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Topological Sigma-Semiring Separation and Ordered Measures in Noetherian Hyperconvexes

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Abstract: The interplay between topological hyperconvex spaces and sigma-finite measures in such spaces gives rise to a set of analytical observations. This paper introduces the Noetherian class of k-finite k-hyperconvex topological subspaces (NHCs) admitting countable finite covers. A sigma-finite measure is constructed in a sigma-semiring in a NHC under a topological ordering of NHCs. The topological ordering relation maintains the irreflexive and anti-symmetric algebraic properties while retaining the homeomorphism of NHCs. The monotonic measure sequence in a NHC determines the convexity and compactness of topological subspaces. Interestingly, the topological ordering in NHCs in two isomorphic topological spaces induces the corresponding ordering of measures in sigma-semirings. Moreover, the uniform topological measure spaces of NHCs need not always preserve the pushforward measures, and a NHC semiring is functionally separable by a set of inner-measurable functions.

Keywords: topological spaces; sigma-semiring; measure spaces; convex; Noetherian class

MSC: 54F05; 54E15; 28C15

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1. Introduction

The interplay between topological spaces, Borel sets, Baire categorization and measurability in a $\sigma - semiring$ structure is interesting as well as complex. The interactions between topology and measure theory are generally formulated by forming the smallest $\sigma - field$, where the compactness of a subspace facilitates the computation of measure [1]. It is known that a metrizable space may be separable or may not be separable, affecting the measures in the $\sigma - semiring$ structures within the spaces. This effectively gives rise to the formation of a Borel hierarchy in metrizable spaces [2]. Note that the Borel sets as well as Baire sets are members of $\sigma-$algebra generated by a set of subspaces in a topological space X. The existence of scattering in a topological space affects Borel classification as well as measures. A topological space X is called a scattered space if $\forall A \subset X, \exists a \in A$ such that $A \cap (U \subset X) = \{a\}$ where $U = U^o$ (here, U^o represents the interior of a respective open set). If we consider a continuous function $f : X \to Y$ from a topological space X to a Hausdorff topological space Y, then an interesting question arises: what is the condition to form a function $f(.)$ in Baire first class? The answer is mainly two-fold in view of topology and convexity: (1) if X is a metric space and Y is a convex subspace of a Banach space, and (2) if X is a normal topological space and $Y = R$, where R is a set of real numbers. Interestingly, the connectedness of a topological space has a role in this case. For example, the properties of Baire first category are preserved by $f : X \to Y$ if the topological space X is normal and the topological space Y is arc-connected [2]. If a space is metrizable, then one can find $\sigma-$discrete bases within the space affecting the measurability. A function $f : X \to Y$ induces the co$-\sigma-$discrete bases in Y, given as $\{f(U^o) : U^o \subset X\}$, if X has respective $\sigma-$discrete bases [3]. Note that the concept of $\sigma-$discrete bases in a space can be extended to the concept of hyper-Borel sets in a space [3]. Interestingly, the continuous

Borel measurable functions between non-separable spaces do not necessarily preserve the structures of $\sigma-$discrete bases. Similarly, the interplay between the topological homeomorphism and Borel isomorphism shows that not all topological properties are retained in Borel isomorphic spaces. A bijection $f : X \rightarrow Y$ is called Borel isomorphic if f, f^{-1} are both Borel. Moreover, every Borel measurable function is analytically measurable (i.e., Souslin measurable) [4]. However, it is found that every topological homeomorphism is a Borel isomorphism, but the converse is not always true [4]. As a consequence, one cannot guarantee that the measurability can be preserved in a generalized topological space, even if the Borel isomorphism is attained by $f : X \rightarrow Y$. Moreover, the Borel measure need not be always real valued in a topological space. For example, the complex-valued Borel measures exist in a Minkowski topological space, which allows computing densities with respect to the Lebesgue measure [5].

Motivation and Contributions

The developments in topological measure theory are propelled by Alexandrov and Varadarajan, considering that the topological spaces are always completely regular as well as Hausdorff [6,7]. The fundamental question in measure theory and its topological variants is the extensibility of $\sigma-$algebras [8]. The approach of Alexandrov is based upon the finitely additive set-valued functions in a topological space, and the approach of Varadarajan is primarily based upon the C^b algebraic forms of bounded continuous real-valued functions in the completely regular spaces. Kirk and Crenshaw further generalized the C^b algebraic approach by introducing the concept of paving $W(X)$ in a space X and then constructing a $\sigma - ring$ based on the paving [6]. However, the concept of paving has a strong flavor of general topology, and the corresponding topological measure is finitely $W - regular$. Moreover, the structure of $\sigma - ring$ depending on $W(X)$ is a modification of a standard $\sigma - semiring$ in a topological space. Furthermore, the algebra-based topological separation of subspaces also depends on $W(X)$. In the case of a completely regular topological space, an extremely disconnected space (i.e., closure of open set is open) exists, where the corresponding Baire sets become reduced and the zero-sets are easy to identify [7]. In other words, the topological determination of measure compactness becomes simpler in this setting. It is shown that topological measures and deficient measures may not always support subadditivity and the properties of linear functionals while admitting the weak convergence of topological measures, which is a variety of Alexandrov weak convergence [9]. Interestingly, if we consider a ring of sets $\sigma(A)$ and a topological vector space X, then the measure $\mu : \sigma(A) \rightarrow X$ may show strong convergence to zero if $\mu(\langle B \rangle_{i=1}^{n}) \rightarrow 0$ in $\sigma(A)$ where the sets in sequence $\langle B \rangle_{i=1}^{n}$ under measure are disjoint [10]. These observations are the motivation to investigate the properties of topological measure in the topologically ordered spaces under an anti-symmetric ordering relation. Moreover, it is interesting to analyze the inherent topological properties, such as invariances and measure sequences, if the topological spaces are hyperconvex Noetherian varieties. The interesting questions are as follows: (1) How do we formulate an irreflexive and anti-symmetric topological ordering relation between two Noetherian classes? (2) What are the properties of topological measures in such Noetherian hyperconvex classes under topological ordering relation? (3) What are the properties of a topological measure sequence in the hyperconvex space? This paper addresses these questions and presents the analytical results by combining the elements of topology and measure theory.

The main contributions made in this paper can be summarized as follows. A Noetherian k-hyperconvex class (NHC) in a Hausdorff topological space is constructed such that every local neighborhood basis is countably coverable, and a fiber can be suitably attached for finite k. A topological ordering relation is introduced between two NHCs, where the ordering relation is irreflexive, anti-symmetric and transitive without affecting the homeomorphism of topological spaces. This paper proposes a set of analytical properties of finite measures in sigma-semirings under the topological ordering relation in NHCs. We show that pushforward measures are not always maintainable, and the sigma-semiring is topologically separable by inner-measurable functions.

The rest of the paper is organized as follows. The preliminary concepts and a set of existing classical results are presented in Section 2. The proposed definitions of topological structures are presented in Section 3. The analytical results are presented in Section 4. Finally, Section 5 concludes the paper.

2. Preliminaries

In general, a real-valued measure is formulated based on the algebraic semiring structure on sets. The generalized algebraic structure of the semiring on a set S is given by $\langle S, +, \cdot \rangle$ where $\langle S, + \rangle$ is a commutative semigroup, $\langle S, \cdot \rangle$ is a semigroup and the multiplication $\cdot : S^2 \rightarrow S$ distributes over $+ : S^2 \rightarrow S$ within the structure. The concept of the zero-set plays important roles in the inter-relationship between the algebraic semiring structure and the topological space, which is defined as follows [11].

Definition 1 (zero-set in topological space). *Let X be a Hausdorff topological space and $f : X \rightarrow R$ be a real-valued function. The zero-set in the topological space $A \subset X$ is defined to be a subset such that $A = f^{-1}(0)$.*

The co-zero set is the complement of the zero-set, which is denoted as $coz(A)$. The set of continuous functions in a topological space can generate a $\sigma-$semiring structure. As a result, we can define the zero-set in a topological space alternatively as presented the following definition [2,12].

Definition 2 (semiring zero-set in topological space). *If X is a topological space and $C(X, (0, \infty])$ denotes a set of continuous functions generating a semiring in the topological space X, then a closed set $Cl(f)$ of a function $f(.)$ is the zero-set such that $Cl(f) = f^{-1}(0)$.*

It is well known that a Hausdorff topological space X is a Tychonoff space if every subspace $B \subseteq X$ and a point $a \in X \backslash B$ are functionally separable, where $B = \overline{B} \neq \phi$. Note that $\overline{B}$ is the closure of the corresponding set. Suppose we consider a *family* of subspaces F in a Tychonoff topological space X. Hence, we can define the concept of measurability, which is given as follows [2].

Definition 3 (measurable topological subspaces). *Let X be a set and Y be a Tychonoff space. A function $f : X \rightarrow Y$ is called $F-$measurable if $[U \subset Y] \Rightarrow [f^{-1}(U) \in F]$ where $U = U^o$.*

It is important to note that not all subspaces are measurable. For example, the Bernstein set, which is a Baire–Lindelöf variety, is not measurable [11]. In a linear space, the convexity of functionals and bounded real-valued linear functions have an interesting relationship in terms of measures. Suppose $\omega : X \rightarrow R \cup \{+\infty\}$ is an increasing functional on the linear space of real-valued functions with convexity [13]. If we consider two functions f and g in the space X, then the convexity of $\omega : X \rightarrow R \cup \{+\infty\}$ satisfies the condition given by $[f \geq g] \Rightarrow [\omega(f) \geq \omega(g)]$. Let us consider that X is a family of continuous real-valued functions on a topological space A represented as $f : A \rightarrow R$. If the measure $\mu : \sigma(X) \rightarrow R$ is finite, then it results in the following theorem [13].

Theorem 1. *Every finite measure $\mu : \sigma(X) \rightarrow R$ is regular and closed. Moreover, if $\langle B_n \rangle_{n=1}^{m}$ is a sequence of compact sets in $\sigma(X)$ such that the measure preserves $\mu(\langle B_n \rangle_{n=1}^{m}) \rightarrow \mu(A)$ then the measure is regular in the corresponding topological measure space.*

The inter-relationship between the measurability and Baire categorization of a topological space X is presented in the following theorem where $zer(A)$ denotes a respective zero-set A within the topological space [11].

Theorem 2. *A real-valued function $f : X \to R$ in a topological space X is Baire first category if, and only if, $f(.)$ is $zer(A)-$measurable.*

There is a relationship between the homeomorphism in topological spaces and the multiplicative isomorphism of a semiring structure under the mapping, which is presented in the following theorem [12].

Theorem 3. *In a topological space X the function $f(x) = 1/x$ induces the homeomorphism between the topological spaces $(0, \infty], [0, \infty)$ and also induces the multiplicative isomorphism between semirings in $(0, \infty], [0, \infty)$.*

The interplay between the convexity of a topological subspace and homeomorphism is illustrated in the following theorem [11].

Theorem 4. *Any completely regular topological space X is homeomorphic to a closed subspace $A \subset X$ if X is convex compact, where A is a set of extreme points of the respective topological space.*

Note that if the set of extreme points of a topological space $A \subset X$ is Lindelöf, then the Baire first category measurable function $f(.)$ exists in $A \subset X$, and it can be extended to X, which is also Baire first category measurable. Moreover, it is important to note that the Zariski topological space can be established within the Noetherian space, admitting a finite as well as signed Borel measure [14].

3. Definitions: Hyperconvexity and Measures

In this paper, $\Lambda \subseteq Z^+$ denotes an index set, and the topological spaces are Hausdorff as well as first countable. If two topological spaces, A, B are isomorphic, then it is denoted by the algebraic relation $A \cong_{isom} B$.

Definition 4 (topological k-hyperconvexity). *Let (X, τ_X) be a Hausdorff topological space and $x_p \in X$ be a point. An open neighborhood of x_p given by $N_p \subset X$ is called topologically hyperconvex if $N_p \subset \bigcap_{i \in \Lambda} \overline{A_i}$ where each $\overline{A_i}$ is convex in X and $i < +\infty$. A hyperconvex open neighborhood of $\{x_p\} \in \tau_X$ in Hausdorff space is denoted by HN_p. A HN_p is called $k - hyperconvex$ if $i \in [1, k]$.*

In this paper, we write the hyperconvex subspace to indicate a $k - hyperconvex$ subspace for $k > 1$. Note that the topological hyperconvexity maintains the countable and finitely boundedness property such that if $i \in I \subset \Lambda$ then $\sup(I) < +\infty$ and $|I| > 1$, in general. However, in this case, the finite intersection property *excludes* the possibilities of attaining $HN_p = \phi$ as well as $\{x_p\} = HN_p$ where $\{x_p\} \in \tau_X$. As a result, the concept of hyperconvex Noetherian class within the topological space (X, τ_X) can be established, which is defined as follows.

Definition 5 (hyperconvex Noetherian class). *Let $x_p \in X$ be a point in Hausdorff first countable topological space (X, τ_X) with a hyperconvex open neighborhood basis $NB_p = \{N_{p(k)} \subset X : k \in \Lambda, HN_{p(k)} \cong N_{p(k)}\}$ within the space. An open convex collection $S_p = \{A_i \subset X : i \in \Lambda, x_p \in A_i\}$ is called a Noetherian hyperconvex class (NHC) at $\{x_p\} \in \tau_X$ if the following properties are satisfied.*

$$\begin{array}{l} \forall A_i \in S_p, A_i = A_i^o, \\ \forall A_i \in S_p, \exists HN_{p(i)} \exists HN_{p(k)}, \overline{HN_{p(i)}} \subset A_i^o \subset \overline{HN_{p(k)}}, \\ [i \leq k] \Rightarrow [A_i \subseteq A_k]. \end{array} \quad (1)$$

A Noetherian hyperconvex class S_p is a relaxed variety such that $\overline{HN_{p(i)}} \neq \overline{A_i}$. In other words, $HN_{p(i)}$ need not be locally dense in subspaces in S_p. The Noetherian hyperconvex class S_p is called finite if $i \in I \subset \Lambda$.

Remark 1. *Note that, in general, S_p is not a proper neighborhood basis of $\{x_p\} \in \tau_X$ although S_p is countable. The reason is that if we consider that X is not compact and $I = \Lambda$, then $\exists k \in \Lambda$ such that $\forall i > k, \overline{A_i} = \overline{A_k}$ in (X, τ_X) admitting a finite Noetherian class. In an alternative view, it is possible that $\underset{\forall i \in \Lambda}{\cup} (A_i \in S_p) \subset X$, where (X, τ_X) is a compact topological space. In summary, the compactness of a topological space does not influence the nature of finite S_p.*

Note that from now on, if we consider two Hausdorff first-countable topological spaces (X, τ_X) and (Y, τ_Y), then the corresponding Noetherian hyperconvex classes at any arbitrary points in two spaces are denoted as S_X and S_Y, respectively. The formation of a neighborhood fiber in a hyperconvex topological subspace at a point $x_p \in X$ in the corresponding Noetherian hyperconvex classes $S_X \equiv S_p$ in (X, τ_X) is defined as follows.

Definition 6 (neighborhood fiber). *Let (X, τ_X) be a first-countable topological space and $\underset{\forall i \in \Lambda}{\cup} A_i \subseteq X$ such that $A_i \in S_X$. A fiber $I \subseteq R, \mu_{p \times I} = \{x_p\} \times I$ at $\{x_p\} \in \tau_X$ is a neighborhood fiber if $HN_p \subset \underset{\forall i \in \Lambda}{\cap} A_i$ is a hyperconvex neighborhood of x_p.*

Remark 2. *Note that the condition given by $\exists \overline{HN_p} \subset \underset{\forall i \in \Lambda}{\cap} A_i$ such that $\exists k \in \Lambda, N_{p(k)} \in NB_p$ and $\overline{HN_p} \subset HN_{p(k)}$ is maintained in $S_X \equiv S_p$, where $k < +\infty$. The neighborhood fiber $\mu_{p \times I}$ is a symmetrically compact fiber in S_X if $\exists (a \in R) \neq 0, I = [-a, a]$.*

If we consider that (X, τ_X) and (Y, τ_Y) are two first-countable Hausdorff topological spaces with respective Noetherian hyperconvex class S_X and the Noetherian class S_Y then it is possible to establish a topological ordering relation $<_f$ between the spaces under the function $f : (X, \tau_X) \to (Y, \tau_Y)$ by considering the closure of subspaces. The definition of topological ordering is defined as follows.

Definition 7 (topological ordering). *The Noetherian hyperconvex class S_X and the Noetherian class S_Y in the respective first-countable Hausdorff topological spaces are topologically ordered if $\forall A_i \in S_X, \exists B_i \in S_Y$ such that $f^{-1}(\overline{E} \subset B_i) \subset (\overline{F} \subset A_i)$. The topological ordering is represented as $S_X <_f S_Y$.*

It can be observed that the topological ordering relation preserves the concept of continuity of $f : (X, \tau_X) \to (Y, \tau_Y)$. Later, we will show that $S_X <_f S_Y$ enforces Noetherian hyperconvexity in the codomain of $f : (X, \tau_X) \to (Y, \tau_Y)$ under homeomorphism. Interestingly, if $\forall A_i \in S_X$ one can find that $A_i = \underset{k \in [1,n]}{\cup} F_k, F_k = F_k^o$ such that $n \in \Lambda, n < +\infty$ then $S_X \cong \sigma_{sr(X)}$, where $\sigma_{sr(X)}$ is a $\sigma - semiring$ in S_X. As a result, one can consider the corresponding topological space as a topological measure space $(X, \sigma_{sr(X)}, \mu_X)$ incorporating a consistent topological measure as defined next.

Definition 8 (NHC measure). *A finite measure $\mu_X : \sigma_{sr(X)} \to [0, +\infty)$ is a topological NHC measure in S_X if the following conditions are maintained.*

$$\begin{array}{l} \forall F_k \in \sigma_{sr(X)}, \mu_X(F_k) < \mu_X(\overline{F_k}), \mu_X(\{\phi\}) = 0, \\ \forall m, n \in \Lambda, [A_m \subseteq A_n] \Rightarrow [\mu_X(\overline{A_m}) \leq \mu_X(\overline{A_n})], \\ [A_m \subset A_n] \Rightarrow [\mu_X(\overline{A_m \cup A_n}) = \mu_X(\overline{A_n})]. \end{array} \quad (2)$$

Note that the NHC topological measure in a $\sigma - semiring$ is measure consistent in local subspaces and also in the global subspaces within the corresponding topological space. The NHC topological measure $\mu_{ae} : \sigma_{sr(X \cup Y)} \to [0, +\infty)$ is called *almost-everywhere* in two topological spaces X, Y if $\forall x_p \in X, \exists y_p \in Y$ it is true that $\exists \overline{N_x} \in \sigma_{sr(X)}, \exists \overline{N_y} \in \sigma_{sr(Y)}$ such that $\mu_{ae}(\overline{N_x}) = \mu_{ae}(\overline{N_y})$, where $x_p \in N_x, y_p \in N_y$ are two respective open neighborhoods and $X \cap Y = \phi$.

Remark 3. *A topological NHC measure in S_X generates a non-zero monotone sequence $\Phi(S_X) = \langle \mu(\overline{A_i}) \rangle_{i=1}^{n}$ determining the compactness as well as convexity of (X, τ_X). For example, if (X, τ_X) is a compact and convex topological space, then $n < +\infty, \exists l \in R^+$ such that $\underset{\forall i \in \Lambda}{\cup} \overline{A_i} = X$ and $\Phi(S_X) \to l > 0$. As a result, $\Phi(S_X)$ is bounded and strongly convergent. Otherwise, the sequence $\Phi(S_X)$ is divergent in nature, where $n \to +\infty$.*

4. Main Results

The analytical results are presented in two parts as follows. First, we illustrate the topological and measure theoretic properties of sigma-semiring measures in NHC in Section 4.1. The topological separability of sigma-semiring structures in a NHC and the properties of measures are presented in Section 4.2.

4.1. Properties of Topological NHC Measures

There is a relationship between the k-hyperconvex topological subspaces and the first-countable property of a Hausdorff topological space. This interrelationship is presented in the following theorem.

Theorem 5. *In a topological space (X, τ_X) if NB_p is a finite hyperconvex neighborhood system at $x_p \in X$ then it is a Noetherian hyperconvex class if (X, τ_X) is a first-countable non-compact topological space.*

Proof. Let (X, τ_X) be a first-countable topological space, where $x_p \in X$ is an arbitrary point. A local hyperconvex neighborhood system at $x_p \in X$ is given by $NB_p = \left\{ N_{p(k)} \subset X : k \in \Lambda, HN_{p(k)} \cong N_{p(k)} \right\}$ such that one can find a bijection $f : Z^+ \to NB_p$. The corresponding Noetherian hyperconvex class is S_X at $x_p \in X$. If we consider that $k \in [1, n < +\infty]$ then we can find a corresponding $l \in [1, m < n]$ such that $\underset{l}{\cup}(A_l \in S_X) \subset X$ if, and only if, (X, τ_X) is non-compact and $X \backslash \underset{l}{\cup} \overline{A_l} \neq \phi$ is open. Moreover, according to the definition $\forall N_{p(k)} \in NB_p, \exists A_l \in S_X$ such that $\overline{N_{p(k)}} \subset A_l$ in (X, τ_X). Inductively, it can be concluded that $[(a \in \Lambda) < (b \in \Lambda)] \Rightarrow [N_{p(a)} \subset N_{p(b)}]$ and $\exists l \in [1, m]$ such that $A_l \subset \overline{N_{p(b)}}$ in non-compact (X, τ_X). Hence, the local neighborhood system NB_p is a Noetherian hyperconvex class where $m < b \leq n$ and $f(.)$ is finitely countable. □

Remark 4. *Note that a first-countable topological space may admit a k-finite k-hyperconvex class. It is important to note that a non-convex Hausdorff topological space (X, τ_X) need not always admit a Noetherian hyperconvex class of NB_p for $k \in \Lambda$ at any arbitrary $\{x_p\} \in \tau_X$ within the space irrespective of the compactness of X. The reason is that if $B_{p(k)} \subset X$ is not a convex neighborhood of $x_p \in X$ in the compact non-convex (X, τ_X) then $\underset{k \in \Lambda}{\cup} \overline{B_{p(k)}} = X$; otherwise if (X, τ_X) is non-convex as well as non-compact, then $\underset{k \in \Lambda}{\cup} \overline{B_{p(k)}} \subset X$. This results in the following corollary, which is a stronger property.*

Corollary 1. *A Noetherian $\left\{ B_{p(n)} \subseteq X, n \in [1, k] \right\}$ admits hyperconvex NB_p in a compact Hausdorff and first-countable (X, τ_X) if, and only if, $Cov(X) = \{X \subset E_i, i \in [1, m]\}$ is a countable finite cover of X , where each $F_i \subset E_i$ is a convex subcover of X.*

The topological ordering relation $S_X <_f S_Y$ between the two spaces maintains the respective NHC structures. However, the relation $<_f$ also preserves the hyperconvexity in the NHC in the codomain of continuous $f(.)$. The following theorem presents this property.

Theorem 6. *If (X, τ_X) and (Y, τ_Y) are first-countable topological spaces with hyperconvex $E \subset X$ and $E <_f (F \subset Y)$, then F is also hyperconvex in Y.*

Proof. Let (X, τ_X) and (Y, τ_Y) be two first-countable topological spaces such that $X \cap Y = \phi$. Suppose $x_p \in X$ is an arbitrary point with the corresponding hyperconvex neighborhood basis NB_p. If S_X is a NHC in (X, τ_X) such that $\forall A_i \in S_X, x_p \in A_i$ then $\exists N_{p(n)}, N_{p(m)} \in NB_p$ within the topological space, maintaining the property that $[n < m] \Rightarrow [\overline{N_{p(n)}} \subset A_i \subset \overline{N_{p(m)}}]$ in (X, τ_X). If we consider a continuous function $f : (X, \tau_X) \to (Y, \tau_Y)$, then $\forall A_i \in S_X, \exists B_i \in S_Y$ such that $[f^{-1}(B_i) \subset A_i] \Rightarrow [f^{-1}(\overline{B_i}) \subset \overline{A_i}]$. However, if $A_i <_f B_i$ topological ordering is preserved by $f : S_X \to S_Y$ in the two respective topological spaces, then $[\overline{E} \subset B_i] \Rightarrow [f^{-1}(\overline{E}) \subset (\overline{F} \subset A_i)]$ where $E = E^o$ and $F = F^o$. Hence, it can be concluded that $E \subset \bigcap_{i \in [1,k]} (N_{q(i)} \subset Y)$ where $k < +\infty$ such that $B_i \subset \overline{N_{q(k)}}$. As a result, $E \subset Y$ is also hyperconvex under $S_X <_f S_Y$. □

Note that the converse of Theorem 6 may not always be satisfied under the anti-symmetric topological ordering relation, and additional conditions are required. The following lemma is a natural extension of the topological ordering property.

Lemma 1. *The topological ordering $S_X <_f S_Y$ preserves homeomorphism of $f : (X, \tau_X) \to (Y, \tau_Y)$.*

There is an interplay between the isomorphisms of the two topological subspaces, topological ordering between the respective NHCs and the corresponding topological measures of the NHCs. The topological ordering in the two NHCs induces an algebraic order between the topological measures in the corresponding NHCs. This property is presented in the following theorem.

Theorem 7. *If $S_X <_f S_Y$ is preserved in topological spaces $X \cong_{isom} Y$ then $(\mu_{ae} \circ f^{-1})(\overline{F}) < \mu_{ae}(\overline{E})$ where $F \subset Y$ and $E \subset X$.*

Proof. Let (X, τ_X) and (Y, τ_Y) be two first-countable Hausdorff topological spaces with respective NHCs S_X, S_Y. Note that the topological spaces are separated as $X \cap Y = \phi$. Suppose we consider $A_i \in S_X$ and $B_i \in S_Y$ preserving $S_X <_f S_Y$, which results in $A_i <_f B_i$. If the topological measure $\mu_{ae} : \sigma_{sr(X \cup Y)} \to [0, +\infty)$ is an *almost-everywhere* variety and $X \cong_{isom} Y$ then the $\mu_{ae}(\overline{A_i} \in \sigma_{sr(X)}) = \mu_{ae}(\overline{B_i} \in \sigma_{sr(Y)})$ condition is maintained. However, due to the topological ordering $(E = E^o) <_f (F = F^o)$ between $\overline{E} \subset A_i$ and $\overline{F} \subset B_i$ one can conclude that $[f^{-1}(\overline{F}) \subset \overline{E}] \Rightarrow [(\mu_{ae} \circ f^{-1})(\overline{F}) < \mu_{ae}(\overline{E})]$. □

The above theorem influences the Baire categorization of topological subspaces as illustrated in the following corollary.

Corollary 2. *In $S_X <_f S_Y$ and $S_X \cong_{isom} S_Y$ if $(\mu_{ae} \circ f^{-1})(\overline{F}) < \mu_{ae}(\overline{E})$ then $E \subset X$ and $F \subset Y$ need not be locally dense in A_i and B_i.*

Proof. The proof is relatively straightforward because μ_{ae} is a measure consistently maintaining algebraic ordering $<$ under topological ordering $<_f$ even if $(E \cup \partial E) \subset \overline{A_i}$ and $(F \cup \partial F) \subset \overline{B_i}$. □

There is an interplay between the topological ordering and pushforward measure in the two NHCs. Suppose the function $g : (X, \sigma_{sr(X)}, \mu_X) \to (Y, \sigma_{sr(Y)}, \mu_Y)$ is a uniformly measurable function in two isomorphic topological measure spaces. It is interesting to note that the topological ordering $<_f$ does not preserve the pushforward measure in NHC under composition with the measurable function $g : (X, \sigma_{sr(X)}, \mu_X) \to (Y, \sigma_{sr(Y)}, \mu_Y)$. This property is presented in the following theorem, where f^{-1} is the inverse of the corresponding function under the topological ordering relation.

Theorem 8. *If* $g : (X, \sigma_{sr(X)}, \mu_X) \to (Y, \sigma_{sr(Y)}, \mu_Y)$ *is uniformly measurable in* $X \cong_{isom} Y$ *then* $(f^{-1} \circ g)$ *is not a pushforward measure in* $S_X <_f S_Y$*.*

Proof. Let $(X, \sigma_{sr(X)}, \mu_X), (Y, \sigma_{sr(Y)}, \mu_Y)$ be two measure spaces in respective topological spaces, where the $S_X <_f S_Y$ condition is maintained between two NHCs. Suppose $g : (X, \sigma_{sr(X)}, \mu_X) \to (Y, \sigma_{sr(Y)}, \mu_Y)$ is a uniformly measurable function with $X \cong_{isom} Y$ such that $\mu_X(\overline{A_i} \in \sigma_{sr(X)}) = \mu_X(g^{-1}(\overline{B_i} \in \sigma_{sr(Y)}))$. However, the topological ordering $S_X <_f S_Y$ induces an inequality in measures under composition $(f^{-1} \circ g)$ which is given by $\mu_Y(\overline{B_i} \in \sigma_{sr(Y)}) > \mu_X((f^{-1} \circ g)(\overline{A_i} \in \sigma_{sr(X)}))$. Hence, the condition of the pushforward measure is not preserved by $(f^{-1} \circ g)$ under $<_f$ between the two NHCs. □

Although the pushforward measure is not preserved by $<_f$ topological ordering between multiple NHCs, the hyperconvex neighborhood system is finitely measurable in each topological measure space, and the topological ordering induces an order in the corresponding measures. This observation is illustrated in the following lemma.

Lemma 2. *In every first-countable* (X, τ_X) *the topological measure space* $(X, \sigma_{sr(X)}, \mu_X)$ *admits finite measures of hyperconvex neighborhood basis and the topological ordering* $S_X <_f S_Y$ *between NHCs induces a corresponding order in the neighborhood measures.*

Proof. Let (X, τ_X) be a first-countable topological space, where $NB_p = \left\{N_{p(k)} \subset X : k \in \Lambda, HN_{p(k)} \cong N_{p(k)}\right\}$ is an open hyperconvex neighborhood basis. Clearly, $\overline{NB_p}$ is countable under the bijection $h : Z^+ \to \overline{NB_p}$ where $\overline{NB_p} = \left\{\overline{N_{p(k)}}\right\}$. As a result, the measure $(\mu_X \circ h) \in (0, +\infty)$ is finite in the corresponding measure space $(X, \sigma_{sr(X)}, \mu_X)$ where $0 < \mu_X(N_{p(k)}) < \mu_X(\overline{N_{p(k)}})$ by the definition of topological NHC measure. Moreover, if (Y, τ_Y) is another first-countable topological space with NB_q for some $y_q \in Y$ then $(\mu_X \circ f^{-1})(\overline{N_{q(k)}}) < \mu_Y(\overline{N_{q(k)}})$ under $<_f$ between the topological measure spaces $(X, \sigma_{sr(X)}, \mu_X), (Y, \sigma_{sr(Y)}, \mu_Y)$. □

4.2. Topological Separation of Sigma-Semiring and Measurability

It is noted earlier in this paper that the increasing convex functional $\omega : X \to R \cup \{+\infty\}$ can be formulated in a linear function space X, where ω is convex. However, the measure of the convex bounded measurable functions in a linear function space is finitely additive with the assumption that the sequential semicontinuity of Borel measurable functions is preserved. Note that the convex functional measure can be extended to be infinite. The relationship between the measures and the hyperconvex topological space presented in this paper consider finite measures under the topological decomposition and separation of measure spaces while at least preserving subadditivity. The Hausdorff topological measure space admitting a NHC is considered to be continuous and simply connected in nature.

Let $A_k, A_{k-1} \in S_X$ be the $k - hyperconvex$ and $(k-1) - hyperconvex$ subspaces, respectively, in a NHC in (X, τ_X). Suppose we consider $E_{((k-1),n)} \subset X$ such that $E_{((k-1),n)} = (A_k \cup A_{k-1}) \backslash \overline{A_k}$ where $\overline{A_k} = A_{k-1}$ and $n \in \Lambda$. If we take the collection $E_{((k-1),n)} = \bigcup_{i \in [1,n]} D_{((k-1),i)}$ such that $D_{((k-1),i)} = D^o_{((k-1),i)}$, then a topological separation of the corresponding $\sigma_X - semiring$ is given by the following equation.

$$\begin{aligned}
&m, n, u \in \Lambda, i \leq u, j \leq u, \\
&\forall E_{((k-m),u)}, [i \neq j] \Rightarrow [D_{((k-m),i)} \cap D_{((k-m),j)} = \phi], \\
&\Omega(\sigma_X) = \{A_k\} \cup \left\{D_{((k-m),u)}\right\}, \\
&\bigcup_{m,u} E_{((k-m),u)} \subset S_X.
\end{aligned} \tag{3}$$

This immediately leads to the following lemma.

Lemma 3. *A* $(k-i)-$ *hyperconvex subspace is locally dense in the respective component in* $\Omega(\sigma_X)$ *if, and only if,* $i=0$.

The proof of the lemma is directly derivable from the structure of the topologically separated $\sigma_X - semiring$. However, it further results in the following theorem.

Theorem 9. *A topologically separated* $\Omega(\sigma_X)$ *is functionally separable by* $\{g_v : \Omega(\sigma_X) \to R, v \in \Lambda\}$ *such that* $\bigcap_v g_v = \phi$ *and every* $g_v(.)$ *is inner-measurable.*

Proof. Let a topologically separated $\Omega(\sigma_X)$ be in (X, τ_X) and a set of real-valued functions be given by $\{g_v : \Omega(\sigma_X) \to R, v \in \Lambda\}$ such that $\bigcap_v g_v = \phi$. Suppose that the functions in the set maintain the property of local continuity in the topological space by open mapping as $\forall v \in \Lambda, g_v(W_v) \subset N_v$ such that $N_v = N_v^o$ and $\forall B \subset N_v, g_v^{-1}(B) \subset W_v$ with $B = B^o$. As a result, it can be concluded that $[v \neq (u \in \Lambda)] \Rightarrow [N_v \cap (N_u \subset R) = \phi]$. Moreover, as $N_v = N_v^o$ and $W_v = W_v^o$, every $g_v(.)$ is pushforward inner-measurable due to $(\mu_X \circ g_v^{-1})(N_v) < \mu_X(\overline{W_v})$. □

Example 1. *Let us consider a topological space in 1D such that* $A_k = (-a, a), A_{k-i} = (-(a + \delta(i)), (a + \delta(i)))$ *where* $\delta(i) > 0, i \in \Lambda$. *In this case, the topological separation of the* $\sigma_X - semiring$ *is given by*

$$\begin{aligned} & i \in \Lambda, n \in Z^+ \cup \{0\}, \\ & \Omega(\sigma_X) = \{(-a, a)\} \cup \\ & \{(-(a + (n+1).\delta(i)), -(a + n.\delta(i))), ((a + n.\delta(i)), (a + (n+1).\delta(i)))\}. \end{aligned} \quad (4)$$

As a result, the topological separation $\Omega(\sigma_X)$ *is also separated by* $g_v : \Omega(\sigma_X) \to R$ *if, and only if, the open neighborhoods under locally continuous mappings are disjoint as* $[i \neq j] \Rightarrow [(N_i \subset R) \cap (N_j \subset R) = \phi]$ where the $\forall (B = B^o) \subset N_i, g_i^{-1}(B) \subset W_i$ *condition is preserved. Moreover, every topological separation in* $\Omega(\sigma_X)$ *is inner-measurable because* $(\mu_R \circ g_i)(W_i) < \mu_R(\overline{N_i})$ *where* μ_R *is a finite positive measure in reals.*

5. Conclusions

In a Hausdorff first-countable topological space, the Noetherian hyperconvex class is a generalization of a neighborhood basis without preserving the open property of the singleton element under the intersection of corresponding neighborhoods of that element. The *k*-finite convex intersection generates a *k*-hyperconvex topological subspace admitting a sigma-semiring, which is finitely measurable. The irreflexive, anti-symmetric and transitive topological ordering between two Noetherian *k*-hyperconvex classes retains the homeomorphism of respective topological spaces and induces the ordering in measures in corresponding sigma-semirings. The measure sequence in a Noetherian *k*-hyperconvex class helps in determining compactness of the topological subspace. The measures under the topological ordering do not always preserve the pushforward property, and the sigma-semirings are topologically separable by a set of inner-measurable functions. The concepts presented in this paper may find applications in the topological analysis of dynamics.

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