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Some New Estimates on Coordinates of Left and Right Convex Interval-Valued Functions Based on Pseudo Order Relation

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Abstract: The relevance of convex and non-convex functions in optimization research is well known. Due to the behavior of its definition, the idea of convexity also plays a major role in the subject of inequalities. The main concern of this paper is to establish new integral inequalities for newly defined left and right convex interval-valued function on coordinates through pseudo order relation and double integral. Some of the Hermite–Hadamard type inequalities for the product of two left and right convex interval-valued functions on coordinates are also obtained. Moreover, Hermite–Hadamard–Fejér type inequalities are also derived for left and right convex interval-valued functions on coordinates. Some useful examples are also presented to prove the validity of this study. The proved results of this paper are generalizations of many known results, which are proved by Dragomir, Latif et al. and Zhao, and can be vied as applications of this study.

Keywords: double integral; left and right convex interval-valued function on coordinates; Hermite–Hadamard inequality; Hermite–Hadamard–Fejér inequality

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1. Introduction

Convex analysis has made major contributions to the improvement of various fields of applied and pure study. In recent decades, there has been a lot of interest in the study and differentiation of many directions of the traditional idea of convexity. There have lately been a slew of convex function extensions and modifications developed. Because the functions discovered in a large number of theoretical and practical economics problems are not classical convex functions, many scholars have been interested in the sweeping generalisation of function convexity in past few decades, such as h-convex functions [1–5], log-convex functions [6–9], log-h-convex functions [10], and especially coordinated convex functions [11]. Many authors have proposed different expansions and generalizations of integral inequalities for coordinated convex functions since 2001 (see [12–17] and the references therein).

Moore's interval analysis theory, which he proposed in a numerical analysis in [18], has advanced rapidly in recent decades. In computational problems, a computer can be programmed to discover an interval that contains the precise solution. Interval analysis also ensures that the solution to the model equation is tightly contained. Interval analysis is also commonly used in chemical and structured engineering, economics, control circuitry design, robotics, beam physics, behavioural ecology, constraint satisfaction, computer graphics, signal processing, asteroid orbits and global optimization and neural network output optimization [19], and many other fields.

Many writers have merged integral inequalities with interval-valued functions (*I-V-Fs*) in recent decades, and many great findings have resulted. Costa proposed Opial-type disparities for *I-V-Fs* in [20]. Chalco-Cano et al. used the generalized Hukuhara derivative to examine Ostrowski-type inequalities for *I-V-Fs* in [20]. The Minkowski-type inequalities and Beckenbach-type inequalities for *I-V-Fs* were developed by Roman-Flores et al. in [21,22]. Zhao et al. [23] discovered the Hermite–Hadamard type inequalities for interval-valued coordinated functions very recently.

In a literature review, we noted that most of authors used inclusion relation to obtain different types of inequalities for interval-valued functions, such as Zhao et al. [24] who, in 2008, developed h-convex *I-V-Fs* (*h*-convex *I-V-Fs*) and demonstrated the following Hermite–Hadamard type inequality (HH type inequality) for *h*-convex *I-V-Fs*, based on the above literature.

Theorem 1. *[24] Let $\Psi : [\mu, v] \subset \mathbb{R} \to \mathbb{R}_I^+$ be an h-convex I-V-F given by $\Psi(\omega) = [\Psi_*(\omega), \Psi^*(\omega)]$ for all $\omega \in [\mu, v]$, with $h : [0, 1] \to \mathbb{R}^+$ and $h\left(\frac{1}{2}\right) \neq 0$, where $\Psi_*(\omega)$ and $\Psi^*(\omega)$ are h-convex and h-concave functions, respectively. If Ψ is interval Riemann integrable (in sort, IR-integrable), then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} \Psi\left(\frac{\mu+v}{2}\right) \supseteq \frac{1}{v-\mu} (IR) \int_\mu^v \Psi(\omega)d\omega \supseteq [\Psi(\mu) + \Psi(v)] \int_0^1 h(\xi)d\xi, \tag{1}$$

where $\xi \in [0, 1]$.

Yanrong An et al. [25] took a step forward by introducing the class of (h_1, h_2)-convex *I-V-Fs* and establishing interval-valued Hermite–Hadamard type inequality for (h_1, h_2)-convex *I-V-Fs*. We suggest that readers consult [26–28] and the references therein for more examination of the literature on the applications and properties of generalized convex functions and HH type integral inequalities.

On the other hand, recently, Zhang et al. [29] introduced pseudo order relation on the space of interval and proposed the new class of convex functions in interval-valued settings by using pseudo order relation, which is known as left and right convex *I-V-Fs* (LR-convex *I-V-Fs*). By using this class, they established continuous Jensen's inequalities and proved that Jensen's inequality defined by Costa and Roman-Flores [30] is a special case of these inequalities. Khan et al. went a step further by providing new convex and extended LR-convex *I-V-F* classes, as well as a new fractional HH type and HH type inequalities for LR-(h_1, h_2)-convex *I-V-F* [31], LR-p-convex *I-V-F* [32], and LR-log-h-convex *I-V-F* [33], and the references therein. We refer the readers to [31–40] and the references therein for a further analysis of the literature on the applications and properties of fuzzy Riemannian integrals, and inequalities and generalized convex fuzzy mappings.

Motivated and inspired by the research work of Dragomir [11], Latif et al. [16], Hao et al. [23] and Zhang et al. [29], this paper is organized as follows: Section 2 consist of some preliminary notions, and some new definitions and results. Section 3 obtains Hermite–Hadamard and Hermite–Hadamard–Fejér inequalities for left and right convex interval-valued functions (LR-convex *I-V-Fs*) on coordinates, and some related inequalities via pseudo order relation and interval double integrals. We finalise with Section 4 of conclusion and future plan.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_I$ be the space of all closed and bounded intervals of $\mathbb{R}$, such that $\varpi \in \mathbb{R}_I$ is defined by

$$\varpi = [\varpi_*,\ \varpi^*] = \{\omega \in \mathbb{R} |\ \varpi_* \leq \omega \leq \varpi^*\}, (\varpi_*,\ \varpi^* \in \mathbb{R}).$$

If $\varpi_* = \varpi^*$, then ϖ is said to be degenerate. If $\varpi_* \geq 0$, then $[\varpi_*,\ \varpi^*]$ is called positive interval. The set of all positive interval is denoted by $\mathbb{R}_I^+$ and defined as $\mathbb{R}_I^+ = \{[\varpi_*,\ \varpi^*] : [\varpi_*,\ \varpi^*] \in \mathbb{R}_I \text{ and } \varpi_* \geq 0\}$. Let $\varrho \in \mathbb{R}$ and $\varrho\varpi$ be defined by

$$\varrho.\varpi = \begin{cases} [\varrho\varpi_*,\ \varrho\varpi^*]\ if\ \varrho > 0, \\ \{0\}\ if\ \varrho = 0, \\ [\varrho\varpi^*, \varrho\varpi_*]\ if\ \varrho < 0. \end{cases} \quad (2)$$

Then, the Minkowski difference $-\varpi$, addition $\varpi + \xi$ and $\varpi \times \xi$ for $\varpi, \xi \in \mathbb{R}_I$ are defined by

$$\begin{aligned} [\xi_*,\ \xi^*] - [\varpi_*,\ \varpi^*] &= [\xi_* - \varpi_*,\ \xi^* - \varpi^*], \\ [\xi_*,\ \xi^*] + [\varpi_*,\ \varpi^*] &= [\xi_* + \varpi_*,\ \xi^* + \varpi^*], \end{aligned} \quad (3)$$

and

$$[\xi_*,\ \xi^*] \times [\varpi_*,\ \varpi^*] = [min\{\xi_*\varpi_*,\ \xi^*\varpi_*,\ \xi_*\varpi^*,\ \xi^*\varpi^*\},\ max\{\xi_*\varpi_*,\ \xi^*\varpi_*,\ \xi_*\varpi^*,\ \xi^*\varpi^*\}].$$

The inclusion " $\supseteq$ " means that

$$\varpi \supseteq \xi \text{ if and only if, } [\varpi_*,\ \varpi^*] \supseteq [\xi_*,\ \xi^*], \text{ and if and only if } \varpi_* \leq \xi_*,\ \xi^* \leq \varpi^*.$$

Remark 1. *[29] (i) The relation " $\leq_p$ " is defined on $\mathbb{R}_I$ by*

$$[\xi_*,\ \xi^*] \leq_p [\varpi_*,\ \varpi^*] \text{ if and only if } \xi_* \leq \varpi_*,\ \xi^* \leq \varpi^*, \quad (4)$$

for all $[\xi_*,\ \xi^*],\ [\varpi_*,\ \varpi^*] \in \mathbb{R}_I$, and it is a pseudo order relation. The relation $[\xi_*,\ \xi^*] \leq_p [\varpi_*,\ \varpi^*]$ coincident to $[\xi_*,\ \xi^*] \leq [\varpi_*,\ \varpi^*]$ on $\mathbb{R}_I$ when it is " $\leq_p$ "

(ii) It can be easily seen that " $\leq_p$ " looks like "left and right" on the real line $\mathbb{R}$, so we call " $\leq_p$ " is "left and right" (or "LR" order, in short).

For $[\xi_*,\ \xi^*],\ [\varpi_*,\ \varpi^*] \in \mathbb{R}_I$, the Hausdorff–Pompeiu distance between intervals $[\xi_*,\ \xi^*]$ and $[\varpi_*,\ \varpi^*]$ is defined by

$$d([\xi_*,\ \xi^*],\ [\varpi_*,\ \varpi^*]) = max\{|\xi_* - \varpi_*|,\ |\xi^* - \varpi^*|\}. \quad (5)$$

It is a familiar fact that $(\mathbb{R}_I, d)$ is a complete metric space.

Now, we recall the same concept of interval integral operators.

Theorem 2. *[18] If $\Psi : [\mu, v] \subset \mathbb{R} \to \mathbb{R}_I$ is an I-V-F given by $(x) = [\Psi_*(x),\ \Psi^*(x)]$, then Ψ is Riemann integrable over $[\mu, v]$ if and only if, Ψ_* and Ψ^* both are Riemann integrable over $[\mu, v]$, such that*

$$(IR)\int_\mu^v \Psi(x)dx = \left[(R)\int_\mu^v \Psi_*(x)dx,\ (R)\int_\mu^{\ v}\ \Psi^*(x)dx\right]. \quad (6)$$

The collection of all Riemann integrable real valued functions and Riemann integrable I-V-F is denoted by $\mathcal{R}_{[\mu,\ v]}$ and $\mathfrak{T}\mathcal{R}_{[\mu,\ v]}$, respectively.

Note that Theorem 3 is also true for interval double integrals. The collection of all double integrable *I-V-F* is denoted $\mathfrak{TO}_{\Delta}$, respectively.

Theorem 3. *[35] Let $\Delta = [c,\ d] \times [\mu,\ v]$. If $\Psi : \Delta \to \mathbb{R}_I$ is an interval-valued double integrable (ID-integrable) on Δ , then we have*

$$(ID)\int_c^d \int_\mu^v \Psi(x,\omega)d\omega dx = (IR)\int_c^d (IR)\int_\mu^v \Psi(x,\omega)d\omega dx.$$

Definition 1. *[11] The non-negative real valued function $\Psi : \Delta = [c,\ d] \times [\mu,\ v] \to \mathbb{R}^+$ is said to be convex function on coordinate Δ if*

$$\begin{gathered} \Psi(\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z) \\ \le \xi\varsigma\Psi(x,y) + \xi(1-\varsigma)\Psi(x,z) + (1-\xi)\varsigma\Psi(\omega,y) + (1-\xi)(1-\varsigma)\Psi(\omega,z), \end{gathered} \tag{7}$$

for all $(x,\ y),\ (\omega,z) \in \Delta,\ \xi$ and $\xi,\varsigma \in [0,\ 1]$. If inequality (7) is reversed, then Ψ is called concave function on coordinate Δ.

Definition 2. *[24] The I-V-F $\Psi : [\mu, v] \to \mathbb{R}_I^+$ is said to be convex I-V-F on $[\mu, v]$ if*

$$\Psi(\xi x + (1-\xi)\omega) \supseteq \xi\Psi(x) + (1-\xi)\Psi(\omega), \tag{8}$$

for all $x,\ \omega \in [\mu, v],\ \xi \in [0,\ 1]$. If Ψ is concave I-V-F on $[\mu, v]$, then inclusion operation in (8) is reversed.

Definition 3. *[35] The I-V-F $\Psi : \Delta \to \mathbb{R}_I^+$ is said to be LR-convex I-V-F on coordinate Δ if*

$$\begin{gathered} \Psi(\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z) \\ \supseteq \xi\varsigma\Psi(x,y) + \xi(1-\varsigma)\Psi(x,z) + (1-\xi)\varsigma\Psi(\omega,y) + (1-\xi)(1-\varsigma)\Psi(\omega,z), \end{gathered} \tag{9}$$

for all $(x,\ y),\ (\omega,z) \in \Delta,$ and $\xi,\varsigma \in [0,\ 1]$. If inequality (9) is reversed, then Ψ is called concave I-V-F on coordinate Δ.

Definition 4. *[29] The I-V-F $\Psi : [\mu, v] \to \mathbb{R}_I^+$ is said to be LR-convex I-V-F on $[\mu, v]$ if*

$$\Psi(\xi x + (1-\xi)\omega) \le_p \xi\Psi(x) + (1-\xi)\Psi(\omega), \tag{10}$$

for all $x,\ \omega \in [\mu, v],\ \xi \in [0,\ 1]$. If Ψ is concave I-V-F on $[\mu, v]$, then inequality (10) is reversed.

Definition 5. *[31] Let $h_1,\ h_2 : [0,\ 1] \subseteq [\mu, v] \to \mathbb{R}^+$, such that $h_1,\ h_2 0$. Then, I-V-F $\Psi : [\mu, v] \to \mathbb{R}_I^+$ is said to be $(h_1,\ h_2)$-convex I-V-F on $[\mu, v]$ if*

$$\Psi(\xi x + (1-\xi)\omega\ ) \le_p h_1(\xi)h_2(1-\xi)\Psi(x) + h_1(1-\xi)h_2(\xi)\Psi(\omega), \tag{11}$$

for all $x,\ \omega \in [\mu, v],\ \xi \in [0,\ 1]$. If Ψ is $(h_1,\ h_2)$-concave on $[\mu, v]$, then inequality (11) is reversed.

Remark 2. *[31] If $h_2(\xi) \equiv 1$, then $(h_1,\ h_2)$-convex I-V-F becomes h_1-convex I-V-F, which is*

$$\Psi(\xi x + (1-\xi)\omega) \le_p h_1(\xi)\Psi(x) + h_1(1-\xi)\Psi(\omega),\ \forall\ x,\ \omega \in [\mu, v],\ \xi \in [0,\ 1]. \tag{12}$$

If $h_1(\xi) = \xi, h_2(\xi) \equiv 1$, then $(h_1,\ h_2)$-convex I-V-F becomes convex I-V-F, which is

$$\Psi(\xi x + (1-\xi)\omega) \le_p \xi\Psi(x) + (1-\xi)\Psi(\omega),\ \forall\ x,\ \omega \in [\mu, v],\ \xi \in [0,\ 1]. \tag{13}$$

If $h_1(\xi) = h_2(\xi) \equiv 1$, then $(h_1,\ h_2)$-convex I-V-F becomes P-convex I-V-F, which is

$$\Psi(\xi x + (1-\xi)\omega) \le_p \Psi(x) + \Psi(\omega),\ \forall\ x,\ \omega \in [\mu, v],\ \xi \in [0,\ 1]. \tag{14}$$

Theorem 4. *[31] Let* $\Psi, \mathcal{H} : [\mu, v] \to \mathbb{R}_I^+$ *be two LR-*(h_1, h_2)*-convex I-V-Fs with* $h_1, h_2 : [0, 1] \to \mathbb{R}^+$ *and* $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$*, such that* $\Psi(x) = [\Psi_*(x), \Psi^*(x)]$ *and* $\mathcal{H}(x) = [\mathcal{H}_*(x), \mathcal{H}^*(x)]$ *for all* $x \in [\mu, v]$*. If* $\Psi \times \mathcal{H}$ *is interval Riemann integrable, then*

$$\begin{aligned} \frac{1}{v-\mu} (IR) \int_\mu^v \Psi(x) \times \mathcal{H}(x)dx \leq_p \mathcal{M}(\mu, v) \int_0^1 [h_1(\xi)h_2(1-\xi)]^2 d\xi \\ +\mathcal{N}(\mu, v) \int_0^1 h_1(\xi)h_2(\xi)h_1(1-\xi)h_2(1-\xi)d\xi, \end{aligned} \quad (15)$$

and,

$$\begin{aligned} &\frac{1}{2\left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2} \Psi\left(\frac{\mu+v}{2}\right) \times \mathcal{H}\left(\frac{\mu+v}{2}\right) \\ &\leq_p \frac{1}{v-\mu} (IR) \int_\mu^v \Psi(x)\mathcal{H}(x)dx + \mathcal{N}(\mu, v) \int_0^1 [h_1(\xi)h_2(1-\xi)]^2 d\xi \\ &+\mathcal{M}(\mu, v) \int_0^1 h_1(\xi)h_2(\xi)h_1(1-\xi)h_2(1-\xi)d\xi, \end{aligned} \quad (16)$$

where $\mathcal{M}(\mu, v) = \Psi(\mu) \times \mathcal{H}(\mu) + \Psi(v) \times \mathcal{H}(v), \mathcal{N}(\mu, v) = \Psi(\mu) \times \mathcal{H}(v) + \Psi(v) \times \mathcal{H}(\mu)$*, and* $\mathcal{M}(\mu, v) = [\mathcal{M}_*(\mu, v), \mathcal{M}^*(\mu, v)]$ *and* $\mathcal{N}(\mu, v) = [\mathcal{N}_*(\mu, v), \mathcal{N}^*(\mu, v)]$*.*

Remark 3. *If* $h_1(\xi) = \xi$ *and* $h_2(\xi) \equiv 1$*, then (15) reduces to the result for convex I-V-F:*

$$\frac{1}{v-\mu} (IR) \int_\mu^v \Psi(x) \times \mathcal{H}(x)dx \leq_p \frac{1}{3}\mathcal{M}(\mu, v) + \frac{1}{6}\mathcal{N}(\mu, v), \quad (17)$$

and

If $h_1(\xi) = \xi$ *and* $h_2(\xi) \equiv 1$*, then (16) reduces to the result for convex I-V-F:*

$$2\,\Psi\left(\frac{\mu+v}{2}\right) \times \mathcal{H}\left(\frac{\mu+v}{2}\right) \leq_p \frac{1}{v-\mu} (IR) \int_\mu^v \Psi(x) \times \mathcal{H}(x)dx + \frac{1}{6}\mathcal{M}(\mu, v) + \frac{1}{3}\mathcal{N}(\mu, v). \quad (18)$$

Theorem 5. *[31] Let* $\Psi : [\mu, v] \to \mathbb{R}_I^+$ *be a convex I-V-F with* $\mu < v$ *, such that* $\Psi(x) = [\Psi_*(x), \Psi^*(x)]$ *for all* $x \in [\mu, v]$ *. If* $\Psi(x)$ *is interval Riemann integrable and* $\mathfrak{D} : [\mu, v] \to \mathbb{R}$*,* $\mathfrak{D}(x) \geq 0$*, symmetric with respect to* $\frac{\mu+v}{2}$*, and* $\int_\mu^v \mathfrak{D}(x)dx > 0$ *, then*

$$\Psi\left(\frac{\mu+v}{2}\right) \leq_p \frac{1}{\int_\mu^v \mathfrak{D}(x)dx} (IR) \int_\mu^v \Psi(x)\mathfrak{D}(x)dx \leq_p \frac{\Psi(\mu) + \Psi(v)}{2}. \quad (19)$$

If Ψ *is concave I-V-F, then inequality (19) is reversed. If* $\mathfrak{D}(x) = 1$ *, then inequality (19) reduces to the following inequality:*

$$\Psi\left(\frac{\mu+v}{2}\right) \leq_p \frac{1}{v-\mu} (IR) \int_\mu^v \Psi(x)dx \leq_p \frac{\Psi(\mu) + \Psi(v)}{2}. \quad (20)$$

LR-Convex Interval-Valued Functions on Coordiantes

Definition 6. *The I-V-F* $\Psi : \Delta \to \mathbb{R}_I^+$ *is said to be LR-convex I-V-F on coordiante* Δ *if*

$$\begin{aligned} \Psi(\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z) \\ \leq_p \xi\varsigma\Psi(x, y) + \xi(1-\varsigma)\Psi(x, z) + (1-\xi)\varsigma\Psi(\omega, y) + (1-\xi)(1-\varsigma)\Psi(\omega, z), \end{aligned} \quad (21)$$

for all $(x, y), (\omega, z) \in \Delta$*, and* $\xi, \varsigma \in [0, 1]$*. If inequality (21) is reversed, then* Ψ *is called concave I-V-F on coordinate* Δ*.*

The proof of Lemma 1 is straightforward and will be omitted in this case.

Lemma 1. *Let* $\Psi : \Delta \to \mathbb{R}_I^+$ *be an I-V-F on coordinate* Δ*. Then,* Ψ *is LR-convex I-V-F on coordinate* Δ*, if and only if there exist two LR-convex I-V-Fs* $\Psi_x : [\mu, v] \to \mathbb{R}_I^+$*,* $\Psi_x(w) = \Psi(x, w)$ *and* $\Psi_\omega : [c, d] \to \mathbb{R}_I^+$*,* $\Psi_\omega(y) = \Psi(y, \omega)$*.*

Proof. From the definition of coordinated *I-V-F*, it can be easily proved. □

From Lemma 1, we can easily note each LR-convex *I-V-F* is an LR-convex *I-V-F* on the coordinate. However, the converse is not true (see Example 1).

Theorem 6. *Let* $\Psi : \Delta \to \mathbb{R}_I^+$ *be a I-V-F on* Δ*, such that*

$$\Psi(x,\omega) = [\Psi_*(x,\omega),\ \Psi^*(x,\omega)], \tag{22}$$

for all $(x,\omega) \in \Delta$*. Then,* Ψ *is LR-convex I-V-F on coordinate* Δ*, if and only if,* $\Psi_*(x,\omega)$ *and* $\Psi^*(x,\omega)$ *are convex functions on coordinate* Δ*.*

Proof. Assume that $\Psi_*(x,\omega)$ and $\Psi^*(x,\omega)$ are convex functions on coordinate Δ. Then, from (7), for all $(x,\ y),\ (\omega,z) \in \Delta,\ \xi$ and $\varsigma \in [0,\ 1]$, we have

$$\begin{gathered}\Psi_*(\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z)\\ \leq \xi\varsigma\Psi_*(x,y) + \xi(1-\varsigma)\Psi_*(x,z) + \varsigma(1-\xi)\Psi_*(\omega,y) + (1-\xi)(1-\varsigma)\Psi_*(\omega,z),\end{gathered}$$

and

$$\begin{gathered}\Psi^*(\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z)\\ \leq \xi\varsigma\Psi_*(x,y) + \xi(1-\varsigma)\Psi^*(x,z) + \varsigma(1-\xi)\Psi^*(\omega,y) + (1-\xi)(1-\varsigma)\Psi^*(\omega,z),\end{gathered}$$

Then, by (22), (2) and (3), we obtain

$$\begin{aligned}&\Psi((\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z))\\ &\quad= [\Psi_*(\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z),\ \Psi^*(\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z)],\\ &\quad\leq_p \xi\varsigma[\Psi_*(x,y),\ \Psi^*(x,y)] + \xi(1-\varsigma)[\Psi_*(x,z),\ \Psi*(x,z)]\\ &\quad+\varsigma(1-\xi)[\Psi_*(\omega,y),\ \Psi^*(\omega,y)] + (1-\xi)(1-\varsigma)[\Psi_*(\omega,z),\ \Psi^*(\omega,z)].\end{aligned}$$

That is

$$\begin{aligned}&\Psi(\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z)\\ &\leq_p \xi\varsigma\Psi(x,y) + \xi(1-\varsigma)\Psi(x,z) + (1-\xi)\varsigma\Psi(\omega,y) + (1-\xi)(1-\varsigma)\Psi(\omega,z),\end{aligned}$$

and hence, Ψ is LR-convex *I-V-F* on coordinate Δ.Conversely, let Ψ be LR-convex *I-V-F* on coordinate Δ. Then, for all $(x,\ y),\ (\omega,z) \in \Delta$, and $\xi, \varsigma \in [0,\ 1]$, we have

$$\begin{aligned}&\Psi(\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z)\\ &\leq_p \xi\varsigma\Psi(x,y) + \xi(1-\varsigma)\Psi(x,z) + (1-\xi)\varsigma\Psi(\omega,y) + (1-\xi)(1-\varsigma)\Psi(\omega,z).\end{aligned}$$

Therefore, again from (22), we have

$$\begin{aligned}&\Psi((\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z))\\ &= [\Psi_*(\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z),\ \Psi^*(x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z)].\end{aligned}$$

From (11) and (13), we obtain

$$\begin{gathered}\xi\varsigma\Psi(x,y) + \xi(1-\varsigma)\Psi(x,z) + (1-\xi)\varsigma\Psi(\omega,y) + (1-\xi)(1-\varsigma)\Psi(\omega,z)\\ = \xi\varsigma[\Psi_*(x,y),\ \Psi^*(x,y)] + \xi(1-\varsigma)[\Psi_*(x,z),\ \Psi^*(x,z)]\\ +\varsigma(1-\xi)[\Psi_*(\omega,y),\ \Psi^*(x,y)] + (1-\xi)(1-\varsigma)[\Psi_*(\omega,z),\ \Psi^*(x,z)],\end{gathered}$$

for all $\varsigma, \xi \in [0,\ 1]$. Then, by LR-convexity on coordinate of Ψ, we have for all $\varsigma, \xi \in [0,\ 1]$, such that

$$\begin{aligned}&\Psi_*(\xi x + (1-\xi)\omega,\ \varsigma y + (1-\varsigma)z)\\ &\leq \xi\varsigma\Psi_*(x,y) + \xi(1-\varsigma)\Psi_*(x,z) + (1-\xi)\varsigma\Psi_*(\omega,y) + (1-\xi)(1-\varsigma)\Psi_*(\omega,z),\end{aligned}$$

and

$$\begin{aligned}&\Psi^*(\xi x+(1-\xi)\omega,\ \varsigma y+(1-\varsigma)z)\\&\leq \xi\varsigma\Psi^*(x,y)+\xi(1-\varsigma)\Psi^*(x,z)+(1-\xi)\varsigma\Psi^*(\omega,y)+(1-\xi)(1-\varsigma)\Psi^*(\omega,z),\end{aligned}$$

Hence, the result follows. □

Remark 4. *If one takes* $\Psi_*(x,\omega)=\Psi^*(x,\omega)$ *, then* Ψ *is known as a function on the coordinate if* Ψ *satisfies the coming inequality*

$$\begin{aligned}&\Psi(\xi x+(1-\xi)\omega,\ \varsigma y+(1-\varsigma)z)\\&\leq \xi\varsigma\Psi(x,y)+\xi(1-\varsigma)\Psi(x,z)+(1-\xi)\varsigma\Psi(\omega,y)+(1-\xi)(1-\varsigma)\Psi(\omega,z),\end{aligned}$$

is valid which is defined by Dragomir [11].

Let one take $\Psi_*(x,\omega)\neq\Psi^*(x,\omega)$ and $\Psi_*(x,\omega)$ is an affine function and $\Psi^*(x,\omega)$ is a concave function. If coming inequality,

$$\begin{aligned}&\Psi(\xi x+(1-\xi)\omega,\ \varsigma y+(1-\varsigma)z)\\&\supseteq \xi\varsigma\Psi(x,y)+\xi(1-\varsigma)\Psi(x,z)+(1-\xi)\varsigma\Psi(\omega,y)+(1-\xi)(1-\varsigma)\Psi(\omega,z),\end{aligned}$$

is valid, then Ψ is named as IVF on the coordinate, which is defined by Zhao et al. ([23], Definition 2 and Example 2).

Example 1. We consider the *I-V-Fs* $\Psi:[0,\ 1]\times[0,\ 1]\to\mathbb{R}_I^+$ defined by,

$$\Psi(x,\omega)=[x\omega,(6+e^x)(6+e^\omega)]$$

Since end point functions $\Psi_*(x,\omega)$, $\Psi^*(x,\omega)$ are convex functions on the coordinates. Hence $\Psi(x,\omega)$ is convex *I-V-F* on *the* coordinate.

From Example 1, it can be easily seen that each LR-convex *I-V-F* on the coordinates is not a LR-convex *I-V-F*.

Theorem 7. *Let* Δ *be a coordinated convex set, and let* $\Psi:\Delta\to\mathbb{R}_I^+$ *be a I-V-F such that*

$$\Psi(x,\omega)=[\Psi_*(x,\omega),\ \Psi^*(x,\omega)], \tag{23}$$

for all $(x,\omega)\in\Delta$*. Then,* Ψ *is LR-concave I-V-F on coordinate* Δ*, if and only if,* $\Psi_*(x,\omega)$ *and* $\Psi^*(x,\omega)$*are concave function on coordinate* Δ*.*

Proof. The demonstration of proof of Theorem 7 is similar to the demonstration proof of Theorem 6. □

Example 2. *We consider the I-V-Fs* $\Psi:[0,\ 1]\times[0,\ 1]\to\mathbb{R}_I^+$ *defined by,*

$$\Psi(x,\omega)=[2(6-e^x)(6-e^\omega),4(6-e^x)(6-e^\omega)\ ]$$

Since end point functions $\Psi_*(x,\omega)$, $\Psi^*(x,\omega)$ *are concave functions on the coordinate. Hence,* $\Psi(x,\omega)$ *is concave I-V-F on the coordinate.*

3. Hermite–Hadamard Inequalities on Coordinates

In this section, we propose HH and HH–Fejér inequalities for LR-convex *I-V-Fs on* coordinates, and verify with the help of some nontrivial example. Throughout in this section, we will not include the symbols (R), (IR), and (ID) before the integral sign.

Theorem 8. *Let* $\Psi:\Delta\to\mathbb{R}_I^+\subset\mathbb{R}_I$ *be a LR-convex I-V-F on coordinate* Δ *such that* $\Psi(x,\omega)=[\Psi_*(x,\omega),\ \Psi^*(x,\omega)]$ *for all* $(x,\omega)\in\Delta$*. Then, following inequality holds:*

$$\begin{aligned}\Psi\left(\tfrac{c+d}{2},\ \tfrac{\mu+v}{2}\right) &\leq_p \frac{1}{2}\left[\tfrac{1}{d-c}\int_c^d \Psi\left(x,\tfrac{\mu+v}{2}\right)dx+\tfrac{1}{v-\mu}\int_\mu^v \Psi\left(\tfrac{c+d}{2},\omega\right)d\omega\right] \\ &\leq_p \tfrac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)d\omega dx \\ &\leq_p \frac{1}{4(d-c)}\left[\int_c^d \Psi(x,\mu)dx+\int_c^d \Psi(x,v)dx\right] \\ &+\frac{1}{4(v-\mu)}\left[\int_\mu^v \Psi(c,\omega)d\omega+\int_\mu^v \Psi(d,\omega)d\omega\right] \\ &\leq_p \tfrac{\Psi(c,\mu)+\Psi(d,\mu)+\Psi(c,v)+\Psi(d,v)}{4}.\end{aligned} \tag{24}$$

if $\Psi(x,\omega)$ concave I-V-F then,

$$\begin{aligned}\Psi\left(\tfrac{c+d}{2},\ \tfrac{\mu+v}{2}\right) &\geq_p \frac{1}{2}\left[\tfrac{1}{d-c}\int_c^d \Psi\left(x,\tfrac{\mu+v}{2}\right)dx+\tfrac{1}{v-\mu}\int_\mu^v \Psi\left(\tfrac{c+d}{2},\omega\right)d\omega\right] \\ &\geq_p \tfrac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)d\omega dx \\ &\geq_p \frac{1}{4(d-c)}\left[\int_c^d \Psi(x,\mu)dx+\int_c^d \Psi(x,v)dx\right] \\ &+\frac{1}{4(v-\mu)}\left[\int_\mu^v \Psi(c,\omega)d\omega+\int_\mu^v \Psi(d,\omega)d\omega\right] \\ &\geq_p \tfrac{\Psi(c,\mu)+\Psi(d,\mu)+\Psi(c,v)+\Psi(d,v)}{4}.\end{aligned} \tag{25}$$

Proof. Let $\Psi:\Delta\to\mathbb{R}_I^+$ be a LR-convex *I-V-F* on coordinate. Then, by hypothesis, we have

$$4\Psi\left(\frac{c+d}{2},\frac{\mu+v}{2}\right)\leq_p \Psi(\xi c+(1-\xi)d,\ \xi\mu+(1-\xi)v)+\Psi((1-\xi)c+\xi d,\ (1-\xi)\mu+\xi v).$$

by using Theorem 6, we have

$$\begin{aligned}4\Psi_*\left(\tfrac{c+d}{2},\tfrac{\mu+v}{2}\right) &\leq \Psi_*(\xi c+(1-\xi)d,\ \xi\mu+(1-\xi)v)+\Psi_*((1-\xi)c+\xi d,\ (1-\xi)\mu+\xi v) \\ 4\Psi^*\left(\tfrac{c+d}{2},\tfrac{\mu+v}{2}\right) &\leq \Psi^*(\xi c+(1-\xi)d,\ \xi\mu+(1-\xi)v)+\Psi^*((1-\xi)c+\xi d,\ (1-\xi)\mu+\xi v)\end{aligned}$$

By using Lemma 1, we have

$$\begin{aligned}2\Psi_*\left(x,\tfrac{\mu+v}{2}\right) &\leq \Psi_*(x,\ \xi\mu+(1-\xi)v)+\Psi_*(x,\ (1-\xi)\mu+\xi v), \\ 2\Psi^*\left(x,\tfrac{\mu+v}{2}\right) &\leq \Psi^*(x,\ \xi\mu+(1-\xi)v)+\Psi^*(x,\ (1-\xi)\mu+\xi v),\end{aligned} \tag{26}$$

and

$$\begin{aligned}2\Psi_*\left(\tfrac{c+d}{2},\omega\right) &\leq \Psi_*(\xi c+(1-\xi)d,\ \omega)+\Psi_*((1-\xi)c+\xi d,\ \omega), \\ 2\Psi^*\left(\tfrac{c+d}{2},\omega\right) &\leq \Psi^*(\xi c+(1-\xi)d,\ \omega)+\Psi^*((1-\xi)c+\xi d,\ \omega).\end{aligned} \tag{27}$$

From (26) and (27), we have

$$\begin{aligned}&2\left[\Psi_*\left(x,\tfrac{\mu+v}{2}\right),\Psi^*\left(x,\tfrac{\mu+v}{2}\right)\right] \\ &\leq_p [\Psi_*(x,\ \xi\mu+(1-\xi)v),\Psi^*(x,\ \xi\mu+(1-\xi)v)] \\ &+[\Psi_*(x,\ (1-\xi)\mu+\xi v),\Psi^*(x,\ (1-\xi)\mu+\xi v)],\end{aligned}$$

and

$$\begin{aligned}&2\left[\Psi_*\left(\tfrac{c+d}{2},\omega\right),\Psi^*\left(\tfrac{c+d}{2},\omega\right)\right] \\ &\leq_p [\Psi_*(\xi c+(1-\xi)d,\ \omega),\Psi^*(\xi c+(1-\xi)d,\ \omega)] \\ &+[\Psi_*(\xi c+(1-\xi)d,\ \omega),\Psi^*(\xi c+(1-\xi)d,\ \omega)],\end{aligned}$$

it follows that

$$\Psi\left(x, \frac{\mu+v}{2}\right) \leq_p \Psi(x,\ \xi\mu+(1-\xi)v)+\Psi(x,\ (1-\xi)\mu+\xi v), \tag{28}$$

and

$$\Psi\left(\frac{c+d}{2}, \omega\right) \leq_p \Psi(\xi c+(1-\xi)d,\ \omega)+\Psi(\xi c+(1-\xi)d,\ \omega) \tag{29}$$

Since $\Psi(x, .)$ and $\Psi(., \omega)$, both are LR-convex- *I-V-Fs* on coordinate, then from inequality (20), inequality (28) and (29) we have

$$\Psi\left(x, \frac{\mu+v}{2}\right) \leq_p \frac{1}{v-\mu} \int_\mu^v \Psi(x,\omega)d\omega \leq_p \frac{\Psi(x,\mu)+\Psi(x,v)}{2}. \tag{30}$$

and

$$\Psi\left(\frac{c+d}{2}, \omega\right) \leq_p \frac{1}{d-c} \int_c^d \Psi(x,\omega)dx \leq_p \frac{\Psi(c,\omega)+\Psi(d,\omega)}{2}. \tag{31}$$

Dividing double inequality (30) by $(d-c)$, and integrating with respect to x over $[c,\ d]$, we have

$$\begin{aligned} \tfrac{1}{d-c} \int_c^d \Psi\left(x, \tfrac{\mu+v}{2}\right)dx &\leq_p \tfrac{1}{(d-c)(v-\mu)} \int_c^d \int_\mu^v \Psi(x,\omega)d\omega dx \\ &\leq_p \frac{1}{2(d-c)}\left[\int_c^d \Psi(x,\mu)dx+\int_c^d \Psi(x,v)dx\right]. \end{aligned} \tag{32}$$

Similarly, dividing double inequality (31) by $(v-\mu)$, and integrating with respect to x over $[\mu,\ v]$, we have

$$\begin{aligned} \tfrac{1}{v-\mu} \int_\mu^v \Psi\left(\tfrac{c+d}{2}, \omega\right)d\omega &\leq_p \tfrac{1}{(d-c)(v-\mu)} \int_c^d \int_\mu^v \Psi(x,\omega)d\omega dx \\ &\leq_p \frac{1}{2(v-\mu)}\left[\int_\mu^v \Psi(c,\omega)d\omega+\int_\mu^v \Psi(d,\omega)d\omega\right]. \end{aligned} \tag{33}$$

By adding (32) and (33), we have

$$\begin{aligned} &\frac{1}{2}\left[\tfrac{1}{d-c} \int_c^d \Psi\left(x, \tfrac{\mu+v}{2}\right)dx+\tfrac{1}{v-\mu} \int_\mu^v \Psi\left(\tfrac{c+d}{2}, \omega\right)d\omega\right] \leq_p \tfrac{1}{(d-c)(v-\mu)} \int_c^d \int_\mu^v \Psi(x,\omega)d\omega dx \\ &\leq_p \frac{1}{4(d-c)}\left[\int_c^d \Psi(x,\mu)dx+\int_c^d \Psi(x,v)dx\right]+\frac{1}{4(v-\mu)}\left[\int_\mu^v \Psi(c,\omega)d\omega+\int_\mu^v \Psi(d,\omega)d\omega\right]. \end{aligned} \tag{34}$$

From the left side of inequality (20), we have

$$\Psi\left(\frac{c+d}{2},\ \frac{\mu+v}{2}\right) \leq_p \frac{1}{d-c} \int_c^d \Psi\left(x, \frac{\mu+v}{2}\right)dx, \tag{35}$$

$$\Psi\left(\frac{c+d}{2},\ \frac{\mu+v}{2}\right) \leq_p \frac{1}{v-\mu} \int_\mu^v \Psi\left(\frac{c+d}{2}, \omega\right)d\omega. \tag{36}$$

Taking addition of inequality (35) with inequality (36), we have

$$\Psi\left(\frac{c+d}{2},\ \frac{\mu+v}{2}\right) \leq_p \frac{1}{2}\left[\frac{1}{d-c} \int_c^d \Psi\left(x, \frac{\mu+v}{2}\right)dx+\frac{1}{v-\mu} \int_\mu^v \Psi\left(\frac{c+d}{2}, \omega\right)d\omega\right] \tag{37}$$

now from right side of inequality (20), we have

$$\frac{1}{d-c} \int_c^d \Psi(x,\mu)dx \leq_p \frac{\Psi(c,\mu)+\Psi(d,\mu)}{2} \tag{38}$$

$$\frac{1}{d-c}\int_c^d \Psi(x,v)dx \leq_p \frac{\Psi(c,v)+\Psi(d,v)}{2} \quad (39)$$

$$\frac{1}{v-\mu}\int_\mu^v \Psi(c,\omega)d\omega \leq_p \frac{\Psi(c,v)+\Psi(c,\mu)}{2} \quad (40)$$

$$\frac{1}{v-\mu}\int_\mu^v \Psi(d,\omega)d\omega \leq_p \frac{\Psi(d,v)+\Psi(d,\mu)}{2} \quad (41)$$

By adding inequalities (38)–(41), we have

$$\frac{1}{4(d-c)}\left[\int_c^d \Psi(x,\mu)dx + \int_c^d \Psi(x,v)dx\right] + \frac{1}{4(v-\mu)}\left[\int_\mu^v \Psi(c,\omega)d\omega + \int_\mu^v \Psi(d,\omega)d\omega\right] \leq_p \frac{\Psi(c,\mu)+\Psi(d,\mu)+\Psi(c,v)+\Psi(d,v)}{4} \quad (42)$$

By combining inequalities (34), (37) and (42), we obtain the desired result. □

Remark 5. *Let one take $\Psi_*(x,\omega)$ as an affine function and $\Psi^*(x,y)$ as a convex function. If $\Psi_*(x,\omega) \neq \Psi^*(x,\omega)$, then from Remark 4 and (24), we acquire the following inequality (see [23]):*

$$\begin{aligned}\Psi\left(\frac{c+d}{2}, \frac{\mu+v}{2}\right) &\supseteq \frac{1}{2}\left[\frac{1}{d-c}\int_c^d \Psi\left(x, \frac{\mu+v}{2}\right)dx + \frac{1}{v-\mu}\int_\mu^v \Psi\left(\frac{c+d}{2},\omega\right)d\omega\right] \\ &\supseteq \frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)d\omega dx \\ &\supseteq \frac{1}{4(d-c)}\left[\int_c^d \Psi(x,\mu)dx + \int_c^d \Psi(x,v)dx\right] \\ &+\frac{1}{4(v-\mu)}\left[\int_\mu^v \Psi(c,\omega)d\omega + \int_\mu^v \Psi(d,\omega)d\omega\right] \\ &\supseteq \frac{\Psi(c,\mu)+\Psi(d,\mu)+\Psi(c,v)+\Psi(d,v)}{4}.\end{aligned}$$

If $\Psi_*(x,\omega) = \Psi^*(x,\omega)$, then from (24), we acquire the coming inequality (see [11]):

$$\begin{aligned}\Psi\left(\frac{c+d}{2}, \frac{\mu+v}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{d-c}\int_c^d \Psi\left(x, \frac{\mu+v}{2}\right)dx + \frac{1}{v-\mu}\int_\mu^v \Psi\left(\frac{c+d}{2},\omega\right)d\omega\right] \\ &\leq \frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)d\omega dx \\ &\leq \frac{1}{4(d-c)}\left[\int_c^d \Psi(x,\mu)dx + \int_c^d \Psi(x,v)dx\right] \\ &+\frac{1}{4(v-\mu)}\left[\int_\mu^v \Psi(c,\omega)d\omega + \int_\mu^v \Psi(d,\omega)d\omega\right] \\ &\leq \frac{\Psi(c,\mu)+\Psi(d,\mu)+\Psi(c,v)+\Psi(d,v)}{4}.\end{aligned}$$

Example 3. We consider the I-V-Fs $\Psi : [0,\ 1] \times [0,\ 1] \to \mathbb{R}_I^+$ defined by,

$$\Psi(x) = [2,\ 6](6+e^x)(6+e^\omega)$$

Since end point functions $\Psi_*(x,\omega)$, $\Psi^*(x,\omega)$ are convex functions on the coordinate, then $\Psi(x,\omega)$ is convex I-V-F on the coordinate.

$$\Psi\left(\frac{c+d}{2},\ \frac{\mu+v}{2}\right) = \left[2\left(5+e^{\frac{1}{2}}\right)^2, 6\left(6+e^{\frac{1}{2}}\right)^2\right],$$

$$\frac{1}{2}\left[\frac{1}{d-c}\int_c^d \Psi\left(x,\ \frac{\mu+v}{2}\right)dx + \frac{1}{v-\mu}\int_\mu^v \Psi\left(\frac{c+d}{2},\omega\right)d\omega\right] = \left[4\left(6+e^{\frac{1}{2}}\right)(5+e), 12\left(6+e^{\frac{1}{2}}\right)(5+e)\right],$$

$$\frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)d\omega dx = \left[2(5+e)^2, 6(5+e)^2\right],$$

$$\frac{1}{4(d-c)}\left[\int_c^d \Psi(x,\mu)dx + \int_c^d \Psi(x,v)dx\right]$$

$$+\frac{1}{4(v-\mu)}\left[\int_\mu^v \Psi(c,\omega)d\omega + \int_\mu^v \Psi(d,\omega)d\omega\right] = [(5+e)(13+e), 3(5+e)(13+e)]$$

$$\frac{\Psi(c,\mu)+\Psi(d,\mu)+\Psi(c,v)+\Psi(d,v)}{4} = \left[\frac{(6+e)(20+e)+49}{2}, \frac{6((6+e)(20+e)+49)}{2}\right].$$

That is

$$\begin{aligned}\left[2\left(5+e^{\frac{1}{2}}\right)^2, 6\left(6+e^{\frac{1}{2}}\right)^2\right] &\leq_p \left[4\left(6+e^{\frac{1}{2}}\right)(5+e), 12\left(6+e^{\frac{1}{2}}\right)(5+e)\right]\\ &\leq_p \left[2(5+e)^2, 6(5+e)^2\right]\\ &\leq_p [(5+e)(13+e), 3(5+e)(13+e)]\\ &\leq_p \left[\frac{(6+e)(20+e)+49}{2}, 3((6+e)(20+e)+49)\right].\end{aligned}$$

Hence, Theorem 8 is verified.

We now give the HH–Fejér inequality for the LR-convex *I-V-Fs* on the coordinate via the pseudo order relation in the following result.

Theorem 9. *Let $\Psi : \Delta = [c,\ d] \times [\mu,\ v] \to \mathbb{R}_I^+$ be a LR-convex I-V-F on coordinate with $c < d$ and $\mu < v$ such that $\Psi(x,\omega) = [\Psi_*(x,\omega),\ \Psi^*(x,\omega)]$ for all $(x,\omega) \in \Delta$. Let $\mathfrak{D} : [c,\ d] \to \mathbb{R}$ with $\mathfrak{D}(x) \geq 0$, $\int_c^d \mathfrak{D}(x)dx > 0$ and $\mathcal{W} : [\mu,\ v] \to \mathbb{R}$ with $\mathcal{W}(\omega) \geq 0$, $\int_\mu^v \mathcal{W}(\omega)d\omega > 0$, be two symmetric functions with respect to $\frac{c+d}{2}$ and $\frac{\mu+v}{2}$, respectively. Then, the following inequality holds:*

$$\begin{aligned}&\Psi\left(\frac{c+d}{2}, \frac{\mu+v}{2}\right) \leq_p \frac{1}{2}\left[\frac{1}{\int_c^d \mathfrak{D}(x)dx}\int_c^d \Psi\left(x, \frac{\mu+v}{2}\right)\mathfrak{D}(x)dx + \frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega}\int_\mu^v \Psi\left(\frac{c+d}{2},\omega\right)\mathcal{W}(\omega)d\omega\right]\\ &\leq_p \frac{1}{\int_c^d \mathfrak{D}(x)dx\int_\mu^v \mathcal{W}(\omega)d\omega}\int_c^d\int_\mu^v \Psi(x,\omega)\mathfrak{D}(x)\mathcal{W}(\omega)d\omega dx\\ &\leq_p \frac{1}{4\int_c^d \mathfrak{D}(x)dx}\left[\int_c^d \Psi(x,\mu)dx + \int_c^d \Psi(x,v)dx\right]\\ &+\frac{1}{4\int_\mu^v \mathcal{W}(\omega)d\omega}\left[\int_\mu^v \Psi(c,\omega)d\omega + \int_\mu^v \Psi(d,\omega)d\omega\right]\\ &\qquad\leq_p \frac{\Psi(c,\mu)+\Psi(d,\mu)+\Psi(c,v)+\Psi(d,v)}{4}.\end{aligned} \tag{43}$$

Proof. Since Ψ both is an LR-convex *I-V-Fs* on coordinate Δ, it follows, then by Lemma 1, that functions there exist:

$$\Psi_x : [\mu,v] \to \mathbb{R}_I^+,\ \Psi_x(\omega) = \Psi(x,\omega),\ \Psi_\omega : [c,d] \to \mathbb{R}_I^+,\ \Psi_\omega(x) = \Psi(x,\omega).$$

Thus, from inequality (19), for each, we have

$$\Psi_x\left(\frac{\mu+v}{2}\right) \leq_p \frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega} \int_\mu^v \Psi_x(\omega)\mathcal{W}(\omega)d\omega \leq_p \frac{\Psi_x(\mu)+\Psi_x(v)}{2},$$

and

$$\Psi_\omega\left(\frac{c+d}{2}\right) \leq_p \frac{1}{\int_c^d \mathfrak{D}(x)dx} \int_c^d \Psi_\omega(x)\mathfrak{D}(x)dx \leq_p \frac{\Psi_\omega(c)+\Psi_\omega(d)}{2}.$$

The above inequalities can be written as

$$\Psi\left(x, \frac{\mu+v}{2}\right) \leq_p \frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega} \int_\mu^v \Psi(x,\omega)\mathcal{W}(\omega)d\omega \leq_p \frac{\Psi(x,\mu)+\Psi(x,\ v)}{2}, \tag{44}$$

and

$$\Psi\left(\frac{c+d}{2}, \omega\right) \leq_p \frac{1}{\int_c^d \mathfrak{D}(x)dx} \int_c^d \Psi(x,\omega)\mathfrak{D}(x)dx \leq_p \frac{\Psi(c,\omega)+\Psi(d,\omega)}{2} \tag{45}$$

Multiplying (44) by $\mathfrak{D}(x)$ and then integrating the resultant with respect to x over $[c,\ d]$, we have

$$\begin{aligned}\int_c^d \Psi\left(x, \tfrac{\mu+v}{2}\right)\mathfrak{D}(x)dx &\leq_p \frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega} \int_c^d \int_\mu^v \Psi(x,\omega)\mathfrak{D}(x)\mathcal{W}(\omega)d\omega dx \\ &\leq_p \int_c^d \tfrac{\Psi(x,\mu)+\Psi(x,\ v)}{2}\mathfrak{D}(x)dx\end{aligned} \tag{46}$$

Now, multiplying (45) by $\mathcal{W}(\omega)$ and then integrating the resultant with respect to ω over $[\mu,\ v]$, we have

$$\begin{aligned}\int_\mu^v \Psi\left(\tfrac{c+d}{2}, \omega\right)\mathcal{W}(\omega)d\omega &\leq_p \frac{1}{\int_c^d \mathfrak{D}(x)dx} \int_c^d \int_\mu^v \Psi(x,\omega)\mathfrak{D}(x)\mathcal{W}(\omega)dxd\omega \\ &\leq_p \int_\mu^v \tfrac{\Psi(c,\omega)+\Psi(d,\omega)}{2}\mathcal{W}(\omega)d\omega.\end{aligned} \tag{47}$$

Since $\int_c^d \mathfrak{D}(x)dx > 0$ and $\int_c^d \mathcal{W}(\omega)d\omega > 0$, then dividing (46) and (47) by $\int_c^d \mathfrak{D}(x)dx > 0$ and $\int_c^d \mathcal{W}(\omega)d\omega > 0$, respectively, we obtain

$$\begin{aligned}&\tfrac{1}{2}\left[\tfrac{1}{\int_c^d \mathfrak{D}(x)dx} \int_c^d \Psi\left(x, \tfrac{\mu+v}{2}\right)\mathfrak{D}(x)dx + \tfrac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega} \int_\mu^v \Psi\left(\tfrac{c+d}{2}, \omega\right)\mathcal{W}(\omega)d\omega\right] \\ &\quad\leq_p \tfrac{1}{\int_c^d \mathfrak{D}(x)dx \int_c^d \mathcal{W}(\omega)d\omega} \int_c^d \int_\mu^v \Psi(x,\omega)\mathfrak{D}(x)\mathcal{W}(\omega)d\omega dx. \\ &\leq_p \left[\tfrac{1}{\int_c^d \mathfrak{D}(x)dx} \int_c^d \tfrac{\Psi(x,\mu)+\Psi(x,\ v)}{4}\mathfrak{D}(x)dx + \frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega} \int_\mu^v \tfrac{\Psi(c,\omega)+\Psi(d,\omega)}{4}\mathcal{W}(\omega)d\omega\right].\end{aligned} \tag{48}$$

Now, from the left part of double inequalities (44) and (45), we obtain

$$\Psi\left(\frac{c+d}{2}, \frac{\mu+v}{2}\right) \leq_p \frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega} \int_\mu^v \Psi\left(\frac{c+d}{2}, \omega\right)\mathcal{W}(\omega)d\omega, \tag{49}$$

and

$$\Psi\left(\frac{c+d}{2}, \frac{\mu+v}{2}\right) \leq_p \frac{1}{\int_c^d \mathfrak{D}(x)dx} \int_c^d \Psi\left(x, \frac{\mu+v}{2}\right)\mathfrak{D}(x)dx \tag{50}$$

Summing the inequalities (49) and (50), we obtain

$$\Psi\left(\frac{c+d}{2}, \frac{\mu+v}{2}\right) \leq_p \frac{1}{2}\left[\frac{1}{\int_c^d \mathfrak{D}(x)dx}\int_c^d \Psi\left(x, \frac{\mu+v}{2}\right)\mathfrak{D}(x)dx + \frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega}\int_\mu^v \Psi\left(\frac{c+d}{2}, \omega\right)\mathcal{W}(\omega)d\omega\right]. \quad (51)$$

Similarly, from the right part of (44) and (45), we can obtain

$$\frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega}\int_\mu^v \Psi(c,\omega)\mathcal{W}(\omega)d\omega \leq_p \frac{\Psi(c,\mu)+\Psi(c,\ v)}{2}, \quad (52)$$

$$\frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega}\int_\mu^v \Psi(d,\omega)\mathcal{W}(\omega)d\omega \leq_p \frac{\Psi(d,\mu)+\Psi(d,\ v)}{2}, \quad (53)$$

and

$$\frac{1}{\int_c^d \mathfrak{D}(x)dx}\int_c^d \Psi(x,\mu)\mathfrak{D}(x)dx \leq_p \frac{\Psi(c,\mu)+\Psi(d,\mu)}{2} \quad (54)$$

$$\frac{1}{\int_c^d \mathfrak{D}(x)dx}\int_c^d \Psi(x,v)\mathfrak{D}(x)dx \leq_p \frac{\Psi(c,v)+\Psi(d,v)}{2} \quad (55)$$

Adding (52)–(55) and dividing by 4, we obtain

$$\begin{aligned}&\frac{1}{4\int_\mu^v \mathcal{W}(\omega)d\omega}\left[\int_\mu^v \Psi(c,\omega)\mathcal{W}(\omega)d\omega + \int_\mu^v \Psi(d,\omega)\mathcal{W}(\omega)d\omega\right]\\ &+\frac{1}{4\int_c^d \mathfrak{D}(x)dx}\left[\int_c^d \Psi(x,\mu)\mathfrak{D}(x)dx + \int_c^d \Psi(x,v)\mathfrak{D}(x)dx\right]\\ &\leq_p \frac{\Psi(c,\mu)+\Psi(c,\ v)+\Psi(d,\mu)+\Psi(d,\ v)}{4}.\end{aligned} \quad (56)$$

Combing inequalities (48), (51) and (56), we obtain

$$\begin{aligned}\Psi\left(\tfrac{c+d}{2}, \tfrac{\mu+v}{2}\right) &\leq_p \tfrac{1}{2}\left[\frac{1}{\int_c^d \mathfrak{D}(x)dx}\int_c^d \Psi\left(x, \tfrac{\mu+v}{2}\right)\mathfrak{D}(x)dx + \frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega}\int_\mu^v \Psi\left(\tfrac{c+d}{2}, \omega\right)\mathcal{W}(\omega)d\omega\right]\\ &\leq_p \frac{1}{\int_c^d \mathfrak{D}(x)dx\int_\mu^v \mathcal{W}(\omega)d\omega}\int_c^d\int_\mu^v \Psi(x,\omega)\mathfrak{D}(x)\mathcal{W}(\omega)d\omega dx\\ &\leq_p \frac{1}{4\int_\mu^v \mathcal{W}(\omega)d\omega}\left[\int_\mu^v \Psi(c,\omega)\mathcal{W}(\omega)d\omega + \int_\mu^v \Psi(d,\omega)\mathcal{W}(\omega)d\omega\right]\\ &+\frac{1}{4\int_c^d \mathfrak{D}(x)dx}\left[\int_c^d \Psi(x,\mu)\mathfrak{D}(x)dx + \int_c^d \Psi(x,v)\mathfrak{D}(x)dx\right]\\ &\leq_p \frac{\Psi(c,\mu)+\Psi(c,\ v)}{2} + \frac{\Psi(d,\mu)+\Psi(d,\ v)}{2} + \frac{\Psi(c,\mu)+\Psi(d,\mu)}{2} + \frac{\Psi(c,v)+\Psi(d,v)}{2}\end{aligned}$$

Hence, this concludes the proof. □

Remark 6. *If one takes $\mathcal{W}(\omega) = 1 = \mathfrak{D}(x)$, then from (43), we achive (24).*

Let one take $\Psi_*(x,\omega)$ is an affine function and $\Psi^*(x,\mathrm{y})$ is a convex function. If $\Psi_*(x,\omega) \neq \Psi^*(x,\omega)$, then from Remark 4 and (43), we obtain the coming inequality (see [23]):

$$\begin{aligned}
\Psi\left(\frac{c+d}{2}, \frac{\mu+v}{2}\right) &\supseteq \frac{1}{2}\left[\frac{1}{\int_c^d \mathfrak{D}(x)dx}\int_c^d \Psi\left(x, \frac{\mu+v}{2}\right)\mathfrak{D}(x)dx + \frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega}\int_\mu^v \Psi\left(\frac{c+d}{2}, \omega\right)\mathcal{W}(\omega)d\omega\right] \\
&\supseteq \frac{1}{\int_c^d \mathfrak{D}(x)dx \int_\mu^v \mathcal{W}(\omega)d\omega}\int_c^d\int_\mu^v \Psi(x,\omega)\mathfrak{D}(x)\mathcal{W}(\omega)d\omega dx \\
&\supseteq \frac{1}{4\int_\mu^v \mathcal{W}(\omega)d\omega}\left[\int_\mu^v \Psi(c,\omega)\mathcal{W}(\omega)d\omega + \int_\mu^v \Psi(d,\omega)\mathcal{W}(\omega)d\omega\right] \\
&\quad + \frac{1}{4\int_c^d \mathfrak{D}(x)dx}\left[\int_c^d \Psi(x,\mu)\mathfrak{D}(x)dx + \int_c^d \Psi(x,v)\mathfrak{D}(x)dx\right] \\
&\supseteq \frac{\Psi(c,\mu)+\Psi(c,\, v)}{2} + \frac{\Psi(d,\mu)+\Psi(d,\, v)}{2} + \frac{\Psi(c,\mu)+\Psi(d,\mu)}{2} + \frac{\Psi(c,v)+\Psi(d,v)}{2}
\end{aligned}$$

If $\Psi_*(x,\omega) = \Psi^*(x,\omega)$, then from (43), we obtain the coming inequality (see [11]):

$$\begin{aligned}
\Psi\left(\frac{c+d}{2}, \frac{\mu+v}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{\int_c^d \mathfrak{D}(x)dx}\int_c^d \Psi\left(x, \frac{\mu+v}{2}\right)\mathfrak{D}(x)dx + \frac{1}{\int_\mu^v \mathcal{W}(\omega)d\omega}\int_\mu^v \Psi\left(\frac{c+d}{2}, \omega\right)\mathcal{W}(\omega)d\omega\right] \\
&\leq \frac{1}{\int_c^d \mathfrak{D}(x)dx \int_\mu^v \mathcal{W}(\omega)d\omega}\int_c^d\int_\mu^v \Psi(x,\omega)\mathfrak{D}(x)\mathcal{W}(\omega)d\omega dx \\
&\leq \frac{1}{4\int_\mu^v \mathcal{W}(\omega)d\omega}\left[\int_\mu^v \Psi(c,\omega)\mathcal{W}(\omega)d\omega + \int_\mu^v \Psi(d,\omega)\mathcal{W}(\omega)d\omega\right] \\
&\quad + \frac{1}{4\int_c^d \mathfrak{D}(x)dx}\left[\int_c^d \Psi(x,\mu)\mathfrak{D}(x)dx + \int_c^d \Psi(x,v)\mathfrak{D}(x)dx\right] \\
&\leq \frac{\Psi(c,\mu)+\Psi(c,\, v)}{2} + \frac{\Psi(d,\mu)+\Psi(d,\, v)}{2} + \frac{\Psi(c,\mu)+\Psi(d,\mu)}{2} + \frac{\Psi(c,v)+\Psi(d,v)}{2}
\end{aligned}$$

We now obtain some HH inequalities for the product of LR-convex *I-V-Fs* on the coordinates. These inequalities are refinements of some known inequalities (see [11–13,16,23]).

Theorem 10. *Let $\Psi, \mathcal{H} : \Delta = [c,\, d] \times [\mu,\, v] \subset \mathbb{R}^2 \to \mathbb{R}_I^+$ be two LR-convex I-V-Fs on coordinate Δ, such that $\Psi(x,\omega) = [\Psi_*(x,\omega),\ \Psi^*(x,\omega)]$ and $\mathcal{H}(x,\omega) = [\mathcal{H}_*(x,\omega),\ \mathcal{H}^*(x,\omega)]$ for all $(x,\omega) \in \Delta$. Then, the following inequality holds:*

$$\begin{aligned}
&\frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega) \times \mathcal{H}(x,\omega)d\omega dx \\
&\leq_p \frac{1}{9}P(c,d,\mu,v) + \frac{1}{18}\mathcal{M}(c,d,\mu,v) + \frac{1}{36}\mathcal{N}(c,d,\mu,v).
\end{aligned} \tag{57}$$

where

$$\begin{aligned}
&P(c,d,\mu,v) = \Psi(c,\mu) \times \mathcal{H}(c,\mu) + \Psi(c,v) \times \mathcal{H}(c,v) \\
&+\Psi(d,\mu) \times \mathcal{H}(d,\mu) + \Psi(d,v) \times \mathcal{H}(d,v), \\
&\mathcal{M}(c,d,\mu,v) = \Psi(c,\mu) \times \mathcal{H}(c,v) + \Psi(c,v) \times \mathcal{H}(c,\mu) + \Psi(d,\mu) \times \mathcal{H}(d,v) + \\
&\Psi(d,v) \times \mathcal{H}(d,\mu) + \Psi(c,\mu) \times \mathcal{H}(d,\mu) + \Psi(d,v) \times \mathcal{H}(c,v) + \Psi(d,\mu) \times \mathcal{H}(c,\mu) + \\
&\Psi(c,v) \times \mathcal{H}(d,v), \\
&\mathcal{N}(c,d,\mu,v) = \Psi(c,\mu) \times \mathcal{H}(d,v) + \Psi(d,\mu) \times \mathcal{H}(c,v) + \Psi(d,v) \times \mathcal{H}(c,\mu) + \\
&\Psi(d,\mu) \times \mathcal{H}(c,v)
\end{aligned}$$

and $P(c,d,\mu,v)$, $\mathcal{M}(c,d,\mu,v)$ *and* $\mathcal{N}(c,d,\mu,v)$ *are defined as follows:*

$$\begin{aligned}P(c,d,\mu,v) &= [P_*(c,d,\mu,v),\ P^*(c,d,\mu,v)],\\ \mathcal{M}(c,d,\mu,v) &= [\mathcal{M}_*(c,d,\mu,v),\ \mathcal{M}^*(c,d,\mu,v)],\\ \mathcal{N}(c,d,\mu,v) &= [\mathcal{N}_*(c,d,\mu,v),\ \mathcal{N}^*(c,d,\mu,v)].\end{aligned}$$

Proof. Let Ψ and $\mathcal{H}$ both are LR-convex *I-V-Fs* on coordinate $[c,\ d]\times[\mu,\ v]$. Then

$$\begin{aligned}&\Psi(\xi c+(1-\xi)d,\ \varsigma\mu+(1-\varsigma)v)\\ &\leq_p \xi\varsigma\Psi(c,\mu)+\xi(1-\varsigma)\Psi(c,v)+(1-\xi)\varsigma\Psi(d,\mu)+(1-\xi)(1-\varsigma)\Psi(d,v),\end{aligned}$$

and

$$\begin{aligned}&\mathcal{H}(\xi c+(1-\xi)d,\ \varsigma\mu+(1-\varsigma)v)\\ &\leq_p \xi\varsigma\mathcal{H}(c,\mu)+\xi(1-\varsigma)\mathcal{H}(c,v)+(1-\xi)\varsigma\mathcal{H}(d,\mu)+(1-\xi)(1-\varsigma)\mathcal{H}(d,v).\end{aligned}$$

Since Ψ and $\mathcal{H}$ both are LR-convex *I-V-Fs* on coordinate then by Lemma 1, there exist

$$\Psi_x:[\mu,v]\to\mathbb{R}_I^+,\ \Psi_x(\omega)=\Psi(x,\omega),\ \mathcal{H}_x:[\mu,v]\to\mathbb{R}_I^+,\ \mathcal{H}_x(\omega)=\mathcal{H}(x,\omega),$$

and

$$\Psi_\omega:[c,d]\to\mathbb{R}_I^+,\ \Psi_\omega(x)=\Psi(x,\omega),\ \mathcal{H}_\omega:[c,d]\to\mathbb{R}_I^+,\ \mathcal{H}_\omega(x)=\mathcal{H}(x,\omega).$$

Since $\Psi_x,\ \mathcal{H}_x,\ \Psi_\omega$ and $\mathcal{H}_\omega$ are *I-V-Fs*, then by inequality (17), we have

$$\begin{aligned}\tfrac{1}{d-c}\int_c^d \Psi_\omega(x)\times\mathcal{H}_\omega(x)dx &\leq_p \tfrac{1}{3}[\Psi_\omega(c)\times\mathcal{H}_\omega(c)+\Psi_\omega(d)\times\mathcal{H}_\omega(d)]\\ &+\tfrac{1}{6}[\ \Psi_\omega(c)\times\mathcal{H}_\omega(d)+\Psi_\omega(d)\times\mathcal{H}_\omega(c)\ ],\end{aligned}$$

and

$$\begin{aligned}\tfrac{1}{v-\mu}\int_\mu^v \Psi_x(\omega)\times\mathcal{H}_x(\omega)d\omega &\leq_p \tfrac{1}{3}[\Psi_x(\mu)\times\mathcal{H}_x(\mu)+\Psi_x(v)\times\mathcal{H}_x(v)]\\ &+\tfrac{1}{6}[\ \Psi_x(\mu)\times\mathcal{H}_x(v)+\Psi_x(\mu)\times\mathcal{H}_x(v)\ ].\end{aligned}$$

The above inequalities can be written as

$$\begin{aligned}\tfrac{1}{d-c}\int_c^d \Psi(x,\omega)\times\mathcal{H}(x,\omega)dx &\leq_p \tfrac{1}{3}[\Psi(c,\omega)\times\mathcal{H}(c,\omega)+\Psi(d,\omega)\times\mathcal{H}(d,\omega)]\\ &+\tfrac{1}{6}[\ \Psi(c,\omega)\times H(d,\omega)+\Psi(d,\omega)\times H(c,\omega)\ ],\end{aligned} \tag{58}$$

and

$$\begin{aligned}\tfrac{1}{v-\mu}\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega &\leq_p \tfrac{1}{3}[\Psi(x,\mu)\times\mathcal{H}(x,\mu)+\Psi(x,v)\times\mathcal{H}(x,v)]\\ &+\tfrac{1}{6}[\ \Psi(x,\mu)\times H(x,\mu)+\Psi(x,v)\times H(x,v)\ ]\end{aligned} \tag{59}$$

Firstly, we solve inequality (58), taking integration on the both sides of inequality with respect to ω over interval $[\mu,\ v]$ and dividing both sides by $v-\mu$, we have

$$\begin{aligned}&\tfrac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx\\ &\leq_p \tfrac{1}{3(v-\mu)}\int_\mu^v[\Psi(c,\omega)\times\mathcal{H}(c,\omega)+\Psi(d,\omega)\times\mathcal{H}(d,\omega)]d\omega\\ &+\tfrac{1}{6(v-\mu)}\int_\mu^v[\Psi(c,\omega)\times\mathcal{H}(d,\omega)+\Psi(d,\omega)\times\mathcal{H}(c,\omega)]d\omega.\end{aligned} \tag{60}$$

Now, again by inequality (17), we have

$$\begin{aligned}\tfrac{1}{(v-\mu)}\int_\mu^v \Psi(c,\omega)\times\mathcal{H}(c,\omega)d\omega &\leq_p \tfrac{1}{3}\int_\mu^v[\Psi(c,\mu)\times\mathcal{H}(c,\mu)+\Psi(c,v)\times\mathcal{H}(c,v)]d\omega\\ &+\tfrac{1}{6}\int_\mu^v[\Psi(c,\mu)\times\mathcal{H}(c,v)+\Psi(c,\mu)\times\mathcal{H}(c,v)]d\omega.\end{aligned} \tag{61}$$

$$\begin{aligned}\tfrac{1}{(v-\mu)}\int_\mu^v \Psi(d,\omega)\times\mathcal{H}(d,\omega)d\omega &\leq_p \tfrac{1}{3}\int_\mu^v[\Psi(d,\mu)\times\mathcal{H}(d,\mu)+\Psi(d,v)\times\mathcal{H}(d,v)]d\omega\\ &+\tfrac{1}{6}\int_\mu^v[\Psi(d,\mu)\times\mathcal{H}(d,v)+\Psi(d,\mu)\times\mathcal{H}(c,v)]d\omega.\end{aligned} \tag{62}$$

$$\frac{1}{(v-\mu)}\int_\mu^v \Psi(c,\omega)\times\mathcal{H}(d,\omega)d\omega \leq_p \frac{1}{3}\int_\mu^v[\Psi(c,\mu)\times\mathcal{H}(d,\mu)+\Psi(c,v)\times\mathcal{H}(d,v)]d\omega + \frac{1}{6}\int_\mu^v[\Psi(c,\mu)\times\mathcal{H}(d,v)+\Psi(c,v)\times\mathcal{H}(d,\mu)]d\omega. \quad (63)$$

$$\frac{1}{(v-\mu)}\int_\mu^v \Psi(d,\omega)\times\mathcal{H}(c,\omega)d\omega \leq_p \frac{1}{3}\int_\mu^v[\Psi(d,\mu)\times\mathcal{H}(c,\mu)+\Psi(d,v)\times\mathcal{H}(c,v)]d\omega + \frac{1}{6}\int_\mu^v[\Psi(d,\mu)\times\mathcal{H}(c,v)+\Psi(d,v)\times\mathcal{H}(c,\mu)]d\omega. \quad (64)$$

From (61)–(64), inequality (60), we have

$$\frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx \leq_p \frac{1}{9}P(c,d,\mu,v)+\frac{1}{18}\mathcal{M}(c,d,\mu,v)+\frac{1}{36}\mathcal{N}(c,d,\mu,v).$$

Hence, this concludes the proof of the theorem. □

Remark 7. *Let one take $\Psi_*(x,\omega)$, $\mathcal{H}_*(x,\omega)$ are an affine function and $\Psi^*(x,y)$, $\mathcal{H}^*(x,y)$ are convex function. If $\Psi_*(x,\omega)\neq\Psi^*(x,\omega)$ and $\mathcal{H}_*(x,\omega)\neq\mathcal{H}^*(x,\omega)$, then from Remark 4 and (57), we obtain the coming inequality (see [23]):*

$$\frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx \supseteq \frac{1}{9}P(c,d,\mu,v)+\frac{1}{18}\mathcal{M}(c,d,\mu,v)+\frac{1}{36}\mathcal{N}(c,d,\mu,v).$$

If $\Psi_*(x,\omega)=\Psi^*(x,\omega)$ and $\mathcal{H}_*(x,\omega)=\mathcal{H}^*(x,\omega)$, then from (57), we obtain the coming inequality (see [16]):

$$\frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx \leq \frac{1}{9}P(c,d,\mu,v)+\frac{1}{18}\mathcal{M}(c,d,\mu,v)+\frac{1}{36}\mathcal{N}(c,d,\mu,v).$$

Theorem 11. *Let $\Psi,\mathcal{H}:\Delta=[c,\ d]\times[\mu,\ v]\subset\mathbb{R}^2\to\mathbb{R}_I^+$ be two d LR-convex I-V-Fs on the coordinate, such that $\Psi(x)=[\Psi_*(x,\omega),\ \Psi^*(x,\omega)]$ and $\mathcal{H}(x)=[\mathcal{H}_*(x,\omega),\ \mathcal{H}^*(x,\omega)]$ for all $(x,\omega)\in\Delta$. Then, the following inequality holds:*

$$4\,\Psi\left(\frac{c+d}{2},\frac{\mu+v}{2}\right)\times\mathcal{H}\left(\frac{c+d}{2},\frac{\mu+v}{2}\right) \leq_p \frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx+\frac{5}{36}P(c,d,\mu,v) + \frac{7}{36}\mathcal{M}(c,d,\mu,v)+\frac{2}{9}\mathcal{N}(c,d,\mu,v). \quad (65)$$

where $P(c,d,\mu,v)$, $\mathcal{M}(c,d,\mu,v)$ and $\mathcal{N}(c,d,\mu,v)$ are given in Theorem 10.

Proof. Since $\Psi,\mathcal{H}:\Delta\to\mathbb{R}_I^+$ be two LR-convex *I-V-Fs*, then from inequality (18), we have

$$2\Psi\left(\frac{c+d}{2},\frac{\mu+v}{2}\right)\times\mathcal{H}\left(\frac{c+d}{2},\frac{\mu+v}{2}\right) \leq_p \frac{1}{d-c}\int_c^d \Psi\left(x,\frac{\mu+v}{2}\right)\times\mathcal{H}\left(x,\frac{\mu+v}{2}\right)dx + \frac{1}{6}\left[\Psi\left(c,\frac{\mu+v}{2}\right)\times\mathcal{H}\left(c,\frac{\mu+v}{2}\right)+\Psi\left(d,\frac{\mu+v}{2}\right)\times\mathcal{H}\left(d,\frac{\mu+v}{2}\right)\right] + \frac{1}{3}\left[\Psi\left(c,\frac{\mu+v}{2}\right)\times\mathcal{H}\left(d,\frac{\mu+v}{2}\right)+\Psi\left(d,\frac{\mu+v}{2}\right)\times\mathcal{H}\left(c,\frac{\mu+v}{2}\right)\right], \quad (66)$$

and

$$2\Psi\left(\frac{c+d}{2},\frac{\mu+v}{2}\right)\times\mathcal{H}\left(\frac{c+d}{2},\frac{\mu+v}{2}\right) \leq_p \frac{1}{v-\mu}\int_\mu^v \Psi\left(\frac{c+d}{2},\omega\right)\times\mathcal{H}\left(\frac{c+d}{2},\omega\right)d\omega + \frac{1}{6}\left[\Psi\left(\frac{c+d}{2},\mu\right)\times\mathcal{H}\left(\frac{c+d}{2},\mu\right)+\Psi\left(\frac{c+d}{2},v\right)\times\mathcal{H}\left(\frac{c+d}{2},v\right)\right] + \frac{1}{3}\left[\Psi\left(\frac{c+d}{2},\mu\right)\times\mathcal{H}\left(\frac{c+d}{2},v\right)+\Psi\left(\frac{c+d}{2},v\right)\times\mathcal{H}\left(\frac{c+d}{2},\mu\right)\right]. \quad (67)$$

Summing the inequalities (66) and (67), then taking the multiplication of the resultant one by 2, we obtain

$$\begin{aligned}
&8\Psi\left(\tfrac{c+d}{2},\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\tfrac{\mu+v}{2}\right)\\
&\leq_p \tfrac{2}{d-c}\int_c^d \Psi\left(x,\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(x,\tfrac{\mu+v}{2}\right)dx+\tfrac{2}{v-\mu}\int_\mu^v \Psi\left(\tfrac{c+d}{2},\omega\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\omega\right)d\omega\\
&+\tfrac{1}{6}\left[2\Psi\left(c,\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(c,\tfrac{\mu+v}{2}\right)+2\Psi\left(d,\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(d,\tfrac{\mu+v}{2}\right)\right]\\
&+\tfrac{1}{6}\left[2\Psi\left(\tfrac{c+d}{2},\mu\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\mu\right)+2\Psi\left(\tfrac{c+d}{2},v\right)\times\mathcal{H}\left(\tfrac{c+d}{2},v\right)\right]\\
&+\tfrac{1}{3}\left[2\Psi\left(c,\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(d,\tfrac{\mu+v}{2}\right)+2\Psi\left(d,\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(c,\tfrac{\mu+v}{2}\right)\right]\\
&+\tfrac{1}{3}\left[2\Psi\left(\tfrac{c+d}{2},\mu\right)\times\mathcal{H}\left(\tfrac{c+d}{2},v\right)+2\Psi\left(\tfrac{c+d}{2},v\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\mu\right)\right].
\end{aligned}\tag{68}$$

Now, with the help of integral inequality (18) for each integral on the right-hand side of (68), we have

$$\begin{aligned}
&2\Psi\left(c,\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(c,\tfrac{\mu+v}{2}\right)\\
&\leq_p \tfrac{1}{v-\mu}\int_\mu^v \Psi(c,\omega)\times\mathcal{H}(c,\omega)d\omega+\tfrac{1}{6}[\Psi(c,\mu)\times\mathcal{H}(c,\mu)+\Psi(c,v)\times\mathcal{H}(c,v)]\\
&+\tfrac{1}{3}[\Psi(c,\mu)\times\mathcal{H}(c,v)+\Psi(c,v)\times\mathcal{H}(c,\mu)].
\end{aligned}\tag{69}$$

$$\begin{aligned}
&2\Psi\left(d,\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(d,\tfrac{\mu+v}{2}\right)\\
&\leq_p \tfrac{1}{v-\mu}\int_\mu^v \Psi(d,\omega)\times\mathcal{H}(d,\omega)d\omega+\tfrac{1}{6}[\Psi(d,\mu)\times\mathcal{H}(d,\mu)+\Psi(d,v)\times\mathcal{H}(d,v)]\\
&+\tfrac{1}{3}[\Psi(d,\mu)\times\mathcal{H}(d,v)+\Psi(d,v)\times\mathcal{H}(d,\mu)].
\end{aligned}\tag{70}$$

$$\begin{aligned}
&2\Psi\left(c,\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(d,\tfrac{\mu+v}{2}\right)\\
&\leq_p \tfrac{1}{v-\mu}\int_\mu^v \Psi(c,\omega)\times\mathcal{H}(d,\omega)d\omega+\tfrac{1}{6}[\Psi(c,\mu)\times\mathcal{H}(d,\mu)+\Psi(c,v)\times\mathcal{H}(d,v)]\\
&+\tfrac{1}{3}[\Psi(c,\mu)\times\mathcal{H}(d,v)+\Psi(c,v)\times\mathcal{H}(d,\mu)].
\end{aligned}\tag{71}$$

$$\begin{aligned}
&2\Psi\left(d,\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(c,\tfrac{\mu+v}{2}\right)\\
&\leq_p \tfrac{1}{v-\mu}\int_\mu^v \Psi(d,\omega)\times\mathcal{H}(c,\omega)d\omega+\tfrac{1}{6}[\Psi(d,\mu)\times\mathcal{H}(c,\mu)+\Psi(d,v)\times\mathcal{H}(c,v)]\\
&+\tfrac{1}{3}[\Psi(d,\mu)\times\mathcal{H}(c,v)+\Psi(d,v)\times\mathcal{H}(c,\mu)].
\end{aligned}\tag{72}$$

$$\begin{aligned}
&2\Psi\left(\tfrac{c+d}{2},\mu\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\mu\right)\\
&\leq_p \tfrac{1}{d-c}\int_c^d \Psi(x,\mu)\times\mathcal{H}(x,\mu)dx+\tfrac{1}{6}[\Psi(c,\mu)\times\mathcal{H}(c,\mu)+\Psi(d,\mu)\times\mathcal{H}(d,\mu)]\\
&+\tfrac{1}{3}\left[\Psi\left(\tfrac{c+d}{2},\mu\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\mu\right)+\Psi\left(\tfrac{c+d}{2},\mu\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\mu\right)\right].
\end{aligned}\tag{73}$$

$$\begin{aligned}
&2\Psi\left(\tfrac{c+d}{2},v\right)\times\mathcal{H}\left(\tfrac{c+d}{2},v\right)\\
&\leq_p \tfrac{1}{d-c}\int_c^d \Psi(x,v)\times\mathcal{H}(x,v)dx+\tfrac{1}{6}[\Psi(c,v)\times\mathcal{H}(c,v)+\Psi(d,v)\times\mathcal{H}(d,v)]\\
&+\tfrac{1}{3}\left[\Psi\left(\tfrac{c+d}{2},v\right)\times\mathcal{H}\left(\tfrac{c+d}{2},v\right)+\Psi\left(\tfrac{c+d}{2},v\right)\times\mathcal{H}\left(\tfrac{c+d}{2},v\right)\right].
\end{aligned}\tag{74}$$

$$\begin{aligned}
&2\Psi\left(\tfrac{c+d}{2},\mu\right)\times\mathcal{H}\left(\tfrac{c+d}{2},v\right)\\
&\leq_p \tfrac{1}{d-c}\int_c^d \Psi(x,\mu)\times\mathcal{H}(x,v)dx+\tfrac{1}{6}[\Psi(c,\mu)\times\mathcal{H}(c,v)+\Psi(d,\mu)\times\mathcal{H}(d,v)]\\
&+\tfrac{1}{3}\left[\Psi\left(\tfrac{c+d}{2},\mu\right)\times\mathcal{H}\left(\tfrac{c+d}{2},v\right)+\Psi\left(\tfrac{c+d}{2},\mu\right)\times\mathcal{H}\left(\tfrac{c+d}{2},v\right)\right].
\end{aligned}\tag{75}$$

$$\begin{aligned}
&2\Psi\left(\tfrac{c+d}{2},v\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\mu\right)\\
&\leq_p \tfrac{1}{d-c}\int_c^d \Psi(x,v)\times\mathcal{H}(x,\mu)dx+\tfrac{1}{6}[\Psi(c,v)\times\mathcal{H}(c,\mu)+\Psi(d,v)\times\mathcal{H}(d,\mu)]\\
&+\tfrac{1}{3}\left[\Psi\left(\tfrac{c+d}{2},v\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\mu\right)+\Psi\left(\tfrac{c+d}{2},v\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\mu\right)\right].
\end{aligned}\tag{76}$$

From (69)–(76), we have

$$
\begin{aligned}
&8\Psi\left(\frac{c+d}{2},\frac{\mu+v}{2}\right)\times\mathcal{H}\left(\frac{c+d}{2},\frac{\mu+v}{2}\right)\\
&\leq_p \frac{2}{d-c}\int_c^d \Psi\left(x,\frac{\mu+v}{2}\right)\times\mathcal{H}\left(x,\frac{\mu+v}{2}\right)dx+\frac{2}{v-\mu}\int_\mu^v \Psi\left(\frac{c+d}{2},\omega\right)\times\mathcal{H}\left(\frac{c+d}{2},\omega\right)dx\\
&+\frac{1}{6(v-\mu)}\int_\mu^v \Psi(c,\omega)\times\mathcal{H}(c,\omega)d\omega+\frac{1}{6(v-\mu)}\int_\mu^v \Psi(d,\omega)\times\mathcal{H}(d,\omega)d\omega\\
&+\frac{1}{6(d-c)}\int_c^d \Psi(x,\mu)\times\mathcal{H}(x,\mu)dx+\frac{1}{6(d-c)}\int_c^d \Psi(x,v)\times\mathcal{H}(x,v)dx\\
&+\frac{1}{3(v-\mu)}\int_\mu^v \Psi(c,\omega)\times\mathcal{H}(d,\omega)d\omega+\frac{1}{3(v-\mu)}\int_\mu^v \Psi(d,\omega)\times\mathcal{H}(c,\omega)d\omega\\
&+\frac{1}{3(d-c)}\int_c^d \Psi(x,\mu)\times\mathcal{H}(x,v)dx+\frac{1}{3(d-c)}\int_c^d \Psi(x,v)\times\mathcal{H}(x,\mu)dx\\
&+\frac{1}{18}P(c,d,\mu,v)+\frac{1}{9}\mathcal{M}(c,d,\mu,v)+\frac{2}{9}\mathcal{N}(c,d,\mu,v).
\end{aligned}
\tag{77}
$$

Now, again with the help of integral inequality (18) for first two integrals on the right-hand side of (77), we have the following relation

$$
\begin{aligned}
&\frac{2}{d-c}\int_c^d \Psi\left(x,\frac{\mu+v}{2}\right)\times\mathcal{H}\left(x,\frac{\mu+v}{2}\right)dx\\
&\leq_p \frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx\\
&+\frac{1}{3(d-c)}\int_c^d[\Psi(x,\mu)\times\mathcal{H}(x,\mu)+\Psi(x,v)\times\mathcal{H}(x,v)]dx\\
&+\frac{1}{6(d-c)}\int_c^d[\Psi(\mu,x)\times\mathcal{H}(x,v)+\Psi(x,v)\times\mathcal{H}(x,\mu)]dx,
\end{aligned}
\tag{78}
$$

$$
\begin{aligned}
&\frac{2}{v-\mu}\int_\mu^v \Psi\left(\frac{c+d}{2},\omega\right)\times\mathcal{H}\left(\frac{c+d}{2},\omega\right)dx\\
&\leq_p \frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx\\
&+\frac{1}{3(v-\mu)}\int_\mu^v[\Psi(c,\omega)\times\mathcal{H}(c,\omega)+\Psi(d,\omega)\times\mathcal{H}(d,\omega)]d\omega\\
&+\frac{1}{6(v-\mu)}\int_\mu^v[\Psi(c,\omega)\times\mathcal{H}(d,\omega)+\Psi(d,\omega)\times\mathcal{H}(c,\omega)]d\omega.
\end{aligned}
\tag{79}
$$

From (78) and (79), we have

$$
\begin{aligned}
&8\Psi\left(\frac{c+d}{2},\frac{\mu+v}{2}\right)\times\mathcal{H}\left(\frac{c+d}{2},\frac{\mu+v}{2}\right)\\
&\leq_p \frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx\\
&+\frac{1}{3(d-c)}\int_c^d[\Psi(x,\mu)\times\mathcal{H}(x,\mu)+\Psi(x,v)\times\mathcal{H}(x,v)]dx\\
&+\frac{1}{6(d-c)}\int_c^d[\Psi(x,\mu)\times\mathcal{H}(x,v)+\Psi(x,v)\times\mathcal{H}(x,\mu)]dx\\
&+\frac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx\\
&+\frac{1}{3(v-\mu)}\int_\mu^v[\Psi(c,\omega)\times\mathcal{H}(c,\omega)+\Psi(d,\omega)\times\mathcal{H}(d,\omega)]d\omega\\
&+\frac{1}{6(v-\mu)}\int_\mu^v[\Psi(c,\omega)\times\mathcal{H}(d,\omega)+\Psi(d,\omega)\times\mathcal{H}(c,\omega)]d\omega,\\
&+\frac{1}{6(v-\mu)}\int_\mu^v \Psi(c,\omega)\times\mathcal{H}(c,\omega)d\omega+\frac{1}{6(v-\mu)}\int_\mu^v \Psi(d,\omega)\times\mathcal{H}(d,\omega)d\omega\\
&+\frac{1}{6(d-c)}\int_c^d \Psi(x,\mu)\times\mathcal{H}(x,\mu)dx+\frac{1}{6(d-c)}\int_c^d \Psi(x,v)\times\mathcal{H}(x,v)dx\\
&+\frac{1}{3(v-\mu)}\int_\mu^v \Psi(c,\omega)\times\mathcal{H}(d,\omega)d\omega+\frac{1}{3(v-\mu)}\int_\mu^v \Psi(d,\omega)\times\mathcal{H}(c,\omega)d\omega\\
&+\frac{1}{3(d-c)}\int_c^d \Psi(x,\mu)\times\mathcal{H}(x,v)dx+\frac{1}{3(d-c)}\int_c^d \Psi(x,v)\times\mathcal{H}(x,\mu)dx\\
&+\frac{1}{18}P(c,d,\mu,v)+\frac{1}{9}\mathcal{M}(c,d,\mu,v)+\frac{2}{9}\mathcal{N}(c,d,\mu,v).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&8\Psi\left(\frac{c+d}{2},\frac{\mu+v}{2}\right)\times\mathcal{H}\left(\frac{c+d}{2},\frac{\mu+v}{2}\right)\\
&\leq_p \frac{2}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx\\
&+\frac{2}{3(d-c)}\int_c^d[\Psi(x,\mu)\times\mathcal{H}(x,\mu)+\Psi(x,v)\times\mathcal{H}(x,v)]dx\\
&+\frac{1}{3(d-c)}\int_c^d[\Psi(x,\mu)\times\mathcal{H}(x,v)+\Psi(x,v)\times\mathcal{H}(x,\mu)]dx\\
&+\frac{2}{3(v-\mu)}\int_\mu^v[\Psi(c,\omega)\times\mathcal{H}(c,\omega)+\Psi(d,\omega)\times\mathcal{H}(d,\omega)]d\omega\\
&+\frac{1}{3(v-\mu)}\int_\mu^v[\Psi(c,\omega)\times\mathcal{H}(d,\omega)+\Psi(d,\omega)\times\mathcal{H}(c,\omega)]d\omega\\
&+\frac{1}{18}P(c,d,\mu,v)+\frac{1}{9}\mathcal{M}(c,d,\mu,v)+\frac{2}{9}\mathcal{N}(c,d,\mu,v).
\end{aligned}
\tag{80}
$$

Now, using integral inequality (17) for integrals on the right-hand side of (78), we have the following relation

$$\begin{aligned}\frac{1}{d-c}\int_c^d \Psi(x,\mu)\times\mathcal{H}(x,\mu)dx \leq_p \tfrac{1}{3}[\Psi(c,\mu)\times\mathcal{H}(c,\mu)+\Psi(d,\mu)\times\mathcal{H}(d,\mu)]\\ +\tfrac{1}{6}[\ \Psi(c,\mu)\times H(d,\mu)+\Psi(d,\mu)\times H(c,\mu)\ ],\end{aligned} \tag{81}$$

$$\begin{aligned}\frac{1}{d-c}\int_c^d \Psi(x,v)\times\mathcal{H}(x,v)dx \leq_p \tfrac{1}{3}[\Psi(c,v)\times\mathcal{H}(c,v)+\Psi(d,v)\times\mathcal{H}(d,v)]\\ +\tfrac{1}{6}[\ \Psi(c,v)\times H(d,v)+\Psi(d,v)\times H(c,v)\ ],\end{aligned} \tag{82}$$

$$\begin{aligned}\frac{1}{d-c}\int_c^d \Psi(x,\mu)\times\mathcal{H}(x,v)dx \leq_p \tfrac{1}{3}[\Psi(c,\mu)\times\mathcal{H}(c,v)+\Psi(d,\mu)\times\mathcal{H}(d,v)]\\ +\tfrac{1}{6}[\ \Psi(c,\mu)\times H(d,v)+\Psi(d,\mu)\times H(c,v)\ ],\end{aligned} \tag{83}$$

$$\begin{aligned}\frac{1}{d-c}\int_c^d \Psi(x,v)\times\mathcal{H}(x,\mu)dx \leq_p \tfrac{1}{3}[\Psi(c,v)\times\mathcal{H}(c,\mu)+\Psi(d,v)\times\mathcal{H}(d,\mu)]\\ +\tfrac{1}{6}[\ \Psi(c,v)\times H(d,\mu)+\Psi(d,v)\times H(c,\mu)\ ],\end{aligned} \tag{84}$$

$$\begin{aligned}\frac{1}{v-\mu}\int_\mu^v \Psi(c,\omega)\times\mathcal{H}(c,\omega)d\omega \leq_p \tfrac{1}{3}[\Psi(c,\mu)\times\mathcal{H}(c,\mu)+\Psi(c,v)\times\mathcal{H}(c,v)]\\ +\tfrac{1}{6}[\ \Psi(c,\mu)\times H(c,v)+\Psi(c,v)\times H(c,\mu)\ ],\end{aligned} \tag{85}$$

$$\begin{aligned}\frac{1}{v-\mu}\int_\mu^v \Psi(d,\omega)\times\mathcal{H}(d,\omega)d\omega \leq_p \tfrac{1}{3}[\Psi(d,\mu)\times\mathcal{H}(d,\mu)+\Psi(d,v)\times\mathcal{H}(d,v)]\\ +\tfrac{1}{6}[\ \Psi(d,\mu)\times H(d,v)+\Psi(d,v)\times H(d,\mu)\ ],\end{aligned} \tag{86}$$

$$\begin{aligned}\frac{1}{v-\mu}\int_\mu^v \Psi(c,\omega)\times\mathcal{H}(d,\omega)d\omega \leq_p \tfrac{1}{3}[\Psi(c,\mu)\times\mathcal{H}(d,\mu)+\Psi(c,v)\times\mathcal{H}(d,v)]\\ +\tfrac{1}{6}[\ \Psi(c,\mu)\times H(d,v)+\Psi(c,v)\times H(d,\mu)\ ],\end{aligned} \tag{87}$$

$$\begin{aligned}\frac{1}{v-\mu}\int_\mu^v \Psi(d,\omega)\times\mathcal{H}(c,\omega)d\omega \leq_p \tfrac{1}{3}[\Psi(d,\mu)\times\mathcal{H}(c,\mu)+\Psi(d,v)\times\mathcal{H}(c,v)]\\ +\tfrac{1}{6}[\ \Psi(d,\mu)\times H(c,v)+\Psi(d,v)\times H(c,\mu)\ ].\end{aligned} \tag{88}$$

From (81)–(88) and inequality (80) we have

$$\begin{aligned}&4\,\Psi\left(\tfrac{c+d}{2},\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\tfrac{\mu+v}{2}\right)\\ &\leq_p \tfrac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx+\tfrac{5}{36}P(c,d,\mu,v)\\ &+\tfrac{7}{36}\mathcal{M}(c,d,\mu,v)+\tfrac{2}{9}\mathcal{N}(c,d,\mu,v).\end{aligned}$$

This concludes the proof. □

Remark 8. *Let one take $\Psi_*(x,\omega)$, $\mathcal{H}_*(x,\omega)$ as an affine function and $\Psi^*(x,\mathrm{y})$, $\mathcal{H}^*(x,\mathrm{y})$ as a convex function. If $\Psi_*(x,\omega)\neq\ \Psi^*(x,\omega)$ and $\mathcal{H}_*(x,\omega)\neq\ \mathcal{H}^*(x,\omega)$, then from Remark 4 and (65), we obtain the coming inequality (see [23]):*

$$\begin{aligned}&4\,\Psi\left(\tfrac{c+d}{2},\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\tfrac{\mu+v}{2}\right)\\ &\supseteq \tfrac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx+\tfrac{5}{36}P(c,d,\mu,v)\\ &+\tfrac{7}{36}\mathcal{M}(c,d,\mu,v)+\tfrac{2}{9}\mathcal{N}(c,d,\mu,v).\end{aligned}$$

if $\Psi_*(x,\omega)=\Psi^*(x,\omega)$ and $\mathcal{H}_*(x,\omega)=\mathcal{H}^*(x,\omega)$, then from (65), we obtain the coming inequality (see [16]):

$$\begin{aligned}&4\,\Psi\left(\tfrac{c+d}{2},\tfrac{\mu+v}{2}\right)\times\mathcal{H}\left(\tfrac{c+d}{2},\tfrac{\mu+v}{2}\right)\\ &\leq \tfrac{1}{(d-c)(v-\mu)}\int_c^d\int_\mu^v \Psi(x,\omega)\times\mathcal{H}(x,\omega)d\omega dx+\tfrac{5}{36}P(c,d,\mu,v)\\ &+\tfrac{7}{36}\mathcal{M}(c,d,\mu,v)+\tfrac{2}{9}\mathcal{N}(c,d,\mu,v).\end{aligned}$$

4. Conclusions

We introduced LR-convex interval-valued functions on coordinates through pseudo order relation. Moreover, we demonstrated various Hermite–Hadamard type inequalities via LR-convexity for interval-valued functions on coordinates. Our findings broaden the scope of several well-known inequalities and will aid in the development of interval integral inequalities and interval convex analysis theory. Inequalities for preinvex interval-valued functions, as well as certain applications in interval nonlinear programming, are the next steps for this study.

Finally, we think that our findings may be applied to other fractional calculus models having Mittag–Liffler functions in their kernels, such as Atangana–Baleanue and Prabhakar fractional operators. This consideration has been kept as an open problem for academics interested in this topic. Researchers that are interested might follow the steps outlined in references [39,40].

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