n+1)(1-\delta)^2(\lambda+\mu k)\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m}{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)k\Gamma(k+\lambda+1)}}. \tag{13}$$

*The result is sharp, with the extremal function $f$ given by (4).*

**Proof.** For the function $f \in \mathcal{A}$, we have to show that:

$$\left|f'(z) - 1\right| < 1 - \delta. \tag{14}$$

By a simple calculation we obtain

$$\left|f'(z) - 1\right| \le \sum_{k=2}^{\infty} k a_k |z|,$$

which is less than $1-\delta$ if

$$\sum_{k=2}^{\infty} \frac{k}{1-\delta} a_k |z| < 1.$$

Function $f \in \mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$ if and only if

$$\frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(k+\lambda+1)} a_k^2 < 1,$$

relation (14) is true if

$$\frac{k}{1-\delta}|z| \le \sum_{k=2}^{\infty} \sqrt{\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(k+\lambda+1)}},$$

or, equivalently,

$$|z| \le \sum_{k=2}^{\infty} \sqrt{\frac{(1-\delta)^2(\lambda+\mu k)\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)k\Gamma(n+1)\Gamma(k+\lambda+1)}},$$

which completes the proof. □

**Theorem 5.** *Consider $f \in \mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$. Then:*

1. *$f$ is starlike of order $\delta$, $0 \le \delta < 1$, in the disc $|z| < r_1$ where:*

$$r_1 = \inf_{k\ge 2} \sqrt{\frac{(1-\delta)^2(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)(k+\delta-2)^2\Gamma(n+1)\Gamma(k+\lambda+1)}}.$$

2. *$f$ is convex of order $\delta$, $0 \le \delta < 1$, in the disc $|z| < r_2$ where:*

$$r_2 = \inf_{k\ge 2} \sqrt{\frac{(1-\delta)^2(\lambda+\mu k)\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)k(k-1)^2\Gamma(n+1)\Gamma(k+\lambda+1)}}.$$

*Each of these results is sharp for the extremal function $f$ given by (4).*

**Proof.** 1. For $0 \le \delta < 1$ we have to prove that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\delta. \quad (15)$$

We obtain

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \left|\frac{\sum_{k=2}^{\infty}(k-1)a_k|z|}{1+\sum_{k=2}^{\infty} a_k|z|}\right|,$$

which is less than $1-\delta$ if

$$\sum_{k=2}^{\infty} \frac{(k+\delta-2)}{1-\delta} a_k|z| < 1.$$

Function $f \in \mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$ if and only if:

$$\frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(k+\lambda+1)} a_k^2 < 1.$$

Relation (15) holds if:

$$\frac{k+\delta-2}{1-\delta}|z| < \sqrt{\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(k+\lambda+1)}},$$

equivalently,

$$|z| < \sqrt{\frac{(1-\delta)^2(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \Gamma(n+k)}{\left(\frac{\beta|\gamma|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)(k+\delta-2)^2\Gamma(n+1)\Gamma(k+\lambda+1)}},$$

which yields the starlikeness of the family.

2. The function $f$ is convex if and only the function $zf'$ is starlike; therefore it is enough to prove (2) with a similar method as that of the proof of (1). Thus, the function $f$ is convex if and only if:

$$|zf''(z)| < 1-\delta. \quad (16)$$

We obtain

$$|zf''(z)| \le \left|\sum_{k=3}^{\infty} k(k-1)a_k|z|\right| < 1-\delta,$$

equivalently

$$\sum_{k=2}^{\infty} \frac{k(k-1)}{1-\delta} a_k|z| < 1.$$

Function $f \in \mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$ if and only if

$$\frac{1}{\Gamma(n+1)}\sum_{k=2}^{\infty}\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m\Gamma(n+k)}{\left(\frac{\beta|\gamma|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(k+\lambda+1)}a_k^2<1$$

and relation (16) is true if

$$\frac{k(k-1)}{1-\delta}|z|<\sqrt{\frac{(\lambda+\mu k)k\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m\Gamma(n+j)}{\left(\frac{\beta|\gamma|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}|z|,$$

equivalently with

$$|z|<\sqrt{\frac{(1-\delta)^2(\lambda+\mu k)\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m\Gamma(n+k)}{\left(\frac{\beta|\gamma|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)k(k-1)^2\Gamma(n+1)\Gamma(k+\lambda+1)}},$$

which yields the convexity of the family. □

## 5. Conclusions

The study presented in this paper followed the line of research regarding introducing new classes of univalent functions using different operators. The operator used for obtaining the original results of this paper is part of the celebrated family of fractional integral operators, much investigated in recent years. Using the operator presented in Definition 5, the new subclass of analytic functions under investigation in this paper, $\mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$, was introduced in Definition 6. The paper presented the results of the studies carried out on coefficients, for finding the distortion bound of the functions in the new class and for establishing domains of starlikeness, convexity and close-to-convexity for the functions in the class $\mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$ and finding radii associated with those domains.

As future lines of study involving the class $\mathcal{IR}(\mu,\lambda,\beta,\gamma,\alpha,l,m,n)$, aspects related to subordination and superordination properties could be investigated. Interesting results related to the relatively new concepts of fuzzy differential subordinations and superordinations might be also obtained.

The symmetry properties of the functions defined by an equation or inequality to obtain solutions with particular properties could be studied. The study of special functions by the differential subordinations method could give interesting results about their symmetry properties. In a future paper, the symmetry properties for different functions using the concept of quantum computing could be studied.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The author declares no conflict of interest.

## References

1. Alb Lupaş, A.; Oros, G.I. Differential Subordination and Superordination Results Using Fractional Integral of Confluent Hypergeometric Function. *Symmetry* **2021**, *13*, 327. [CrossRef]
2. Alb Lupaş, A.; Oros, G.I. On Special Differential Subordinations Using Fractional Integral of Sălăgean and Ruscheweyh Operators. *Symmetry* **2021**, *13*, 1553. [CrossRef]
3. Alb Lupaş, A.; Cătaş, A. An Application of the Principle of Differential Subordination to Analytic Functions Involving Atangana–Baleanu Fractional Integral of Bessel Functions. *Symmetry* **2021**, *13*, 971. [CrossRef]

4. Ghanim, F.; Al-Janaby, H.F.; Bazighifan, O. Some New Extensions on Fractional Differential and Integral Properties for Mittag-Leffler Confluent Hypergeometric Function. *Fractal Fract.* **2021**, *5*, 143. [CrossRef]
5. Rashid, S.; Khalid, A.; Bazighifan, O.; Oros, G.I. New Modifications of Integral Inequalities via $\gamma$-Convexity Pertaining to Fractional Calculus and Their Applications. *Mathematics* **2021**, *9*, 1753. [CrossRef]
6. Sahoo, S.K.; Tariq, M.; Ahmad, H.; Kodamasingh, B.; Shaikh, A.A.; Botmart, T.; El-Shorbagy, M.A. Some Novel Fractional Integral Inequalities over a New Class of Generalized Convex Function. *Fractal Fract.* **2022**, *6*, 42. [CrossRef]
7. Srivastava, H.M.; Kashuri, A.; Mohammed, P.O.; Nonlaopon, K. Certain Inequalities Pertaining to Some New Generalized Fractional Integral Operators. *Fractal Fract.* **2021**, *5*, 160. [CrossRef]
8. Akdemir, A.O.; Butt, S.I.; Nadeem, M.; Ragusa, M.A. New General Variants of Chebyshev Type Inequalities via Generalized Fractional Integral Operators. *Mathematics* **2021**, *9*, 122. [CrossRef]
9. Rashid, S.; Hammouch, Z.; Ashraf, R.; Chu, Y.-M. New computation of unified bounds via a more general fractional operator using generalized Mittag-Leffler function in the kernel. *Comput. Model. Eng. Sci.* **2020**, *126*, 359–378. [CrossRef]
10. Cătaş, A. On certain class of p-valent functions defined by a new multiplier transformations. In Proceedings of the Book of the International Symposium G.F.T.A., Istanbul, Turkey, 20–24 August 2007; Istanbul Kultur University: Istanbul, Turkey, 2007; pp. 241–250.
11. Al-Oboudi, F.M. On univalent functions defined by a generalized Sălăgean operator. *Ind. J. Math. Math. Sci.* **2004**, *27*, 1429–1436. [CrossRef]
12. Sălăgean, G.S. Subclasses of univalent functions. In *Lecture Notes in Math*; Springer: Berlin, Germany, 1983; Volume 1013, pp. 362–372.
13. Ruscheweyh, S. New criteria for univalent functions. *Proc. Am. Math. Soc.* **1975**, *49*, 109–115. [CrossRef]
14. Alb Lupaş, A. About some differential sandwich theorems using a multiplier transformation and Ruscheweyh derivative. *J. Comput. Anal. Appl.* **2016**, *21*, 1218–1224.
15. Owa, S. On the distortion theorems I. *Kyungpook Math. J.* **1978**, *18*, 53–59.
16. Owa, S.; Srivastava, H.M. Univalent and starlike generalized hypergeometric functions. *Can. J. Math.* **1987**, *39*, 1057–1077. [CrossRef]
17. Najafzadeh, S.H.; Pezeshki, E. Some aspects of univalent holomorphic functions involving Ruscheweyh and Salagean operator. *Analele Universităţii Oradea Fasc. Mat.* **2013**, *XX*, 61–70.