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Generalized Nonlocal Symmetries of Two-Component Camassa–Holm and Hunter–Saxton Systems

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Abstract: The two-component Camassa–Holm system and two-component Hunter–Saxton system are completely integrable models. In this paper, it is shown that these systems admit nonlocal symmetries by their geometric integrability. As an application, we obtain the recursion operator and conservation laws by using this kind of nonlocal symmetries.

Keywords: nonlocal symmetries; two-component Camassa–Holm system; two-component Hunter–Saxton system; integrability property

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1. Introduction

This paper mainly discusses nonlocal symmetries, conservation laws, and recursion operators of the two-component Camassa–Holm system [1,2] and the two-component Hunter–Saxton system [1,2]. These systems have Lax-pairs and bi-Hamiltonian structures, which are completely integrable systems. It has long been known that integrable systems have nonlocal symmetries, which is an interesting trait. The nonlocal symmetries of an integrable equation are related to the conservation law of the equations, the exact solutions, the Darboux transformation, and the integrability of the equations. Therefore, it is very important to study the nonlocal symmetries of integrable equations. As an application, we obtain the recursion operator and conservation laws by using this kind of nonlocal symmetries.

The famous Camassa–Holm equation [3,4]

$$u_t - u_{xxt} = -3u_x u + uu_{xxx} + 2u_x u_{xx} \tag{1}$$

was first derived by Fokas and Fuchssteiner and Camassa and Holm. This model describes the unidirectional propagation of shallow water waves over a flat bottom. In fact, Fokas and Fuchssteiner [4] obtained the Camassa–Holm equation through the integrability of the KdV equation. Interestingly, Olver and Rosenau also obtained the Camassa–Holm equation by using the tri-Hamiltonian duality method [1].

If we introduce $m = u - u_{xx}$, then Equation (1) can be rewritten in the following form:

$$\begin{aligned} m_t &= -m_x u - 2mu_x, \\ m &= u - u_{xx}. \end{aligned} \tag{2}$$

Olver, Rosenau and Chen Liu Zhang obtained a two-component Camassa–Holm system, which is an extension of the Camassa–Holm equation,

$$\begin{aligned} &m_t + 2mu_x + um_x - \rho\rho_x = 0, \\ &\rho_t + (\rho u)_x = 0, \end{aligned} \tag{3}$$

where $m = u - u_{xx}$. They point out that this system admits bi-Hamiltonian structure and Lax-pairs. The two-component Camassa–Holm system is also derived from the Green–Naghdi equation by Constantin and Ivanov [5]. The two-component Camassa–Holm system is a geodesic flow concerning the $H1$ metric on the semidirect product space $Diffs(S1) \ltimes C1(S1)$ [6,7]. Wave breaking phenomena of system (2) that have a certain initial value have been studied extensively. Various properties of the Camassa–Holm equation have been studied extensively [8–10]. Reyes shows that the Camassa–Holm equation has geometric integrability [11–14]. The Camassa–Holm equation can be obtained from a non-stretching invariant plane curve flow in Centro-Affine differential geometry by Chou and Qu in [15]. In Reference [16], Misiolek shows that the Camassa–Holm equation is a geodesic flow of a right-invariant on the Virasoro group. The two-component Camassa–Holm and Hunter–Saxton systems also have drawn much attention and have multi-peakon solitons [5]. As an extension of the Camassa–Holm equation, μ Camassa–Holm type equations also have geometric integrability and a bi-Hamiltonian structure, drawing much attention [17–22].

The outline of this paper is as follows. In Section 2, the generalized symmetries and their commutators of the two-component Camassa–Holm system are constructed. A recursion operator for the two-component Camassa–Holm system is obtained in Section 3. In Section 4, we construct generalized nonlocal symmetries and an infinite number of the two-component Hunter–Saxton system. Section 5 presents a concluding remark on this work.

2. Nonlocal Symmetries of the Two-Component Camassa–Holm System

2.1. Pseudo-Spherical Surface

Definition 1 ([23])**.** *A scalar differential equation $F(x, t, u, u_x, \cdots, u_{x^n,t^m}) = 0$ in two independent variables x, t is of pseudo-spherical type (or if it describes pseudo-spherical surfaces) if there exist one-forms $\omega^\alpha = 0$, defined as*

$$\omega^\alpha = f_{\alpha 1}(x, t, u, \cdots, u_{x^r,t^p})dx + f_{\alpha 2}(x, t, u, \cdots, u_{x^s,t^q})dt, \quad \alpha = 1, 2, 3, \tag{4}$$

for which the coefficients $f_{\alpha,\beta}$ are smooth functions that depend on x, t and a finite number of derivatives of u, such that the one-forms $\overline{\omega}^\alpha = \omega^\alpha(u(x, t))$ satisfy the structure equations given by

$$\begin{aligned} d\overline{\omega}^1 &= \omega^3 \wedge \omega^2, \\ d\overline{\omega}^2 &= \omega^1 \wedge \omega^3, \\ d\overline{\omega}^3 &= \omega^1 \wedge \omega^2, \end{aligned}$$

whenever $u = u(x, t)$ is a solution of $F(x, t, u, u_x, \cdots, u_{x^n,t^m}) = 0$.

Definition 2 ([23])**.** *An equation $F(x, t, u, u_x, \cdots, u_{x^n,t^m}) = 0$ is geometrically integrable if it describes a nontrial one-parameter family of pseudo-spherical surfaces.*

Proposition 1 ([23])**.** *Let $F(x, t, u, u_x, \cdots, u_{x^n,t^m}) = 0$ be a differential equation describing pseudo-spherical surfaces with associated one-forms ω^α. The following two Pfaffian systems are completely integrable whenever $u(x, t)$ is a solution of $F(x, t, u, u_x, \cdots, u_{x^n,t^m}) = 0$,*

$$-2d\Gamma = \omega^3 + \omega^2 - 2\Gamma\omega^1 + \Gamma^2(\omega^3 - \omega^2), \tag{5}$$
$$-2d\Omega = \omega^3 - \omega^2 - 2\Omega\omega^1 + \Omega^2(\omega^3 + \omega^2). \tag{6}$$

Moreover, the one-forms

$$\theta = \omega^1 - \Gamma(\omega^3 - \omega^2), \tag{7}$$
$$\hat{\theta} = -\omega^1 + \Gamma(\omega^3 + \omega^2) \tag{8}$$

are closed whenever $u(x,t)$ *is a solution of* $F(x,t,u,u_x,\cdots,u_{x^n,t^m})=0$ *and* Γ *(respectively,* Ω*) is a solution of* (5) *(respectively,* (6)*).*

Proposition 2 ([23])**.** *The two-component Camassa–Holm system describes pseudo-spherical surfaces; therefore, it is geometrically integrable.*

For Proposition 2, we have the one-form [8]

$$\begin{aligned}\omega_1 =&\left[\left(\rho^2-\frac{1}{4}\right)\lambda^2+\lambda m+\frac{5}{4}\right]\mathrm{d}x+\left[\left(\frac{1}{4}-\rho^2\right)u\lambda^2\right.\\&\left.+\left(\frac{1}{2}\rho^2-um+\frac{1}{2}u_x-\frac{1}{8}\right)\lambda-\frac{3}{4}u+\frac{5}{8}\lambda^{-1}\right]\mathrm{d}t,\\ \omega_2 =&\lambda\mathrm{d}x-\left(\lambda u+u_x-\frac{1}{2}\right)\mathrm{d}t,\\ \omega_3 =&\left[\left(\frac{1}{4}-\rho^2\right)\lambda^2-\lambda m+\frac{3}{4}\right]\mathrm{d}x\\&+\left[\left(\rho^2-\frac{1}{4}\right)u\lambda^2-\left(\frac{1}{2}\rho^2-um+\frac{1}{2}u_x-\frac{1}{8}\right)\lambda\right.\\&\left.-\frac{5}{4}u+\frac{3}{8}\lambda^{-1}\right]\mathrm{d}t,\end{aligned} \tag{9}$$

which is associated with two-component Camassa–Holm system.

Theorem 1 ([8])**.** *The two-component Camassa–Holm system admits a quadratic pseudo-potential* α*, which is defined by the system*

$$\begin{aligned}\alpha_x &= -\alpha^2+\lambda^2\rho^2+\lambda m+\tfrac{1}{4}\\ \alpha_t &= \left[\left(\tfrac{1}{2\lambda}-u\right)\alpha+\tfrac{1}{2}u_x\right]_x\end{aligned} \tag{10}$$

where $\lambda\neq 0$, $m=u-u_{xx}$

As an application, an infinite number of conservation laws with Equation (10) can be obtained. We expand α as

$$\alpha=\sum_{n=-2}^{\infty}\alpha_n(x,t)\lambda^{-\frac{n}{2}} \tag{11}$$

Furthermore, we substitute the above equation into the conservation law described by Equation (10) and possess the parameter λ. The following system on α_j has been obtained:

$$\begin{aligned}&\alpha_{-2}^2(x,t)=\rho,\\ &-2\alpha_{-1}(x,t)\alpha_{-2}(x,t)=0,\\ &\alpha_{-2,x}^2=-\alpha_{-1}^2(x,t)-2\alpha_0(x,t)\alpha_{-2}(x,t)+m,\\ &\quad\cdots\end{aligned} \tag{12}$$

Thus, we calculate an infinite number of conservation laws of the two-component Camassa–Holm equation. The first three conservation densities are given by

$$\begin{aligned}H_1&=\int\rho\mathrm{d}x,\\ H_2&=\int\frac{m}{\rho}\mathrm{d}x\\ H_3&=\int\frac{\rho_x^3+\rho^2-\rho m^2}{\rho^3}\mathrm{d}x\end{aligned} \tag{13}$$

Now, if we set

$$\alpha = \sum_{n=0}^{\infty} \alpha_n(x,t)\lambda^{\frac{n}{2}}, \tag{14}$$

then Equation (10) can be rewritten as

$$\left(\sum_{n=0}^{\infty} \alpha_n(x,t)\lambda^{\frac{n}{2}}\right)_x = -\left(\sum_{n=0}^{\infty} \alpha_n(x,t)\lambda^{\frac{n}{2}}\right)^2 + \lambda m + \lambda^2\rho^2 + \frac{1}{4} \tag{15}$$

The following system on $\alpha_j(x,t)$ has been obtained:

$$\alpha_{0,x} = -\alpha_0(x,t)^2 + \frac{1}{4}, \tag{16}$$
$$\alpha_{1,x} = -2\alpha_0(x,t)\alpha_1(x,t), \tag{17}$$
$$\alpha_{2,x} = -2\alpha_0(x,t)\alpha_2(x,t) - \alpha_1(x,t)^2 + m, \tag{18}$$
$$\cdots \tag{19}$$

As a result, the first three conservation functionals are constructed,

$$H_1 = \int u \,\mathrm{d}x, \tag{20}$$
$$H_2 = \int \left(\rho^2 - u^2 - u_x^2\right)\mathrm{d}x, \tag{21}$$
$$H_3 = \int (p^4 - 2p^2u^2 + 4p^2uu_x - 2p^2u_x^2 + u^4 + 6u^2u_x^2 \tag{22}$$
$$- 4uu_x^3 + u_x^4 + p^2u - p^2u_x - u^3 - 3uu_x^2 + u_x^3)\mathrm{d}_x. \tag{23}$$

We obtain the same conservation densities as in [8], where the pseudo-potential function expands on "λ^n" and "λ^{-n}".

2.2. Nonlocal Symmetries for the Two-Component Camassa–Holm System

Definition 3 ([11])**.** *Let N be a nonzero integer or $N = \infty$. An $N-$dimensional covering π of a (system of) partial differential equation(s) $F_a = 0, a = 1, \cdots, k$, is a triplet*

$$(\alpha^b, b = 1, \cdots, N; X_{ib}, b = 1, \cdots, N, i = 1, \cdots, N; \tilde{D}_i, i = 1, \cdots, n) \tag{24}$$

of variables α^b, smooth functions X_{ib} depending on x^i, u^α, α_b, and a finite number of partial derivatives of u^α, and linear operators

$$\tilde{D}_i = D_i + \sum_{b=1}^{N} X_{ib}\frac{\partial}{\partial \alpha^b}, \tag{25}$$

such that equations

$$\tilde{D}_i(X_{jb}) = \tilde{D}_j(X_{ib}), \quad i,j = 1, \cdots, n, b = 1, \cdots, N \tag{26}$$

hold whenever $u^\alpha(X^i)$ is a solution of $F_a = 0$.

Definition 4 ([11])**.** *Let $F_a = 0, a = 1, \cdots, k$ be a system of partial differential equations, with $\pi = (\alpha^b, X_{i,b}, \tilde{D}_i)$ a covering of $F_a = 0$. A nonlocal π-symmetry of $F_a = 0$ is a generalized symmetry*

$$X = \sum_a G^a \frac{\partial}{\partial u^a} + \sum_b H^b \frac{\partial}{\partial u^b} \tag{27}$$

of the augmented system

$$F_a = 0, \tag{28}$$

$$\frac{\partial \alpha^b}{\partial x^i} = X_{ib}, \tag{29}$$

Now, a pseudo-potential α in Equation (10) has been defined. Then, the potential function δ is defined as

$$\delta_x = \alpha, \tag{30}$$

$$\delta_t = \left(\frac{1}{2\lambda} - u\right)\alpha + \frac{1}{2}u_x \tag{31}$$

Based on the infinitesimal criteria [24] for symmetries, the two-component Camassa–Holm system admits the following evolutionary vector field,

$$\begin{aligned} W =& \alpha e^{2\delta}\frac{\partial}{\partial u} - \left[2\lambda^2\rho\rho_x + \lambda m_x + \left(4\lambda^2\rho^2 + 4\lambda m\right)\alpha\right]e^{2\delta}\frac{\partial}{\partial m}, \\ & - \lambda(\rho_x + 2\alpha\rho)e^{2\delta}\frac{\partial}{\partial \rho}. \end{aligned} \tag{32}$$

If we introduce $\beta(x,t)$, which satisfies the following system,

$$\beta_x = -e^{2\delta}\left(2\lambda^3\rho^2 + \lambda^2 m\right), \tag{33}$$

$$\beta_t = e^{2\delta}\left(-\frac{\alpha^2}{2} + 2\lambda^3 u\rho^2 + \lambda^2 um - \frac{1}{2}\lambda^2\rho^2 + \frac{1}{8}\right). \tag{34}$$

then the two-component Camassa–Holm system admits the following nonlocal symmetry:

$$\begin{aligned} V =& \alpha e^{2\delta}\frac{\partial}{\partial u} - \left[2\lambda^2\rho\rho_x + \lambda m_x + \left(4\lambda^2\rho^2 + 4\lambda m\right)\alpha\right]e^{2\delta}\frac{\partial}{\partial m} \\ & - \lambda(\rho_x + 2\rho\alpha)e^{2\delta}\frac{\partial}{\partial \rho} - \left(2\lambda^3\rho^2 + \lambda^2 m\right)e^{2\delta}\frac{\partial}{\partial \alpha} \\ & + \beta\frac{\partial}{\partial \delta} + \left[\beta^2 + \left(3\lambda^4\rho^2 + \lambda^3 m\right)e^{4\delta}\right]\frac{\partial}{\partial \beta}. \end{aligned} \tag{35}$$

This result has been proved in [8].

Theorem 2. *The generalized symmetries for the augmented two-component Camassa–Holm system* (10), (30), *and* (36) *admit the following vector fields,*

$$
\begin{aligned}
V_1 =& \ (-2u_x m - u m_x - \rho\rho_x)\frac{\partial}{\partial m} - u_t\frac{\partial}{\partial u} - (\rho u)_x\frac{\partial}{\partial \rho} \\
&+ \left(\left(\frac{1}{2\lambda} - u\right)\alpha + \frac{1}{2}u_x\right)_x\frac{\partial}{\partial \alpha} + \left(\left(\frac{1}{2\lambda} - u\right)\alpha + \frac{1}{2}u_x\right)\frac{\partial}{\partial \delta} \\
&+ e^{2\delta}\left(-\frac{\alpha^2}{2} + 2\lambda^3 u\rho^2 + \lambda^2 u m - \frac{1}{2}\lambda^2\rho^2 + \frac{1}{8}\right)\frac{\partial}{\partial \beta}, \\
V_2 =& \ m_x\frac{\partial}{\partial m} + u_x\frac{\partial}{\partial u} + \rho_x\frac{\partial}{\partial \rho} + \left(-\alpha^2 + \lambda^2\rho^2 + \lambda m + \frac{1}{4}\right)\frac{\partial}{\partial \alpha} \\
&+ \alpha\frac{\partial}{\partial \delta} + \left(-e^{2\delta}\left(2\lambda^3\rho^2 + \lambda^2 m\right)\right)\frac{\partial}{\partial \beta}, \\
V_3 =& \ \frac{\partial}{\partial \delta} + 2\beta\frac{\partial}{\partial \beta}, \\
V_4 =& \ \frac{\partial}{\partial \beta}, \\
V_5 =& \ \alpha e^{2\delta}\frac{\partial}{\partial u} - \left[2\lambda^2\rho\rho_x + \lambda m_x + \left(4\lambda^2\rho^2 + 4\lambda m\right)\alpha\right]e^{2\delta}\frac{\partial}{\partial m} \\
&- \lambda(\rho_x + 2\rho\alpha)e^{2\delta}\frac{\partial}{\partial \rho} - \left(2\lambda^3\rho^2 + \lambda^2 m\right)e^{2\delta}\frac{\partial}{\partial \alpha} \\
&+ \beta\frac{\partial}{\partial \delta} + \left[\beta^2 + \left(3\lambda^4\rho^2 + \lambda^3 m\right)e^{4\delta}\right]\frac{\partial}{\partial \beta}.
\end{aligned}
\tag{36}
$$

The proof of the above theorem is a straightforward computation.

Corollary 1. *The five nonlocal symmetries (36) generate a Lie algebra $\mathfrak{E}$, and their commutators are presented in Table 1.*

Table 1. The commutation table of the two-component Camassa–Holm system's nonlocal symmetry algebra.

	V_1	V_2	V_3	V_4	V_5
V_1	0	0	0	0	0
V_2	0	0	0	0	0
V_3	0	0	0	$-2V_4$	$2V_5$
V_4	0	0	$2V_4$	0	V_3
V_5	0	0	$-2V_5$	$-V_3$	0

3. Recursion Operators for the Two-Component Camassa–Holm System

Set α as

$$\alpha = \frac{\psi_x}{\psi} = (\ln\psi)_x, \tag{37}$$

where α is determined by the pseudo-potential Equation (10). The function ψ satisfies the second-order linear problem

$$\psi_{xx} = \psi\lambda^2\rho^2 + \psi\lambda m + \frac{\psi}{4}. \tag{38}$$

The potential δ can be written as

$$\delta_x = \alpha, \tag{39}$$

then, we have

$$\delta = \ln\psi. \tag{40}$$

Now, the function $G = \alpha e^{2\delta}$ can be rewritten as

$$G(x,t) = \psi_x \psi. \tag{41}$$

Furthermore, we know that the shadow $G(x,t)$ satisfies

$$D_x^{-1} G = \frac{\psi}{2}, \tag{42}$$
$$D_x G = \psi_{xx}\psi + \psi_x^2, \tag{43}$$
$$D_{xx} G = \psi_{xxx}\psi + 3\psi_{xxx}\psi_x. \tag{44}$$

Using Equation (38), we obtain

$$2\lambda\psi_{xx}\psi_x = \psi_x\psi m + \frac{\lambda\psi_x\psi}{2}. \tag{45}$$

In contrast, from Equation (43), $\psi_{xx}\psi$ can be written as

$$\lambda\psi_{xx}\psi_x = \frac{G_x}{6} - \frac{\lambda\psi_{xxx}\psi}{3}. \tag{46}$$

In the above equation, the term $\psi\psi_{xxx}$ can be obtained by taking the x-derivative of Equation (38) and multiplying the resulting equation by $2\lambda\psi_x$,

$$2\lambda\psi_{xxx}\psi = \psi_x\psi m + \psi^2 m_x + \frac{\lambda\psi_x\psi}{2} \tag{47}$$

Then, we have

$$2\lambda\psi_{xx}\psi_x = \frac{G_{xx}}{3} - \frac{(D_x^{-1}G(x,t))m_x}{3\lambda} - \frac{G(x,t)m}{6\lambda} - \frac{G(x,t)}{12}. \tag{48}$$

The function $G(x,t)$ satisfies the following equation by replacing (45) in Equation (48):

$$0 = \frac{G_{xx}}{3} - \frac{(D_x^{-1}G(x,t))m_x}{3\lambda} - \frac{G(x,t)m}{6\lambda} - \frac{G(x,t)}{12} - \frac{G(x,t)(\lambda+2m)}{4\lambda} \tag{49}$$

As a result, Equation (49) can lead to

$$\lambda G = \left(D_x^2 - 1\right)^{-1}\left(m_x D_x^{-1} + 2m\right)G. \tag{50}$$

The pseudo-differential operator

$$Q = \left(D_x^2 - 1\right)^{-1}\left(m_x D_x^{-1} + 2m\right) \tag{51}$$

is precisely the recursion operator for the two-component Camassa–Holm system.

4. Nonlocal Symmetries of the Two-Component Hunter–Saxton System

The two-component Hunter–Saxton system

$$\begin{aligned} &m_t + 2mu_x + um_x - \rho\rho_x = 0, \\ &\rho_t + (\rho u)_x = 0, \\ &m = u - u_{xx} \end{aligned} \tag{52}$$

describes a pseudo-spherical surface with the following associated one-form:

$$\begin{aligned}
\omega_1 =& \left[\left(\rho^2 - \frac{1}{4}\right)\lambda^2 + \lambda m + 1\right]\mathrm{d}x + \left[\left(\frac{1}{4} - \rho^2\right)u \right.\\
&\left. + \left(\frac{1}{2}\rho^2 - um + \frac{1}{2}u_x - \frac{1}{8}\right)\lambda - u + \frac{1}{2}\lambda^{-1}\right]\mathrm{d}t \\
\omega_2 =& \lambda \mathrm{d}x - \left(\lambda u + u_x - \frac{1}{2}\right)\mathrm{d}t \\
\omega_3 =& \left[\left(\frac{1}{4} - \rho^2\right)\lambda^2 - \lambda m + 1\right]\mathrm{d}x \\
&+ \left[\left(\rho^2 - \frac{1}{4}\right)u\lambda^2 - \left(\frac{1}{2}\rho^2 - um + \frac{1}{2}u_x - \frac{1}{8}\right)\lambda \right.\\
&\left. - u + \frac{1}{2}\lambda^{-1}\right]\mathrm{d}t.
\end{aligned} \tag{53}$$

Theorem 3. *The two-component Hunter–Saxton system admits the quadratic pseudo-potential α defined by the system*

$$\begin{aligned}
\alpha_x =& -\alpha^2 + \lambda^2\rho^2 + \lambda m, \\
\alpha_t =& \left(u - \frac{1}{2\lambda}\right)\alpha^2 - u_x\alpha - \lambda^2 u\rho^2 + \lambda\left(\frac{\rho^2}{2} - um\right).
\end{aligned} \tag{54}$$

where $m = -u_{xx}$, and parameter$\lambda \neq 0$.

Now, we construct an infinite number of conservation laws of the two-component Hunter–Saxton system with the pseudo-potential α. Set

$$\alpha = \sum_{n=-2}^{\infty} \alpha_n(x,t)\lambda^{-\frac{n}{2}}. \tag{55}$$

Substituting this into Equation (52), one obtains the equations for the coefficient function $\alpha_{i,x}, i = -2, -1, 0, \cdots,$

$$\sum_{n=-2}^{\infty} (\alpha_n)_x \lambda^{-\frac{n}{2}} = -\left(\sum_{n=-2}^{\infty} \alpha_n(x,t)\lambda^{-\frac{n}{2}}\right)^2 + \lambda m + \lambda^2\rho^2 \tag{56}$$

Comparing the coefficients of λ^{-i}, we obtain the following equations for $\alpha_i, i = -2, -1, 0, \cdots$:

$$\begin{aligned}
&\alpha^2_{-2}(x,t) = \rho, \\
&-2\alpha_{-1}(x,t)\alpha_{-2}(x,t) = 0, \\
&\alpha^2_{-2,x} = -\alpha^2_{-1}(x,t) - 2\alpha_0(x,t)\alpha_{-2}(x,t) + m, \\
&\quad \cdots
\end{aligned} \tag{57}$$

Then, the $\alpha_i, i = -2, -1, 0, \cdots$ can be determined recursively. The first three conservation densities are presented as follows:

$$\begin{aligned}
&H_1 = \int \rho \mathrm{d}x, \\
&H_2 = \int \frac{m}{\rho}\mathrm{d}x \\
&H_3 = \int \frac{\rho_x^2 - \rho m^2}{\rho^3}\mathrm{d}x.
\end{aligned} \tag{58}$$

If we set

$$\alpha = \sum_{n=0}^{\infty} \alpha_n(x,t)\lambda^{\frac{n}{2}}, \tag{59}$$

then one pbtains the equations for the coefficient functions $\alpha_{i,x}, i = 0, 1, 2, \cdots$,

$$\left(\sum_{n=0}^{\infty} \alpha_n(x,t)\lambda^{\frac{n}{2}}\right)_x = -\left(\sum_{n=0}^{\infty} \alpha_n(x,t)\lambda^{\frac{n}{2}}\right)^2 + \lambda m + \lambda^2\rho^2, \tag{60}$$

Comparing the coefficients of λ^i, the following equations for α_i are obtained:

$$\begin{aligned} \alpha_{0,x} &= -\alpha_0(x,t)^2, \\ \alpha_{1,x} &= -2\alpha_0(x,t)\alpha_1(x,t), \\ \alpha_{2,x} &= -2\alpha_0(x,t)\alpha_2(x,t) - \alpha_1(x,t)^2 + m, \\ &\cdots. \end{aligned}$$

Then, the $\alpha_i, i = 0, 1, 2, \cdots$ can be determined, and conservation densities are presented as follows:

$$\begin{aligned} H_1 &= \int \left(\rho^2 - u_x^2\right) \mathrm{d}x, \\ H_2 &= \int (p^4 - 2p^2u^2 + 4p^2uu_x - 2p^2u_x^2 + u^4 + 6u^2u_x^2 \\ &\quad - 4uu_x^3 + p^2u - p^2u_x - 3uu_x^2 + u_x^3)\mathrm{d}_x \\ &\cdots. \end{aligned}$$

Now, we consider the two-component Hunter–Saxton system's nonlocal symmetries. System (52) can be rewritten as

$$\begin{aligned} \alpha_x &= -\alpha^2 + \lambda^2\rho^2 + \lambda m + \tfrac{1}{4} \\ \alpha_t &= \left[\left(\tfrac{1}{2\lambda} - u\right)\alpha + \tfrac{1}{2}u_x\right]_x \end{aligned} \tag{61}$$

where $m = -u_{xx}$. Furthermore, the potential function $\delta(x,t)$ is defined as

$$\delta_x = \alpha, \tag{62}$$

$$\delta_t = \left(\frac{1}{2\lambda} - u\right)\alpha + \frac{1}{2}u_x \tag{63}$$

Theorem 4. *The following vector fields are the generalized symmetries for the augmented two-component Hunter–Saxton system* (52) *and* (61)–(63).

$$
\begin{aligned}
V_1 = &\ (-2u_x m - u m_x - \rho\rho_x)\frac{\partial}{\partial m} - u_t\frac{\partial}{\partial u} - (\rho u)_x\frac{\partial}{\partial \rho} \\
&+ \left(\left(\frac{1}{2\lambda} - u\right)\alpha + \frac{1}{2}u_x\right)_x\frac{\partial}{\partial \alpha} + \left(\left(\frac{1}{2\lambda} - u\right)\alpha + \frac{1}{2}u_x\right)\frac{\partial}{\partial \delta} \\
&+ e^{2\delta}\left(-\frac{\alpha^2}{2} + 2\lambda^3 u\rho^2 + \lambda^2 u m - \frac{1}{2}\lambda^2\rho^2\right)\frac{\partial}{\partial \beta}, \\
V_2 = &\ m_x\frac{\partial}{\partial m} + u_x\frac{\partial}{\partial u} + \rho_x\frac{\partial}{\partial \rho} + \left(-\alpha^2 + \lambda^2\rho^2 + \lambda m\right)\frac{\partial}{\partial \alpha} \\
&+ \alpha\frac{\partial}{\partial \delta} + \left(-e^{2\delta}\left(2\lambda^3\rho^2 + \lambda^2 m\right)\right)\frac{\partial}{\partial \beta}, \\
V_3 = &\ \frac{\partial}{\partial \delta} + 2\beta\frac{\partial}{\partial \beta}, \\
V_4 = &\ \frac{\partial}{\partial \beta}, \\
V_5 = &\ \alpha e^{2\delta}\frac{\partial}{\partial u} - \left[2\lambda^2\rho\rho_x + \lambda m_x + \left(4\lambda^2\rho^2 + 4\lambda m\right)\alpha\right]e^{2\delta}\frac{\partial}{\partial m} \\
&- \lambda(\rho_x + 2\rho\alpha)e^{2\delta}\frac{\partial}{\partial \rho} - \left(2\lambda^3\rho^2 + \lambda^2 m\right)e^{2\delta}\frac{\partial}{\partial \alpha} \\
&+ \beta\frac{\partial}{\partial \delta} + \left[\beta^2 + \left(3\lambda^4\rho^2 + \lambda^3 m\right)e^{4\delta}\right]\frac{\partial}{\partial \beta}.
\end{aligned}
\tag{64}
$$

where

$$\beta_x = -e^{2\delta}\left(2\lambda^3\rho^2 + \lambda^2 m\right), \tag{65}$$

$$\beta_t = e^{2\delta}\left(-\frac{\alpha^2}{2} + 2\lambda^3 u\rho^2 + \lambda^2 u m - \frac{1}{2}\lambda^2\rho^2\right). \tag{66}$$

In addition, the following Corollary can be obtained.

Corollary 2. *The nonlocal symmetries* (64) *generate a Lie algebra* $\mathfrak{E}$, *and their commutators are presented in Table* 2.

Table 2. The commutation table of the two-component Hunter–Saxton system's nonlocal symmetry algebra.

	$\boldsymbol{V_1}$	$\boldsymbol{V_2}$	$\boldsymbol{V_3}$	$\boldsymbol{V_4}$	$\boldsymbol{V_5}$
V_1	0	0	0	0	0
V_2	0	0	0	0	0
V_3	0	0	0	$-2V_4$	$2V_5$
V_4	0	0	$2V_4$	0	V_3
V_5	0	0	$-2V_5$	$-V_3$	0

5. Concluding Remarks

We have shown that the two-component Camassa–Holm system and the two-component Hunter–Saxton system admit a class of nonlocal symmetries, and the recursion operator of the two-component Camassa–Holm system is constructed by using its potential variables. Thus, these kinds of nonlocal symmetries and recursion operators are related to the systems' integrability. It is well-known that the Novikov equation and Degasperis–Procesi equation are geometric integrable models, and their spectral matrix is 3×3. It is difficult to define a pseudo-potential function using the 3×3 spectral matrix. However, it is interesting to investigate the existence of nonlocal symmetries and recursion operators, which will be our future study.

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References

1. Olver, P.J.; Rosenau, P. Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support. *Phys. Rev.* **1996**, *53*, 1900. [CrossRef] [PubMed]
2. Chen, M.; Liu, S.; Zhang, Y. A two-component generalization of the Camassa-Holm equation and its solutions. *Lett. Math. Phys.* **2006**, *75*, 1. [CrossRef]
3. Camassa, R.; Holm, D.D. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* **1993**, **71**, 1661–1664. [CrossRef] [PubMed]
4. Fokas, A.; Fuchssteiner, B. The hierarchy of the Benjamin-Ono equation. *Physics D* **1981**, *4*, 47. [CrossRef]
5. Constantin, A.; Ivanov, R.I. On an integrable two-component Camassa-Holm shallow water system. *Phys. Lett. A* **2008**, *372*, 7129. [CrossRef]
6. Guha, P.; Olver, P. Geodesic flow and two (super) component analog of the Camassa-Holm equation. *SIGMA Symmetry Integr. Geom. Methods Appl.* **2006**, *2*, 54. [CrossRef]
7. Zuo, D. A two-component μ-Hunter-Saxton equation. *Inverse Probl.* **2010**, *26*, 085003. [CrossRef]
8. Song, J.F.; Qu, C.Z. Geometric integrability of two-component Camassa-Holm and Hunter-Saxton systems. *Commun. Theor. Phys.* **2011**, *55*, 955.
9. Yan, L.; Song, J.F.; Qu, C.Z. Nonlocal symmetries and geometric integrability of multi-component Camassa-Holm and Hunter-Saxton Systems. *Chin. Phys. Lett.* **2011**, *28*, 050204. [CrossRef]
10. Lenells, J.; Misiolek, G.; Tiglay, F. Integrable evolution equations on spaces of tensor densities and their Peakon solutions. *Commun. Math. Phys.* **2010**, *299*, 129. [CrossRef]
11. Reyes, E.G. Geometric integrability of the Camassa-Holm equation. *Lett. Math. Phys.* **2012**, *59*, 17. [CrossRef]
12. Heredero, R.; Reyes, E. Nonlocal symmetries and a Darboux transformation for the Camassa-Holm equation. *J. Phys. A Math. Theor.* **2009**, *42*, 182002. [CrossRef]
13. Reyes, E. Nonlocal symmetries and Kaup-Kupershmidt equation. *J. Math. Phys.* **2005**, *46*, 073507. [CrossRef]
14. Heredero, R.; Reyes, E. Geometric integrability of the Camassa-Holm equation II. *Int. Math. Res. Not.* **2012**, *2012*, 3089–3125. [CrossRef]
15. Chou, K.S.; Qu, C.Z. Integrable equations arising from the motion of plane curves. *Physica D* **2002**, *162*, 9–33. [CrossRef]
16. Khesin, B.; Lenells, J.; Misiolek, G. Generalized Hunter-Saxton equation and the geometry of the group of circle diffeomorphisms. *Math. Ann.* **2008**, *342*, 617. [CrossRef]
17. Christov, O. On the nonlocal symmetries of the μ-Camassa-Holm equation. *J. Nonlinear Math. Phys.* **2012**, *3*, 1250025. [CrossRef]
18. Qu, C.Z.; Fu, Y.; Liu, Y. Well-posedness, wave breaking and peakons for a modified μ-Camassa-Holm equation. *J. Funct. Anal.* **2014**, *266*, 433. [CrossRef]
19. Qu, C.Z.; Fu, Y.; Liu, Y. Blow-up solutions and peakons to a generalized μ-Camassa-Holm integrable equation. *Commun. Math. Phys.* **2014**, *266*, 375. [CrossRef]
20. Qu, C.Z.; Zhang, Y.; Liu, X.C.; Liu, Y. Orbital stability of periodic peakons to a generalized μ-Camassa-Holm equation. *Arch. Ration. Mech. Anal.* **2014**, *211*, 593. [CrossRef]
21. Liu, X.C.; Liu, Y.; Olver, P.J.; Qu, C.Z. Orbital stability of peakons for a generalization of the modified Camassa-Holm equation. *Nonlinearity* **2014**, *27*, 2297. [CrossRef]
22. Liu, Y.; Olver, P.J.; Qu, C.Z.; Zhang, S.H. On the blow-up of solutions to the integrable modified Camassa-Holm equation. *Anal. Appl.* **2014**, *12*, 355. [CrossRef]
23. Chern, S.S.; Tenenblat, K. Pseudo-spherical surfaces and evolution equations. *Stud. Appl. Math.* **1986**, *74*, 55. [CrossRef]
24. Bluman, G.W.; Kumei, S. *Symmetries and Differential Equations*; Springer: Berlin/Heidelberg, Germany, 1989.