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Multi-Parameter Quantum Integral Identity Involving Raina's Function and Corresponding q-Integral Inequalities with Applications

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Abstract: Convexity performs its due role in the theoretical field of inequalities according to the nature and conduct of the properties it displays. A correlation connectivity, which is visible between the two variables symmetry and convexity, enhances its importance. In this paper, we derive a new multi-parameter quantum integral identity involving Raina's function. Applying this generic identity as an auxiliary result, we establish some new generalized quantum estimates of certain integral inequalities pertaining to the class of $\mathcal{R}_s$-convex functions. Moreover, we give quantum integral inequalities for the product of $\mathcal{R}_{s_1}$- and $\mathcal{R}_{s_2}$-convex functions as well as another quantum result for a function that satisfies a special condition. In order to demonstrate the efficiency of our main results, we offer many important special cases for suitable choices of parameters and finally for $\mathcal{R}_s$-convex functions that are absolute-value bounded.

Keywords: convexity; $\mathcal{R}_s$-convex; quantum derivatives; quantum integrals; Hölder's inequality

JEL Classification: 05A30; 26A33; 26A51; 34A08; 26D07; 26D10; 26D15

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1. Introduction and Preliminaries

In recent years, the classical concept of convexity has been extended and generalized in different directions using novel and innovative ideas. Cortez et al. [1] presented a new generalization of convexity classes as follows:

Definition 1 ([1])**.** *Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ be a bounded sequence of positive real numbers. A non-empty set $\mathcal{I}$ is said to be generalized convex if*

$$v_1 + \tau \mathcal{R}_{\rho,\lambda,\sigma}(v_2 - v_1) \in \mathcal{I}, \quad \forall\, v_1, v_2 \in \mathcal{I}, \tau \in [0,1].$$

Here, $\mathcal{R}_{\rho,\lambda,\sigma}(\cdot)$ is Raina's function, which is defined as follows:

$$\mathcal{R}_{\rho,\lambda,\sigma}(z) = \mathcal{R}^{\sigma(0),\sigma(1),\dots}_{\rho,\lambda}(z) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} z^k, \tag{1}$$

where $|z| < R$ and $\Gamma(\cdot)$ is the well-known Gamma function. For more details, see [2].

Definition 2 ([1]). *Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ be a bounded sequence of positive real numbers. A function $\Psi : \mathcal{I} \to \mathbb{R}$ is said to be generalized convex if*

$$\Psi(v_1 + \tau \mathcal{R}_{\rho,\lambda,\sigma}(v_2 - v_1)) \leq (1-\tau)\Psi(v_1) + \tau\Psi(v_2), \quad \forall\, v_1, v_2 \in \mathcal{I}, \tau \in [0,1].$$

In addition to its many applications, another fascinating aspect of convexity is its close relation with theories of inequalities. We can obtain several classical and new inequalities using convexity and its generalization (see, e.g., [3–5] and references therein). The most studied results pertaining to the convexity properties of functions are Hermite–Hadamard's inequality, Ostrowski's inequality and Simpson's inequality. For more details, see [6].

Cortez et al. [1] derived a new version of Hermite–Hadamard's inequality using the class of generalized convex functions. This result reads as follows:

Theorem 1. *Let $\Psi : [c_1, c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)] \to \mathbb{R}$ be a generalized convex function. Then,*

$$\Psi\left(\frac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}\right) \leq \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \Psi(x)\mathrm{d}x \leq \frac{\Psi(c_1) + \Psi(c_2)}{2}.$$

Note that if we take $\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1) = c_2 - c_1$, then we can recapture the classical Hermite–Hadamard inequality from the above inequality for convex functions, which reads as follows:

Theorem 2. *Let $\Psi : [c_1, c_2] \to \mathbb{R}$ be a convex function. Then,*

$$\Psi\left(\frac{c_1 + c_2}{2}\right) \leq \frac{1}{c_2 - c_1} \int_{c_1}^{c_2} \Psi(x)\mathrm{d}x \leq \frac{\Psi(c_1) + \Psi(c_2)}{2}.$$

We now recall the following two basic concepts regarding quantum calculus that are helpful to us in obtaining the main results of the paper.

Definition 3 ([7,8]). *Let $\Psi : [c_1, c_2] \to \mathbb{R}$ be an arbitrary function. Then, the q-derivative of Ψ on $[c_1, c_2]$ at τ is defined as*

$${}_{c_1}D_q\Psi(\tau) = \frac{\Psi(\tau) - \Psi(q\tau + (1-q)c_1)}{(1-q)(\tau - c_1)}, \ \tau \neq c_1 \text{ and } D_q\Psi(c_1) = \lim_{\tau \to c_1} {}_{c_1}D_q\Psi(\tau),$$

where $0 < q < 1$ is a constant.

Definition 4 ([7,8]). *Let $\Psi : [c_1, c_2] \to \mathbb{R}$ be an arbitrary function. Then, the q-integral of Ψ on $[c_1, c_2]$ is defined as*

$$\int_{c_1}^{x} \Psi(\tau)\, {}_{c_1}\mathrm{d}_q\tau = (1-q)(x - c_1) \sum_{n=0}^{\infty} q^n \Psi(q^n x + (1-q^n)c_1),$$

for all $x \in [c_1, c_2]$, where $0 < q < 1$ is a constant.

Recently, Tariboon and Ntouyas [7,8] utilized the concepts of quantum calculus in obtaining the quantum analogues of inequalities involving convexity. These ideas and techniques of Tariboon and his coauthor attracted many researchers, particulary those working in the field of inequalities involving convexity and its generalizations. Since then, numerous new quantum analogues of classical inequalities have been obtained in the literature. For example, Noor et al. [9] and Sudsutad et al. [10] obtained q-analogues of Hermite–Hadamard's inequality using the class of convex functions. Noor et al. [11] obtained

q-Hermite–Hadamard inequalities using the class of pre-invex functions. Alp et al. [12] gave a refined q-analogue of Hermite–Hadamard's inequality. Zhang et al. [13] obtained a generalized quantum integral identity and obtained several new q-analogues of certain integral inequalities. Very recently, Du et al. [14] obtained another fascinating q-integral identity and obtained various q-analogues of certain integral inequalities. For more information about quantum calculus and its applications, see [15–26].

Before we move towards our main results, we first define the class of $\mathcal{R}_s$-convex functions.

Definition 5. *Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ be a bounded sequence of positive real numbers. A function $\Psi : \mathcal{I} \to \mathbb{R}$ is said to be $\mathcal{R}_s$-convex if*

$$\Psi(v_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(v_2 - v_1)) \leq (1-\tau)^s\Psi(v_1) + \tau^s\Psi(v_2), \quad \forall\, v_1, v_2 \in \mathcal{I}, \tau \in [0,1], s \in (0,1].$$

Note that we can recapture Definition 2 by taking $s = 1$ in Definition 5 above.

Inspired by the above results and literature, in Section 2, we derive a new multi-parameter quantum integral identity. Using this generic identity as an auxiliary result, we obtain some new generalized quantum estimates of certain integral inequalities pertaining to the class of $\mathcal{R}_s$-convex functions. In Section 3, we give quantum integral inequalities for the product of $\mathcal{R}_{s_1}$- and $\mathcal{R}_{s_2}$-convex functions as well as another quantum result for a function that satisfies a special condition. As for applications, in Section 4, we discuss several important special cases of the established results for suitable choices of parameters and also for $\mathcal{R}_s$-convex functions that are absolute-value bounded. In Section 5, some conclusions and future research are given.

2. Main Results

In this section, we first present the following multi-parameter quantum integral identity, which will be a main tool to derive our main results. For brevity, we denote $\mathcal{B} := [c_1, c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)]$, where $\mathcal{B}^\circ$ is the interior set and $\delta, \varepsilon \in \mathbb{R}$.

Lemma 1. *Let $\Psi : \mathcal{B} \to \mathbb{R}$ be a q-differentiable function on $\mathcal{B}^\circ$ with $\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1) > 0$. If ${}_{c_1}\mathrm{D}_q\Psi$ is integrable on $\mathcal{B}$ and $0 < q < 1$, then*

$$\begin{aligned}\mathcal{M}(\Psi;\mathcal{R}) :=\,& \varepsilon\Psi(c_1) + (\delta - \varepsilon)\Psi\left(\frac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}\right) + (1-\delta)\Psi(c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) \\ & - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \Psi(x)\, {}_{c_1}\mathrm{d}_q x \\ =\,& \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\left[\int_0^{\frac{1}{2}} (q\tau - \varepsilon)\, {}_{c_1}\mathrm{D}_q\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))\, {}_0\mathrm{d}_q\tau \right. \\ & \left. + \int_{\frac{1}{2}}^1 (q\tau - \delta)\, {}_{c_1}\mathrm{D}_q\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))\, {}_0\mathrm{d}_q\tau\right]. \end{aligned} \tag{2}$$

Proof. Let

$$\begin{aligned}\mathcal{S}_1 &:= \int_0^{\frac{1}{2}} (q\tau - \varepsilon)\, {}_{c_1}\mathrm{D}_q\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))\, {}_0\mathrm{d}_q\tau, \\ \mathcal{S}_2 &:= \int_{\frac{1}{2}}^1 (q\tau - \delta)\, {}_{c_1}\mathrm{D}_q\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))\, {}_0\mathrm{d}_q\tau.\end{aligned}$$

A direct computation gives

$$\begin{aligned}
\mathcal{S}_1 &= \int_0^{\frac{1}{2}} q \frac{\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) - \Psi(c_1 + q\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))}{(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \, {}_0\mathrm{d}_q\tau \\
&\quad - \varepsilon \int_0^{\frac{1}{2}} \frac{\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) - \Psi(c_1 + q\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))}{\tau(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \, {}_0\mathrm{d}_q\tau \\
&= \frac{1}{2}\sum_{n=0}^{\infty} q^{n+1} \frac{\Psi\left(c_1 + \frac{1}{2}q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right) - \Psi\left(c_1 + \frac{1}{2}q^{n+1}\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \\
&\quad - \varepsilon \sum_{n=0}^{\infty} \frac{\Psi\left(c_1 + \frac{1}{2}q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right) - \Psi\left(c_1 + \frac{1}{2}q^{n+1}\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \\
&= \frac{q\sum\limits_{n=0}^{\infty} q^n\Psi\left(c_1 + \frac{1}{2}q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right) - \sum\limits_{n=1}^{\infty} q^n\Psi\left(c_1 + \frac{1}{2}q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right)}{2\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \\
&\quad - \varepsilon \frac{\sum\limits_{n=0}^{\infty} \Psi\left(c_1 + \frac{1}{2}q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right) - \sum\limits_{n=1}^{\infty} \Psi\left(c_1 + \frac{1}{2}q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \\
&= \frac{1}{2}\left[\frac{\Psi\left(\frac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}\right)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} - (1-q)\sum_{n=0}^{\infty} q^n \frac{\Psi\left(c_1 + \frac{1}{2}q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}\right] - \varepsilon \cdot \frac{\Psi\left(\frac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}\right) - \Psi(c_1)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \\
&= \left(\frac{1}{2} - \varepsilon\right)\frac{\Psi\left(\frac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}\right)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} + \frac{\varepsilon}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}\Psi(c_1) - \frac{1}{2}(1-q)\sum_{n=0}^{\infty} q^n \frac{\Psi\left(c_1 + \frac{1}{2}q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \\
&= \left(\frac{1}{2} - \varepsilon\right)\frac{\Psi\left(\frac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}\right)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} + \frac{\varepsilon}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}\Psi(c_1) - \frac{1}{\mathcal{R}^2_{\rho,\lambda,\sigma}(c_2 - c_1)}\int_{c_1}^{c_1 + \frac{1}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \Psi(x) \, {}_0\mathrm{d}_q x.
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
\mathcal{S}_2 &= \int_{\frac{1}{2}}^{1} q\tau \, {}_{c_1}\mathrm{D}_q\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) \, {}_0\mathrm{d}_q\tau - \delta\int_{\frac{1}{2}}^{1} {}_{c_1}\mathrm{D}_q\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) \, {}_0\mathrm{d}_q\tau \\
&= \int_0^1 q\tau \, {}_{c_1}\mathrm{D}_q\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) \, {}_0\mathrm{d}_q\tau - \delta\int_0^1 {}_{c_1}\mathrm{D}_q\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) \, {}_0\mathrm{d}_q\tau \\
&\quad - \left(\int_0^{\frac{1}{2}} q\tau \, {}_{c_1}\mathrm{D}_q\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) \, {}_0\mathrm{d}_q\tau - \delta\int_0^{\frac{1}{2}} {}_{c_1}\mathrm{D}_q\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) \, {}_0\mathrm{d}_q\tau\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^1 q\tau\, {}_{c_1}D_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))\,{}_0d_q\tau-\delta\int_0^1 {}_{c_1}D_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))\,{}_0d_q\tau\\
&=\int_0^1 q\frac{\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))-\Psi(c_1+q\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))}{(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\,{}_0d_q\tau\\
&\quad-\delta\int_0^1 \frac{\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))-\Psi(c_1+q\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))}{\tau(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\,{}_0d_q\tau\\
&=\sum_{n=0}^{\infty}q^{n+1}\frac{\Psi\big(c_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)-\Psi\big(c_1+q^{n+1}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\\
&\quad-\delta\sum_{n=0}^{\infty}\frac{\Psi\big(c_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)-\Psi\big(c_1+q^{n+1}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\\
&=\frac{q\sum\limits_{n=0}^{\infty}q^n\Psi\big(c_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)-\sum\limits_{n=1}^{\infty}q^n\Psi\big(c_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\\
&\quad-\delta\frac{\sum\limits_{n=0}^{\infty}\Psi\big(c_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)-\sum\limits_{n=0}^{\infty}\Psi\big(c_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\\
&=\frac{\Psi\big(c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}-\sum_{n=0}^{\infty}(1-q)q^n\frac{\Psi\big(c_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}-\delta\cdot\frac{\Psi\big(c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)-\Psi(c_1)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\\
&=\frac{(1-\delta)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\cdot\Psi\big(c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)+\frac{\delta}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(c_1)-\sum_{n=0}^{\infty}(1-q)q^n\frac{\Psi\big(c_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\\
&=\frac{(1-\delta)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi\big(c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)+\frac{\delta}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(c_1)-\frac{1}{\mathcal{R}^2_{\rho,\lambda,\sigma}(c_2-c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(x)\,{}_0d_qx,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}} q\tau\, {}_{c_1}D_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))\,{}_0d_q\tau-\delta\int_0^{\frac{1}{2}} {}_{c_1}D_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))\,{}_0d_q\tau\\
&=\frac{1}{2}\left[\frac{\Psi\left(\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}-(1-q)\sum_{n=0}^{\infty}q^n\frac{\Psi\big(c_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\right]-\delta\cdot\frac{\Psi\left(\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)-\Psi(c_1)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\\
&=\left(\frac{1}{2}-\delta\right)\cdot\frac{\Psi\left(\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}+\frac{\delta}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(c_1)-\frac{1}{\mathcal{R}^2_{\rho,\lambda,\sigma}(c_2-c_1)}\int_{c_1}^{c_1+\frac{1}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(x)\,{}_0d_qx,
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathcal{S}_2=&\frac{(1-\delta)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi\big(c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\big)-\left(\frac{1}{2}-\delta\right)\cdot\frac{\Psi\left(\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\\
&-\frac{1}{\mathcal{R}^2_{\rho,\lambda,\sigma}(c_2-c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(x)\,{}_0d_qx+\frac{1}{\mathcal{R}^2_{\rho,\lambda,\sigma}(c_2-c_1)}\int_{c_1}^{c_1+\frac{1}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(x)\,{}_0d_qx.
\end{aligned}$$

This completes the proof. □

Corollary 1. *In Lemma 1, taking* $\varepsilon = \dfrac{1}{6}$ *and* $\delta = \dfrac{5}{6}$*, we have*

$$\begin{aligned}
&\frac{\Psi(c_1)}{6} + \frac{2\Psi\left(\dfrac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}\right)}{3} + \frac{\Psi(c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))}{6} - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \Psi(x)\, {}_{c_1}\mathrm{d}_q x \\
&= \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1) \left[\int_0^{\frac{1}{2}} \left(q\tau - \frac{1}{6}\right) {}_{c_1}\mathrm{D}_q \Psi(c_1 + \tau \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))\, {}_0\mathrm{d}_q \tau \right. \\
&\quad \left. + \int_{\frac{1}{2}}^{1} \left(q\tau - \frac{5}{6}\right) {}_{c_1}\mathrm{D}_q \Psi(c_1 + \tau \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))\, {}_0\mathrm{d}_q \tau \right].
\end{aligned}$$

Corollary 2. *In Lemma 1, choosing* $\varepsilon = \delta = \dfrac{q}{1+q}$*, we obtain*

$$\begin{aligned}
&\frac{q}{1+q}\Psi(c_1) + \frac{1}{1+q}\Psi(c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \Psi(x)\, {}_{c_1}\mathrm{d}_q x \\
&= q\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1) \int_0^1 \left(\tau - \frac{1}{1+q}\right) {}_{c_1}\mathrm{D}_q \Psi(c_1 + \tau \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))\, {}_0\mathrm{d}_q \tau.
\end{aligned}$$

Corollary 3. *In Lemma 1, taking* $\varepsilon = \dfrac{1}{4}$ *and* $\delta = \dfrac{3}{4}$*, we obtain*

$$\begin{aligned}
&\frac{\Psi(c_1)}{4} + \frac{\Psi\left(\dfrac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}\right)}{2} + \frac{\Psi(c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))}{2} - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \Psi(x)\, {}_{c_1}\mathrm{d}_q x \\
&= \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1) \left[\int_0^{\frac{1}{2}} \left(q\tau - \frac{1}{4}\right) {}_{c_1}\mathrm{D}_q \Psi(c_1 + \tau \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))\, {}_0\mathrm{d}_q \tau \right. \\
&\quad \left. + \int_{\frac{1}{2}}^{1} \left(q\tau - \frac{3}{4}\right) {}_{c_1}\mathrm{D}_q \Psi(c_1 + \tau \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))\, {}_0\mathrm{d}_q \tau \right].
\end{aligned}$$

Using Lemma 1, we can obtain our main results.

Theorem 3. *Let* $\Psi : \mathcal{B} \to \mathbb{R}$ *be a q-differentiable function on* $\mathcal{B}^\circ$ *with* $\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1) > 0$*. If* $|{}_{c_1}\mathrm{D}_q \Psi|$ *is an integrable* $\mathcal{R}_s$*-convex function with* $s \in (0,1]$ *and* $0 < q < 1$*, then*

$$\begin{aligned}
|\mathcal{M}(\Psi;\mathcal{R})| \leq \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1) &\Big[|{}_{c_1}\mathrm{D}_q \Psi(c_1)| \Big(2^{1-s}(A_1(\varepsilon;q) + A_2(\delta;q)) - (A_3(s,\varepsilon;q) + A_4(s,\delta;q)) \Big) \\
&+ |{}_{c_1}\mathrm{D}_q \Psi(c_2)| (A_3(s,\varepsilon;q) + A_4(s,\delta;q)) \Big], \qquad (3)
\end{aligned}$$

where

$$A_1(\varepsilon;q) := \int_0^{\frac{1}{2}} |q\tau - \varepsilon|\,{}_0\mathrm{d}_q\tau = \begin{cases} \dfrac{8\varepsilon^2 - 2\varepsilon(1+q) + q}{4(1+q)}, & 0 \le \dfrac{\varepsilon}{q} \le \dfrac{1}{2}, \\ \dfrac{2\varepsilon(1+q) - q}{4(1+q)}, & \dfrac{1}{2} < \dfrac{\varepsilon}{q}, \end{cases}$$

$$A_2(\delta;q) := \int_{\frac{1}{2}}^{1} |q\tau - \delta|\,{}_0\mathrm{d}_q\tau = \begin{cases} \dfrac{3q - 2\delta(1+q)}{4(1+q)}, & 0 \le \dfrac{\delta}{q} \le \dfrac{1}{2}, \\ \dfrac{8\delta^2 - 6\delta(1+q) + 5q}{4(1+q)}, & \dfrac{1}{2} < \dfrac{\delta}{q} \le 1, \\ \dfrac{2\delta(1+q) - 3q}{4(1+q)}, & 1 < \dfrac{\delta}{q}, \end{cases}$$

$$A_3(s,\varepsilon;q) := \int_0^{\frac{1}{2}} \tau^s |q\tau - \varepsilon|\,{}_0\mathrm{d}_q\tau = \begin{cases} \dfrac{2\varepsilon^{s+2}(1-q)^2}{(1-q^{s+1})(1-q^{s+2})} \\ +\dfrac{(1-q)(q-\varepsilon) + (q-1)(1-\varepsilon)q^{s+2}}{2^{s+2}(1-q^{s+1})(1-q^{s+2})}, & ,0 \le \dfrac{\varepsilon}{q} \le \dfrac{1}{2}, \\ -\dfrac{(1-q)(q-\varepsilon) + (q-1)(1-\varepsilon)q^{s+2}}{2^{s+2}(1-q^{s+1})(1-q^{s+2})}, & \dfrac{1}{2} < \dfrac{\varepsilon}{q}, \end{cases}$$

and

$$A_4(s,\delta;q) := \int_{\frac{1}{2}}^{1} \tau^s |q\tau - \delta|\,{}_0\mathrm{d}_q\tau = \begin{cases} \dfrac{\delta(1-q)(1-2^{s+1})}{2^{s+1}(1-q^{s+1})} + \dfrac{q(1-q)(2^{s+2}-1)}{2^{s+2}(1-q^{s+2})}, & 0 \le \dfrac{\delta}{q} \le \dfrac{1}{2}, \\ -\dfrac{\delta(1-q)(1+2^{s+1})}{2^{s+1}(1-q^{s+1})} + \dfrac{q(1-q)(1+2^{s+2})}{2^{s+2}(1-q^{s+2})} \\ +\dfrac{2\delta^{s+2}(1-q)^2}{(1-q^{s+1})(1-q^{s+2})}, & \dfrac{1}{2} < \dfrac{\delta}{q} \le 1, \\ -\dfrac{\delta(1-q)(1-2^{s+1})}{2^{s+1}(1-q^{s+1})} + \dfrac{q(1-q)(2^{s+2}-1)}{2^{s+2}(1-q^{s+2})}, & 1 < \dfrac{\delta}{q}. \end{cases}$$

Proof. Using Lemma 1 and $\mathcal{R}_s$-convexity of $|{}_{c_1}\mathrm{D}_q\Psi|$ and applying inequality $(1-\tau)^s \le 2^{1-s} - \tau^s$, for $\tau \in (0,1)$, we have

$$
\begin{aligned}
|\mathcal{M}(\Psi;\mathcal{R})| &= \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\Bigg|\int_0^{\frac{1}{2}}(q\tau-\varepsilon)\,{}_{c_1}\mathrm{D}_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))\,{}_0\mathrm{d}_q\tau \\
&\quad+\int_{\frac{1}{2}}^{1}(q\tau-\delta)\,{}_{c_1}\mathrm{D}_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))\,{}_0\mathrm{d}_q\tau\Bigg| \\
&\le \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\Bigg[\int_0^{\frac{1}{2}}|q\tau-\varepsilon||\,{}_{c_1}\mathrm{D}_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))|\,{}_0\mathrm{d}_q\tau \\
&\quad+\int_{\frac{1}{2}}^{1}|q\tau-\delta||\,{}_{c_1}\mathrm{D}_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))|\,{}_0\mathrm{d}_q\tau\Bigg] \\
&\le \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\Bigg[\int_0^{\frac{1}{2}}|q\tau-\varepsilon|\Big((1-\tau)^s|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|+\tau^s|\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|\Big)\,{}_0\mathrm{d}_q\tau \\
&\quad+\int_{\frac{1}{2}}^{1}|q\tau-\delta|\Big((1-\tau)^s|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|+\tau^s|\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|\Big)\,{}_0\mathrm{d}_q\tau\Bigg] \\
&= \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\Bigg[|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|\Bigg(\int_0^{\frac{1}{2}}(1-\tau)^s|q\tau-\varepsilon|\,{}_0\mathrm{d}_q\tau+\int_{\frac{1}{2}}^{1}(1-\tau)^s|q\tau-\delta|\,{}_0\mathrm{d}_q\tau\Bigg) \\
&\quad+|\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|\Bigg(\int_0^{\frac{1}{2}}\tau^s|q\tau-\varepsilon|\,{}_0\mathrm{d}_q\tau+\int_{\frac{1}{2}}^{1}\tau^s|q\tau-\delta|\,{}_0\mathrm{d}_q\tau\Bigg)\Bigg] \\
&\le \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\Bigg[|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|\Bigg(\int_0^{\frac{1}{2}}(2^{1-s}-\tau^s)|q\tau-\varepsilon|\,{}_0\mathrm{d}_q\tau+\int_{\frac{1}{2}}^{1}(2^{1-s}-\tau^s)|q\tau-\delta|\,{}_0\mathrm{d}_q\tau\Bigg) \\
&\quad+|\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|\Bigg(\int_0^{\frac{1}{2}}\tau^s|q\tau-\varepsilon|\,{}_0\mathrm{d}_q\tau+\int_{\frac{1}{2}}^{1}\tau^s|q\tau-\delta|\,{}_0\mathrm{d}_q\tau\Bigg)\Bigg] \\
&= \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\Bigg[|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|\Big(2^{1-s}(A_1(\varepsilon;q)+A_2(\delta;q))-(A_3(s,\varepsilon;q)+A_4(s,\delta;q))\Big) \\
&\quad+|\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|\Big(A_3(s,\varepsilon;q)+A_4(s,\delta;q)\Big)\Bigg].
\end{aligned}
$$

This completes the proof. □

Theorem 4. *Let $\Psi:\mathcal{B}\to\mathbb{R}$ be a q-differentiable function on $\mathcal{B}^\circ$ with $\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)>0$. If $|\,{}_{c_1}\mathrm{D}_q\Psi|^r$ is an integrable $\mathcal{R}_s$-convex function with $s\in(0,1]$ and $0<q<1$, then for $r>1$ and $p^{-1}+r^{-1}=1$, we have*

$$|\mathcal{M}(\Psi,\mathcal{R})| \leq \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\Bigg(K_1^{\frac{1}{p}}(\varepsilon,q;p)\big(|{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r(2^{-s}-K_2(s;q))+|{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r K_2(s;q)\big)^{\frac{1}{r}}$$

$$+K_3^{\frac{1}{p}}(\delta,q;p)\big(|{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r(2^{-s}-K_4(s;q))+|{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r K_4(s;q)\big)^{\frac{1}{r}}\Bigg), \tag{4}$$

where

$$K_1(\varepsilon,q;p) := \int_0^{\frac{1}{2}} |q\tau-\varepsilon|^p \,{}_0\mathrm{d}_q\tau = \begin{cases} \frac{1-q}{2}\sum_{n=0}^{\infty} q^n\left(\frac{q^{n+1}}{2}-\varepsilon\right)^p \\ +\frac{2(1-q)\varepsilon^{p+1}}{q}\sum_{n=0}^{\infty} q^n(1-q^n)^p, & 0\leq \frac{\varepsilon}{q}\leq \frac{1}{2}, \\ \frac{1-q}{2}\sum_{n=0}^{\infty} q^n\left(\frac{q^{n+1}}{2}-\varepsilon\right)^p, & \frac{1}{2}<\frac{\varepsilon}{q}. \end{cases}$$

$$K_3(\delta,q;p) := \int_{\frac{1}{2}}^{1} |q\tau-\delta|^p \,{}_0\mathrm{d}_q\tau$$

$$= \begin{cases} (1-q)\sum_{n=0}^{\infty} q^n\left(q^{n+1}-\delta\right)^p - \frac{1-q}{2}\sum_{n=0}^{\infty} q^n\left(\frac{q^{n+1}}{2}-\delta\right)^p, & 0\leq\frac{\delta}{q}\leq\frac{1}{2}, \\ \frac{2(1-q)\delta^{p+1}}{q}\sum_{n=0}^{\infty} q^n(1-q^n)^p \\ +(1-q)\sum_{n=0}^{\infty} q^n\left(q^{n+1}-\delta\right)^p + \frac{1-q}{2}\sum_{n=0}^{\infty} q^n\left(\frac{q^{n+1}}{2}-\delta\right)^p, & \frac{1}{2}<\frac{\delta}{q}\leq 1, \\ (1-q)\sum_{n=0}^{\infty} q^n\left(q^{n+1}-\delta\right)^p + \frac{1-q}{2}\sum_{n=0}^{\infty} q^n\left(\frac{q^{n+1}}{2}-\delta\right)^p, & 1<\frac{\delta}{q}, \end{cases}$$

$$K_2(s;q) := \int_0^{\frac{1}{2}} \tau^s \,{}_0\mathrm{d}_q\tau = \frac{1-q}{2^{s+1}(1-q^{s+1})},$$

and

$$K_4(s;q) := \int_{\frac{1}{2}}^{1} \tau^s \,{}_0\mathrm{d}_q\tau = \frac{(1-q)(2^{s+1}-1)}{2^{s+1}(1-q^{s+1})}.$$

Proof. Using Lemma 1, Hölder's inequality and $\mathcal{R}_s$-convexity of $|{}_{c_1}\mathrm{D}_q\Psi|^r$ and applying inequality $(1-\tau)^s \leq 2^{1-s}-\tau^s$, for $\tau\in(0,1)$, we have

$$
\begin{aligned}
&|\mathcal{M}(\Psi;\mathcal{R})| \\
&= \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\left|\int_0^{\frac{1}{2}}(q\tau-\varepsilon)\,{}_{c_1}\mathrm{D}_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))\,{}_0\mathrm{d}_q\tau\right. \\
&\quad \left.+\int_{\frac{1}{2}}^1(q\tau-\delta)\,{}_{c_1}\mathrm{D}_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))\,{}_0\mathrm{d}_q\tau\right| \\
&\le \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\left[\int_0^{\frac{1}{2}}|q\tau-\varepsilon||\,{}_{c_1}\mathrm{D}_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))|\,{}_0\mathrm{d}_q\tau\right. \\
&\quad \left.+\int_{\frac{1}{2}}^1|q\tau-\delta||\,{}_{c_1}\mathrm{D}_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))|\,{}_0\mathrm{d}_q\tau\right] \\
&\le \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\varepsilon|^p\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\,{}_{c_1}\mathrm{D}_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))|^r\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\right. \\
&\quad \left.+\left(\int_{\frac{1}{2}}^1|q\tau-\delta|^p\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|\,{}_{c_1}\mathrm{D}_q\Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))|^r\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\right] \\
&\le \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\left[K_1^{\frac{1}{p}}(\varepsilon,q;p)\left(|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r\int_0^{\frac{1}{2}}(1-\tau)^s\,{}_0\mathrm{d}_q\tau+|\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r\int_0^{\frac{1}{2}}\tau^s\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\right. \\
&\quad \left.+K_3^{\frac{1}{p}}(\delta,q;p)\left(|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r\int_{\frac{1}{2}}^1(1-\tau)^s\,{}_0\mathrm{d}_q\tau+|\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r\int_{\frac{1}{2}}^1\tau^s\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\right] \\
&\le \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\left[K_1^{\frac{1}{p}}(\varepsilon,q;p)\left(|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r\int_0^{\frac{1}{2}}(2^{1-s}-\tau^s)\,{}_0\mathrm{d}_q\tau+|\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r\int_0^{\frac{1}{2}}\tau^s\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\right. \\
&\quad \left.+K_3^{\frac{1}{p}}(\delta,q;p)\left(|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r\int_{\frac{1}{2}}^1(2^{1-s}-\tau^s)\,{}_0\mathrm{d}_q\tau+|\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r\int_{\frac{1}{2}}^1\tau^s\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\right] \\
&= \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\left[K_1^{\frac{1}{p}}(\varepsilon,q;p)\big(|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r(2^{-s}-K_2(s;q))+|\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|^rK_2(s;q)\big)^{\frac{1}{r}}\right. \\
&\quad \left.+K_3^{\frac{1}{p}}(\delta,q;p)\big(|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r(2^{-s}-K_4(s;q))+|\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|^rK_4(s;q)\big)^{\frac{1}{r}}\right].
\end{aligned}
$$

This completes the proof. □

Remark 1. *Using Lemma 1, many new and interesting results via Hölder–İşcan, Chebyshev, Markov, Young and Minkowski inequalities using different classes of convex functions can be established. We omit their proofs here and the details are left to the interested reader.*

3. Further Results

Our next results are given below.

Theorem 5. *Let $\Psi, g : \mathcal{B} \to \mathbb{R}$ be continuous and non-negative functions on $\mathcal{B}$. If Ψ and g are respectively $\mathcal{R}_{s_1}$- and $\mathcal{R}_{s_2}$-convex functions on $\mathcal{B}$, then for $s \in (0,1]$ and $0 < q < 1$, we have*

$$\begin{aligned}
&\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)} \Psi(x)g(x)\,{}_{c_1}\mathrm{d}_q x \\
&\le \Psi(c_1)g(c_1)\left(2^{1-s_1-s_2} - \frac{1-q}{1-q^{s_1+s_2+1}}\right) + \Psi(c_1)g(c_2)\left(2^{1-s_1}\frac{1-q}{1-q^{s_2+1}} - \frac{1-q}{1-q^{s_1+s_2+1}}\right) \\
&\quad + \Psi(c_2)g(c_1)\left(2^{1-s_2}\frac{1-q}{1-q^{s_1+1}} - \frac{1-q}{1-q^{s_1+s_2+1}}\right) + \Psi(c_2)g(c_2)\frac{1-q}{1-q^{s_1+s_2+1}}.
\end{aligned}$$

Proof. Since Ψ and g are respectively $\mathcal{R}_{s_1}$- and $\mathcal{R}_{s_2}$-convex functions, we have

$$\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) \le (1-\tau)^{s_1}\Psi(c_1) + \tau^{s_1}\Psi(c_2) \tag{5}$$

$$g(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) \le (1-\tau)^{s_2}g(c_1) + \tau^{s_2}g(c_2). \tag{6}$$

Multiplying both sides of (5) and (6), we obtain

$$\begin{aligned}
&\Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))g(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) \\
&\le (1-\tau)^{s_1+s_2}\Psi(c_1)g(c_1) + \tau^{s_2}(1-\tau)^{s_1}\Psi(c_1)g(c_2) \\
&\quad + \tau^{s_1}(1-\tau)^{s_2}\Psi(c_2)g(c_1) + \tau^{s_1+s_2}\Psi(c_2)g(c_2).
\end{aligned} \tag{7}$$

Taking the q-integral for (7) with respect to τ and using the inequality $(1-\tau)^s \le 2^{1-s} - \tau^s$, for $\tau \in (0,1)$, we obtain

$$\begin{aligned}
&\int_0^1 \Psi(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))g(c_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))\,{}_0\mathrm{d}_q\tau \\
&\le \Psi(c_1)g(c_1)\int_0^1 (1-\tau)^{s_1+s_2}\mathrm{d}_{p,q}\tau + \Psi(c_1)g(c_2)\int_0^1 \tau^{s_2}(1-\tau)^{s_1}\,{}_{c_1}\mathrm{d}_q\tau \\
&\quad + \Psi(c_2)g(c_1)\int_0^1 \tau^{s_1}(1-\tau)^{s_2}\,{}_0\mathrm{d}_q\tau + \Psi(c_2)g(c_2)\int_0^1 \tau^{s_1+s_2}\,{}_{c_1}\mathrm{d}_q\tau \\
&\le \Psi(c_1)g(c_1)\int_0^1 (2^{1-s_1-s_2} - \tau^{s_1+s_2})\,{}_0\mathrm{d}_q\tau + \Psi(c_1)g(c_2)\int_0^1 \tau^{s_2}(2^{1-s_1} - \tau^{s_1})\,{}_0\mathrm{d}_q\tau \\
&\quad + \Psi(c_2)g(c_1)\int_0^1 \tau^{s_1}(2^{1-s_2} - \tau^{s_2})\,{}_0\mathrm{d}_q\tau + \Psi(c_2)g(c_2)\int_0^1 \tau^{s_1+s_2}\,{}_0\mathrm{d}_q\tau \\
&= \Psi(c_1)g(c_1)\left(2^{1-s_1-s_2} - \frac{1-q}{1-q^{s_1+s_2+1}}\right) \\
&\quad + \Psi(c_1)g(c_2)\left(2^{1-s_1}\frac{1-q}{1-q^{s_2+1}} - \frac{1-q}{1-q^{s_1+s_2+1}}\right) \\
&\quad + \Psi(c_2)g(c_1)\left(2^{1-s_2}\frac{1-q}{1-q^{s_1+1}} - \frac{1-q}{1-q^{s_1+s_2+1}}\right) + \Psi(c_2)g(c_2)\frac{1-q}{1-q^{s_1+s_2+1}}.
\end{aligned}$$

In addition,

$$\int_0^1 \Psi(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))g(c_1+\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))\,{}_0\mathrm{d}_q\tau$$
$$=\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)} \Psi(x)g(x)\,{}_{c_1}\mathrm{d}_q x.$$

This completes the proof. □

Before we present our next result, let us recall Condition 1, which was introduced by Noor and Noor [27].

Condition 1. *Assume that the function $\mathcal{R}_{\rho,\lambda,\sigma}(\cdot)$ satisfies the following condition:*

$$\mathcal{R}_{\rho,\lambda,\sigma}(\theta\mathcal{R}_{\rho,\lambda,\sigma}(v-u))=\theta\mathcal{R}_{\rho,\lambda,\sigma}(v-u),\quad \theta\in\mathbb{R}.$$

Theorem 6. *Let $\Psi:\mathcal{B}\to\mathbb{R}$ be an $\mathcal{R}_s$-convex function. If $\mathrm{h}:\mathcal{B}\to\mathbb{R}$ is a non-negative and integrable function on $\mathcal{B}$ and symmetric about $\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}$, where $\mathcal{R}_{\rho,\lambda,\sigma}(\cdot)$ satisfies Condition 1, then for $s\in(0,1]$ and $0<q<1$, we have*

$$\Psi\left(\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)} \mathrm{h}(x)\,{}_{c_1}\mathrm{d}_q x\le 2^{1-s}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)} \Psi(x)\mathrm{h}(x)\,{}_{c_1}\mathrm{d}_q x.$$

Proof. Using $\mathcal{R}_s$-convexity of Ψ, for every $x=c_1+\frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)$ and $y=c_1+\frac{1-\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)$ with $\tau=[-1,1]$, we have

$$\Psi\left(x+\frac{\mathcal{R}(y-x)}{2}\right)\le 2^{-s}\Psi(x)+(1-2^{-1})^s\Psi(y).$$

Using Condition 1, we obtain

$$\Psi\left(c_1+\frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)+\frac{\mathcal{R}_{\rho,\lambda,\sigma}(c_1+\frac{1-\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)-c_1-\frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))}{2}\right)$$
$$=\Psi\left(c_1+\frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)-\frac{\tau\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)$$
$$=\Psi\left(\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)$$
$$\le 2^{-s}\Psi\left(c_1+\frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\right)+(1-2^{-1})^s\Psi\left(c_1+\frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\right).$$

Multiplying both sides of the above inequality by $\mathrm{h}\left(c_1 + \frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right)$ and integrating with respect to τ on $[-1, 1]$, we obtain

$$\Psi\left(\frac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}\right)\int_{-1}^{1} \mathrm{h}\left(c_1 + \frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right) {}_{c_1}\mathrm{d}_q\tau$$
$$\leq 2^{-s}\int_{-1}^{1} \Psi\left(c_1 + \frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right)\mathrm{h}\left(c_1 + \frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right) {}_{c_1}\mathrm{d}_q\tau$$
$$+ (2^{1-s} - 2^{-s})\int_{-1}^{1} \Psi\left(c_1 + \frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right)\mathrm{h}\left(c_1 + \frac{1+\tau}{2}\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\right) {}_{c_1}\mathrm{d}_q\tau.$$

Since h is symmetric about $\frac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}$, we have

$$\Psi\left(\frac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}\right)\frac{2}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)} \mathrm{h}(x) \, {}_{c_1}\mathrm{d}_q x$$
$$\leq 2^{-s}\frac{2}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)} \Psi(x)\mathrm{h}(x) \, {}_{c_1}\mathrm{d}_q x$$
$$+ (2^{1-s} - 2^{-s})\frac{2}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)} \Psi(x)\mathrm{h}(c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1) - x) \, {}_{c_1}\mathrm{d}_q x$$
$$= 2^{-s}\frac{2}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)} \Psi(x)\mathrm{h}(x) \, {}_{c_1}\mathrm{d}_q x$$
$$+ (2^{1-s} - 2^{-s})\frac{2}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)} \Psi(x)\mathrm{h}(x) \, {}_{c_1}\mathrm{d}_q x$$
$$= 2\frac{2^{1-s}}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)} \Psi(x)\mathrm{h}(x) \, {}_{c_1}\mathrm{d}_q x.$$

This completes the proof. □

Corollary 4. *Taking* $\mathrm{h}(x) \equiv 1$ *and letting* $q \to 1^-$*, we have the left-hand side of Hermite–Hadamard's inequality for* $\mathcal{R}_s$*-convex functions.*

Remark 2. *Taking* $s = 1$ *in our results, we obtain quantum integral inequalities via generalized convex functions. Moreover, choosing* $\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1) = c_2 - c_1$*, we obtain quantum integral inequalities via s-convex functions. We omit their proofs here and the details are left to the interested reader.*

4. Applications

We now discuss some important special cases as applications of our main results.

Corollary 5. *In Theorem 3, taking* $\varepsilon = \frac{1}{6}$ *and* $\delta = \frac{5}{6}$*, we have*

$$\left|\frac{\Psi(c_1)}{6} + \frac{2\Psi\left(\frac{2c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)}{2}\right)}{3} + \frac{\Psi(c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1))}{6} - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \Psi(x)\, {}_{c_1}\mathrm{d}_q x\right|$$
$$\leq \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\left[|{}_{c_1}\mathrm{D}_q\Psi(c_1)|\left(2^{1-s}\left(B_1\left(\frac{1}{6};q\right) + B_2\left(\frac{5}{6};q\right)\right) - \left(B_3\left(s,\frac{1}{6};q\right) + B_4\left(s,\frac{5}{6};q\right)\right)\right)\right.$$
$$\left. + |{}_{c_1}\mathrm{D}_q\Psi(c_2)|\left(B_3\left(s,\frac{1}{6};q\right) + B_4\left(s,\frac{5}{6};q\right)\right)\right],$$

where

$$B_1\left(\frac{1}{6};q\right) := \int_0^{\frac{1}{2}} \left|q\tau - \frac{1}{6}\right| {}_0\mathrm{d}_q\tau = \begin{cases} \frac{6q-1}{36(1+q)}, & 0 \leq \frac{1}{6q} \leq \frac{1}{2}, \\ \frac{1-2q}{12(1+q)}, & \frac{1}{2} < \frac{1}{6q}, \end{cases}$$

$$B_2\left(\frac{5}{6};q\right) := \int_{\frac{1}{2}}^{1} \left|q\tau - \frac{5}{6}\right| {}_0\mathrm{d}_q\tau = \begin{cases} \frac{4q-5}{12(1+q)}, & 0 \leq \frac{5}{6q} \leq \frac{1}{2}, \\ \frac{5}{36(1+q)}, & \frac{1}{2} < \frac{5}{6q} \leq 1, \\ \frac{5-q}{12(1+q)}, & 1 < \frac{5}{6q}, \end{cases}$$

$$B_3\left(s,\frac{1}{6};q\right) := \int_0^{\frac{1}{2}} \tau^s\left|q\tau - \frac{1}{6}\right| {}_0\mathrm{d}_q\tau = \begin{cases} \frac{2(1-q)^2}{6^{s+2}(1-q^{s+1})(1-q^{s+2})} \\ + \frac{(1-q)(6q-1) + 5(q-1)q^{s+2}}{6.2^{s+2}(1-q^{s+1})(1-q^{s+2})}, & ,0 \leq \frac{1}{6q} \leq \frac{1}{2}, \\ -\frac{(1-q)(6q-1) + 5(q-1)q^{s+2}}{6.2^{s+2}(1-q^{s+1})(1-q^{s+2})}, & \frac{1}{2} < \frac{1}{6q}, \end{cases}$$

$$B_4\left(s,\frac{5}{6};q\right) := \int_{\frac{1}{2}}^{1} \tau^s\left|q\tau - \frac{5}{6}\right| {}_0\mathrm{d}_q\tau = \begin{cases} \frac{5(1-q)(1-2^{s+1})}{6.2^{s+1}(1-q^{s+1})} + \frac{q(1-q)(2^{s+2}-1)}{2^{s+2}(1-q^{s+2})}, & 0 \leq \frac{5}{6q} \leq \frac{1}{2}, \\ -\frac{5(1-q)(1+2^{s+1})}{6.2^{s+1}(1-q^{s+1})} + \frac{q(1-q)(1+2^{s+2})}{2^{s+2}(1-q^{s+2})} \\ + \frac{25^{s+2}(1-q)^2}{6^{s+2}(1-q^{s+1})(1-q^{s+2})}, & \frac{1}{2} < \frac{5}{6q} \leq 1, \\ -\frac{5(1-q)(1-2^{s+1})}{6.2^{s+1}(1-q^{s+1})} + \frac{q(1-q)(2^{s+2}-1)}{2^{s+2}(1-q^{s+2})}, & 1 < \frac{5}{6q}. \end{cases}$$

Corollary 6. *In Theorem 3, choosing* $\varepsilon = \delta = \dfrac{q}{1+q}$, *we obtain*

$$\begin{aligned}
&\left|\frac{q}{1+q}\Psi(c_1) + \frac{1}{1+q}\Psi(c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)) - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \Psi(x)\, {}_{c_1}\mathrm{d}_q x\right| \\
&\leq \mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)\bigg[|\,{}_{c_1}\mathrm{D}_q\Psi(c_1)|\bigg(2^{1-s}(C_1(q) + C_2(q)) - (C_3(s;q) + C_4(s;q))\bigg) \\
&\quad + |\,{}_{c_1}\mathrm{D}_q\Psi(c_2)|\bigg(C_3(s;q) + C_4(s;q)\bigg)\bigg],
\end{aligned}$$

where

$$C_1(q) := \int_0^{\frac{1}{2}} \left|q\tau - \frac{q}{1+q}\right| {}_0\mathrm{d}_q\tau = \begin{cases} \dfrac{8q^2 - q(1+q)^2}{4(1+q)^3}, & 0 \leq \dfrac{1}{1+q} \leq \dfrac{1}{2}, \\ \dfrac{q}{4(1+q)}, & \dfrac{1}{2} < \dfrac{1}{1+q}, \end{cases}$$

$$C_2(q) := \int_{\frac{1}{2}}^{1} \left|q\tau - \frac{q}{1+q}\right| {}_0\mathrm{d}_q\tau = \begin{cases} \dfrac{q}{4(1+q)}, & 0 \leq \dfrac{1}{1+q} \leq \dfrac{1}{2}, \\ \dfrac{5q^3 + 12q^2 - q}{4(1+q)^3}, & \dfrac{1}{2} < \dfrac{1}{1+q} \leq 1, \\ \dfrac{-q}{4(1+q)}, & 1 < \dfrac{1}{1+q}, \end{cases}$$

$$C_3(s;q) := \int_0^{\frac{1}{2}} \tau^s \left|q\tau - \frac{q}{1+q}\right| {}_0\mathrm{d}_q\tau = \begin{cases} \dfrac{2q^{s+2}(1-q)^2}{(1+q)^{s+2}(1-q^{s+1})(1-q^{s+2})} \\ + \dfrac{q^2(1-q) + q(q-1)q^{s+2}}{2^{s+2}(1+q)(1-q^{s+1})(1-q^{s+2})}, , 0 \leq \dfrac{1}{1+q} \leq \dfrac{1}{2}, \\ -\dfrac{q^2(1-q) + (q-1)q^{s+3}}{2^{s+2}(1+q)(1-q^{s+1})(1-q^{s+2})}, & \dfrac{1}{2} < \dfrac{1}{1+q}. \end{cases}$$

$$C_4(s;q) := \int_{\frac{1}{2}}^{1} \tau^s \left|q\tau - \frac{q}{1+q}\right| {}_0\mathrm{d}_q\tau = \begin{cases} \dfrac{q(1-q)(1-2^{s+1})}{2^{s+1}(1+q)(1-q^{s+1})} + \dfrac{q(1-q)(2^{s+2}-1)}{2^{s+2}(1-q^{s+2})}, & 0 \leq \dfrac{1}{1+q} \leq \dfrac{1}{2}, \\ -\dfrac{q(1-q)(1+2^{s+1})}{2^{s+1}(1+q)(1-q^{s+1})} + \dfrac{q(1-q)(1+2^{s+2})}{2^{s+2}(1-q^{s+2})} \\ + \dfrac{2q^{s+2}(1-q)^2}{(1+q)^{s+2}(1-q^{s+1})(1-q^{s+2})}, & \dfrac{1}{2} < \dfrac{1}{1+q} \leq 1, \\ -\dfrac{q(1-q)(1-2^{s+1})}{2^{s+1}(1+q)(1-q^{s+1})} + \dfrac{q(1-q)(2^{s+2}-1)}{2^{s+2}(1-q^{s+2})}, & 1 < \dfrac{1}{1+q}. \end{cases}$$

Corollary 7. *In Theorem 3, taking* $\varepsilon = \dfrac{1}{4}$ *and* $\delta = \dfrac{3}{4}$, *we obtain*

$$\left|\frac{\Psi(c_1)}{4}+\frac{\Psi\left(\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)}{2}+\frac{\Psi(c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))}{2}-\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(x)\,{}_{c_1}\mathrm{d}_q x\right|$$
$$\leq \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\Bigg[|{}_{c_1}\mathrm{D}_q\Psi(c_1)|\left(2^{1-s}\left(L_1\left(\frac{1}{4};q\right)+L_2\left(\frac{3}{4};q\right)\right)-\left(L_3\left(s,\frac{1}{4};q\right)+L_4\left(s,\frac{3}{4};q\right)\right)\right)$$
$$+|{}_{c_1}\mathrm{D}_q\Psi(c_2)|\left(L_3\left(s,\frac{1}{4};q\right)+L_4\left(s,\frac{3}{4};q\right)\right)\Bigg],$$

where

$$L_1\left(\frac{1}{4};q\right):=\int_0^{\frac{1}{2}}\left|q\tau-\frac{1}{4}\right|{}_0\mathrm{d}_q\tau=\begin{cases}\frac{q}{8(1+q)}, & 0\leq\frac{1}{4q}\leq\frac{1}{2},\\ \frac{1-q}{8(1+q)}, & \frac{1}{2}<\frac{1}{4q},\end{cases}$$

$$L_2\left(\frac{3}{4};q\right):=\int_{\frac{1}{2}}^{1}\left|q\tau-\frac{3}{4}\right|{}_0\mathrm{d}_q\tau=\begin{cases}\frac{5q-3}{8(1+q)}, & 0\leq\frac{3}{4q}\leq\frac{1}{2},\\ \frac{q}{8(1+q)}, & \frac{1}{2}<\frac{3}{4q}\leq 1,\\ \frac{3-3q}{8(1+q)}, & 1<\frac{3}{4q},\end{cases}$$

$$L_3\left(s,\frac{1}{4};q\right):=\int_0^{\frac{1}{2}}\tau^s\left|q\tau-\frac{1}{4}\right|{}_0\mathrm{d}_q\tau=\begin{cases}\frac{(1-q)^2}{2^{2s+3}(1-q^{s+1})(1-q^{s+2})}\\ +\frac{(1-q)+3(q-1)q^{s+2}}{2^{s+4}(1-q^{s+1})(1-q^{s+2})}, & 0\leq\frac{1}{4q}\leq\frac{1}{2},\\ -\frac{(1-q)+3(q-1)q^{s+2}}{2^{s+4}(1-q^{s+1})(1-q^{s+2})}, & \frac{1}{2}<\frac{1}{4q},\end{cases}$$

$$L_4\left(s,\frac{3}{4};q\right):=\int_{\frac{1}{2}}^{1}\tau^s\left|q\tau-\frac{3}{4}\right|{}_0\mathrm{d}_q\tau=\begin{cases}\frac{3(1-q)(1-2^{s+1})}{2^{s+3}(1-q^{s+1})}+\frac{q(1-q)(2^{s+2}-1)}{2^{s+2}(1-q^{s+2})}, & 0\leq\frac{3}{4q}\leq\frac{1}{2},\\ -\frac{3(1-q)(1+2^{s+1})}{2^{s+3}(1-q^{s+1})}+\frac{q(1-q)(1+2^{s+2})}{2^{s+2}(1-q^{s+2})}\\ +\frac{3^{s+2}(1-q)^2}{2^{2s+3}(1-q^{s+1})(1-q^{s+2})}, & \frac{1}{2}<\frac{3}{4q}\leq 1,\\ -\frac{3(1-q)(1-2^{s+1})}{2^{s+3}(1-q^{s+1})}+\frac{q(1-q)(2^{s+2}-1)}{2^{s+2}(1-q^{s+2})}, & 1<\frac{3}{4q}.\end{cases}$$

Corollary 8. *In Theorem 4, choosing* $\varepsilon=\frac{1}{6}$ *and* $\delta=\frac{5}{6}$*, we have*

$$\left|\frac{\Psi(c_1)}{6}+\frac{2\Psi\left(\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)}{3}+\frac{\Psi(c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))}{6}-\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(x)\,{}_{c_1}\mathrm{d}_q x\right|$$
$$\leq \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\left(K_1^{\frac{1}{p}}\left(\frac{1}{6},q;p\right)\left(|{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r(2^{-s}-K_2(s;q))+|{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r K_2(s;q)\right)^{\frac{1}{r}}\right.$$
$$\left.+K_3^{\frac{1}{p}}\left(\frac{5}{6},q;p\right)\left(|{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r(2^{-s}-K_4(s;q))+|{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r K_4(s;q)\right)^{\frac{1}{r}}\right).$$

Corollary 9. *In Theorem 4, taking* $\varepsilon=\delta=\frac{q}{1+q}$*, we obtain*

$$\left|\frac{q}{1+q}\Psi(c_1)+\frac{1}{1+q}\Psi(c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))-\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(x)\,{}_{c_1}\mathrm{d}_q x\right|$$
$$\leq \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\left(K_1^{\frac{1}{p}}(q;p)\left(|{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r(2^{-s}-K_2(s;q))+|{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r K_2(s;q)\right)^{\frac{1}{r}}\right.$$
$$\left.+K_3^{\frac{1}{p}}(q;p)\left(|{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r(2^{-s}-K_4(s;q))+|{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r K_4(s;q)\right)^{\frac{1}{r}}\right).$$

Corollary 10. *In Theorem 4, choosing* $\varepsilon=\frac{1}{4}$ *and* $\delta=\frac{3}{4}$*, we obtain*

$$\left|\frac{\Psi(c_1)}{4}+\frac{\Psi\left(\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)}{2}+\frac{\Psi(c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))}{2}-\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(x)\,{}_{c_1}\mathrm{d}_q x\right|$$
$$\leq \mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\left(K_1^{\frac{1}{p}}\left(\frac{1}{4},q;p\right)\left(|{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r(2^{-s}-K_2(s;q))+|{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r K_2(s;q)\right)^{\frac{1}{r}}\right.$$
$$\left.+K_3^{\frac{1}{p}}\left(\frac{3}{4},q;p\right)\left(|{}_{c_1}\mathrm{D}_q\Psi(c_1)|^r(2^{-s}-K_4(s;q))+|{}_{c_1}\mathrm{D}_q\Psi(c_2)|^r K_4(s;q)\right)^{\frac{1}{r}}\right).$$

We now discuss applications regarding absolute-value bounded functions of the results obtained from our main results. We suppose that the following condition is satisfied:

$$|{}_{c_1}\mathrm{D}_q\Psi|\leq\Delta,$$

and $0<q<1$ is a constant.

Applying the above condition, we have the following results.

Corollary 11. *Under the assumptions of Theorem 3, the following inequality holds:*

$$\begin{aligned}&\Bigg|\varepsilon\Psi(c_1)+(\delta-\varepsilon)\Psi\left(\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)+(1-\delta)\Psi(c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))\\&\quad-\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(x)\,{}_{c_1}\mathrm{d}_qx\Bigg|\\&\leq 2^{1-s}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\Delta(A_1(\varepsilon;q)+A_2(\delta;q)).\end{aligned}$$

Corollary 12. *Under the assumptions of Theorem 4, the following inequality holds:*

$$\begin{aligned}&\Bigg|\varepsilon\Psi(c_1)+(\delta-\varepsilon)\Psi\left(\frac{2c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}{2}\right)+(1-\delta)\Psi(c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1))\\&\quad-\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\int_{c_1}^{c_1+\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)}\Psi(x)\,{}_{c_1}\mathrm{d}_qx\Bigg|\\&\leq 2^{-\frac{s}{r}}\mathcal{R}_{\rho,\lambda,\sigma}(c_2-c_1)\Delta\left(K_1^{\frac{1}{p}}(\varepsilon,q;p)+K_3^{\frac{1}{p}}(\delta,q;p)\right).\end{aligned}$$

5. Conclusions

In this paper, we derive a new multi-parameter quantum integral identity. Applying this generic identity as an auxiliary result, we establish some new generalized quantum estimates of certain integral inequalities pertaining to the class of $\mathcal{R}_s$-convex functions. Furthermore, we obtain quantum integral inequalities for the product of $\mathcal{R}_{s_1}$- and $\mathcal{R}_{s_2}$-convex functions as well as another new quantum result for a function that satisfies Condition M. We also offer some applications of the obtained results for suitable choices of parameters included in the identity found. Finally, two results for $\mathcal{R}_s$-convex functions that are absolute-value bounded are given. In any case, we hope that these results continue to sharpen our understanding of the nature of quantum calculus and its huge applications in different fields. For future developments, we will derive several new and interesting inequalities via the Hölder–İşcan, Chebyshev, Markov, Young and Minkowski inequalities using quantum calculus for different classes of convex functions.

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