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The Spectrum of Second Order Quantum Difference Operator

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Abstract: In this study, we give another generalization of second order backward difference operator ∇^2 by introducing its quantum analog ∇_q^2. The operator ∇_q^2 represents the third band infinite matrix. We construct its domains $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$ in the spaces c_0 and c of null and convergent sequences,respectively, and establish that the domains $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$ are Banach spaces linearly isomorphic to c_0 and c, respectively, and obtain their Schauder bases and α-, β- and γ-duals. We devote the last section to determine the spectrum, the point spectrum, the continuous spectrum and the residual spectrum of the operator ∇_q^2 over the Banach space c_0 of null sequences.

Keywords: q-calculus; quantum difference operator; α-, β- and γ-duals; point spectrum; continuous spectrum; residual spectrum

MSC: 46A45; 46B45; 47A10

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Citation: Yaying, T.; Hazarika, B.; Chandra Tripathy, B.; Mursaleen, M. The Spectrum of Second Order Quantum Difference Operator. *Symmetry* **2022**, *14*, 557. https://doi.org/10.3390/sym14030557

Academic Editors: Alexander Zaslavski and Sergei D. Odintsov

Received: 9 December 2021
Accepted: 7 March 2022
Published: 10 March 2022

1. Introduction Furthermore, Preliminaries

The q-analog of a mathematical expression means the generalization of that expression by using the parameter q. The generalized expression returns the original expression when q approaches 1^-. The study of q-calculus can be traced back to the time of Euler. It is a wide and interesting area of research in recent times. Due to the vast applications in mathematics, physics and engineering sciences of q-calculus, numerous researchers are engaged in the field. In the field of mathematics, it is widely used by researchers in approximation theory, combinatorics, hypergeometric functions, operator theory, special functions, quantum algebras, etc.

Let $q \in (0,1)$. Then the q-number $[n]_q$ is defined by

$$[n]_q = \begin{cases} \sum_{k=0}^{n-1} q^k & , \quad n = 1,2,3,\ldots, \\ 0 & , \quad n = 0. \end{cases}$$

One can notice that $[n]_q = n$ whenever $q \to 1^-$.

Definition 1. *The q-analog $\binom{n}{k}_q$ of binomial coefficient $\binom{n}{k}$ is defined by*

$$\binom{n}{k}_q = \begin{cases} \dfrac{[n]_q!}{[n-k]_q![k]_q!} & , \quad n \geq k, \\ 0 & , \quad k > n, \end{cases}$$

where q-factorial $[n]_q!$ of n is given by

$$[n]_q! = \begin{cases} \prod_{k=1}^{n} [k]_q & , \quad n = 1,2,3,\ldots, \\ 1 & , \quad n = 0. \end{cases}$$

Furthermore, $\binom{0}{0}_q = \binom{n}{0}_q = \binom{n}{n}_q = 1$. Further $\binom{n}{n-k}_q = \binom{n}{k}_q$ which is natural q-analog of its ordinary version $\binom{n}{n-k} = \binom{n}{k}$. For detailed studies in q-calculus, readers may see [1]. Since the q-calculus and time scale calculus are correlated, readers can see [2–5] and references therein for more information.

1.1. Sequence Space

The set of all real- or complex-valued sequences is denoted by ω. Sequence space is a linear subspace of ω. Some of the well-known examples of sequence spaces are the space of absolutely summable sequences, the space of bounded sequences, the space of convergent sequences and the space of null sequences, denoted by ℓ_1, ℓ_∞, c and c_0, respectively. The spaces of all bounded and convergent series, respectively, are denoted by bs and cs. Any Banach sequence space with continuous coordinates is called a BK-space. The spaces c_0 and c are BK-spaces equipped with the supremum norm $\|z\|_c = \sup_{k\in\mathbb{N}_0} |z_k|$, where $\mathbb{N}_0$ is the set of non-negative integers.

Let $A = (a_{n,k})$ be an infinite matrix of real or complex entries. Let $A_n = (a_{n,k})_{k\in\mathbb{N}_0}$. The A-transform of a sequence $z = (z_k)$ is a sequence $Az = \{(Az)_n\} = \{\sum_{k=0}^{\infty} a_{n,k} z_k\}$, provided that the series $\sum_{k=0}^{\infty} a_{n,k} z_k$ converges for each $n \in \mathbb{N}_0$. Additionally, if Z and U are two sequence spaces and $Az \in U$ for every sequence $z \in Z$, then the matrix A is said to define a matrix mapping from Z to U. The notation (Z, U) represents the family of all matrices that map from Z to U. Furthermore, a triangle matrix $A = (a_{n,k})$ means, when $a_{n,n} \neq 0$ and $a_{n,k} = 0$ for $n < k$.

The domain Z_A of the matrix A in the space Z is a sequence space. It is defined by

$$Z_A = \{z \in \omega : Az \in Z\}. \tag{1}$$

Additionally, if Z is a BK-space and A is a triangle matrix, then the sequence space Z_A is also a BK-space endowed with the norm $\|z\|_{Z_A} = \|Az\|_Z$. Recently several authors used special matrices for constructing new sequence spaces (or BK-spaces) using their domain in classical sequence spaces. One may refer to the papers [6–11] related to this area.

1.2. Difference Operators and Sequence Spaces

Define the operators Δ and ∇ by $(\Delta z)_k = z_k - z_{k+1}$ and $(\nabla z)_k = z_k - z_{k-1}$ for all $k \in \mathbb{N}_0$, which are well known as forward and backward difference operators of first order, respectively. Here and in what follows, we assume that $z_k = 0$ for $k < 0$. Difference operators are extensively used in summability theory, spectral theory, approximation theory, quantum or post-quantum theory, etc. Sequences such as $(z_k) = (k)$ is not convergent in the ordinary sense. However, the sequence $((\Delta z)_k)$ defined by $(\Delta z)_k = -1$ for all k, is convergent.

In the present literature, several difference operators and their generalizations (cf. [8,12–19]) can be found, studied by many researchers from all around the globe. The domains $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are studied by Kızmaz [19]. Later on, the operators Δ and ∇ were extended to Δ^2 and ∇^2 defined by $(\Delta^2 z)_k = (\Delta z)_k - (\Delta z)_{k+1}$ and $(\nabla^2 z)_k = (\nabla z)_k - (\nabla z)_{k-1}$, respectively, (cf. [15,17]). The studies on difference operators and their domains in the classical sequence spaces further attracted more researchers after the introduction of generalized difference operators $B(r,s)$ [13], $B(r,s,t)$ [14], Δ^r [18], ∇^r [16], $B^{(m)}$ [12] and $B_v^{(m)}$ [8] defined by

$$(B(r,s)z)_k = rz_k + sz_{k-1};\ (B(r,s,t)z)_k = rz_k + sz_{k-1} + tz_{k-2};\ (\Delta^r z)_k = (\Delta(\Delta^{r-1}z))_k;$$
$$(\nabla^r z)_k = (\nabla(\nabla^{r-1}z))_k;\ (B^{(r)}z)_k = \sum_{j=0}^{m}\binom{m}{j} r^{m-j}s^j z_{k-j};\ (B_v^{(r)}z)_k = \sum_{j=0}^{m}\binom{m}{j} r^{m-j}s^j v_{k-j}z_{k-j};$$

respectively. Readers may also refer these high-quality papers [7,12,20–24] concerning sequence spaces constructed using the difference operators.

1.3. Motivation

Several studies can be found in the literature dealing with the quantum generalization of well-known operators including Hausdorff matrices, difference operators, etc. Quite recently, Demiriz and Şahin [25], Yaying et al. [11] and Bakery and Mohamed [26] studied quantum generalizations of Cesàro sequence spaces. Moreover, studies on the (p,q)-analogue of Euler sequence spaces [27] is a recent addition in the field of sequence spaces and summability theory. The spectrum of q-Cesàro matrix in the space c_0 is discussed by Yıldırım [28]. However, to date, no studies have been carried out on the determination of the spectrum of quantum difference operators.

Motivated by the above studies, we present quantum difference operator of the second order and study its domain in the spaces c and c_0 of convergent and null sequences, respectively, and determine its spectrum, point spectrum, residual spectrum, and continuous spectrum over the space c_0.

2. Operator ∇_q^2 and Sequence Spaces $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$

Define the difference operator $\nabla_q^2 : \omega \to \omega$ by

$$(\nabla_q^2 z)_k = z_k - (1+q)z_{k-1} + qz_{k-2},$$

where $k \in \mathbb{N}_0$ and any term of the sequence with negative indices are assumed to be zero. The operator $\nabla_q^2 = (\delta_{n,k}^{2;q})$ can also be expressed in the form of a triangle matrix as follows:

$$\delta_{n,k}^{2;q} = \begin{cases} (-1)^{n-k} q^{\binom{n-k}{2}} \binom{2}{n-k}_q, & (k \le n), \\ 0, & (k > n), \end{cases}$$

for all $n,k \in \mathbb{N}_0$. We pressumed that $\binom{n}{k}_q = 0$ for $k > n$. Equivalently

$$\nabla_q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -(1+q) & 1 & 0 & 0 & \cdots \\ q & -(1+q) & 1 & 0 & \cdots \\ 0 & q & -(1+q) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We observe that the operator ∇_q^2 reduces to ∇^2 when $q \to 1^-$. Furthermore, we noticed that the operator $\nabla_q^2 \neq \nabla_q \circ \nabla_q$, unlike its ordinary version ∇^2. In fact

$$(\nabla_q^2 z)_k = (\nabla_q z)_k - q(\nabla_q z)_{k-1}.$$

Using some elementary calculation, we derive the inverse of the operator ∇_q^2 as

$$(\nabla_q^{-2})_{n,k} = \begin{cases} \binom{n-k+1}{n-k}_q &, \quad 0 \le k \le n, \\ 0 &, \quad k > n. \end{cases}$$

Now, we define the q-difference sequence spaces $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$ by

$$\begin{aligned} c_0(\nabla_q^2) &:= \left\{ z = (z_k) \in \omega : \nabla_q^2 z \in c_0 \right\}, \\ c(\nabla_q^2) &:= \left\{ z = (z_k) \in \omega : \nabla_q^2 z \in c \right\}. \end{aligned}$$

We observe that the spaces $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$ are reduced to the second order backward difference sequence spaces $c_0(\nabla^2)$ and $c(\nabla^2)$, respectively, as $q \to 1^-$. Thus, the inclusions $c_0 \subseteq c_0(\nabla^2) \subseteq c_0(\nabla_q^2)$ and $c \subseteq c(\nabla^2) \subseteq c(\nabla_q^2)$ are evident. Moreover, the inclusion $c_0(\nabla_q^2) \subseteq c(\nabla_q^2)$ is trivial. In the light of the notation (1), the above sequence spaces are redefined as

$$c_0(\nabla_q^2) = (c_0)_{\nabla_q^2} \text{ and } c(\nabla_q^2) = c_{\nabla_q^2}.$$

Furthermore, a sequence space X is said to be symmetric if $z_{\phi(k)} \in X$ whenever $(z_k) \in X$, where $\phi(k)$ is a permutation on $\mathbb{N}_0$ (cf. [21]). Consider the sequence $(z_k) = (k)_{k\in\mathbb{N}_0}$, then $(z_k) \in c(\nabla_q^2)$. Now, consider the rearranged sequence $(z'_k) = (z_0, z_2, z_3, z_1, z_5, z_6, z_4, z_8, \ldots)$. Then, $(z'_k) \notin c(\nabla_q^2)$. Consequently, $c(\nabla_q^2)$ is not a symmetric space.

We define sequence $u = (u_n)$ as the ∇_q^2-transform of the sequence $z = (z_k)$, i.e.,

$$u_n = (\nabla_q^2 z)_n = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{2}{k}_q z_{n-k} = z_n - (1+q)z_{n-1} + qz_{n-2} \tag{2}$$

for each $n \in \mathbb{N}_0$. In the remaining part of the article, the sequences u and z are related by (2). Moreover, by using (2), we observe that

$$z_k = \sum_{v=0}^{k} \binom{k-v+1}{k-v}_q u_v \tag{3}$$

for each $k \in \mathbb{N}_0$.

Now, we state our first result:

Theorem 1. *$c_0(\nabla_q^2)$ and $c(\nabla_q^2)$ are BK-spaces endowed with the norm*

$$\|z\|_{c_0(\nabla_q^2)} = \|z\|_{c(\nabla_q^2)} = \sup_{n\in\mathbb{N}_0} \left| (\nabla_q^2 z)_n \right|.$$

Proof. The proof is straightforward. So, we omit details. □

Theorem 2. *$c_0(\nabla_q^2) \cong c_0$ and $c(\nabla_q^2) \cong c$.*

Proof. Define the mapping $\pi : c_0(\nabla_q^2) \to c_0$ by $\pi z = u = \nabla_q^2 z$ for all $z \in c_0(\nabla_q^2)$. Clearly, π is linear and 1-1. Let the sequence $z = (z_k)$ be defined as in (3) and $u = (u_n)$ be any arbitrary sequence in c_0. Then, we have

$$\lim_{n\to\infty} (\nabla_q^2 z)_n = \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{2}{k}_q z_{n-k} = \lim_{n\to\infty} u_n = 0.$$

This implies $z \in c_0(\nabla_q^2)$ and the mapping π is onto, and norm preserving. Hence, $c_0(\nabla_q^2) \cong c_0$.

The proof for the space $c(\nabla_q^2)$ can be obtained in a similar fashion. Hence the proof. □

Now we construct bases for the spaces $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$. We recall that, for a triangle matrix A, the matrix domain Z_A has a basis if and only if Z has a basis, (cf. Jarrah and Malkowsky [29] (Theorem 2.3)). Thus, by using Theorem 2, we immediately arrive at the following result:

Theorem 3. *For every fixed $k \in \mathbb{N}_0$, define the sequences $f^{(k)}(q) = \left(f_n^{(k)}(q)\right)$ by*

$$\begin{aligned} f_n^{(-1)}(q) &= \sum_{k=0}^{n} \binom{k+1}{k}_q, \; n = 0, 1, 2, \cdots; \\ f_n^{(k)}(q) &= \begin{cases} \binom{n-k+1}{n-k}_q &, \; k \le n, \\ 0 &, \; k > n. \end{cases} \end{aligned}$$

Then

(a) *the set $\left\{f^{(0)}(q), f^{(1)}(q), f^{(2)}(q), \ldots\right\}$ forms the basis for the space $c_0(\nabla_q^2)$ and every $z \in c_0(\nabla_q^2)$ has a unique representation of the form $z = \sum_{k=0}^{\infty} u_k f^{(k)}(q)$.*

(b) *the set $\left\{f^{(-1)}(q), f^{(0)}(q), f^{(1)}(q), f^{(2)}(q), \ldots\right\}$ forms the basis for the space $c(\nabla_q^2)$ and every $z \in c(\nabla_q^2)$ can be uniquely expressed in the form $z = re + \sum_{k=0}^{\infty}(u_k - r) f^{(k)}(q)$, where $u_k = (\nabla_q^2 z)_k \to r$ as $k \to \infty$ and e denote the unit sequence.*

3. α-, β-and γ-Duals

In the present section, we determine the α-, β-and γ-duals of the spaces $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$. Since the computation of duals is similar for both the spaces, we omit the proof for the space $c(\nabla_q^2)$. Before proceeding, we recall the definitions of α-, β-and γ-duals.

Definition 2. *The α-, β- and γ-duals Z^α, Z^β and Z^γ of a sequence space Z are defined by*

$$\begin{aligned} Z^\alpha &:= \{d = (d_k) \in \omega : dz = (d_k z_k) \in \ell_1 \text{ for all } z \in Z\}, \\ Z^\beta &:= \{d = (d_k) \in \omega : dz = (d_k z_k) \in cs \text{ for all } z \in Z\}, \\ Z^\gamma &:= \{d = (d_k) \in \omega : dz = (d_k z_k) \in bs \text{ for all } z \in Z\}, \end{aligned}$$

respectively.

Chandra and Tripathy [30] investigated the generalized duals of sequence spaces. We present the following lemma which is essential to compute the dual spaces. In what follows, we denote the collection of all finite subsets of $\mathbb{N}_0$ by $\mathcal{N}$.

Lemma 1. *[31] The following statements hold:*

(i) *$A = (a_{n,k}) \in (c_0, \ell_1) = (c, \ell_1)$ if and only if*

$$\sup_{K \in \mathcal{N}} \sum_{n=0}^{\infty} \left| \sum_{k \in K} a_{n,k} \right| < \infty. \tag{4}$$

(ii) *$A = (a_{n,k}) \in (c_0, c)$ if and only if*

$$\sup_{n \in \mathbb{N}_0} \sum_{k=0}^{\infty} |a_{n,k}| < \infty \tag{5}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n \to \infty} a_{n,k} = \alpha_k \text{ for each } k \in \mathbb{N}_0. \tag{6}$$

(iii) *$A = (a_{n,k}) \in (c_0, \ell_\infty) = (c, \ell_\infty)$ if and only if (5) holds.*

Theorem 4. *The set*

$$D_1(q) := \left\{ d = (d_n) \in \omega : \sup_{N \in \mathcal{N}} \sum_{k=0}^{\infty} \left| \sum_{n \in N} \binom{n-k+1}{n-k}_q d_n \right| < \infty \right\}.$$

is the α-dual of the spaces $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$.

Proof. Consider

$$d_n z_n = \sum_{k=0}^{n} \binom{n-k+1}{n-k}_q d_n u_k = (G(q)u)_n \tag{7}$$

for all $n \in \mathbb{N}_0$, where the matrix $G(q) = \left(g_{n,k}^q\right)$ is defined by

$$g_{n,k}^q = \begin{cases} \binom{n-k+1}{n-k}_q d_n & , \quad 0 \le k \le n, \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}_0$. By using (7), it follows that $dz = (d_n z_n) \in \ell_1$ whenever $z \in c_0(\nabla_q^2)$ if and only if $G(q)u \in \ell_1$ whenever $u \in c_0$. Thus, we deduce that $d = (d_n)$ is a sequence in $\left[c_0(\nabla_q^2)\right]^\alpha$ if and only if the matrix $G(q)$ belongs to the class (c_0, ℓ_1). Thus, we conclude from Part (i) of Lemma 1 that $\left[c_0(\nabla_q^2)\right]^\alpha = D_1(q)$.

This completes the proof. □

Theorem 5. *The following statements hold:*

(a) $\left[c_0(\nabla_q^2)\right]^\beta = D_2(q) \cap D_3(q)$,

(b) $\left[c(\nabla_q^2)\right]^\beta = D_2(q) \cap D_3(q) \cap D_4(q)$,

where the sets $D_2(q)$, $D_3(q)$ and $D_4(q)$ are defined by

$$\begin{aligned} D_2(q) &:= \left\{ d = (d_n) \in \omega : \sum_{n=k}^{\infty} \binom{n-k+1}{n-k}_q d_k \text{ exists for each } k \in \mathbb{N}_0 \right\}, \\ D_3(q) &:= \left\{ d = (d_n) \in \omega : \sup_{n \in \mathbb{N}_0} \sum_{k=0}^{n} \left| \sum_{v=k}^{n} \binom{v-k+1}{v-k}_q d_v \right| < \infty \right\}, \\ D_4(q) &:= \left\{ d = (d_n) \in \omega : \lim_{n \to \infty} \sum_{k=0}^{n} \sum_{v=k}^{n} \binom{v-k+1}{v-k}_q d_v \ \text{ exists} \right\}. \end{aligned}$$

Proof. Consider

$$\begin{aligned} \sum_{k=0}^{n} d_k z_k &= \sum_{k=0}^{n} \left[\sum_{v=0}^{k} \binom{k-v+1}{k-v}_q u_v \right] d_k \\ &= \sum_{k=0}^{n} \left[\sum_{v=k}^{n} \binom{v-k+1}{v-k}_q d_v \right] u_k \\ &= (H(q)u)_n \end{aligned} \tag{8}$$

for each $n \in \mathbb{N}_0$, where the matrix $H(q) = (h_{n,k}^q)$ is defined by

$$h_{n,k}^q = \begin{cases} \sum_{v=k}^{n} \binom{v-k+1}{v-k}_q d_v & , \quad 0 \le k \le n, \\ 0 & , \quad k > n \end{cases}$$

for all $n,k \in \mathbb{N}_0$. Thus, on using (8), it follows that $dz = (d_n z_n) \in cs$, whenever $z = (z_n) \in c_0(\nabla_q^2)$ if and only if $H(q)u \in c$ whenever $u = (u_k) \in c_0$. This yields that $d = (d_n)$ is a sequence in $\left[c_0(\nabla_q^2)\right]^\beta$ if and only the matrix $H(q)$ belongs to the class (c_0, c). This in turn implies by using Part (ii) of Lemma 1 that

$$\sup_{n\in\mathbb{N}_0} \sum_{k=0}^{n} \left|h_{n,k}^q\right| < \infty \text{ and } \lim_{n\to\infty} h_{n,k}^q \text{ exists for each } k \in \mathbb{N}_0.$$

Thus, $\left[c_0(\nabla_q^2)\right]^\beta = D_2(q) \cap D_3(q)$.

This completes the proof. □

Theorem 6. *The set $D_3(q)$ is the γ-dual of the spaces $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$.*

Proof. The proof is similar to Theorem 5 by using Part (iii) instead of Part (ii) of Lemma 1. We omit details to avoid the repetition of similar statements. □

4. Matrix Transformations

In the present section, we determine necessary and sufficient conditions for matrix mappings from the spaces $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$ to any one of the space ℓ_∞, c, c_0, or ℓ_1. The following theorem, which is immediate from Kirişçi and Başar [7], is fundamental in our investigation.

Theorem 7. *Let Z be any one of the space c or c_0 and $U \subset \omega$. Define $T^{(n)} = (t_{m,k}^{(n)})$ and $T = (t_{n,k})$ by*

$$\begin{aligned} t_{m,k}^{(n)} &= \begin{cases} 0 & (k > m), \\ \sum_{v=k}^{m} \binom{v-k+1}{v-k}_q a_{n,v} & (0 \le k \le m), \end{cases} \\ t_{nk} &= \sum_{v=k}^{m} \binom{v-k+1}{v-k}_q a_{n,v} \end{aligned}$$

for all $n,k \in \mathbb{N}_0$. Then, $A = (a_{n,k}) \in (Z(\nabla_q^2), U)$ if and only if $T^{(n)} = (t_{m,k}^{(n)}) \in (Z, c)$ for each $n \in \mathbb{N}_0$, and $T = (t_{n,k}) \in (Z, U)$.

Proof. Let $A \in (Z(\nabla_q^2), U)$ and $z \in Z(\nabla_q^2)$. Then, we obtain the following equality

$$\sum_{k=0}^{m} a_{n,k} z_k = \sum_{k=0}^{n} \sum_{v=0}^{k} \binom{k-v+1}{k-v}_q u_v a_{n,k} = \sum_{k=0}^{m} \sum_{v=k}^{m} \binom{v-k+1}{v-k}_q a_{n,v} u_k = \sum_{k=0}^{m} t_{m,k}^{(n)} u_k \tag{9}$$

for all $m,n \in \mathbb{N}$. Since Az exists, so $T^{(n)} \in (Z, c)$. Again as $m \to \infty$ in (9), we obtain $Az = Tu$. Since $Az \in U$, so $Tu \in U$ which yields the consequence that $T \in (Z, U)$.

Conversely, assume that $T^{(n)} = (t_{m,k}^{(n)}) \in (Z, c)$ for all $n \in \mathbb{N}$, and $T = (t_{n,k}) \in (Z, U)$. Let $z \in Z(\nabla_q^2)$. Then, for each $n \in \mathbb{N}$, $(a_{n,k})_{k\in\mathbb{N}} \in Z^\beta$ which in turn implies the fact that $(a_{n,k})_{k\in\mathbb{N}} \in \left[Z(\nabla_q^2)\right]^\beta$ for each $n \in \mathbb{N}$. Again from (9), $Az = Tu$ as $m \to \infty$. This implies that $A \in (Z(\nabla_q^2), U)$. □

Now, by using the matrix mapping charaterization results given in Stieglitz and Tietz [31] together with Theorem 7, we obtain the following results:

Corollary 1. *The following statements hold:*

1. $A \in (c_0(\nabla_q^2), \ell_\infty)$ *if and only if*

$$\sup_{m \in \mathbb{N}_0} \sum_{k=0}^{\infty} \left| t_{m,k}^{(n)} \right| < \infty, \tag{10}$$

$$\lim_{m \to \infty} t_{m,k}^{(n)} \text{ exists for all } k \in \mathbb{N}_0 \tag{11}$$

hold and

$$\sup_{n \in \mathbb{N}_0} \sum_{k=0}^{\infty} \left| t_{n,k} \right| < \infty, \tag{12}$$

also holds.

2. $A \in (c_0(\nabla_q^2), c)$ *if and only if (10), (11) and (12) hold, and*

$$\lim_{n \to \infty} t_{n,k} \text{ exists for all } k \in \mathbb{N}_0, \tag{13}$$

also holds.

3. $A \in (c_0(\nabla_q^2), c_0)$ *if and only if (10), (11) and (12) hold, and*

$$\lim_{n \to \infty} t_{n,k} = 0 \text{ for all } k \in \mathbb{N}_0, \tag{14}$$

also holds.

4. $A \in (c_0(\nabla_q^2), \ell_1)$ *if and only if (10) and (11) hold, and*

$$\sup_{N \in \mathcal{N}} \sum_{k=0}^{\infty} \left| \sum_{n \in N} t_{n,k} \right| < \infty, \tag{15}$$

also holds.

Corollary 2. *The following statements hold:*

1. $A \in (c(\nabla_q^2), \ell_\infty)$ *if and only if (10), (11) and (12) hold, and*

$$\lim_{m \to \infty} \sum_{k=0}^{\infty} t_{m,k}^{(n)} \text{ exists for each } n, \tag{16}$$

also holds.

2. $A \in (c(\nabla_q^2), c)$ *if and only if (10), (11), (12), (13) and (16) hold, and*

$$\lim_{n \to \infty} \sum_{k=0}^{n} t_{n,k} \text{ exists}$$

also holds.

3. $A \in (c(\nabla_q^2), c_0)$ *if and only if (10), (11), (12), (14) and (16) hold, and*

$$\lim_{n \to \infty} \sum_{k=0}^{n} t_{n,k} = 0$$

also holds.

4. $A \in (c(\nabla_q^2), \ell_1)$ *if and only if (10), (11), (15) and (16) hold.*

5. Spectrum of ∇_q^2 in C_0

The point spectrum, continuous spectrum and residual spectrum of the operator ∇_q^2 in the space c_0 are determined in this section.

Let $Z \neq \{\theta\}$ be a complex normed space and $\mathsf{L} : D(\mathsf{L}) \to Z$ be a linear operator, where $D(\mathsf{L})$ stands for the domain of L. Moreover L^*, $R(\mathsf{L})$ and $B(Z)$ denote the adjoint of

L, the range of L and the set of all bounded linear operators on Z into itself, respectively. Let $\mathbb{C}$ represents the set of all complex numbers. Then, the operator $\mathsf{L}_\alpha^{-1} = (\mathsf{L} - \alpha I)^{-1}$ is called the resolvent operator of L, given that L_α is invertible, where $\alpha \in \mathbb{C}$ and I is the identity operator on $D(\mathsf{L})$. Further $\alpha \in \mathbb{C}$ is called a regular value of L if it suffices the following conditions:

(S1) L_α^{-1} exists;
(S2) L_α^{-1} is bounded;
(S3) L_α^{-1} is defined on a set which is dense in Z.

The resolvent set of L is the set $r(\mathsf{L}, Z)$ containing all the regular values of L. The spectrum of L is defined by the set $s(\mathsf{L}, Z) = \mathbb{C} \setminus r(\mathsf{L}, Z)$. The spectrum $s(\mathsf{L}, Z)$ is partitioned into three disjoint sets:

(a) **Point spectrum** $s_p(\mathsf{L}, Z) = \{\alpha \in \mathbb{C} :$ (S1) does not hold$\}$.
(b) **Continuous spectrum** $s_c(\mathsf{L}, Z) = \{\alpha \in \mathbb{C} :$ (S1) and (S3) hold but (S2) does not hold$\}$.
(c) **Residual spectrum** $s_r(\mathsf{L}, Z) = \{\alpha \in \mathbb{C} :$ (S1) holds (S3) does not hold, (S2) may or may not hold$\}$.

In the field of functional analysis, the studies on determining the spectrum of special operators over different sequence spaces has become an active area of research. Several studies can be found in the literature dealing with the investigation of spectrum of well known matrices over various sequence spaces. However, we shall briefly highlight on the studies related to the determination of spectrum of difference operators only. The spectrum of the first order difference operator Δ over the spaces ℓ_p and bv_p was studied by Akhmedov and Başar [32,33], over the spaces c_0, c and ℓ_p $(0 < p < 1)$ was studied by Altay and Başar [6,34], and over the sequence spaces ℓ_1 and bv was examined by Kayaduman and Furkan [35]. The spectrum of second order backward difference operator ∇^2 over the space c_0 was studied by Dutta and Baliarsingh [15]. The spectrum of the generalized difference operator $B(r, s)$ over the spaces ℓ_p and bv_p was studied by Bilgiç and Furkan [13] and over the spaces ℓ_1 and bv was studied by Furkan et al. [36]. Further, the fine spectrum of the difference operator $B(r, s, t)$ over the spaces ℓ_1 and bv was studied by Bilgiç and Furkan [14] and over the spaces c_0, c, ℓ_p and bv_p by Furkan et al. [37,38]. Furthermore, the spectra of the difference operator $D(r, 0, 0, s)$ over the sequence spaces c_0, c, ℓ_p and bv_p have been investigated by Tripathy and Paul [39,40]. The spectra of the rth order backward difference operator ∇^r over the Banach space c is studied by Baliarsingh et al. [41]. Baliarsingh and Dutta [16,42] further studied the spectrum of the generalized difference operator ∇_v^r over the spaces ℓ_1 and c_0. Moreover, the spectrum of the more generalized difference operator $B_v^{(m)}$ over the space ℓ_1 and c_0 are investigated by Meng and Mei [8] and Baliarsingh [43], respectively, which is a symmetric spectrum. Natural numbers occurring as levels of a symmetric spectrum or as dimensions of homotopy groups are really placeholders for sphere coordinates. The role of the symmetric group actions on the spaces of a symmetric spectrum is to keep track of how such coordinates are shuffled. Permutations will come up over and over again in constructions and results about symmetric spectra, and there is a very useful small set of rules which predict when to expect permutations.

Lemma 2 ([44], p. 129)**.** *The matrix $A = (a_{n,k})$ gives rise to a bounded linear operator $\mathsf{L} \in B(c_0)$ if and only if $\lim_{n\to\infty} a_{n,k} = 0$ for each $k \in \mathbb{N}_0$ and $\sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} |a_{n,k}| < \infty$. Further, $\|\mathsf{L}\| = \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} |a_{n,k}|$.*

Lemma 3 ([45], p. 59))**.** *A linear operator L has a dense range $\Leftrightarrow$ the adjoint L^* is* 1-1.

Theorem 8. *The operator $\nabla_q^2 : c_0 \to c_0$ is a linear operator and* $\left\|\nabla_q^2\right\|_{(c_0,c_0)} = 2(1+q)$.

Proof. The result immediately follows from the fact that

$$\binom{2}{0}_q + \binom{2}{1}_q + q\binom{2}{2}_q = 2(1+q).$$

□

Theorem 9. *Let $q \in (0,1)$. Then $s(\nabla_q^2, c_0) = \{\alpha \in \mathbb{C} : |1-\alpha| \le 1+2q\}$.*

Proof. Let $\alpha \in \mathbb{C}$ satisfying $|1-\alpha| > 1+2q$. Since $(\nabla_q^2 - \alpha I) = (u_{n,k})$ is a triangle matrix, and therefore has an inverse $(\nabla_q^2 - \alpha I)^{-1} = (v_{n,k})$ defined by

$$v_{n,k} = \begin{bmatrix} \frac{1}{1-\alpha} & 0 & 0 & 0 & \cdots \\ \frac{1+q}{(1-\alpha)^2} & \frac{1}{1-\alpha} & 0 & 0 & \cdots \\ \frac{(1+q)^2}{(1-\alpha)^3} - \frac{q}{(1-\alpha)^2} & \frac{1+q}{(1-\alpha)^2} & \frac{1}{1-\alpha} & 0 & \cdots \\ \frac{(1+q)^3}{(1-\alpha)^4} - \frac{2q(1+q)}{(1-\alpha)^3} & \frac{(1+q)^2}{(1-\alpha)^3} - \frac{q}{(1-\alpha)^2} & \frac{1+q}{(1-\alpha)^2} & \frac{1}{1-\alpha} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Thus, for each $k \in \mathbb{N}_0$, the general expression for the entries $v_{n,k}$ may be computed as

$$\begin{aligned} v_{k,k} &= \frac{1}{1-\alpha}; \\ v_{k,k-1} &= \frac{1+q}{(1-\alpha)^2}; \\ v_{k,k-2} &= \frac{(1+q)^2}{(1-\alpha)^3} - \frac{q}{(1-\alpha)^2}; \\ v_{k,k-3} &= \frac{(1+q)^3}{(1-\alpha)^4} - \frac{2q(1+q)}{(1-\alpha)^3}; \\ &\vdots \end{aligned}$$

and so on.

Now we proceed to show that the matrix $(v_{n,k})$ is bounded on c_0, that is $(v_{n,k}) \in B(c_0)$. Consider the following equality

$$\begin{aligned} S_n &= \sum_{k=0}^{\infty} |v_{n,k}| = \sum_{k=0}^{n} |v_{n,k}| \\ &= |v_{n,n}| + |v_{n,n-1}| + |v_{n,n-2}| + |v_{n,n-3}| + \cdots + |v_{n,0}| \\ &= \left|\frac{1}{1-\alpha}\right| + \left|\frac{1+q}{(1-\alpha)^2}\right| + \left|\frac{(1+q)^2}{(1-\alpha)^3} - \frac{q}{(1-\alpha)^2}\right| + \left|\frac{(1+q)^3}{(1-\alpha)^4} - \frac{2q(1+q)}{(1-\alpha)^3}\right| + \cdots. \\ &\le \frac{1}{1+q}\left\{\left|\frac{1+q}{1-\alpha}\right| + \left|\frac{1+q}{1-\alpha}\right|^2 + \left|\frac{1+q}{1-\alpha}\right|^3 + \frac{q}{1+q}\left|\frac{1+q}{1-\alpha}\right|^2 + \left|\frac{1+q}{1-\alpha}\right|^4 + \frac{2q}{1+q}\left|\frac{1+q}{1-\alpha}\right|^3 + \right. \\ &\qquad \left. \left|\frac{1+q}{1-\alpha}\right|^5 + \frac{3q}{1+q}\left|\frac{1+q}{1-\alpha}\right|^4 + \frac{q^2}{(1+q)^2}\left|\frac{1+q}{1-\alpha}\right|^3 \cdots\right\} \\ &= \frac{1}{1+q}\left\{\left|\frac{1+q}{1-\alpha}\right| + \left|\frac{1+q}{1-\alpha}\right|^2\left\{1 + \frac{q}{1+q}\right\} + \left|\frac{1+q}{1-\alpha}\right|^3\left\{1 + \frac{2q}{1+q} + \frac{q^2}{(1+q)^2}\right\} + \cdots\right\} \\ &= \frac{1}{1+q}\left\{\left|\frac{1+q}{1-\alpha}\right| + \frac{1+2q}{1+q}\left|\frac{1+q}{1-\alpha}\right|^2 + \left(\frac{1+2q}{1+q}\right)^2\left|\frac{1+q}{1-\alpha}\right|^3 + \cdots\right\} \\ &\le \frac{1}{|1-\alpha|}\left\{1 + \left|\frac{1+2q}{1-\alpha}\right| + \left|\frac{1+2q}{1-\alpha}\right|^2 + \cdots\right\}. \end{aligned}$$

Since $\left|\dfrac{1+2q}{1-\alpha}\right| < 1$,

$$\lim_{n\to\infty} S_n \leq \frac{1}{|1-\alpha|}\left\{\frac{1}{1-\left|\dfrac{1+2q}{1-\alpha}\right|}\right\} = \frac{1}{|1-\alpha|-|1+2q|} < \infty.$$

Since, $\lim_{n\to\infty} S_n < \infty$ and (S_n) is a sequence of positive reals, we conclude that $\sup_n S_n < \infty$. Moreover, this is obvious from the assumption that $\lim_n v_{n,k} = 0$.

Thus, we realize that

(i) The series $S_n = \sum_{k=0}^{n} v_{n,k}$ converges and $\sup_n S_n$ exists, for each $n \in \mathbb{N}_0$.
(ii) $\lim_n |v_{n,k}| = 0$ for each $k \in \mathbb{N}_0$.

Thus, $(v_{n,k}) \in B(c_0)$, whenever $|1-\alpha| \leq 1+2q$. This implies

$$s(\nabla_q^2, c_0) \subseteq \{\alpha \in \mathbb{C} : |1-\alpha| \leq 1+2q\}. \tag{17}$$

Conversely, we need to show that

$$\{\alpha \in \mathbb{C} : |1-\alpha| \leq 1+2q\} \subseteq s(\nabla_q^2, c_0).$$

We notice, when $\alpha = 1$, the operator $\nabla_q^2 - \alpha I$ is not invertible. Again, when $\alpha \neq 1$ and $|1-\alpha| \leq 1+2q$, we observe that (S_n) is unbounded. This implies $(\nabla_q^2 - \alpha I)^{-1} \notin B(c_0)$ whenever $|1-\alpha| \leq 1+2q$. Hence,

$$\{\alpha \in \mathbb{C} : |1-\alpha| \leq 1+2q\} \subseteq s(\nabla_q^2, c_0). \tag{18}$$

Thus, using (17) and (18), we conclude that

$$s(\nabla_q^2, c_0) = \{\alpha \in \mathbb{C} : |1-\alpha| \leq 1+2q\}.$$

□

Theorem 10. $s_p(\nabla_q^2, c_0) = \emptyset$.

Proof. Suppose $s_p(\nabla_q^2, c_0) \neq \emptyset$. Then there exists at least one non-zero sequence $z = (z_k) \in c_0$ with $\nabla_q^2 z = \alpha z$. This leads us to the following system of equations:

$$\begin{aligned}
z_0 &= \alpha\, z_0 \\
-(1+q)z_0 + z_1 &= \alpha\, z_1 \\
qz_0 - (1+q)z_1 + z_2 &= \alpha\, z_2 \\
qz_1 - (1+q)z_2 + z_3 &= \alpha\, z_3 \\
&\vdots \\
qz_{m-2} - (1+q)z_{m-1} + z_m &= \alpha\, z_m \\
&\vdots \quad .
\end{aligned}$$

Let z_m be the first non-zero component of z, then we get $\alpha = 1$. Substituting $\alpha = 1$ in the next equation, we get $-(1+q)z_m + z_{m+1} = z_{m+1}$. This implies $z_m = 0$, which is a contradiction to our assumption that z_m is the first non-zero component of z. Thus, $s_p(\nabla_q^2, c_0) = \emptyset$. □

Theorem 11. $s_p(\nabla_q^{2,*}, \ell_1) = \{\alpha \in \mathbb{C} : |1-\alpha| \leq 1+q\}$.

Proof. Let $z = (z_k)$ be a non-zero sequence. Then, the matrix equation $\nabla_q^{2,*} z = \alpha z$ yields the following system of linear equations:

$$\begin{aligned} z_0 - (1+q)z_1 + qz_2 &= \alpha z_0, \\ z_1 - (1+q)z_2 + qz_3 &= \alpha z_1, \\ z_2 - (1+q)z_3 + qz_4 &= \alpha z_2, \\ &\vdots \quad . \end{aligned}$$

Using a simple calculation, we obtain

$$\begin{aligned} |z_0| &= \left| \frac{1}{1-\alpha}((1+q)z_1 - qz_2) \right|, \\ |z_1| &= \left| \frac{1}{1-\alpha}((1+q)z_2 - qz_3) \right|, \text{and so on.} \end{aligned}$$

In general,

$$|z_k| = \left| \frac{1}{1-\alpha}((1+q)z_{k+1} - qz_{k+2}) \right|, \; k \geq 0.$$

Clearly, for each $k \in \mathbb{N}_0$, $z = (0, 0, \ldots, z_k, (1+q)z_{k+1}, (1+q)z_{k+1}, 0, 0, \ldots)$ is an eigen vector corresponding to the eigen value α satisfying $|1-\alpha| \leq 1+q$. This is clear from the following statement

$$\begin{aligned} |z_k| &= \left| \frac{1}{1-\alpha} \right| \left| (1+q)^2 z_{k+1} - q(1+q)z_{k+1} \right|, \\ &= \left| \frac{1+q}{1-\alpha} \right| |(1+q)z_{k+1} - qz_{k+1}| \\ &\geq z_{k+1}|1+q-q| = z_{k+1}. \end{aligned}$$

Thus, $z_k \geq z_{k+1}$ for each $k \in \mathbb{N}_0$. This implies that $z \in \ell_1$.
Conversely, it is trivial that if $z \in \ell_1$, then $|1-\alpha| \leq 1+q$.
This completes the proof. □

Theorem 12. $s_r(\nabla_q^2, c_0) = \{\alpha \in \mathbb{C} : |1-\alpha| \leq 1+q\}$.

Proof. Let $|1-\alpha| < 1+q$. It is easy to notice that the operator $\nabla_q^2 - \alpha I$ is invertible. Further, Theorem 11 implies that the operator $\nabla_q^{2,*} - \alpha I$ is not $1-1$ for $|1-\alpha| \leq 1+q$. Hence it is immediate from Lemma 3 that $R(\nabla_q^2 - \alpha I) \neq c_0$.

Thus, $s_r(\nabla_q^2, c_0) = \{\alpha \in \mathbb{C} : |1-\alpha| \leq 1+q\}$. □

Theorem 13. $s_c(\nabla_q^2, c_0) = \{\alpha \in \mathbb{C} : 1+q < |1-\alpha| < 1+2q\}$.

Proof. The result is immediate from Theorems 9, 10 and 12, and the relation $s(\nabla_q^2, c_0) = s_p(\nabla_q^2, c_0) \cup s_r(\nabla_q^2, c_0) \cup s_c(\nabla_q^2, c_0)$. □

6. Conclusions

In this study, we provided an instance wherein quantum calculus has been applied to construct sequence spaces. We constructed quantum difference sequence spaces $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$ defined as the domain of the second order quantum difference operator ∇_q^2 in the spaces c_0 and c, respectively. We further determine the spectrum, the point spectrum, the residual spectrum, and the continuous spectrum of the operator ∇_q^2 in the space c_0. We observed that the operator ∇_q^2 reduces to ∇^2 (cf. [15]) as q tends to 1^-. Thus, our study strengthens the results of Dutta and Baliarsingh [15]. This study will pave the way for various pieces of research in this field. For instance, as a potential research direction, one may study the domain of the difference operator ∇_q^2 in the sequence space ℓ_p of absolutely p-summable sequences and compute the spectrum of ∇_q^2 over ℓ_p (or ℓ_1) (cf. [6,14,16]).

Author Contributions: T.Y., B.H., B.C.T. and M.M. contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful to the anonymous reviewers for carefully studying the article and suggesting necessary changes that improved the presentation of the article. The research of the first author (T. Yaying) is supported by SERB, DST, New Delhi, India, under the grant number EEQ/2019/000082.

Conflicts of Interest: The authors declare no conflict of interest.

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