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Symmetric and Non-Oscillatory Characteristics of the Neutral Differential Equations Solutions Related to *p*-Laplacian Operators

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Abstract: The main purpose of this research was to use the comparison approach with a first-order equation to derive criteria for non-oscillatory solutions of fourth-order nonlinear neutral differential equations with p Laplacian operators. We obtained new results for the behavior of solutions to these equations, and we showed their symmetric and non-oscillatory characteristics. These results complement some previously published articles. To find out the effectiveness of these results and validate the proposed work, two examples were discussed at the end of the paper.

Keywords: symmetric solutions; neutral delay; p-Laplacian operators; fourth-order differential equations; non-oscillatory solutions

MSC: 34K11; 34C10

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1. Introduction

Our goal in this research was finding the non-oscillatory and some symmetric characteristics of the differential equations related to p-Laplacian operators:

$$\left(\varphi(\imath)\left(y'''(\imath)\right)^{p-1}\right)' + \varpi_1(\imath)w^{p-1}(\varpi_2(\imath)) = 0, \tag{1}$$

where $\imath \geq \imath_0$ and $y(\imath) := w(\imath) + \varsigma(\imath)w(\varpi_3(\imath))$. In this work, we assume:

Hypothesis 1. *$p > 1$, $\varphi \in C^1([\imath_0, \infty))$, $\varphi(\imath) > 0$, $\varphi'(\imath) \geq 0$ and*

$$\phi(\imath_0) := \int_{\imath_0}^{\infty} \varphi^{-1/(p-1)}(s)\mathrm{d}s < \infty; \tag{2}$$

Hypothesis 2. *$\varsigma, \varpi_1 \in C([\imath_0, \infty))$, $\varpi_1(\imath) > 0$, $0 \leq \varsigma(\imath) < \varsigma_0 < \infty$,*

Hypothesis 3. *$\varpi_3 \in C^4([\imath_0, \infty))$, $\varpi_2 \in C([\imath_0, \infty))$, $\varpi_3'(\imath) > 0$, $\varpi_3(\imath) \leq \imath$ and $\lim_{\imath\to\infty} \varpi_3(\imath) = \lim_{\imath\to\infty} \varpi_2(\imath) = \infty$.*

Definition 1. *A solution w of Equation (1) is called non-oscillatory whenever it is ultimately positive or negative; otherwise, it is called oscillatory.*

Definition 2. *Equation (1) is said to be oscillatory if all its solutions are oscillatory, otherwise, it is called non-oscillatory.*

Delay differential equations contribute to many scientific applications in life, such as medicine, engineering, physics, and biology. We therefore find that oscillation and symmetric properties play an important role in vibrational motion in flight, interpretation of human self-balancing, problems of automatic control, and in many other areas, see [1–4].

Nonlinear differential equations have played an important role in many sciences, so there has been a research movement on the work of oscillatory as well as non-oscillatory solutions to these equations, see [5–10]. Moreover, the authors in [11–14] discussed the qualitative criteria for differential equations of different orders and used some techniques to find these solutions. Some applications related to this work and oscillatory nonlinear systems can be found in [15,16]. Li et al. [17] studied the oscillatory characteristics of the equation

$$\left(\varphi(\imath)\left(y^{(n-1)}(\imath)\right)^{p-1}\right)' + \varpi_1(\imath)f(w(\varpi_2(\imath))) = 0, \tag{3}$$

under condition

$$\int_{\imath_0}^{\infty} \varphi^{-1/(p-1)}(s)\mathrm{d}s = \infty, \tag{4}$$

and the authors applied the comparison method to obtain some oscillation properties for the same presented equation. In [18], by using the integral average technique, the authors stated that they were interested in discussing the oscillation conditions of the following higher order equation:

$$\left(\varphi(\imath)\left(y^{(n-1)}(\imath)\right)^{p-1}\right)' + \varpi_1(\imath)w^{p-1}(\varpi_2(\imath)) = 0.$$

Bazighifan [19] worked on the asymptotic conditions of solutions of the following equation:

$$\left(\varphi(\imath)\left(y^{(n-1)}(\imath)\right)^{p_1-1}\right)' + \varpi_1(\imath)w^{p_2-1}(\varpi_2(\imath)) = 0,$$

and under condition

$$\int_{\imath_0}^{\infty} \varphi^{-1/(p_1-1)}(s)\mathrm{d}s = \infty.$$

In [20], new oscillatory results for equations related to p-Laplacian-like operators

$$\left(\varphi(\imath)(y'''(\imath))^{p-1}\right)' + \varpi_1(\imath)f(w(\varpi_2(\imath))) = 0,$$

are established.

Our motivation for this work is to continue the results in paper [19]. In fact, in this work, we discuss the properties of non-oscillatory solutions of neutral differential equations by applying the comparison method and using a first-order differential equation.

2. Non-Oscillatory Criteria

In the following, we will express certain lemmas that will help us to demonstrate our primary conclusions:

Lemma 1 ([21])**.** *If $w^{(j)}(\imath) > 0$ and $w^{(i+1)}(\imath) < 0, j = 0, 1, \dots, i$, then*

$$\frac{w(\imath)}{w'(\imath)} \geq \frac{\imath^i/i!}{\imath^{i-1}/(i-1)!}.$$

Lemma 2 ([22], Lemma 2.2.3)**.** *Let* $w \in C^i([\imath_0,\infty),(0,\infty))$, $w^{(i-1)}(\imath)w^{(i)}(\imath) \le 0$ *and* $\lim_{\imath\to\infty} w(\imath) \neq 0$, *then,*

$$w(\imath) \ge \frac{v}{(i-1)!}\imath^{i-1}\left|w^{(i-1)}(\imath)\right| \text{ for } \imath \ge \imath_v, v \in (0,1).$$

Lemma 3 ([23])**.** *Let* β *be a ratio of two odd numbers, then,*

$$Dw - Cw^{(\beta+1)/\beta} \le \frac{\beta^\beta}{(\beta+1)^{\beta+1}}\frac{D^{\beta+1}}{C^\beta},\ C,\ D > 0.$$

Lemma 4. *Assume that* w *is an eventually non-negative and non-zero solution of Equation (1). Then,* $\varphi(\imath)(y'''(\imath))^{p-1}$ *is non-increasing. In addition, one could obtain the following:*

$$\begin{array}{ll} (S_1): & y'(\imath) > 0, y''(\imath) > 0, y'''(\imath) > 0 \text{ and } y^{(4)}(\imath) < 0; \\ (S_2): & y'(\imath) > 0, y''(\imath) < 0, y'''(\imath) > 0 \text{ and } y^{(4)}(\imath) < 0; \\ (S_3): & y'(\imath) > 0, y''(\imath) > 0 \text{ and } y'''(\imath) < 0; \\ (S_4): & y'(\imath) < 0, y''(\imath) > 0 \text{ and } y'''(\imath) < 0. \end{array}$$

Lemma 5. *Suppose that* w *is a non-negative and non-zero solution of Equation (1), such that at least one of* (S_1) *and* (S_2) *is valid. Then, the following equation*

$$z'(\imath) + (1-\varsigma_0)^{p-1}\frac{\varpi_1(\imath)}{\varphi(\varpi_2(\imath))}\left(\frac{v}{6}\varpi_2^3(\imath)\right)^{p-1} z(\varpi_2(\imath)) = 0, \quad (5)$$

would have a non-oscillatory solution.

Proof. Assume that $w > 0$ in Equation (1) with property (S_1) or (S_2). Then, we obtain

$$y'(\imath) > 0,\ y'''(\imath) > 0 \text{ and } y^{(4)}(\imath) < 0.$$

Consequently, by Lemma 2, one could obtain

$$y(\imath) \ge \frac{v}{6}\imath^3 y'''(\imath). \quad (6)$$

From definition of y, we see that

$$w(\imath) \ge (1-\varsigma_0)y(\imath),$$

which, with Equation (1), gives

$$\left(\varphi(\imath)\left(y'''(\imath)\right)^{p-1}\right)' + (1-\varsigma_0)^{p-1}\varpi_1(\imath)y^{p-1}(\varpi_2(\imath)) \le 0. \quad (7)$$

Hence, from (6), if we set $z := \varphi(y''')^{p-1} > 0$, then the following

$$z'(\imath) + (1-\varsigma_0)^{p-1}\frac{\varpi_1(\imath)}{\varphi(\varpi_2(\imath))}\left(\frac{v}{6}\varpi_2^3(\imath)\right)^{p-1} z(\varpi_2(\imath)) \le 0.$$

In ([24], First Col.), one can obtain (5) is additionally will have a non-negative and non-zero solution, and it finishes the proof. □

Lemma 6. *Suppose that* w *represents a non-negative and non-zero solution of Equation (1), where* (S_3) *is satisfied. Then, we have the following equation*

$$\left(\varphi(\imath)\left(x'(\imath)\right)^{p-1}\right)' + (1-\varsigma_0)^{p-1}\varpi_1(\imath)\left(\frac{v}{2}\varpi_2^2(\imath)\right)^{p-1} x^{p-1}(\imath) = 0, \quad (8)$$

would have a non-oscillatory solution.

Proof. Assume that w is a non-negative and non-zero solution of Equation (1), where (S_3) is satisfied. Applying the above mentioned Lemma 2, one could obtain

$$y(\imath) \geq \frac{v}{2}\imath^2 y''(\imath). \tag{9}$$

The same argument that is used in the above proof of the Lemma 6, one could obtain (7). Now, as we make $\psi = \varphi(y'''/y'')^{p-1} < 0$, one may see that

$$\psi'(\imath) \leq -(1-\varsigma_0)^{p-1}\varpi_1(\imath)\frac{y^{p-1}(\varpi_2(\imath))}{(y''(\imath))^{p-1}} - (p-1)\varphi^{-1/(p-1)}(\imath)\psi^{1+(1/(p-1))}(\imath).$$

Thus, using $y''' < 0$ as well as (9), one might obtain that

$$\psi'(\imath) + (1-\varsigma_0)^{p-1}\varpi_1(\imath)\left(\frac{v}{2}\varpi_2^2(\imath)\right)^{p-1} + (p-1)\varphi^{-1/(p-1)}(\imath)\psi^{1+(1/(p-1))}(\imath) \leq 0. \tag{10}$$

Consequently, there is indeed a function $\psi \in C^1([\imath_0,\infty),\mathbb{R})$ and in such a way, (10) is valid. Consequently, we can see from [25] that (8) will have also a non-oscillatory solution, and it finishes proof. □

Theorem 1. *Suppose that (5) and (8) are both oscillatory. This leads to the fact that all the non-oscillatory solution of Equation (1) are tending to zero, when we have the following*

$$\int_{\imath_0}^{\infty}\left(\frac{1}{\varphi(u)}\int_{\imath_0}^{\imath}\varpi_1(s)\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}u = \infty. \tag{11}$$

Proof. Using the contradiction hypothesis, we suppose that w is a non-negative and non-zero solution of Equation (1) having $\lim_{\imath\to\infty} w(\imath) \neq 0$. From Lemma 4, we have cases $(S_1)-(S_4)$. Using Lemmas 5 and 6, and having both of (5) and (8) are oscillatory, we observe that w is valid for (S_4). Now, as we have y as a non-zero and non-negative decreasing function, one could see that $\lim_{\imath\to\infty} y(\imath) = c \geq 0$. Now, assume the opposite, such that $c > 0$. Then, for each $\nu > 0$ and for $\imath$ sufficiently large, one could have that $c \leq y(\imath) < c+\nu$. Picking $\nu < (1-\varsigma_0)(c/\varsigma_0)$, one could have that

$$\begin{aligned} w(\imath) &= y(\imath) - \varsigma_0(\imath)w(\varpi_3(\imath)) > c - \varsigma_0 y(\varpi_3(\imath)) \\ &> L(\gamma+\nu) > Ly(\imath), \end{aligned} \tag{12}$$

where $L = (c - \varsigma_0(c+\nu))/(c+\nu) > 0$. Hence, from (1), we have

$$\begin{aligned} \left(\varphi(\imath)(y'''(\imath))^{p-1}\right)' &= -\varpi_1(\imath)w^{p-1}(\varpi_2(\imath)) \leq -L^{p-1}\varpi_1(\imath)y^{p-1}(\varpi_2(\imath)) \\ &\leq -L^{p-1}\nu^{p-1}\varpi_1(\imath). \end{aligned}$$

Integrating this inequality from $\imath_1$ to $\imath$, we obtain

$$y'''(\imath) \leq -L\nu\left(\frac{1}{\varphi(\imath)}\int_{\imath_1}^{\imath}\varpi_1(s)\mathrm{d}s\right)^{1/(p-1)}.$$

By integrating from $\imath_1$ to $\imath$, we obtain

$$y''(\imath) \leq y''(\imath_1) - L\nu\int_{\imath_1}^{\imath}\left(\frac{1}{\varphi(u)}\int_{\imath_1}^{\imath}\varpi_1(s)\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}u.$$

Letting $\imath \to \infty$, and taking into account (11), we obtain that $\lim_{\imath\to\infty} y''(\imath) = -\infty$. In fact, this is in direct opposition to the reality that $y''(\imath) > 0$. Consequently, $c = 0$; additionally, $w(\imath) \le y(\imath)$ implies $\lim_{\imath\to\infty} w(\imath) = 0$, which contradicts the given assumption, and this finishes the proof. □

Corollary 1. *Let us suppose that (11) is valid. Then all the non-oscillatory solutions of Equation (1) are tending to zero if* $\int_{\imath_0}^{\infty} \varpi_1(s)\mathrm{d}s = \infty$,

$$\liminf_{\imath\to\infty} \int_{\varpi_2(\imath)}^{\imath} \frac{\varpi_1(s)\varpi_2^{3(p-1)}(s)}{\varphi(\varpi_2(s))}\mathrm{d}s > \frac{6^{p-1}}{\mathrm{e}v^{p-1}(1-\varsigma_0)^{p-1}}, \tag{13}$$

and

$$\limsup_{\imath\to\infty} \int_{\imath_0}^{\imath} \left((1-\varsigma_0)^{p-1}\phi^{p-1}(s)\varpi_1(s)\left(\frac{v}{2}\varpi_2^2(s)\right)^{p-1} - \left(\frac{p-1}{p}\right)^p \frac{1}{\varphi^{1/p-1}(s)\phi(s)} \right)\mathrm{d}s > 0. \tag{14}$$

Proof. Obviously, both of ([26], Corollary 2.8) and ([27], Theorem 2) are demonstrating that both (13) and (14) are indicating oscillation of (5) and (8), respectively. □

Lemma 7. *Suppose that w is an eventually non-negative and non-zero solution of Equation (1). If we have the knowledge that y is an increasing and*

$$\varsigma(\imath) \sum_{m=0}^{(i-1)/2} \prod_{n=1}^{2m} \varsigma\left(\varpi_3^j(\imath)\right) < 1, \tag{15}$$

then,

$$w(\imath) \ge (1-\widehat{\varsigma}(\imath))y(\imath), \tag{16}$$

for each non-negative odd integer i, such that

$$\widehat{\varsigma}(\imath) := \varsigma(\imath) \sum_{m=0}^{(i-1)/2} \prod_{n=1}^{2m} \varsigma\left(\varpi_3^j(\imath)\right).$$

Proof. Using the previously mentioned definition of $y(\imath)$, one could obtain the following:

$$\begin{aligned} w(\imath) &= y(\imath) - \varsigma(\imath)w(\varpi_3(\imath)) \\ &= y(\imath) - \varsigma(\imath)y(\varpi_3(\imath)) + \varsigma(\imath)\varsigma(\varpi_3(\imath))w\left(\varpi_3^2(\imath)\right) \\ &= y(\imath) - \varsigma(\imath)y(\varpi_3(\imath)) - \varsigma(\imath)\varsigma(\varpi_3(\imath))\varsigma\left(\varpi_3^2(\imath)\right)y\left(\varpi_3^3(\imath)\right) \\ &\quad + \varsigma(\imath)\varsigma(\varpi_3(\imath))\varsigma\left(\varpi_3^2(\imath)\right)\varsigma\left(\varpi_3^3(\imath)\right)w\left(\varpi_3^4(\imath)\right) \\ &\ge y(\imath) - \sum_{m=0}^{(i-1)/2} \prod_{n=0}^{2m} \varsigma\left(\varpi_3^j(\imath)\right)y\left(\varpi_3^{2m+1}(\imath)\right) + \prod_{n=0}^{i} \varsigma\left(\varpi_3^j(\imath)\right)w\left(\varpi_3^{i+1}(\imath)\right) \end{aligned} \tag{17}$$

$$\ge y(\imath) - \sum_{m=0}^{(i-1)/2} \prod_{n=0}^{2m} \varsigma\left(\varpi_3^j(\imath)\right)y\left(\varpi_3^{2m+1}(\imath)\right), \tag{18}$$

for $\imath \ge \imath_2$, where $\imath_2 \ge \imath_0$ is large enough, as well as each odd non-negative integer i. As of $\varpi_3^{2m+1}(\imath) \le \varpi_3^{2m}(\imath)$, we find

$$y\left(\varpi_3^j(\imath)\right) \le y(\imath), \quad \text{for } j = 0, 1, \ldots, i,$$

which, with (18), gives

$$w(\imath) \geq \left(1 - \sum_{m=0}^{(i-1)/2} \prod_{n=0}^{2m} \varsigma\left(\varpi_3^j(\imath)\right)\right) y(\imath).$$

The proof is complete. □

We may derive the below corollary by substituting $\widehat{\varsigma}(\imath)$ for ς in the following results:

Corollary 2. *Suppose that (11) is valid. We have that all the non-oscillatory solutions in Equation (1) go to zero whenever* $\int_{\imath_0}^{\infty} \varpi_1(s)\mathrm{d}s = \infty$,

$$\liminf_{\imath\to\infty} \int_{\varpi_2(\imath)}^{\imath} (1 - \widehat{\varsigma}(\varpi_2(s)))^{p-1} \frac{\varpi_1(s)\varpi_2^{3(p-1)}(s)}{\varphi(\varpi_2(s))} \mathrm{d}s > \frac{6^{p-1}}{\upsilon^{p-1}\mathrm{e}}$$

and

$$\limsup_{\imath\to\infty} \int_{\imath_0}^{\imath} \left((1 - \widehat{\varsigma}(\varpi_2(s)))^{p-1}\phi^{p-1}(s)\varpi_1(s)\left(\frac{\upsilon}{2}\varpi_2^2(s)\right)^{p-1} - \left(\frac{p-1}{p}\right)^p \frac{1}{\varphi^{1/p-1}(s)\phi(s)}\right)\mathrm{d}s > 0.$$

Example 1. *Let us take the following equation:*

$$\left(\imath^2\left(w(\imath) + 16w\left(\frac{\imath}{2}\right)\right)'''\right)' + \varpi_0 w\left(\frac{\imath}{2}\right) = 0, \imath \geq 1,\ \varpi_0 > 0, \tag{19}$$

where $p = 2$, $\varphi(\imath) = \imath^2$, $\varsigma(\imath) = 16$, $\varpi_3(\imath) = \varpi_2(\imath) = 1/2\imath$ *and* $\varpi_1(\imath) = \varpi_0$.

Moreover, we see that

$$\begin{aligned} &\int_{\imath_0}^{\infty}\left(\frac{1}{\varphi(u)}\int_{\imath_0}^{\imath}\varpi_1(s)\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}u \\ =& \int_{\imath_0}^{\infty}\left(\frac{1}{\imath^2}\int_{\imath_0}^{\imath}\varpi_0\mathrm{d}s\right)\mathrm{d}u \\ =& \infty. \end{aligned}$$

Thus, by Theorem 1, we can observe that in (19), all the non-oscillatory solutions are tending to zero.

Figure 1 depicts multiple solutions of the equation presented in (19) having for the values of $w(1) = 1, w'(1) = 0, \pm 1, \pm 2$, the value $\varpi_0 = 1/2$, such that a non-oscillatory behavior can be seen, as follows:

Example 2. *Let us take the following equation*

$$\left(\imath^2\left(w(\imath) + 4w\left(\frac{\imath}{2}\right)\right)'''\right)' + \varpi_0 w\left(\frac{\imath}{2}\right) = 0, \tag{20}$$

where $\imath \geq 1$, $\varpi_0 > 0$. *We note that* $\varphi(\imath) = \imath^2$, $\varsigma(\imath) = 4$, $\varpi_3(\imath) = \varpi_2(\imath) = 1/2\imath$ *and* $\varpi_1(\imath) = \varpi_0$. *Thus, it's easy to see that*

$$\begin{aligned} &\int_{\imath_0}^{\infty}\varpi_1(s)\mathrm{d}s \\ =& \int_{\imath_0}^{\infty}\varpi_0\mathrm{d}s = \infty. \end{aligned}$$

Furthermore, the conditions (13) and (14) hold. Thus, by Corollary 1, we find that all non-oscillatory solutions of (19) go to zero.

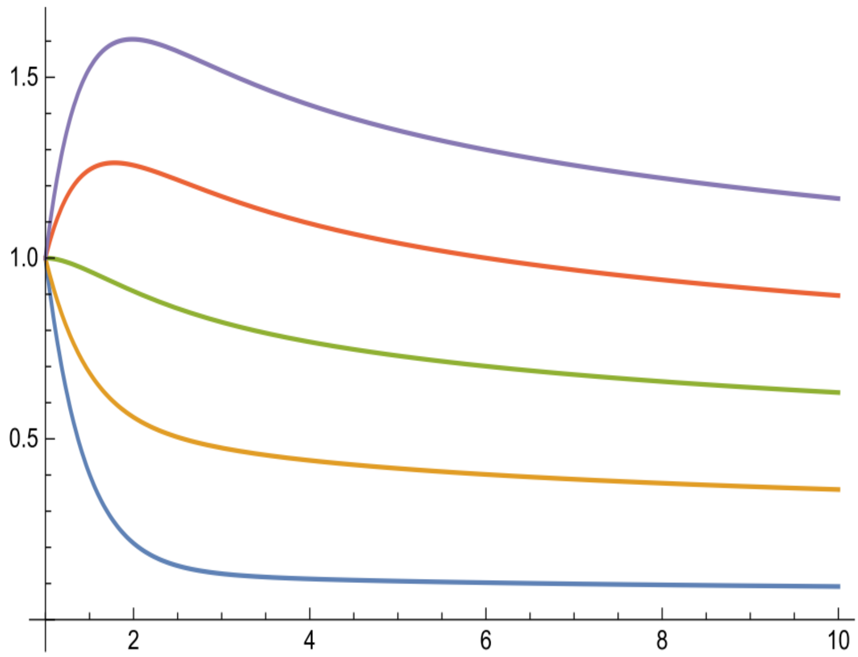

Figure 1. Some solutions of the equation in (19) taking $\varpi_0 = 1/2$.

3. Conclusions

In this research, we intensively studied the criteria for non-oscillatory solutions of fourth-order nonlinear neutral differential equations . Relying on the comparison method with a first-order equation, new asymptotic conditions for Equation (1) is presented. These results complement some previously published articles, where here we discussed two examples. Moreover, some oscillation characteristics of n-order differential equation will be the main focus in the future research.

$$\left(\varphi(\imath)\left(y^{(n-1)}(\imath)\right)^{p_1-1}\right)' + \varpi_1(\imath)w^{p_2-1}(\varpi_2(\imath)) = 0,$$

if $y(\imath) := w(\imath) - \varsigma(\imath)w(\varpi_3(\imath))$.

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