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New Simpson's Type Estimates for Two Newly Defined Quantum Integrals

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Abstract: In this paper, we give some correct quantum type Simpson's inequalities via the application of q-Hölder's inequality. The inequalities of this study are compatible with famous Simpson's 1/8 and 3/8 quadrature rules for four and six panels, respectively. Several special cases from our results are discussed in detail. A counter example is presented to explain the limitation of Hölder's inequality in the quantum framework.

Keywords: Simpson's inequality; convex function; q-Hölder's inequality; quantum derivatives; quantum integrals

MSC: 26B25; 26D10; 26D15

1. Introduction

The numerical integration and the numerical estimations of definite integrals is a vital piece of applied sciences. Simpson's rules are momentous among the numerical techniques. The procedure is credited to Thomas Simpson (1710–1761). Johannes Kepler worked on a similar estimation technique about a century ago, so the algorithm is sometimes referred to as Kepler's formula. Simpson's formula uses the three-step Newton–Cotes quadrature rule, so estimations based on three-step quadratic kernel are sometimes termed as Newton type results. The following are the rules devised by Simpson.

1.   Simpson's 1/3 rule:

$$\int_{\Lambda_1}^{\Lambda_2} \omega(\varrho)d\varrho = \frac{h}{3}\left[\omega_1 + 4\omega_2 + 2\omega_3 + 4\omega_4 + 2\omega_5 + \ldots + 4\omega_n + \omega_{n+1}\right] - \frac{(\Lambda_2 - \Lambda_1)}{180}h^4\omega^{(4)}(\Lambda), \quad (1)$$

where $\Lambda_1 \leq \Lambda \leq \Lambda_2$ and the number of panels are even.

2.   Simpson's 3/8 rule:

$$\int_{\Lambda_1}^{\Lambda_2} \omega(\varrho)d\varrho = \frac{3h}{8}\left[\omega_1 + 3\omega_2 + 3\omega_3 + 2\omega_4 + 3\omega_5 + 3\omega_6 + \ldots + 3\omega_n + \omega_{n+1}\right] - \frac{(\Lambda_2 - \Lambda_1)}{80}h^4\omega^{(4)}(\Lambda), \quad (2)$$

where $\Lambda_1 \leq \Lambda \leq \Lambda_2$ and the number of panels are integral multiples of 3.

An exceptionally popular estimation relating to the above rules is called Simpson's inequality and is presented as follows:

Theorem 1. *Let $\varpi : [\Lambda_1, \Lambda_2] \to \mathbb{R}$ be a fourth-order differentiable function on (Λ_1, Λ_2), where $\left\| \varpi^{(4)} \right\|_{\infty} := \sup_{\varrho \in (\Lambda_1, \Lambda_2)} \left| \varpi^{(4)}(\varrho) \right| < \infty$, then*

$$\left| \frac{1}{6} \left[\varpi(\Lambda_1) + 4\varpi\left(\frac{\Lambda_1 + \Lambda_2}{2} \right) + \varpi(\Lambda_2) \right] - \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \varpi(\varrho) d\varrho \right| \leq \frac{1}{2880} \left\| \varpi^{(4)} \right\|_{\infty} (\Lambda_2 - \Lambda_1)^2. \quad (3)$$

Lately, many researchers have zeroed in on Simpson's type inequalities for differentiable classes of convex functions. In particular, a few mathematicians have chipped away at Simpson and Newton type results for convex mappings, as convexity hypotheses are powerful and a solid strategy for tackling incredible number of issues which emerge in numerous parts of applied sciences. For example, Dragomir et al. [1] introduced Simpson type inequalities alongside their applications to quadrature formula in numerical integration. After this beginning, Simpson type inequalities via *s*-convex functions were developed by Alomari et al. [2]. Sarikaya et al. [3] proved the variants of Simpson type inequalities dependent upon *s*-convexity. Noor et al. [4,5] established some Newton type inequalities for harmonic convex and *p*-harmonic convex functions. Alongside these, some new Newton type inequalities for functions whose local fractional derivatives are generalized convex are proven by Iftikhar et al. [6].

In recent decades, several attempts have been made by researchers to obtain the variants and applications of integral inequalities. In the last couple of years, an assortment of novel methodologies has been used by scientists in summing up the traditional results about integral inequalities. One of those approaches is using the ideas of quantum calculus. It is very notable to everybody that quantum math is calculus without limits. Generally, the subject of quantum calculus (shortly, *q*-calculus) can be followed back to Euler and Jacobi, yet in the many years hence, it has encountered a quick development [7]. The theory gained popularity during the 20th century after the work of Jackson (1910) on defining an integral later known as the *q*-Jackson integral (see previous studies [7–11]). In *q*-calculus, the classical derivative is replaced by the *q*-difference operator in order to deal with non-differentiable functions (see Almeida and Torres and Cresson et al. [12,13]). Applications of *q*-calculus can be found in various branches of mathematics and physics, and the interested readers should consider [14–22] for valuable information. The development of quantum calculus enhanced the interest of researchers to add new ideas in the ongoing theory. Tariboon and Ntouyas [23] presented the idea of quantum integral over the finite interval, acquired a few *q*-analogues of traditional mathematical objects and opened a new setting of exploration. For example, the *q*-analogue of Hölder's inequality, Hermite–Hadamard inequality, Ostrowski inequality, Cauchy–Bunyakovsky-Schwarz, Grüss, Grüss–Cebyshev, and other integral inequalities have been developed. Sudsutated et al. [24] and Noor et al. [25] acquired some extensions in the *q*-trapezoid type inequalities via first time *q*-differentiability. Liu and Zhuang [26] determined some *q*-analogues of trapezoid-like inequalities for twice quantum differentiable functions. Zhuang et al. [27] proved some more general *q*-analogues of trapezoid-like inequalities for first order quantum differentiable functions. Budak et al. [28] established some new simpson's inequalities for *q*-integrals. For some more nitty gritty review with the application point of perspective, we allude to (see [24–45] and the references therein).

Adding motivation to these outcomes, particularly the outcomes created in [28], we notice that it is feasible to treat quantum integral operators introduced in [23,41] to jointly create some new Simpson type inequalities as in [28]. For this reason, we aim to achieve the following objectives.

1. To obtain a new Simpson's type inequality depending upon the two newly defined quantum integrals given in Definitions 3 and 5 which is analogous to 1/3 quadrature formula given by (1) for four panels.

2. To extend Simpson's 3/8 quadrature Formula (2) for six panels in the quantum calculus via indicated quantum integrals.

3. To present a counter example which explains the limiting nature of Hölder's inequality in the quantum framework of calculus.

4. To re-capture the classical results involving the classical Hölder's inequality and making comparison with the results due to q-Hölder's inequality.

2. Preliminaries

Throughout the paper, let $\mathcal{W} := [\Lambda_1, \Lambda_2] \subseteq \mathbb{R}$ with $0 \leq \Lambda_1 < \Lambda_2$ be an interval and $\mathcal{W}^\circ$ be the interior of $\mathcal{W}$. Assume further that $0 < q_r < 1$ be a constant.

This section is devoted to the basic and fundamental results in the q-calculus. We start by collating foundational results and definitions suitable for ongoing study.

Definition 1. *A function $\varpi : \mathcal{W} \to \mathbb{R}$ is called convex, if the inequality*

$$\varpi(\varrho\beta_1 + (1 - \varrho)\beta_2) \leq \varrho\varpi(\beta_1) + (1 - \varrho)\varpi(\beta_2), \tag{4}$$

is satisfied for all $\beta_1, \beta_2 \in \mathcal{W}$, and $\varrho \in [0, 1]$.

Recall that the all-time famous Jackson integral [11] from 0 to an arbitrary real number Λ characterized as follows:

$$\int_0^\Lambda \varpi(\varsigma) \, d_{q_r}\varsigma = (1 - q_r)\Lambda \sum_{\delta=0}^\infty q^\delta \varpi(\Lambda q_r^\delta), \tag{5}$$

provided the series on the right side converges absolutely. Moreover, he gave the integral for an arbitrary finite interval $[\Lambda_1, \Lambda_2]$ as

$$\int_{\Lambda_1}^{\Lambda_2} \varpi(\varsigma) \, d_{q_r}\varsigma = \int_0^{\Lambda_2} \varpi(\varsigma) \, d_{q_r}\varsigma - \int_0^{\Lambda_1} \varpi(\varsigma) \, d_{q_r}\varsigma. \tag{6}$$

In [23], the authors, while developing some classical inequalities in the quantum frame work, studied the concept of q_r-differentiation and q_r-integration over the finite interval.

Definition 2. *For a continuous function $\varpi : \mathcal{W} \to \mathbb{R}$ and $0 < q_r < 1$, then $q_{r\Lambda_1}$-derivative of ϖ at $\Lambda \in \mathcal{W}$ is expressed by the quotient:*

$$_{\Lambda_1}D_{q_r}\varpi(\Lambda) = \frac{\varpi(\Lambda) - \varpi(q_r\Lambda + (1 - q_r)\Lambda_1)}{(1 - q_r)(\Lambda - \Lambda_1)}, \quad \Lambda \neq \Lambda_1. \tag{7}$$

The function ϖ is called $q_{r\Lambda_1}$-differentiable on $\mathcal{W}$, if $_{\Lambda_1}D_{q_r}\varpi(u)$ exists for all $u \in \mathcal{W}$. It is evident that

$$_{\Lambda_1}D_{q_r}\varpi(\Lambda_1) = \lim_{\Lambda \to \Lambda_1} {_{\Lambda_1}D_{q_r}\varpi(\Lambda)}. \tag{8}$$

If $\Lambda_1 = 0$, then the q_r-derivative in classical sense [7] is obtained:

$$D_{q_r}\varpi(\Lambda) = \frac{\varpi(\Lambda) - \varpi(q_r\Lambda)}{\Lambda - q_r\Lambda}. \tag{9}$$

Definition 3. *Let $\varpi : \mathcal{W} \to \mathbb{R}$ be a continuous function and $0 < q_r < 1$. The definite $q_{r\Lambda_1}$-integral of the function ϖ is characterized by the expression*

$$\int_{\Lambda_1}^\alpha \varpi(\varsigma) \, _{\Lambda_1}d_{q_r}\varsigma = (1 - q_r)(\alpha - \Lambda_1) \sum_{\delta=0}^\infty q_r^\delta \varpi(q_r^\delta\alpha + (1 - q_r^\delta)\Lambda_1), \quad \alpha \in \mathcal{W}. \tag{10}$$

In the same paper, the authors also proved the following q_r-Hölder inequality.

Theorem 2. *Let $\omega_1, \omega_2 : \mathcal{W} \to \mathbb{R}$ be two continuous functions. Then, the inequality*

$$
\int_{\Lambda_1}^{y} |\omega_1(\varrho)\omega_2(\varrho)| \,_{\Lambda_1}d_{q_r}\varrho \leq \left(\int_{\Lambda_1}^{y} |\omega_1(\varrho)|^{k_1} \,_{\Lambda_1}d_{q_r}\varrho \right)^{\frac{1}{k_1}} \left(\int_{\Lambda_1}^{y} |\omega_2(\varrho)|^{k_2} \,_{\Lambda_1}d_{q_r}\varrho \right)^{\frac{1}{k_2}}, \quad (11)
$$

holds for all $y \in \mathcal{W}$ and $k_1, k_2 > 1$ with $k_1^{-1} + k_2^{-1} = 1$.

In [41], the authors presented an analogous notion of q_r–derivatives and q_r–integrals by introducing the $q_r^{\Lambda_2}$–derivative and $q_r^{\Lambda_2}$–integrals over the finite real interval $\mathcal{W}$.

Definition 4. *For a continuous function $\omega : \mathcal{W} \to \mathbb{R}$ and $0 < q_r < 1$, then $q_r^{\Lambda_2}$–derivative of ω at $\Lambda \in \mathcal{W}$ is defined by the quotient:*

$$
{}^{\Lambda_2}D_{q_r}\omega(\Lambda) = \frac{\omega(\Lambda) - \omega(q_r\Lambda + (1 - q_r)\Lambda_2)}{(1 - q_r)(\Lambda - \Lambda_2)}, \quad \Lambda \neq \Lambda_2, \quad (12)
$$

Definition 5. *Let $\omega : \mathcal{W} \to \mathbb{R}$ be a continuous function and $0 < q_r < 1$. The definite $q_r^{\Lambda_2}$-integral of the function ω is characterized by the expression*

$$
\int_{\beta}^{\Lambda_2} \omega(\varsigma) \,^{\Lambda_2}d_{q_r}\varsigma = (1 - q_r)(\Lambda_2 - \beta) \sum_{\delta=0}^{\infty} q_r^{\delta}\omega(q_r^{\delta}\beta + (1 - q_r^{\delta})\Lambda_2), \quad \beta \in \mathcal{W}. \quad (13)
$$

Remark 1. *It is worth mentioning that*

(i) *the left $q_{r\Lambda_1}$–derivatives and right $q_r^{\Lambda_2}$–derivatives are not same for general functions defined over the finite real interval $[\Lambda_1, \Lambda_2]$. Indeed, if $\omega(\Lambda) = \Lambda^2$, then*

$$
{}^{\Lambda_2}D_{q_r}\omega(\Lambda) = (1 + q_r)\Lambda + (1 - q_r)\Lambda_2 \neq (1 + q_r)\Lambda + (1 - q_r)\Lambda_1 = {}_{\Lambda_1}D_{q_r}\omega(\Lambda).
$$

However,

$$
{}^{\Lambda_2}D_{q_r}\omega(\Lambda) = \omega'(\Lambda) = {}_{\Lambda_1}D_{q_r}\omega(\Lambda)
$$

provided that $q_r \to 1^{-}$.

(ii) *The q_r–integrals $\int_{k_1}^{\Lambda_2} \omega(\Lambda) \,^{\Lambda_2}d_{q_r}\Lambda$ and $\int_{k_1}^{\Lambda_2} \omega(\Lambda) \,_{k_1}d_{q_r}\Lambda$ are different for general functions. For instance,*

$$
\int_{k_1}^{\Lambda_2} \Lambda \,^{\Lambda_2}d_{q_r}\Lambda = \frac{\Lambda_2 - k_1}{1 + q_r}[k_1 + q_r\Lambda_2] \neq \frac{\Lambda_2 - k_1}{1 + q_r}[k_1 q_r + \Lambda_2] = \int_{k_1}^{\Lambda_2} \Lambda \,_{k_1}d_{q_r}\Lambda.
$$

Furthermore,

$$
\int_{k_1}^{\Lambda_2} \Lambda \,^{\Lambda_2}d_{q_r}\Lambda = \frac{\Lambda_2^2 - k_1^2}{2} = \int_{k_1}^{\Lambda_2} \Lambda \,_{k_1}d_{q_r}\Lambda,
$$

subject to the condition that $q_r \to 1^{-}$.

It is also important to notice that, for an integer n, the quantum analogue is

$$
[n]_{q_r} = \frac{1 - q_r^n}{1 - q_r} = 1 + q_r + q_r^2 + \ldots + q_r^{n-1}, \quad 1 \neq q_r.
$$

Clearly, the limiting value is n for $q_r \to 1^-$. For a detailed survey about the quantum analogues of integers and polynomials we refer to [7].

In [28], the authors presented the following Simpson's type inequalities.

Theorem 3. *Let $\varpi : \mathcal{W} \to \mathbb{R}$ be a continuous and $q_r^{\Lambda_2}$-differentiable function on $\mathcal{W}^\circ$ and $0 < q_r < 1$. If $\left|{}^{\Lambda_2}D_{q_r}\varpi\right|$ is convex and integrable on $\mathcal{W}$, then*

$$
\left| \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \varpi(\wp)\,{}^{\Lambda_2}d_{q_r}\wp - \frac{1}{6}\left[\varpi(\Lambda_1) + 4\varpi\left(\frac{\Lambda_1 + \Lambda_2}{2}\right) + \varpi(\Lambda_2)\right] \right|
$$
$$
\leq (\Lambda_2 - \Lambda_1)\left\{ [M_1(q_r) + M_2(q_r)]\left|{}^{\Lambda_2}D_{q_r}\varpi(\Lambda_1)\right| + [M_3(q_r) + M_4(q_r)]\left|{}^{\Lambda_2}D_{q_r}\varpi(\Lambda_2)\right| \right\},
\tag{14}
$$

where

$$
M_1(q_r) := \int_0^{\frac{1}{2}} \left| q_r\wp - \frac{1}{6} \right| \wp\, d_{q_r}\wp = \begin{cases} \dfrac{1 - 2q_r - 2q_r^2}{24(1+q_r)(1+q_r+q_r^2)}, & \text{if } 0 < q_r < \frac{1}{3}, \\[2mm] \dfrac{18q_r^2 + 18q_r - 7}{216(1+q_r)(1+q_r+q_r^2)}, & \text{if } \frac{1}{3} \leq q_r < 1, \end{cases}
\tag{15}
$$

$$
M_2(q_r) := \int_0^{\frac{1}{2}} \left| q_r\wp - \frac{1}{6} \right|(1 - \wp)\, d_{q_r}\wp = \begin{cases} \dfrac{1 - 4q^3}{24(1+q_r)(1+q+q^2)}, & \text{if } 0 < q_r < \frac{1}{3}, \\[2mm] \dfrac{36q_r^3 + 12q_r^2 + 12q_r + 1}{216(1+q_r)(1+q_r+q_r^2)}, & \text{if } \frac{1}{3} \leq q_r < 1, \end{cases}
\tag{16}
$$

$$
M_3(q_r) := \int_{\frac{1}{2}}^1 \left| q_r\wp - \frac{5}{6} \right| \wp\, d_{q_r}\wp = \begin{cases} \dfrac{15 - 6q_r - 6q_r^2}{24(1+q_r)(1+q_r+q_r^2)}, & \text{if } 0 < q_r < \frac{5}{6}, \\[2mm] \dfrac{18q_r^2 + 18q_r - 7}{216(1+q_r)(1+q_r+q_r^2)}, & \text{if } \frac{5}{6} \leq q_r < 1, \end{cases}
\tag{17}
$$

and

$$
M_4(q_r) := \int_{\frac{1}{2}}^1 \left| q_r\wp - \frac{5}{6} \right|(1 - \wp)\, d_{q_r}\wp = \begin{cases} \dfrac{-5 + 8q_r + 8q_r^2 - 8q_r^3}{24(1+q_r)(1+q_r+q_r^2)}, & \text{if } 0 < q_r < \frac{5}{6}, \\[2mm] \dfrac{12q_r^2 + 12q_r + 5}{216(1+q_r)(1+q_r+q_r^2)}, & \text{if } \frac{5}{6} \leq q_r < 1. \end{cases}
\tag{18}
$$

Theorem 4. *Let $\varpi : \mathcal{W} \to \mathbb{R}$ be a continuous and $q_r^{\Lambda_2}$-differentiable function on $\mathcal{W}^\circ$ and $0 < q_r < 1$. If $\left|{}^{\Lambda_2}D_{q_r}\varpi\right|$ is convex and integrable on $\mathcal{W}$, then*

$$
\left| \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \varpi(\wp)\,{}^{\Lambda_2}d_{q_r}\wp - \frac{1}{6}\left[\varpi(\Lambda_1) + 3\varpi\left(\frac{2\Lambda_1 + \Lambda_2}{3}\right) + 3\varpi\left(\frac{\Lambda_1 + 2\Lambda_2}{3}\right) + \varpi(\Lambda_2)\right] \right|
$$
$$
\leq (\Lambda_2 - \Lambda_1)\left\{ [P_1(q_r) + P_3(q_r) + P_5(q_r)]\left|{}^{\Lambda_2}D_{q_r}\varpi(\Lambda_1)\right| + [P_2(q_r) + P_4(q_r) + P_6(q_r)]\left|{}^{\Lambda_2}D_{q_r}\varpi(\Lambda_2)\right| \right\},
\tag{19}
$$

where

$$
P_1(q_r) := \int_0^{\frac{1}{3}} \left| q_r\wp - \frac{1}{8} \right| \wp\, d_{q_r}\wp = \begin{cases} \dfrac{3 - 5q_r - 5q_r^2}{216(1+q_r)(1+q_r+q_r^2)}, & \text{if } 0 < q_r < \frac{3}{8}, \\[2mm] \dfrac{160q_r^2 + 160q_r - 69}{6912(1+q_r)(1+q_r+q_r^2)}, & \text{if } \frac{3}{8} \leq q_r < 1, \end{cases}
\tag{20}
$$

$$
P_2(q_r) := \int_0^{\frac{1}{3}} \left| q_r\wp - \frac{1}{8} \right|(1 - \wp)\, d_{q_r}\wp = \begin{cases} \dfrac{6 - q_r - q_r^2 - 15q^3}{216(1+q_r)(1+q+q^2)}, & \text{if } 0 < q_r < \frac{3}{8}, \\[2mm] \dfrac{480q_r^3 + 248q_r^2 + 248q_r - 3}{6912(1+q_r)(1+q_r+q_r^2)}, & \text{if } \frac{3}{8} \leq q_r < 1, \end{cases}
\tag{21}
$$

$$
P_3(q_r) := \int_{\frac{1}{3}}^{\frac{2}{3}} \left| q_r\wp - \frac{1}{2} \right| \wp\, d_{q_r}\wp = \begin{cases} \dfrac{9 - 5q_r - 5q_r^2}{54(1+q_r)(1+q_r+q_r^2)}, & \text{if } 0 < q_r < \frac{3}{4}, \\[2mm] \dfrac{6q_r^2 + 6q_r - 3}{108(1+q_r)(1+q_r+q_r^2)}, & \text{if } \frac{3}{4} \leq q_r < 1, \end{cases}
\tag{22}
$$

$$P_4(q_r) := \int_{\frac{1}{3}}^{\frac{2}{3}} \left| q_r \wp - \frac{1}{2} \right| (1 - \wp) \, d_{q_r} \wp = \begin{cases} \frac{5q_r + 5q_r^2 - 9q^3}{54(1+q_r)(1+q+q^2)}, & \text{if } 0 < q_r < \frac{3}{4}, \\ \frac{6q_r^3 + 3}{108(1+q_r)(1+q_r+q_r^2)}, & \text{if } \frac{3}{4} \le q_r < 1, \end{cases} \tag{23}$$

$$P_5(q_r) := \int_{\frac{2}{3}}^{1} \left| q_r \wp - \frac{7}{8} \right| \wp \, d_{q_r} \wp = \begin{cases} \frac{105 - 47q_r - 47q_r^2}{216(1+q_r)(1+q_r+q_r^2)}, & \text{if } 0 < q_r < \frac{7}{8}, \\ \frac{224q_r^2 + 224q_r + 525}{6912(1+q_r)(1+q_r+q_r^2)}, & \text{if } \frac{7}{8} \le q_r < 1, \end{cases} \tag{24}$$

and

$$P_6(q_r) := \int_{0}^{\frac{1}{3}} \left| q_r \wp - \frac{7}{8} \right| (1 - \wp) \, d_{q_r} \wp = \begin{cases} \frac{-42 + 53q_r + 53q_r^2 - 57q^3}{216(1+q_r)(1+q+q^2)}, & \text{if } 0 < q_r < \frac{7}{8}, \\ \frac{-96q_r^3 + 184q_r^2 + 184q_r - 21}{6912(1+q_r)(1+q_r+q_r^2)}, & \text{if } \frac{7}{8} \le q_r < 1. \end{cases} \tag{25}$$

Remark 2. *If $q_r \to 1^-$ in inequality (14), then*

$$\left| \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \varpi(\wp) d\wp - \frac{1}{6} \left[\varpi(\Lambda_1) + 4\varpi\left(\frac{\Lambda_1 + \Lambda_2}{2} \right) + \varpi(\Lambda_2) \right] \right|$$
$$\le \frac{5(\Lambda_2 - \Lambda_1)}{72} \left\{ |\varpi'(\Lambda_1)| + |\varpi'(\Lambda_2)| \right\}. \tag{26}$$

Remark 3. *If $q_r \to 1^-$ in inequality (19), then*

$$\left| \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \varpi(\wp) d\wp - \frac{1}{8} \left[\varpi(\Lambda_1) + 3\varpi\left(\frac{2\Lambda_1 + \Lambda_2}{3} \right) + 3\varpi\left(\frac{\Lambda_1 + 2\Lambda_2}{2} \right) + \varpi(\Lambda_2) \right] \right|$$
$$\le \frac{25(\Lambda_2 - \Lambda_1)}{576} \left\{ |\varpi'(\Lambda_1)| + |\varpi'(\Lambda_2)| \right\}. \tag{27}$$

3. Auxiliary Results

We are ready to prove our main results. At start, we present two multi-parameter identities for $q_{r\Lambda_1}$ and $q_r^{\Lambda_2}$-differentiable functions which provide some useful inequalities of Simpson's type.

Lemma 1. *Let $\varpi : \mathcal{W} \to \mathbb{R}$ be a $q_{r\Lambda_1}-$ and $q_r^{\Lambda_2}$–differentiable function on $\mathcal{W}^\circ$ with $0 < q_r < 1$. If $_{\Lambda_1}D_{q_r}\varpi$ and $^{\Lambda_2}D_{q_r}\varpi$ are continuous and $q_{r\Lambda_1}-, q_r^{\Lambda_2}$–integrable functions on $\mathcal{W}$, then the following identity holds:*

$$\Omega_1(\Lambda_1, \Lambda_2; q_r) = \frac{\Lambda_2 - \Lambda_1}{4} \left[\int_0^1 R_1(q_r, \varrho) \left(_{\Lambda_1}D_{q_r}\varpi\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1 \right) \right. \right.$$
$$\left. \left. - ^{\Lambda_2}D_{q_r}\varpi\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2 \right) \right) d_{q_r}\varrho \right], \tag{28}$$

where

$$R_1(q_r, \varrho) := \begin{cases} q_r\varrho - \frac{1}{6}, & \text{if } 0 \le \varrho < \frac{1}{2}, \\ q_r\varrho - \frac{5}{6}, & \text{if } \frac{1}{2} \le \varrho \le 1, \end{cases} \tag{29}$$

and

$$\Omega_1(\Lambda_1, \Lambda_2; q_r) := \frac{1}{12} \left[\varpi(\Lambda_1) + 4\varpi\left(\frac{3\Lambda_1 + \Lambda_2}{4} \right) + 2\varpi\left(\frac{\Lambda_1 + \Lambda_2}{2} \right) + 4\varpi\left(\frac{\Lambda_1 + 3\Lambda_2}{4} \right) + \varpi(\Lambda_2) \right]$$
$$- \frac{1}{\Lambda_2 - \Lambda_1} \left[\int_{\Lambda_1}^{\frac{\Lambda_1 + \Lambda_2}{2}} \varpi(\wp) \, _{\Lambda_1}d_{q_r}\wp + \int_{\frac{\Lambda_1 + \Lambda_2}{2}}^{\Lambda_2} \varpi(\wp) \, ^{\Lambda_2}d_{q_r}\wp \right]. \tag{30}$$

Proof. By utilizing the property of q_r-integrals given in Equation (6), we have

$$
\int_0^1 R_1(q_r, \varrho) \left({}_{\Lambda_1} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - {}^{\Lambda_2} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) d_{q_r} \varrho
$$

$$
= \int_0^1 q_r \varrho \left({}_{\Lambda_1} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - {}^{\Lambda_2} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) d_{q_r} \varrho
$$

$$
+ \frac{2}{3} \int_0^{\frac{1}{2}} \left({}_{\Lambda_1} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - {}^{\Lambda_2} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) d_{q_r} \varrho
$$

$$
- \frac{5}{6} \int_0^1 \left({}_{\Lambda_1} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - {}^{\Lambda_2} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) d_{q_r} \varrho. \tag{31}
$$

By the application of Definitions 2 and 4 of q_r-derivatives, we find

$$
{}_{\Lambda_1} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - {}^{\Lambda_2} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right)
$$

$$
= \left[\begin{array}{c} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \omega \left(\frac{q_r \varrho}{2} \Lambda_2 + \left(1 - \frac{q_r \varrho}{2} \right) \Lambda_1 \right) \\ + \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) - \omega \left(\frac{q_r \varrho}{2} \Lambda_1 + \left(1 - \frac{q_r \varrho}{2} \right) \Lambda_2 \right) \end{array} \right] \div \frac{(\Lambda_2 - \Lambda_1) \varrho (1 - q_r)}{2}.
$$

Now, incorporating the property of Jackson's integral given by (5), we have

$$
\int_0^{\frac{1}{2}} \left({}_{\Lambda_1} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - {}^{\Lambda_2} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) d_{q_r} \varrho
$$

$$
= \frac{2}{\Lambda_2 - \Lambda_1} \left[\sum_{\delta=0}^\infty \omega \left(\frac{q_r^\delta}{4} \Lambda_2 + \left(1 - \frac{q_r^\delta}{4} \right) \Lambda_1 \right) - \sum_{\delta=0}^\infty \omega \left(\frac{q_r^{\delta+1}}{4} \Lambda_2 + \left(1 - \frac{q_r^{\delta+1}}{4} \right) \Lambda_1 \right) \right.
$$

$$
\left. + \sum_{\delta=0}^\infty \omega \left(\frac{q_r^\delta}{4} \Lambda_1 + \left(1 - \frac{q_r^\delta}{4} \right) \Lambda_2 \right) - \sum_{\delta=0}^\infty \omega \left(\frac{q_r^{\delta+1}}{4} \Lambda_1 + \left(1 - \frac{q_r^{\delta+1}}{4} \right) \Lambda_2 \right) \right]
$$

$$
= \frac{2}{\Lambda_2 - \Lambda_1} \left[\omega \left(\frac{3\Lambda_1 + \Lambda_2}{4} \right) + \omega \left(\frac{\Lambda_1 + 3\Lambda_2}{4} \right) - \omega(\Lambda_1) - \omega(\Lambda_2) \right]. \tag{32}
$$

Similarly,

$$
\int_0^1 \left({}_{\Lambda_1} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - {}^{\Lambda_2} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) d_{q_r} \varrho
$$

$$
= \frac{2}{\Lambda_2 - \Lambda_1} \left[2\omega \left(\frac{\Lambda_1 + \Lambda_2}{2} \right) - \omega(\Lambda_1) - \omega(\Lambda_2) \right]. \tag{33}
$$

Finally, we have

$$
\int_0^1 q_r \varrho \left({}_{\Lambda_1} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - {}^{\Lambda_2} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) d_{q_r} \varrho
$$

$$
= \frac{2}{\Lambda_2 - \Lambda_1} \left[2\omega \left(\frac{\Lambda_1 + \Lambda_2}{2} \right) - (1 - q_r) \sum_{\delta=0}^\infty q_r^\delta \omega \left(\frac{q_r^\delta}{2} \Lambda_2 + \left(1 - \frac{q_r^\delta}{2} \right) \Lambda_1 \right) \right.
$$

$$
\left. - (1 - q_r) \sum_{\delta=0}^\infty q_r^\delta \omega \left(\frac{q_r^\delta}{2} \Lambda_1 + \left(1 - \frac{q_r^\delta}{2} \right) \Lambda_2 \right) \right]
$$

$$
= \frac{4}{\Lambda_2 - \Lambda_1} \omega \left(\frac{\Lambda_1 + \Lambda_2}{2} \right) - \frac{4}{(\Lambda_2 - \Lambda_1)^2} \left(\int_{\Lambda_1}^{\frac{\Lambda_1 + \Lambda_2}{2}} \omega(\tau) \, {}_{\Lambda_1} d_{q_r} \tau + \int_{\frac{\Lambda_1 + \Lambda_2}{2}}^{\Lambda_2} \omega(\tau) \, {}^{\Lambda_2} d_{q_r} \tau \right). \tag{34}
$$

The desired equality (28) is obtained by utilizing the Equations (32)–(34) in (31) and multiplying the outcome with $\frac{\Lambda_2 - \Lambda_1}{4}$. $\square$

Lemma 2. *Let $\omega : \mathcal{W} \to \mathbb{R}$ be a $q_{r\Lambda_1}-$ and $q_r^{\Lambda_2}$–differentiable function on $\mathcal{W}^\circ$ with $0 < q_r < 1$. If $_{\Lambda_1}D_{q_r}\omega$ and $^{\Lambda_2}D_{q_r}\omega$ are continuous and $q_{r\Lambda_1}-, q_r^{\Lambda_2}$–integrable functions on $\mathcal{W}$, then the following identity holds:*

$$\Omega_2(\Lambda_1, \Lambda_2; q_r) = \frac{\Lambda_2 - \Lambda_1}{4} \left[\int_0^1 R_2(q_r, \varrho) \left({_{\Lambda_1}}D_{q_r}\omega\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1 \right) \right. \right.$$
$$\left. \left. - {^{\Lambda_2}}D_{q_r}\omega\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2 \right) \right) d_{q_r}\varrho \right],$$

where

$$R_2(q_r, \varrho) := \begin{cases} q_r\varrho - \frac{1}{8}, & \text{if} \quad 0 \le \varrho < \frac{1}{3}, \\ q_r\varrho - \frac{1}{2}, & \text{if} \quad \frac{1}{3} \le \varrho < \frac{2}{3}, \\ q_r\varrho - \frac{7}{8}, & \text{if} \quad \frac{2}{3} \le \varrho \le 1, \end{cases} \tag{35}$$

and

$$\Omega_2(\Lambda_1, \Lambda_2; q_r) := \frac{1}{16}\left[\omega(\Lambda_1) + 3\omega\left(\frac{5\Lambda_1 + \Lambda_2}{6} \right) + 3\omega\left(\frac{2\Lambda_1 + \Lambda_2}{3} \right) + 2\omega\left(\frac{\Lambda_1 + \Lambda_2}{2} \right) \right.$$
$$\left. + 3\omega\left(\frac{\Lambda_1 + 2\Lambda_2}{3} \right) + 3\omega\left(\frac{\Lambda_1 + 5\Lambda_2}{6} \right) + \omega(\Lambda_2) \right]$$
$$- \frac{1}{\Lambda_2 - \Lambda_1} \left[\int_{\Lambda_1}^{\frac{\Lambda_1 + \Lambda_2}{2}} \omega(\wp) \, _{\Lambda_1}d_{q_r}\wp + \int_{\frac{\Lambda_1 + \Lambda_2}{2}}^{\Lambda_2} \omega(\wp) \, ^{\Lambda_2}d_{q_r}\wp \right].$$

Proof. By applying the property of q_r-integrals given in equation (6), we have

$$\int_0^1 R_2(q_r, \varrho)\left({_{\Lambda_1}}D_{q_r}\omega\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1 \right) - {^{\Lambda_2}}D_{q_r}\omega\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2 \right) \right) d_{q_r}\varrho$$

$$= \frac{3}{8}\left[\int_0^{\frac{1}{3}} \left({_{\Lambda_1}}D_{q_r}\omega\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1 \right) - {^{\Lambda_2}}D_{q_r}\omega\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2 \right) \right) d_{q_r}\varrho \right.$$

$$\left. + \int_0^{\frac{2}{3}} \left({_{\Lambda_1}}D_{q_r}\omega\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1 \right) - {^{\Lambda_2}}D_{q_r}\omega\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2 \right) \right) d_{q_r}\varrho \right] \tag{36}$$

$$+ \int_0^1 \left| q_r\varrho - \frac{7}{8} \right| \left({_{\Lambda_1}}D_{q_r}\omega\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1 \right) \right.$$
$$\left. - {^{\Lambda_2}}D_{q_r}\omega\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2 \right) \right) d_{q_r}\varrho.$$

The rest of the proof follows the same approach as in Lemma 1. $\square$

4. Simpson's Type Inequalities Related to Simpson's 1/3 Quadrature Rule Via Four Panels

In this section, we give Simpson's type inequalities related to Simpson's 1/3 quadrature rule via four panels. We start with the following main result.

Theorem 5. *Let* $\varpi : \mathcal{W} \to \mathbb{R}$ *satisfy the assumptions of Lemma 1. In addition, if* $\left|_{\Lambda_1} D_{q_r} \varpi\right|$ *and* $\left|^{\Lambda_2} D_{q_r} \varpi\right|$ *are convex functions, then*

$$
|\Omega_1(\Lambda_1, \Lambda_2; q_r)| \leq \frac{\Lambda_2 - \Lambda_1}{4} \Big\{ \Big(\left|_{\Lambda_1} D_{q_r}\varpi(\Lambda_2)\right| + \left|^{\Lambda_2} D_{q_r}\varpi(\Lambda_1)\right|\Big) \mathcal{I}_1(q_r)
$$
$$
+ \Big(\left|_{\Lambda_1} D_{q_r}\varpi(\Lambda_1)\right| + \left|^{\Lambda_2} D_{q_r}\varpi(\Lambda_2)\right|\Big) \mathcal{I}_2(q_r) \Big\}, \tag{37}
$$

where

$$
\mathcal{I}_1(q_r) := \int_0^{\frac{1}{2}} \left| q_r\varrho - \frac{1}{6} \right| \frac{\varrho}{2}\, d_{q_r}\varrho + \int_{\frac{1}{2}}^1 \left| q_r\varrho - \frac{5}{6} \right| \frac{\varrho}{2}\, d_{q_r}\varrho
$$
$$
= \begin{cases} -\frac{q_r^2 + q_r - 2}{6(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if}\quad 0 < q_r < \frac{1}{3}, \\[2mm] -\frac{9q_r^2 + 9q_r - 32}{108(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if}\quad \frac{1}{3} < q_r < \frac{5}{6}, \\[2mm] \frac{2q_r^2 + 2q_r + 1}{24(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if}\quad \frac{5}{6} < q_r < 1, \end{cases} \tag{38}
$$

and

$$
\mathcal{I}_2(q_r) := \int_0^{\frac{1}{2}} \left| q_r\varrho - \frac{1}{6} \right| \left(1 - \frac{\varrho}{2}\right) d_{q_r}\varrho + \int_{\frac{1}{2}}^1 \left| q_r\varrho - \frac{5}{6} \right| \left(1 - \frac{\varrho}{2}\right) d_{q_r}\varrho
$$
$$
= \begin{cases} \frac{-3q_r^3 + q_r^2 + q_r + 1}{6(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if}\quad 0 < q_r < \frac{1}{3}, \\[2mm] \frac{-18q_r^3 + 33q_r^2 + 33q_r + 10}{108(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if}\quad \frac{1}{3} < q_r < \frac{5}{6}, \\[2mm] \frac{12q_r^3 + 14q_r^2 + 14q_r + 5}{72(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if}\quad \frac{5}{6} < q_r < 1. \end{cases} \tag{39}
$$

Proof. Consider Lemma 1. By taking modulus on both sides of the identity (28), we obtain

$$
|\Omega_1(\Lambda_1, \Lambda_2; q_r)|
$$
$$
\leq \frac{\Lambda_2 - \Lambda_1}{4} \Bigg\{ \int_0^{\frac{1}{2}} \left| q_r\varrho - \frac{1}{6} \right| \Big[\left|_{\Lambda_1} D_{q_r}\varpi\Big(\frac{\varrho}{2}\Lambda_2 + \Big(1 - \frac{\varrho}{2}\Big)\Lambda_1\Big)\right| + \left|^{\Lambda_2} D_{q_r}\varpi\Big(\frac{\varrho}{2}\Lambda_1 + \Big(1 - \frac{\varrho}{2}\Big)\Lambda_2\Big)\right| \Big] d_{q_r}\varrho
$$
$$
+ \int_{\frac{1}{2}}^1 \left| q_r\varrho - \frac{5}{6} \right| \Big[\left|_{\Lambda_1} D_{q_r}\varpi\Big(\frac{\varrho}{2}\Lambda_2 + \Big(1 - \frac{\varrho}{2}\Big)\Lambda_1\Big)\right| \left|^{\Lambda_2} D_{q_r}\varpi\Big(\frac{\varrho}{2}\Lambda_1 + \Big(1 - \frac{\varrho}{2}\Big)\Lambda_2\Big)\right| \Big] d_{q_r}\varrho \Bigg\}. \tag{40}
$$

As $\left|_{\Lambda_1} D_{q_r}\varpi\right|$ and $\left|^{\Lambda_2} D_{q_r}\varpi\right|$ are convex functions, therefore

$$
\int_0^{\frac{1}{2}} \left| q_r\varrho - \frac{1}{6} \right| \Big[\left|_{\Lambda_1} D_{q_r}\varpi\Big(\frac{\varrho}{2}\Lambda_2 + \Big(1 - \frac{\varrho}{2}\Big)\Lambda_1\Big)\right| + \left|^{\Lambda_2} D_{q_r}\varpi\Big(\frac{\varrho}{2}\Lambda_1 + \Big(1 - \frac{\varrho}{2}\Big)\Lambda_2\Big)\right| \Big] d_{q_r}\varrho
$$
$$
\leq \Big(\left|_{\Lambda_1} D_{q_r}\varpi(\Lambda_2)\right| + \left|^{\Lambda_2} D_{q_r}\varpi(\Lambda_1)\right|\Big) \int_0^{\frac{1}{2}} \left| q_r\varrho - \frac{1}{6} \right| \frac{\varrho}{2} d_{q_r}\varrho \tag{41}
$$
$$
+ \Big(\left|_{\Lambda_1} D_{q_r}\varpi(\Lambda_1)\right| + \left|^{\Lambda_2} D_{q_r}\varpi(\Lambda_2)\right|\Big) \int_0^{\frac{1}{2}} \left| q_r\varrho - \frac{1}{6} \right| \Big(1 - \frac{\varrho}{2}\Big) d_{q_r}\varrho.
$$

Similarly,

$$\int\limits_{\frac{1}{2}}^{1} \left| q_r \varrho - \frac{5}{6} \right| \left[\left| {}_{\Lambda_1} D_{q_r} \varpi \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) \right| + \left| {}^{\Lambda_2} D_{q_r} \varpi \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right| \right] d_{q_r} \varrho$$

$$\leq \left(\left| {}_{\Lambda_1} D_{q_r} \varpi (\Lambda_2) \right| + \left| {}^{\Lambda_2} D_{q_r} \varpi (\Lambda_1) \right| \right) \int\limits_{\frac{1}{2}}^{1} \left| q_r \varrho - \frac{5}{6} \right| \frac{\varrho}{2} d_{q_r} \varrho \tag{42}$$

$$+ \left(\left| {}_{\Lambda_1} D_{q_r} \varpi (\Lambda_1) \right| + \left| {}^{\Lambda_2} D_{q_r} \varpi (\Lambda_2) \right| \right) \int\limits_{\frac{1}{2}}^{1} \left| q_r \varrho - \frac{5}{6} \right| \left(1 - \frac{\varrho}{2} \right) d_{q_r} \varrho.$$

Using inequality (41) and (42) in (40), we have the desired inequality (37). □

Before proving the next main result, we give a counter example that q_r-Hölder's inequality has some limitations.

Example 1. *Let* $\varpi_1(x) = \sqrt{x}$ *and* $\varpi_2(x) = \sqrt{x^3}$. *Suppose further that* $\kappa_1 = 2 = \kappa_2$. *Now,*

$$\int\limits_{\frac{1}{2}}^{1} \sqrt{x}\sqrt{x^3} \, d_{q_r} x = \int\limits_{\frac{1}{2}}^{1} x^2 \, d_{q_r} x = \int\limits_{0}^{1} x^2 \, d_{q_r} x - \int\limits_{0}^{\frac{1}{2}} x^2 \, d_{q_r} x = \frac{7}{8(q_r^2 + q_r + 1)}. \tag{43}$$

Also,

$$\int\limits_{\frac{1}{2}}^{1} x \, d_{q_r} x = \frac{3}{4(q_r + 1)} \quad and \quad \int\limits_{\frac{1}{2}}^{1} x^3 \, d_{q_r} x = \frac{15}{16(q_r^3 + q_r^2 + q_r + 1)}. \tag{44}$$

Finally, we check the inequality for specific values for q_r.
L.H.S for $q_r = \frac{15}{100} < 1$,

$$\int\limits_{\frac{1}{2}}^{1} \sqrt{x}\sqrt{x^3} \, d_{q_r} x = \frac{7}{8\left(\left(\frac{15}{100} \right)^2 + \frac{15}{100} + 1 \right)} = 0.74627. \tag{45}$$

R.H.S for $q_r = \frac{15}{100} < 1$,

$$\left(\int\limits_{\frac{1}{2}}^{1} x \, d_{q_r} x \right)^{\frac{1}{2}} \left(\int\limits_{\frac{1}{2}}^{1} x^3 \, d_{q_r} x \right)^{\frac{1}{2}} = \left(\frac{3}{4(q_r + 1)} \right)^{\frac{1}{2}} \left(\frac{15}{16(q_r^3 + q_r^2 + q_r + 1)} \right)^{\frac{1}{2}} = 0.72109, \tag{46}$$

which justifies our claim.

The example suggests that the inequality

$$\int\limits_{\mu}^{y} |\varpi_1(\varrho)\varpi_2(\varrho)| \, {}_{\Lambda_1} d_{q_r} \varrho \leq \left(\int\limits_{\mu}^{y} |\varpi_1(\varrho)|^{\kappa_1} \, {}_{\Lambda_1} d_{q_r} \varrho \right)^{\frac{1}{\kappa_1}} \left(\int\limits_{\mu}^{y} |\varpi_2(\varrho)|^{\kappa_2} \, {}_{\Lambda_1} d_{q_r} \varrho \right)^{\frac{1}{\kappa_2}} \tag{47}$$

generally not true for $\mu < \Lambda_1$. In other words, the q_r-Hölder's inequality is true if $\mu = \Lambda_1$.

Theorem 6. *Let $\omega : \mathcal{W} \to \mathbb{R}$ satisfies the assumptions of Lemma 1. In addition, if $\left|{}_{\Lambda_1} D_{q_r} \omega\right|^{\kappa_2}$ and $\left|{}^{\Lambda_2} D_{q_r} \omega\right|^{\kappa_2} (\kappa_2 > 1)$ are convex functions on $\mathcal{W}$, then*

$$|\Omega_1(\Lambda_1, \Lambda_2; q_r)| \leq \frac{\Lambda_2 - \Lambda_1}{12 \sqrt[\kappa_2]{4(1 + q_r)}} \left\{ \left[\left(\left|{}_{\Lambda_1} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2} + (3 + 4q_r) \left|{}_{\Lambda_1} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2} \right)^{\frac{1}{\kappa_2}} \right. \right.$$

$$+ \left(\left|{}^{\Lambda_2} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2} + (3 + 4q_r) \left|{}^{\Lambda_2} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2} \right)^{\frac{1}{\kappa_2}} \right]$$

$$+ 3 (\mathcal{I}_{b,\kappa_1}(q_r))^{\frac{1}{\kappa_1}} \left[\left(2 \left|{}_{\Lambda_1} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2} + (2 + 4q_r) \left|{}_{\Lambda_1} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2} \right)^{\frac{1}{\kappa_2}} \right.$$

$$+ \left. \left. \left(2 \left|{}^{\Lambda_2} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2} + (2 + 4q_r) \left|{}^{\Lambda_2} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2} \right)^{\frac{1}{\kappa_2}} \right] \right\}, \tag{48}$$

where $\kappa_1^{-1} + \kappa_2^{-1} = 1$, and

$$\mathcal{I}_{b,\kappa_1}(q_r) := \int_0^1 \left| q_r \varrho - \frac{5}{6} \right|^{\kappa_1} d_{q_r} \varrho = \begin{cases} (1 - q_r) \sum\limits_{\delta=0}^{\infty} q_r^{\delta} \left(\frac{5}{6} - q_r^{\delta+1} \right)^{\kappa_1}, & \text{if } 0 < q_r < \frac{5}{6}, \\[2mm] \left[(1 - q_r) \left(\frac{5}{6} \right)^{\kappa_1 + 1} \sum\limits_{\delta=0}^{\infty} q_r^{\delta} \left(1 - q_r^{\delta+1} \right)^{\kappa_1} \right. \\[1mm] + (1 - q_r) \sum\limits_{\delta=0}^{\infty} q_r^{\delta} \left(q_r^{\delta+1} - \frac{5}{6} \right)^{\kappa_1} \\[1mm] \left. - (1 - q_r) \left(\frac{5}{6} \right)^{\kappa_1 + 1} \sum\limits_{\delta=0}^{\infty} q_r^{\delta} \left(q_r^{\delta+1} - 1 \right)^{\kappa_1} \right], & \text{if } \frac{5}{6} < q_r < 1. \end{cases} \tag{49}$$

Proof. By utilizing the properties of q_r-integrals, we also have identically

$$\Omega_1(\Lambda_1, \Lambda_2; q_r)$$

$$= \frac{\Lambda_2 - \Lambda_1}{4} \left\{ \frac{2}{3} \int_0^{\frac{1}{2}} \left({}_{\Lambda_1} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - {}^{\Lambda_2} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) d_{q_r} \varrho \right.$$

$$+ \left. \int_0^1 \left(q_r \varrho - \frac{5}{6} \right) \left[{}_{\Lambda_1} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - {}^{\Lambda_2} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right] d_{q_r} \varrho \right\}. \tag{50}$$

By the applications of modulus, q_r-Hölder's inequality and the convexity of $\left|{}_{\Lambda_1} D_{q_r} \omega\right|^{\kappa_2}$ and $\left|{}^{\Lambda_2} D_{q_r} \omega\right|^{\kappa_2}$, we have

$$\int_0^1 \left| q_r \varrho - \frac{5}{6} \right| \left| {}_{\Lambda_1} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) \right| d_{q_r} \varrho$$

$$+ \int_0^1 \left| q_r \varrho - \frac{5}{6} \right| \left| {}^{\Lambda_2} D_{q_r} \omega \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right| d_{q_r} \varrho$$

$$\leq \left(\int_0^1 \left| q_r \varrho - \frac{5}{6} \right|^{\kappa_1} \right)^{\frac{1}{\kappa_1}} \left[\left(\int_0^1 \frac{\varrho}{2} \left|{}_{\Lambda_1} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2} d_{q_r} \varrho + \int_0^1 \left(1 - \frac{\varrho}{2} \right) \left|{}_{\Lambda_1} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2} d_{q_r} \varrho \right)^{\frac{1}{\kappa_2}} \right.$$

$$+ \left. \left(\int_0^1 \frac{\varrho}{2} \left|{}^{\Lambda_2} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2} d_{q_r} \varrho + \int_0^1 \left(1 - \frac{\varrho}{2} \right) \left|{}^{\Lambda_2} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2} d_{q_r} \varrho \right)^{\frac{1}{\kappa_2}} \right]$$

$$\leq (\mathcal{I}_{b,\kappa_1}(q_r))^{\frac{1}{\kappa_1}} \left[\left(\frac{2 \left|{}_{\Lambda_1} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2} + 2(1 + 2q_r) \left|{}_{\Lambda_1} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2}}{4(1 + q_r)} \right)^{\frac{1}{\kappa_2}} \right.$$

$$+ \left. \left(\frac{2 \left|{}^{\Lambda_2} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2} + 2(1 + 2q_r) \left|{}^{\Lambda_2} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2}}{4(1 + q_r)} \right)^{\frac{1}{\kappa_2}} \right]. \tag{51}$$

In a similar way, we find

$$
\int_0^{\frac{1}{2}} \left| {}_{\Lambda_1} D_{q_r} \varpi \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) \right| d_{q_r} \varrho
$$

$$
+ \int_0^{\frac{1}{2}} \left| {}^{\Lambda_2} D_{q_r} \varpi \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right| d_{q_r} \varrho
$$

$$
\leq \left(\int_0^{\frac{1}{2}} 1 \, d_{q_r} \varrho \right)^{\frac{1}{\kappa_1}} \left[\left(\int_0^{\frac{1}{2}} \left| {}_{\Lambda_1} D_{q_r} \varpi \left(\frac{\varrho}{2} \Lambda_2 + \left(1 - \frac{\varrho}{2} \right) \Lambda_1 \right) \right|^{\kappa_2} d_{q_r} \varrho \right)^{\frac{1}{\kappa_2}} \right.
$$

$$
\left. + \left(\int_0^{\frac{1}{2}} \left| {}^{\Lambda_2} D_{q_r} \varpi \left(\frac{\varrho}{2} \Lambda_1 + \left(1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right|^{\kappa_2} d_{q_r} \varrho \right)^{\frac{1}{\kappa_2}} \right]
$$

$$
\leq \left(\frac{1}{2} \right)^{\frac{1}{\kappa_1}} \left[\left(\frac{\left| {}_{\Lambda_1} D_{q_r} \varpi(\Lambda_2) \right|^{\kappa_2} + (3 + 4q_r) \left| {}_{\Lambda_1} D_{q_r} \varpi(\Lambda_1) \right|^{\kappa_2}}{8(1 + q_r)} \right)^{\frac{1}{\kappa_2}} \right.
$$

$$
\left. + \left(\frac{\left| {}^{\Lambda_2} D_{q_r} \varpi(\Lambda_1) \right|^{\kappa_2} + (3 + 4q_r) \left| {}^{\Lambda_2} D_{q_r} \varpi(\Lambda_2) \right|^{\kappa_2}}{8(1 + q_r)} \right)^{\frac{1}{\kappa_2}} \right]. \tag{52}
$$

The required inequality (48) is thus obtained by utilizing inequalities (51) and (52) in (50). □

Theorem 7. *Let* $\varpi : \mathcal{W} \to \mathbb{R}$ *satisfies the assumptions of Lemma 1. In addition, if* $\left| {}_{\Lambda_1} D_{q_r} \varpi \right|^{\kappa_2}$ *and* $\left| {}^{\Lambda_2} D_{q_r} \varpi \right|^{\kappa_2}$ *are convex functions on* $\mathcal{W}$ *with* $\kappa_2 \geq 1$*, then*

$$
|\Omega_1(\Lambda_1, \Lambda_2; q)| \leq \frac{\Lambda_2 - \Lambda_1}{12 \sqrt[\kappa_2]{4(1 + q_r)}} \left\{ \left[\left(\left| {}_{\Lambda_1} D_{q_r} \varpi(\Lambda_2) \right|^{\kappa_2} + (3 + 4q_r) \left| {}_{\Lambda_1} D_{q_r} \varpi(\Lambda_1) \right|^{\kappa_2} \right)^{\frac{1}{\kappa_2}} \right. \right.
$$

$$
+ \left(\left| {}^{\Lambda_2} D_{q_r} \varpi(\Lambda_1) \right|^{\kappa_2} + (3 + 4q_r) \left| {}^{\Lambda_2} D_{q_r} \varpi(\Lambda_2) \right|^{\kappa_2} \right)^{\frac{1}{\kappa_2}} \right]
$$

$$
+ 3 \sqrt[\kappa_2]{4(1 + q_r)} (\mathcal{I}_{b,1}(q_r))^{1 - \frac{1}{\kappa_2}} \left\{ \left(\left| {}_{\Lambda_1} D_{q_r} \varpi(\Lambda_2) \right|^{\kappa_2} \mathcal{I}_c(q_r) + \left| {}_{\Lambda_1} D_{q_r} \varpi(\Lambda_1) \right|^{\kappa_2} \mathcal{I}_d(q_r) \right)^{\frac{1}{\kappa_2}} \right.
$$

$$
\left. + \left(\left| {}^{\Lambda_2} D_{q_r} \varpi(\Lambda_1) \right|^{\kappa_2} \mathcal{I}_c(q_r) + \left| {}^{\Lambda_2} D_{q_r} \varpi(\Lambda_2) \right|^{\kappa_2} \mathcal{I}_d(q_r) \right)^{\frac{1}{\kappa_2}} \right\}, \tag{53}
$$

where

$$
\mathcal{I}_c(q_r) := \int_0^1 \left| q_r \varrho - \frac{5}{6} \right| \frac{\varrho}{2} d_{q_r} \varrho = \begin{cases} -\frac{q_r^2 + q_r - 5}{12(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if} \quad 0 < q_r < \frac{5}{6}, \\ \frac{18q_r^2 + 18q_r + 35}{216(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if} \quad \frac{5}{6} < q_r < 1, \end{cases} \tag{54}
$$

$$
\mathcal{I}_d(q_r) := \int_0^1 \left| q_r \varrho - \frac{5}{6} \right| \left(1 - \frac{\varrho}{2} \right) d_{q_r} \varrho = \begin{cases} \frac{-2q_r^3 + 9q_r^2 + 9q_r + 5}{12(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if} \quad 0 < q_r < \frac{5}{6}, \\ \frac{36q_r^3 + 138q_r^2 + 138q_r + 85}{216(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if} \quad \frac{5}{6} < q_r < 1, \end{cases} \tag{55}
$$

and $\mathcal{I}_{b,1}(q_r)$ *is obtained from* $\mathcal{I}_{b,\kappa_1}(q_r)$*, which is same as given in Theorem 6.*

Proof. Consider again Lemma 1. The property of modulus, power mean's inequality, and the convexity of $\left|_{\Lambda_1} D_{q_r} \omega\right|^{\kappa_2}$ and $\left|^{\Lambda_2} D_{q_r} \omega\right|^{\kappa_2}$ leads to

$$
\int_0^1 \left|q_r\varrho - \frac{5}{6}\right| \left|_{\Lambda_1} D_{q_r} \omega\left(\frac{\varrho}{2}\Lambda_2 + \left(1-\frac{\varrho}{2}\right)\Lambda_1\right)\right| d_{q_r}\varrho
$$

$$
+ \int_0^1 \left|q_r\varrho - \frac{5}{6}\right| \left|^{\Lambda_2} D_{q_r} \omega\left(\frac{\varrho}{2}\Lambda_1 + \left(1-\frac{\varrho}{2}\right)\Lambda_2\right)\right| d_{q_r}\varrho
$$

$$
\leq \left(\int_0^1 \left|q_r\varrho - \frac{5}{6}\right|\right)^{1-\frac{1}{\kappa_2}} \left[\left(\int_0^1 \left|q_r\varrho - \frac{5}{6}\right|\left|_{\Lambda_1} D_{q_r} \omega\left(\frac{\varrho}{2}\Lambda_2 + \left(1-\frac{\varrho}{2}\right)\Lambda_1\right)\right|^{\kappa_2} d_{q_r}\varrho\right)^{\frac{1}{\kappa_2}}\right.
$$

$$
+ \left.\left(\int_0^1 \left|q_r\varrho - \frac{5}{6}\right|\left|^{\Lambda_2} D_{q_r} \omega\left(\frac{\varrho}{2}\Lambda_1 + \left(1-\frac{\varrho}{2}\right)\Lambda_2\right)\right|^{\kappa_2} d_{q_r}\varrho\right)^{\frac{1}{\kappa_2}}\right]
$$

$$
\leq \left(\mathcal{I}_{b,\kappa_1}(q_r)\right)^{1-\frac{1}{\kappa_2}} \left[\left(\left|_{\Lambda_1} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2} \mathcal{I}_c(q_r) + \left|_{\Lambda_1} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2} \mathcal{I}_d(q_r)\right)^{\frac{1}{\kappa_2}}\right.
$$

$$
+ \left.\left(\left|^{\Lambda_2} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2} \mathcal{I}_c(q_r) + \left|^{\Lambda_2} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2} \mathcal{I}_d(q_r)\right)^{\frac{1}{\kappa_2}}\right]. \tag{56}
$$

By some parallel calculations, we obtain

$$
\int_0^{\frac{1}{2}} \left|_{\Lambda_1} D_{q_r} \omega\left(\frac{\varrho}{2}\Lambda_2 + \left(1-\frac{\varrho}{2}\right)\Lambda_1\right)\right| d_{q_r}\varrho
$$

$$
+ \int_0^{\frac{1}{2}} \left|^{\Lambda_2} D_{q_r} \omega\left(\frac{\varrho}{2}\Lambda_1 + \left(1-\frac{\varrho}{2}\right)\Lambda_2\right)\right| d_{q_r}\varrho
$$

$$
\leq \left(\frac{1}{2}\right)^{1-\frac{1}{\kappa_2}} \left[\left(\frac{\left|_{\Lambda_1} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2} + (3+4q_r)\left|_{\Lambda_1} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2}}{8(1+q_r)}\right)^{\frac{1}{\kappa_2}}\right.
$$

$$
+ \left.\left(\frac{\left|^{\Lambda_2} D_{q_r} \omega(\Lambda_1)\right|^{\kappa_2} + (3+4q_r)\left|^{\Lambda_2} D_{q_r} \omega(\Lambda_2)\right|^{\kappa_2}}{8(1+q_r)}\right)^{\frac{1}{\kappa_2}}\right]. \tag{57}
$$

The required inequality (53) is thus obtained by utilizing (56) and (57) in (50). □

5. Quantum Analogues of Simpson's Inequalities Related to Simpson's 3/8 Rule

This section is devoted to the extension of Simpson's 3/8 rule. Here we present our results utilizing six panels.

Theorem 8. *Let* $\omega : \mathcal{W} \to \mathbb{R}$ *satisfies the assumptions of Lemma 2. In addition, if* $\left|_{\Lambda_1} D_{q_r} \omega\right|$ *and* $\left|^{\Lambda_2} D_{q_r} \omega\right|$ *are convex functions on* $\mathcal{W}$, *then*

$$
|\Omega_2(\Lambda_1, \Lambda_2; q_r)| \leq \frac{\Lambda_2 - \Lambda_1}{4} \left\{\left(\left|_{\Lambda_1} D_{q_r} \omega(\Lambda_2)\right| + \left|^{\Lambda_2} D_{q_r} \omega(\Lambda_1)\right|\right) \mathcal{I}_M(q_r)\right.
$$

$$
+ \left.\left(\left|_{\Lambda_1} D_{q_r} \omega(\Lambda_1)\right| + \left|^{\Lambda_2} D_{q_r} \omega(\Lambda_2)\right|\right) \mathcal{I}_N(q_r)\right\}, \tag{58}
$$

where

$$
\mathcal{I}_M(q_r) := \sum_{\gamma=3}^{5} \mathcal{I}_\gamma(q_r) = \begin{cases} \frac{-q_r^2-q_r+2}{6(q_r^3+2q_r^2+2q_r+1)}, & \text{if } 0 < q_r < \frac{3}{8}, \\ \frac{-1984q_r^2-1984q_r+4443}{13824(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{3}{8} < q_r < \frac{3}{4}, \\ \frac{-320q_r^2-320q_r+1033}{4608(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{3}{4} < q_r < \frac{7}{8}, \\ \frac{32q_r^2+32q_r+11}{576(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{7}{8} < q_r < 1, \end{cases}
\tag{59}
$$

$$
\mathcal{I}_N(q_r) := \sum_{\nu=6}^{8} \mathcal{I}_\nu(q_r) = \begin{cases} \frac{-3q_r^3+q_r^2+q_r+1}{6(q_r^3+2q_r^2+2q_r+1)}, & \text{if } 0 < q_r < \frac{3}{8}, \\ \frac{4992q_r^3+3184q_r^2+3184q_r+1749}{13824(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{3}{8} < q_r < \frac{3}{4}, \\ \frac{-640q_r^3+976q_r^2+976q_r+263}{4608(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{3}{4} < q_r < \frac{7}{8}, \\ \frac{64q_r^3+68q_r^2+68q_r+25}{576(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{7}{8} < q_r < 1, \end{cases}
\tag{60}
$$

and

$$
\mathcal{I}_3(q_r) := \int_0^{\frac{1}{3}} \left| q_r\varrho - \frac{1}{8} \right| \frac{\varrho}{2}\, d_{q_r}\varrho = \begin{cases} \frac{-5q_r^2-5q_r+3}{432(q_r^3+2q_r^2+2q_r+1)}, & \text{if } 0 < q_r < \frac{3}{8}, \\ \frac{160q_r^2+160q_r-69}{13824(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{3}{8} < q_r < 1, \end{cases}
\tag{61}
$$

$$
\mathcal{I}_4(q_r) := \int_{\frac{1}{3}}^{\frac{2}{3}} \left| q_r\varrho - \frac{1}{2} \right| \frac{\varrho}{2}\, d_{q_r}\varrho = \begin{cases} \frac{-5q_r^2-5q_r+9}{108(q_r^3+2q_r^2+2q_r+1)}, & \text{if } 0 < q_r < \frac{3}{4}, \\ \frac{6q_r^2+6q_r-3}{216(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{3}{4} < q_r < 1, \end{cases}
\tag{62}
$$

$$
\mathcal{I}_5(q_r) := \int_{\frac{2}{3}}^{1} \left| q_r\varrho - \frac{7}{8} \right| \frac{\varrho}{2}\, d_{q_r}\varrho = \begin{cases} \frac{-47q_r^2-47q_r+105}{432(q_r^3+2q_r^2+2q_r+1)}, & \text{if } 0 < q_r < \frac{7}{8}, \\ \frac{224q_r^2+224q_r+525}{13824(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{7}{8} < q_r < 1, \end{cases}
\tag{63}
$$

$$
\mathcal{I}_6(q_r) := \int_0^{\frac{1}{3}} \left| q_r\varrho - \frac{1}{8} \right| \left(1 - \frac{\varrho}{2}\right) d_{q_r}\varrho = \begin{cases} \frac{-30q_r^2-7q_r^2-7q_r+15}{432(q_r^3+2q_r^2+2q_r+1)}, & \text{if } 0 < q_r < \frac{3}{8}, \\ \frac{960q_r^3+656q_r^2+656q_r-75}{13824(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{3}{8} < q_r < 1, \end{cases}
\tag{64}
$$

$$
\mathcal{I}_7(q_r) := \int_{\frac{1}{3}}^{\frac{2}{3}} \left| q_r\varrho - \frac{1}{2} \right| \left(1 - \frac{\varrho}{2}\right) d_{q_r}\varrho = \begin{cases} \frac{-18q_r^3+5q_r^2+5q_r+9}{108(q_r^3+2q_r^2+2q_r+1)}, & \text{if } 0 < q_r < \frac{3}{4}, \\ \frac{4q_r^3+2q_r^2+2q_r+1}{72(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{3}{4} < q_r < 1, \end{cases}
\tag{65}
$$

$$
\mathcal{I}_8(q_r) := \int_{\frac{2}{3}}^{1} \left| q_r\varrho - \frac{7}{8} \right| \left(1 - \frac{\varrho}{2}\right) d_{q_r}\varrho = \begin{cases} \frac{-114q_r^3+59q_r^2+59q_r+21}{432(q_r^3+2q_r^2+2q_r+1)}, & \text{if } 0 < q_r < \frac{7}{8}, \\ \frac{-192q_r^3+592q_r^2+592q_r+483}{13824(q_r^3+2q_r^2+2q_r+1)}, & \text{if } \frac{7}{8} < q_r < 1. \end{cases}
\tag{66}
$$

Proof. The proof is skipped as it follows the same lines as used in the previous Theorem 5 by utilizing the identity given in Lemma 2. □

Theorem 9. *Assume that $\omega : \mathcal{W} \to \mathbb{R}$ satisfies the assumptions of Lemma 2. In addition, if $\left|_{\Lambda_1} D_{q_r}\omega\right|^{\kappa_2}$ and $\left|^{\Lambda_2} D_{q_r}\omega\right|^{\kappa_2} (\kappa_2 > 1)$ are convex functions on $\mathcal{W}$, then*

$$
|\Omega_2(\Lambda_1, \Lambda_2; q_r)| \leq \frac{\Lambda_2 - \Lambda_1}{32 \sqrt[\kappa_2]{6(1 + q_r)}} \left\{ \left(|_{\Lambda_1} D_{q_r}\omega(\Lambda_2)|^{\kappa_2} + (5 + 6q_r)|_{\Lambda_1} D_{q_r}\omega(\Lambda_1)|^{\kappa_2} \right)^{\frac{1}{\kappa_2}} \right.
$$
$$
+ \left(\left|^{\Lambda_2} D_{q_r}\omega(\Lambda_1)\right|^{\kappa_2} + (5 + 6q_r)\left|^{\Lambda_2} D_{q_r}\omega(\Lambda_2)\right|^{\kappa_2} \right)^{\frac{1}{\kappa_2}}
$$
$$
+ 2\left(2|_{\Lambda_1} D_{q_r}\omega(\Lambda_2)|^{\kappa_2} + (4 + 6q_r)|_{\Lambda_1} D_{q_r}\omega(\Lambda_1)|^{\kappa_2} \right)^{\frac{1}{\kappa_2}}
$$
$$
+ 2\left(2\left|^{\Lambda_2} D_{q_r}\omega(\Lambda_1)\right|^{\kappa_2} + (4 + 6q_r)\left|^{\Lambda_2} D_{q_r}\omega(\Lambda_2)\right|^{\kappa_2} \right)^{\frac{1}{\kappa_2}}
$$
$$
+ 8(\mathcal{I}_{e,\kappa_1}(q_r))^{\frac{1}{\kappa_1}} \left[\left(3|_{\Lambda_1} D_{q_r}\omega(\Lambda_2)|^{\kappa_2} + (3 + 6q_r)|_{\Lambda_1} D_{q_r}\omega(\Lambda_1)|^{\kappa_2} \right)^{\frac{1}{\kappa_2}} \right.
$$
$$
\left. \left. + \left(3\left|^{\Lambda_2} D_{q_r}\omega(\Lambda_1)\right|^{\kappa_2} + (3 + 6q_r)\left|^{\Lambda_2} D_{q_r}\omega(\Lambda_2)\right|^{\kappa_2} \right)^{\frac{1}{\kappa_2}} \right] \right\}, \tag{67}
$$

where $\kappa_1^{-1} + \kappa_2^{-1} = 1$, and

$$
\mathcal{I}_{e,\kappa_1}(q_r) := \int_0^1 \left| q_r\varrho - \frac{7}{8} \right|^{\kappa_1} d_{q_r}\varrho =
\begin{cases}
(1 - q_r) \sum_{\delta=0}^{\infty} q_r^{\delta} \left(\frac{7}{8} - q_r^{\delta+1} \right)^{\kappa_1}, & \text{if } \quad 0 < q_r < \frac{7}{8}, \\[2mm]
\left[(1 - q_r)\left(\frac{7}{8}\right)^{\kappa_1+1} \sum_{\delta=0}^{\infty} q_r^{\delta} \left(1 - q_r^{\delta+1} \right)^{\kappa_1} \right. \\
\left. + (1 - q_r) \sum_{\delta=0}^{\infty} q_r^{\delta} \left(q_r^{\delta+1} - \frac{7}{8} \right)^{\kappa_1} \right. \\
\left. - (1 - q_r)\left(\frac{7}{8}\right)^{\kappa_1+1} \sum_{\delta=0}^{\infty} q_r^{\delta} \left(q_r^{\delta+1} - 1 \right)^{\kappa_1} \right], & \text{if } \quad \frac{7}{8} < q_r < 1.
\end{cases}
\tag{68}
$$

Proof. The required inequality (67) can be achieved if we consider Lemma 2 and follow the approach used in the proof of Theorem 6. □

Theorem 10. *Assume that $\omega : \mathcal{W} \to \mathbb{R}$ satisfies the assumptions of Lemma 2. In addition, if $\left|_{\Lambda_1} D_{q_r}\omega\right|^{\kappa_2}$ and $\left|^{\Lambda_2} D_{q_r}\omega\right|^{\kappa_2}$ are convex functions on $\mathcal{W}$ with $\kappa_2 \geq 1$, then*

$$
|\Omega(\Lambda_1, \Lambda_2; q_r)| \leq \frac{\Lambda_2 - \Lambda_1}{32 \sqrt[\kappa_2]{6(1 + q_r)}} \left\{ \left(|_{\Lambda_1} D_{q_r}\omega(\Lambda_2)|^{\kappa_2} + (5 + 6q_r)|_{\Lambda_1} D_{q_r}\omega(\Lambda_1)|^{\kappa_2} \right)^{\frac{1}{\kappa_2}} \right.
$$
$$
+ \left(\left|^{\Lambda_2} D_{q_r}\omega(\Lambda_1)\right|^{\kappa_2} + (5 + 6q_r)\left|^{\Lambda_2} D_{q_r}\omega(\Lambda_2)\right|^{\kappa_2} \right)^{\frac{1}{\kappa_2}}
$$
$$
+ 2\left(2|_{\Lambda_1} D_{q_r}\omega(\Lambda_2)|^{\kappa_2} + 2(2 + 3q_r)|_{\Lambda_1} D_{q_r}\omega(\Lambda_1)|^{\kappa_2} \right)^{\frac{1}{\kappa_2}}
$$
$$
+ 2\left(2\left|^{\Lambda_2} D_{q_r}\omega(\Lambda_1)\right|^{\kappa_2} + 2(2 + 3q_r)\left|^{\Lambda_2} D_{q_r}\omega(\Lambda_2)\right|^{\kappa_2} \right)^{\frac{1}{\kappa_2}}
$$
$$
+ 8 \sqrt[\kappa_2]{6(1 + q_r)}(\mathcal{I}_{e,1}(q_r))^{1 - \frac{1}{\kappa_2}} \left[\left(|_{\Lambda_1} D_{q_r}\omega(\Lambda_2)|^{\kappa_2}\mathcal{I}_g(q_r) + |_{\Lambda_1} D_{q_r}\omega(\Lambda_1)|^{\kappa_2}\mathcal{I}_h(q_r) \right)^{\frac{1}{\kappa_2}} \right.
$$
$$
\left. \left. + \left(\left|^{\Lambda_2} D_{q_r}\omega(\Lambda_1)\right|^{\kappa_2}\mathcal{I}_g(q_r) + \left|^{\Lambda_2} D_{q_r}\omega(\Lambda_2)\right|^{\kappa_2}\mathcal{I}_h(q_r) \right)^{\frac{1}{\kappa_2}} \right] \right\}, \tag{69}
$$

where

$$
\mathcal{I}_g(q_r) := \int_0^1 \left| q_r\varrho - \frac{7}{8} \right| \frac{\varrho}{2} d_{q_r}\varrho =
\begin{cases}
\frac{-q_r^2 - q_r + 7}{16(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if } \quad 0 < q_r < \frac{7}{8}, \\[2mm]
\frac{32q_r^2 + 32q_r + 119}{512(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if } \quad \frac{7}{8} < q_r < 1,
\end{cases}
\tag{70}
$$

$$
\mathcal{I}_h(q_r) := \int_0^1 \left| q_r\varrho - \frac{7}{8} \right| \left(1 - \frac{\varrho}{2} \right) d_{q_r}\varrho =
\begin{cases}
\frac{-2q_r^3 + 13q_r^2 + 13q_r + 7}{16(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if } \quad 0 < q_r < \frac{7}{8}, \\[2mm]
\frac{64q_r^3 + 368q_r^2 + 368q_r + 217}{512(q_r^3 + 2q_r^2 + 2q_r + 1)}, & \text{if } \quad \frac{7}{8} < q_r < 1,
\end{cases}
\tag{71}
$$

and $\mathcal{I}_{e,1}(q_r)$ is achieved from $\mathcal{I}_{e,k_1}(q_r)$, which is given in Theorem 9.

Proof. The proof is skipped as it is similar to the Theorem 7. □

6. Simpson's Type Inequalities Associated with Classical Integrals

This section is devoted to some classical versions of the inequalities developed for q_r-integrals. Some results are deduced from the previous section. The rest of the results are proved to examine the variation of two theories.

Corollary 1. *Let $\omega : \mathcal{W} \to \mathbb{R}$ be a differentiable function on $\mathcal{W}^{\circ}$. If ω' is integrable function on $\mathcal{W}$, then the following identity holds:*

$$\frac{1}{12}\left[\omega(\Lambda_1) + 4\omega\left(\frac{3\Lambda_1 + \Lambda_2}{4}\right) + 2\omega\left(\frac{\Lambda_1 + \Lambda_2}{2}\right) + 4\omega\left(\frac{\Lambda_1 + 3\Lambda_2}{4}\right) + \omega(\Lambda_2)\right]$$

$$- \frac{1}{\Lambda_2 - \Lambda_1}\int_{\Lambda_1}^{\Lambda_2}\omega(\wp)d\wp. \tag{72}$$

$$= \frac{\Lambda_2 - \Lambda_1}{4}\left[\int_0^1 R_1(\varrho)\left(\omega'\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1\right) - \omega'\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2\right)\right)d\varrho\right],$$

where

$$R_1(\varrho) := \begin{cases} \varrho - \frac{1}{6}, & if \quad 0 \le \varrho < \frac{1}{2}, \\ \varrho - \frac{5}{6}, & if \quad \frac{1}{2} \le \varrho \le 1. \end{cases} \tag{73}$$

Proof. Consider Lemma 1. If $q_r \to 1^-$, then we find the desired identity. □

Corollary 2. *Let $\omega : \mathcal{W} \to \mathbb{R}$ satisfies the assumptions of Corollary 1. In addition, if $|\omega'|$ is convex function, then*

$$\left|\frac{1}{12}\left[\omega(\Lambda_1) + 4\omega\left(\frac{3\Lambda_1 + \Lambda_2}{4}\right) + 2\omega\left(\frac{\Lambda_1 + \Lambda_2}{2}\right) + 4\omega\left(\frac{\Lambda_1 + 3\Lambda_2}{4}\right) + \omega(\Lambda_2)\right]\right.$$

$$\left. - \frac{1}{\Lambda_2 - \Lambda_1}\int_{\Lambda_1}^{\Lambda_2}\omega(\wp)\,d\wp\right|$$

$$\le \frac{5(\Lambda_2 - \Lambda_1)}{144}\left[|\omega'(\Lambda_1)| + |\omega'(\Lambda_2)|\right]. \tag{74}$$

Proof. Consider Theorem 5. If we let $q_r \to 1^-$, then the above inequality is obtained. □

Corollary 3. *Let $\omega : \mathcal{W} \to \mathbb{R}$ satisfies the assumptions of Corollary 1. In addition, if $|\omega'|^{\kappa_2}$ $(\kappa_2 > 1)$, is convex function, then*

$$\left|\frac{1}{12}\left[\omega(\Lambda_1) + 4\omega\left(\frac{3\Lambda_1 + \Lambda_2}{4}\right) + 2\omega\left(\frac{\Lambda_1 + \Lambda_2}{2}\right) + 4\omega\left(\frac{\Lambda_1 + 3\Lambda_2}{4}\right) + \omega(\Lambda_2)\right]\right.$$

$$\left. - \frac{1}{\Lambda_2 - \Lambda_1}\int_{\Lambda_1}^{\Lambda_2}\omega(\wp)\,d\wp\right|$$

$$\le \frac{\Lambda_2 - \Lambda_1}{4\sqrt[k_2]{16}}\left(\frac{1 + 2^{\kappa_1+1}}{6^{\kappa_1+1}(\kappa_1+1)}\right)^{\frac{1}{\kappa_1}}\left\{\left(|\omega'(\Lambda_2)|^{\kappa_2} + 7|\omega'(\Lambda_1)|^{\kappa_2}\right)^{\frac{1}{\kappa_2}}\right.$$

$$+ \left(|\omega'(\Lambda_1)|^{\kappa_2} + 7|\omega'(\Lambda_2)|^{\kappa_2}\right)^{\frac{1}{\kappa_2}} + \left(3|\omega'(\Lambda_2)|^{\kappa_2} + 5|\omega'(\Lambda_1)|^{\kappa_2}\right)^{\frac{1}{\kappa_2}}$$

$$\left. + \left(3|\omega'(\Lambda_1)|^{\kappa_2} + 5|\omega'(\Lambda_2)|^{\kappa_2}\right)^{\frac{1}{\kappa_2}}\right\}, \tag{75}$$

where $\kappa_1^{-1} + \kappa_2^{-1} = 1$.

Proof. Firstly, we note that

$$\int_0^{\frac{1}{2}} \left| \varrho - \frac{1}{6} \right|^{\kappa_1} d\varrho = \frac{1 + 2^{\kappa_1+1}}{6^{\kappa_1+1}(\kappa_1 + 1)} = \int_{\frac{1}{2}}^1 \left| \varrho - \frac{5}{6} \right|^{\kappa_1} d\varrho. \tag{76}$$

By the use of modulus on both sides of the identity (72), we have

$$\left| \frac{1}{12} \left[\varpi(\Lambda_1) + 4\varpi\left(\frac{3\Lambda_1 + \Lambda_2}{4} \right) + 2\varpi\left(\frac{\Lambda_1 + \Lambda_2}{2} \right) + 4\varpi\left(\frac{3\Lambda_1 + \Lambda_2}{4} \right) + \varpi(\Lambda_2) \right] \right.$$

$$\left. - \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \varpi(\wp)\, d\wp \right|$$

$$\leq \frac{\Lambda_2 - \Lambda_1}{4} \left\{ \int_0^{\frac{1}{2}} \left| \varrho - \frac{1}{6} \right| \left[\left| \varpi'\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1 \right) \right| + \left| \varpi'\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2 \right) \right| \right] d\varrho \right.$$

$$\left. + \int_{\frac{1}{2}}^1 \left| \varrho - \frac{5}{6} \right| \left[\left| \varpi'\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1 \right) \right| + \left| \varpi'\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2 \right) \right| \right] d\varrho \right\}. \tag{77}$$

If we apply the Hölder's inequality and utilize the convexity of $|\varpi'|^{\kappa_2}$, we find

$$\int_0^{\frac{1}{2}} \left| \varrho - \frac{1}{6} \right| \left[\left| \varpi'\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1 \right) \right| + \left| \varpi'\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2 \right) \right| \right] d\varrho$$

$$\leq \left(\int_0^{\frac{1}{2}} \left| \varrho - \frac{1}{6} \right|^{\kappa_1} d\varrho \right)^{\frac{1}{\kappa_1}} \left[\left(|\varpi'(\Lambda_2)|^{\kappa_2} \int_0^{\frac{1}{2}} \frac{\varrho}{2} d\varrho + |\varpi'(\Lambda_1)|^{\kappa_2} \int_0^{\frac{1}{2}} \left(1 - \frac{\varrho}{2}\right) d\varrho \right)^{\frac{1}{\kappa_2}} \right.$$

$$\left. + \left(|\varpi'(\Lambda_1)|^{\kappa_2} \int_0^{\frac{1}{2}} \frac{\varrho}{2} d\varrho + |\varpi'(\Lambda_2)|^{\kappa_2} \int_0^{\frac{1}{2}} \left(1 - \frac{\varrho}{2}\right) d\varrho \right)^{\frac{1}{\kappa_2}} \right]. \tag{78}$$

Similarly,

$$\int_{\frac{1}{2}}^1 \left| \varrho - \frac{5}{6} \right| \left[\left| \varpi'\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1 \right) \right| + \left| \varpi'\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2 \right) \right| \right] d\varrho$$

$$\leq \left(\int_{\frac{1}{2}}^1 \left| \varrho - \frac{5}{6} \right|^{\kappa_1} d\varrho \right)^{\frac{1}{\kappa_1}} \left[\left(|\varpi'(\Lambda_2)|^{\kappa_2} \int_{\frac{1}{2}}^1 \frac{\varrho}{2} d\varrho + |\varpi'(\Lambda_1)|^{\kappa_2} \int_{\frac{1}{2}}^1 \left(1 - \frac{\varrho}{2}\right) d\varrho \right)^{\frac{1}{\kappa_2}} \right.$$

$$\left. + \left(|\varpi'(\Lambda_1)|^{\kappa_2} \int_{\frac{1}{2}}^1 \frac{\varrho}{2} d\varrho + |\varpi'(\Lambda_2)|^{\kappa_2} \int_{\frac{1}{2}}^1 \left(1 - \frac{\varrho}{2}\right) d\varrho \right)^{\frac{1}{\kappa_2}} \right]. \tag{79}$$

The desired inequality is thus obtained by utilizing inequalities (76), (78), and (79) in (77). □

Corollary 4. *Let* $\omega : \mathcal{W} \to \mathbb{R}$ *be a differentiable function on* $\mathcal{W}^{\circ}$. *If* ω' *is integrable function on* $\mathcal{W}$, *then the following identity holds:*

$$
\frac{1}{16}\left[\omega(\Lambda_1) + 3\omega\left(\frac{5\Lambda_1 + \Lambda_2}{6}\right) + 3\omega\left(\frac{2\Lambda_1 + \Lambda_2}{3}\right) + 2\omega\left(\frac{\Lambda_1 + \Lambda_2}{2}\right)\right.
$$

$$
\left. +3\omega\left(\frac{\Lambda_1 + 2\Lambda_2}{3}\right) + 3\omega\left(\frac{\Lambda_1 + 5\Lambda_2}{6}\right) + \omega(\Lambda_2)\right] - \frac{1}{\Lambda_2 - \Lambda_1}\int_{\Lambda_1}^{\Lambda_2}\omega(\wp)d_{q_r}\wp. \tag{80}
$$

$$
= \frac{\Lambda_2 - \Lambda_1}{4}\left[\int_0^1 R_2(\varrho)\left(\omega'\left(\frac{\varrho}{2}\Lambda_2 + \left(1 - \frac{\varrho}{2}\right)\Lambda_1\right) - \omega'\left(\frac{\varrho}{2}\Lambda_1 + \left(1 - \frac{\varrho}{2}\right)\Lambda_2\right)\right)d\varrho\right],
$$

where

$$
R_2(\varrho) := \begin{cases} \varrho - \frac{1}{8}, & \text{if } 0 \le \varrho < \frac{1}{3}, \\ \varrho - \frac{1}{2}, & \text{if } \frac{1}{3} \le \varrho < \frac{2}{3}, \\ \varrho - \frac{7}{8}, & \text{if } \frac{2}{3} \le \varrho \le 1. \end{cases} \tag{81}
$$

Proof. If we assume $q_r \to 1^-$ in Lemma 2, then we find the identity (80) as a consequence. □

Corollary 5. *Assume that* $\omega : \mathcal{W} \to \mathbb{R}$ *satisfies the assumptions of Corollary 4. In addition, if* $|\omega'|$ *is convex function, then*

$$
\left|\frac{1}{16}\left[\omega(\Lambda_1) + 3\omega\left(\frac{5\Lambda_1 + \Lambda_2}{6}\right) + 3\omega\left(\frac{2\Lambda_1 + \Lambda_2}{3}\right) + 2\omega\left(\frac{\Lambda_1 + \Lambda_2}{2}\right)\right.\right.
$$

$$
\left.\left. +3\omega\left(\frac{\Lambda_1 + 2\Lambda_2}{3}\right) + 3\omega\left(\frac{\Lambda_1 + 5\Lambda_2}{6}\right) + \omega(\Lambda_2)\right] - \frac{1}{\Lambda_2 - \Lambda_1}\int_{\Lambda_1}^{\Lambda_2}\omega(\wp)d\wp\right|
$$

$$
\le \frac{25(\Lambda_2 - \Lambda_1)}{1152}\left[|\omega'(\Lambda_1)| + |\omega'(\Lambda_2)|\right]. \tag{82}
$$

Proof. Consider Theorem 8. If we let $q_r \to 1^-$, then as a consequence, we obtain the desired inequality. □

Corollary 6. *Assume that* $\omega : \mathcal{W} \to \mathbb{R}$ *satisfies the assumptions of Corollary 4. In addition, if* $|\omega'|^{\kappa_2}(\kappa_2 > 1)$, *is convex function, then*

$$
\left|\frac{1}{16}\left[\omega(\Lambda_1) + 3\omega\left(\frac{5\Lambda_1 + \Lambda_2}{6}\right) + 3\omega\left(\frac{2\Lambda_1 + \Lambda_2}{3}\right) + 2\omega\left(\frac{\Lambda_1 + \Lambda_2}{2}\right)\right.\right.
$$

$$
\left.\left. +3\omega\left(\frac{\Lambda_1 + 2\Lambda_2}{3}\right) + 3\omega\left(\frac{\Lambda_1 + 5\Lambda_2}{6}\right) + \omega(\Lambda_2)\right] - \frac{1}{\Lambda_2 - \Lambda_1}\int_{\Lambda_1}^{\Lambda_2}\omega(\wp)d\wp\right|
$$

$$
\le \frac{\Lambda_2 - \Lambda_1}{1152\sqrt[\kappa_1]{2(\kappa_1 + 1)}\sqrt[\kappa_2]{3}}\left\{\sqrt[\kappa_1]{3^{\kappa_1+1} + 5^{\kappa_1+1}}\left[\left(|\omega'(\Lambda_2)|^{\kappa_2} + 11|\omega'(\Lambda_1)|^{\kappa_2}\right)^{\frac{1}{\kappa_2}}\right.\right.
$$

$$
+ \left(|\omega'(\Lambda_1)|^{\kappa_2} + 11|\omega'(\Lambda_2)|^{\kappa_2}\right)^{\frac{1}{\kappa_2}}\right] + 4\sqrt[\kappa_1]{8}\left[\left(3|\omega'(\Lambda_2)|^{\kappa_2} + 9|\omega'(\Lambda_1)|^{\kappa_2}\right)^{\frac{1}{\kappa_2}}\right.
$$

$$
+ \left(3|\omega'(\Lambda_1)|^{\kappa_2} + 9|\omega'(\Lambda_2)|^{\kappa_2}\right)^{\frac{1}{\kappa_2}}\right] + \sqrt[\kappa_1]{3^{\kappa_1+1} + 5^{\kappa_1+1}}\left[\left(5|\omega'(\Lambda_2)|^{\kappa_2} + 7|\omega'(\Lambda_1)|^{\kappa_2}\right)^{\frac{1}{\kappa_2}}\right.
$$

$$
\left.\left. + \left(5|\omega'(\Lambda_1)|^{\kappa_2} + 7|\omega'(\Lambda_2)|^{\kappa_2}\right)^{\frac{1}{\kappa_2}}\right]\right\}, \tag{83}
$$

where $\kappa_1^{-1} + \kappa_2^{-1} = 1$.

Proof. The proof is skipped as it is on the same lines which are followed in the proof of Corollary 3. □

7. Conclusions

The current study discusses the Simpson's type quantum inequalities. The study brings into the spotlight some new estimates for Simpson's type quadrature rules keeping four and six panels. The inequalities due to this study proves to be analogous to the classical rules. Our inequality (37) is analogous to the Simpson's 1/3 rule (1) for four panels while our inequality (58) given in Theorem 8 is a quantum version of Simpson's 3/8 rule (2) for six panels.

We also notice that Corollary 1 is a special case of Lemma 1 while Corollary 2 is a special case of Theorem 5. Corollary 4 gives an identity for six panels for classical integrals and is a special case of identity given in Lemma 2. Corollary 5 is obtained as a special case from Theorem 8.

It is worth mentioning that the theory of quantum differentiable and integrable functions is not completely parallel to the classical calculus. Indeed, the q_r-Hölder's inequality is weak compared to classical cases. Our results in Corollarys 3 and 6 are not special cases of Theorems 6 and 9, respectively. We have presented a counter example to explain the limiting nature of Hölder's inequality in quantum framework of calculus. In some recent papers, the researchers have used the q_r-Hölder's inequality in the strong form which should be carefully re-examined. Finally, we found that the application of q_r-Hölder's inequality gives some new variants of the classical results. Now, due to this varying nature, researchers should not only address the special feature.

Finally, we remark that the result in Corollarys 2 and 5 reveals, in comparison with the established inequalities given in Remarks 2 and 3, that if the number of panels are doubled, then the error reduces half of the previous.

We feel that this study will inspire the researchers working in the area of quantum calculus.

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