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Numerical Investigation of the Two-Dimensional Fredholm Integral Equations of the Second Kind by Bernstein Operators

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Abstract: In this study, the numerical solutions of linear two-dimensional Fredholm integral equations of the second kind via Bernstein operators are considered. The method is presented with illustrative examples for regularized-equal and Chebyshev collocation points. The obtained numerical results from illustrative examples show that the proposed numerical algorithm is accurate and efficient for solving linear two-dimensional Fredholm integral equation of the second kind.

Keywords: Fredholm integral equations; Bernstein approximation; Chebyshev collocation points

1. Introduction

Let $D := [0,1] \times [0,1]$. Two dimensional linear Fredholm integral equations of the second kind

$$\alpha(x,y)u(x,y) - \beta(x,y)\int_0^1\int_0^1 \kappa(x,y;s,t)u(s,t)dsdt = f(x,y), (x,y) \in D \quad (1)$$

is considered, where $\alpha(x,y)$, $\beta(x,y)$ and $f(x,y)$ are given non-zero continuos functions on D. $\kappa(x,y;s,t) \in C(D \times D)$ (i.e., continuous space on $D \times D$) is the kernel function, and $u(x,y)$ is the undetermined function. Fredholm and Volterra integral equations often appear in science and engineering problems. For example Fredholm integral equations arise in mathematical economics (see [1]), in fluid mechanics problems involving hydrodynamic interactions near finite-sized elastic interfaces (see [2,3]) and in physics; for the mass distribution of polymers in a polymeric melt [4]. Volterra integral equations are important for initial value problems of differential equations (see [5]). See also [6–8] for more applications of Fredholm and Volterra integral equations in mathematical physics. Obtaining the analytical solutions of integral equations is difficult, so the numerical solutions of integral equations are necessary.

There are some numerical methods for solving integral equations of the second kind, such as the Bernstein piecewise polynomial method, integral mean value method, Taylor series method and least square method as presented in [9] for one dimensional Fredholm integral equations. A simple numerical method based on Berstein's approximation for Volterra integral equation is presented in [10] for one dimension, and authors approximate unknown function by Bernstein's polynomial approximation. Bivariate generalized Bernstein operators are used to solve Fredholm integral equations in [11] with applications. Lagrange polynomial approximation, Barycentric Lagrange polynomial approximation, and modified Lagrange polynomial approximation were also used for numerical solutions of Volterra and Fredholm integral equations in [12] for one dimension, and in [13] for two dimensions. Cosine-trigonometric basis functions are developed to solve two dimensional Fredholm integral equations of the second kind, with accurate solution by [14]. Weakly singular kernel using spectral collocation is another technique for solving Fredholm integral equations of the second kind [15]. Modified Berstein–Kantorovich operators were

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Citation: Iyikal, O.C. Numerical Investigation of the Two-Dimensional Fredholm Integral Equations of the Second Kind by Bernstein Operators. *Symmetry* **2022**, *14*, 625. https://doi.org/10.3390/sym14030625

Academic Editor: Theodore E. Simos

Received: 25 February 2022
Accepted: 16 March 2022
Published: 21 March 2022

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used for numerical solutions of the Fredholm and Volterra integral equations, see [16]. Two-dimensional Volterra–Fredholm integral equations of the second kind are solved by Berstein operators, based on approximating unknown function with Bernstein polynomials [17]. Two dimensional Bernstein polynomial approximation method for mixed Fredholm-Volterra integral equations is presented in [18] with operational matrix. The method has several advantages in reducing computational burden with good accuracy. Moreover, Bernstein operators are useful to find the approximate solutions of the integral equations, see also [19–23].

This paper presents a two dimensional Bernstein polynomial approximation method with regularized-equal collocation points and with Chebyshev collocation points to solve two dimensional linear Fredholm integral equation of the second kind.

The sections of the paper are organized as follows: In Section 2, the two dimensional Bernstein polynomial approximation method is considered with regularized equally-spaced and with Chebyshev collocation points. A numerical approach based on the method given by [24] is developed for solving Fredholm integral equations of the second kind. In Section 3, the method is applied on some examples from literature. These numerical examples illustrate the efficiency and applicability of the method. The interpretations of the obtained results are given in Section 4.

2. Bernstein Polynomial Approximation Method

2.1. Two-Dimensional Bernstein Polynomial Approximation Method

Two dimesional Bernstein polynomials of degree $m \times n$ on the square $D := [0,1] \times [0,1]$ are

$$\begin{aligned} b_{(k,m),(p,n)}(x,y) &= \binom{m}{k}\binom{n}{p} x^k (1-x)^{m-k} y^j (1-y)^{n-p}, \ (x,y) \in D, \\ k &= 0,1,\ldots,m, \ p = 0,1,\ldots,n \in \mathbb{N}. \end{aligned} \tag{2}$$

Then, the Bernstein approximation $B_{m,n}(f(x,y))$ to a function $f(x,y) : D \to \mathbb{R}$ is the polynomial

$$B_{m,n}(f(x,y)) = \sum_{k=0}^{m} \sum_{p=0}^{n} f\left(\frac{k}{m}, \frac{p}{n}\right) b_{(k,m),(p,n)}(x,y). \tag{3}$$

See [25] for the properties of Bernstein polynomials on D and convergency for fractional integration and see [17] for the properties of Bernstein polynomials on D and convergency of this approximation method for Volterra–Fredholm integral equations.

2.2. Discretization of the Integral Equations by Bernstein's Approximation

Consider the following integral equation,

$$\alpha(x,y)u(x,y) - \beta(x,y) \int_0^1 \int_0^1 \kappa(x,y;s,t)u(s,t)dsdt = f(x,y), \ (x,y) \in D, \tag{4}$$

where, $\alpha(x,y)$, $\beta(x,y)$ and $f(x,y)$ are non-zero continuous functions on D. $\kappa(x,y;s,t) \in C\ (D \times D)$ is the kernel function and $u(x,y)$ is the undetermined function. The two dimensional Bernstein's approximation $B_{m,n}(u(x,y))$ of the $m \times n$ degree are defined on D as,

$$B_{m,n}(u(x,y)) = \sum_{k=0}^{m} \sum_{p=0}^{n} u\left(\frac{k}{m}, \frac{p}{n}\right) b_{(k,m),(p,n)}(x,y), \tag{5}$$

where,

$$\begin{aligned} b_{(k,m),(p,n)}(x,y) &= \binom{m}{k}\binom{n}{p} x^k (1-x)^{m-k} y^p (1-y)^{n-p}, \ (x,y) \in D, \\ k &= 0,1,\ldots,m, \ p = 0,1,\ldots,n \in \mathbb{N}. \end{aligned} \tag{6}$$

In order to find $u(\frac{k}{m}, \frac{p}{n})$, $k = 0, 1, \ldots, m$, $p = 0, 1, \ldots, n$ convert (4) into a linear form by using the collocation points x_i , $i = 0, 1, \ldots, m$, and y_j , $j = 0, 1, \ldots, n$ and approximating the unknown function $u(x, y)$ by (5). That is,

$$\begin{aligned} & \alpha(x_i, y_j) \sum_{k=0}^{m} \sum_{p=0}^{n} u(\frac{k}{m}, \frac{p}{n}) b_{(k,m),(p,n)}(x_i, y_j) \\ & -\beta(x_i, y_j) \int_0^1 \int_0^1 \kappa(x_i, y_j; s, t) \sum_{k=0}^{m} \sum_{p=0}^{n} u(\frac{k}{m}, \frac{p}{n}) b_{(k,m),(p,n)}(s, t) ds dt \\ = \; & f(x_i, y_j), (x_i, y_j) \in D. \end{aligned} \tag{7}$$

Choose collocation points x_i, $i = 0, 1, \ldots, m$ and y_j, $j = 0, 1, \ldots, n$ as regularized-equally spaced

$$x_i = \begin{cases} \varepsilon + \frac{i}{m}, \text{for } i \neq m \\ 1 - \varepsilon, \text{for } i = m \end{cases} \quad \text{and } y_j = \begin{cases} \varepsilon + \frac{j}{n}, \text{for } j \neq n \\ 1 - \varepsilon, \text{for } j = n \end{cases} \tag{8}$$

for arbitrary small ε to get rid of the undetermined computations as 0^0 in the Equation (7) [10]. On the other hand, in the practical calculation, the second Chebyshev collocation points (x_i, y_j), $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n$ are taken as;

$$x_i = \begin{cases} \frac{1-\cos\frac{i\pi}{m}}{2} + \varepsilon, \text{for } i \neq m, i \neq 0 \\ 1 - \varepsilon, \text{for } i = m \end{cases} \quad \text{and } y_j = \begin{cases} \frac{1-\cos\frac{j\pi}{n}}{2} + \varepsilon, \text{for } j \neq n, j \neq 0 \\ 1 - \varepsilon, \text{for } j = n \end{cases}. \tag{9}$$

Equivalently, (7) can be written in the matrix form

$$AX = Y, \tag{10}$$

where, the coefficient matrix $A \in \mathbb{R}^{(m+1)(n+1)\times(m+1)(n+1)}$ is

$$\begin{aligned} A &= \left[\alpha(x_i, y_j) b_{(k,m),(p,n)}(x_i, y_j) - \beta(x_i, y_j) \int_0^1 \int_0^1 \kappa(x_i, y_j; s, t) b_{(k,m),(p,n)}(s, t) ds dt \right] \\ i &= 0, 1, \ldots, m, j = 0, 1, \ldots, n, k = 0, 1, \ldots, m, p = 0, 1, \ldots, n, \end{aligned} \tag{11}$$

right side vector is

$$Y = \left[f(x_i, y_j) \right]^T, i = 0, 1, \ldots, m, j = 0, 1, \ldots, n,$$

and the unknown vector X given as

$$X = \left[u_{m,n}(\frac{k}{m}, \frac{p}{n}) \right]^T.$$

Algorithm 1: Numerical solution of the two dimensional Fredholm integral equation of the second kind by using Bernstein polynomial approximation $B_{m,n}(u(x,y))$, is obtained as follows:

STEP 1. Put the m and n values.
STEP 2. Set the collocation points x_i, $i = 0, 1, \ldots, m$ and y_j, $j = 0, 1, \ldots, n$, as in (8) or (9).
STEP 3. Use STEP 1 and STEP 2 by Equation (11) to find matrix A.
STEP 4. Calculate $Y = \left[f(x_i, y_j)\right]^T$, $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n$.
STEP 5. Solve the system (10) and denote the numerical solution by $u_{m,n}(\frac{k}{m}, \frac{p}{n})$.
STEP 6. Substitute $u_{m,n}(\frac{k}{m}, \frac{p}{n})$ in Equation (5) and compute $B_{m,n}(u_{m,n}(\frac{k}{m}, \frac{p}{n}))$.
STEP 7. Calculate the error function $E(x,y) = u(x,y) - B_{m,n}(u_{m,n}(x,y))$.

3. Numerical Results

In this numerical section, three test problems are used with the following errors. Let x_p, $p = 0, 1, \ldots, N_1$ and y_q, $q = 0, 1, \ldots, N_2$ be the selected points, where N_1 and N_2 are not necessarily equal to m and n, respectively. i.e., Error can be calculated for different number of selected points We define maximum and root mean square errors by

$$E(x,y) = u(x,y) - B_{m,n}(u_{m,n}(x,y)) \tag{12}$$

$$e_{p,q} = E(x_p, y_q) \tag{13}$$

$$e = \max_{p,q} |e_{p,q}| \tag{14}$$

$$RMSE = \sqrt{\frac{1}{(N_1+1)(N_2+1)} \sum_{q=0}^{N_2} \sum_{p=0}^{N_1} e_{p,q}^2} \tag{15}$$

respectively. We use the notation $cond(A)$ to present the condition number of matrix A. Mathematica in double precision is used, to solve the examples. The exact solutions are known and used to show that the numerical solutions obtained by Bernstein polynomial approximation is correct. Moreover, "GaussKronrodRule" is used for numerical integration.

Example 1. *Let $D := [0,1] \times [0,1]$. We consider the integral equation,*

$$u(x,y) = \sin(x+y) \int_0^1 \int_0^1 u(s,t)dtds + f(x,y), (x,y) \in D, \tag{16}$$

where $f(x,y) = [1 + \sin(2) - 2\sin(1)]\sin(x+y)$, has the exact solution $u(x,y) = \sin(x+y)$ [13]. The Table 1 presents the errors and $cond(A)$ in (11) of the linear system (10) by the given algorithm for regularized-equal collocation points in (8). The Table 2 presents the errors and $cond(A)$ in (11) of the linear system (10) by the given algorithm for Chebyshev collocation points in (9).

The values of the approximate solution $u_{m,n}(x_i, y_j)$ at Chebyshev collocation points (9) are more accurate than at regularized equal collocation points. Usually, accuracy of the method is higher by Chebyshev collocation points than regularized equi-spaced points. Analogous results when Lagrange and Brycentric Lagrange methods were used to approximate the solution of two dimensional Fredholm integral equation of the second kind were also obtained in [13].

The $cond(A)$ of the coefficient matrix in (11) for Example 1 is increasing when the values m and n are increasing. This leads to ill-conditioning of the coefficient matrix A. Preconditioning techniques can be used for reducing the condition numbers see [26–29]. For example, a particular class of regular splittings of symmetric M-matrices was used to

precondition the conjugate gradient (CG) method in [29]. Errors $|e_{p,q}|$ of Berstein polynomial approximation method for regularized-equal collocation points and for Chebyshev collocation points when $n = 4$ and $m = 4$ can be seen in Figures 1 and 2, respectively.

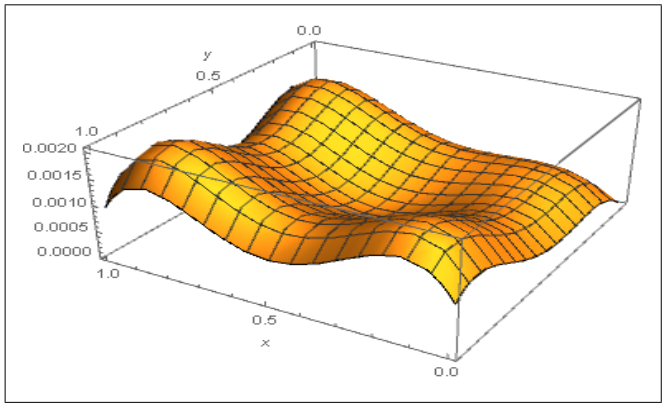

Figure 1. Error $|e_{p,q}|$ for the Bernstein method with regularized-equal collocation points for Example 1.

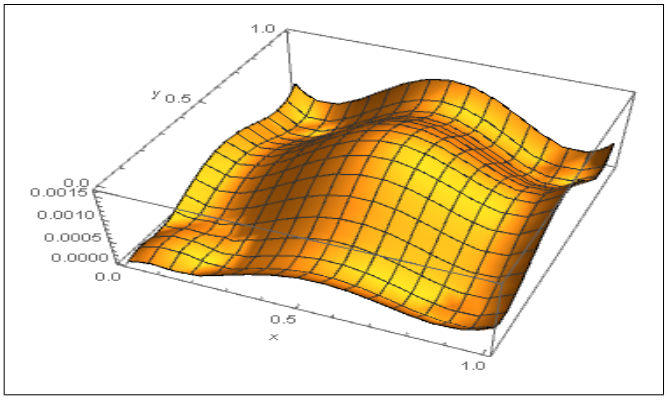

Figure 2. Error $|e_{p,q}|$ for the Bernstein method with Chebyshev collocation points for Example 1.

Example 2. *Let* $D_2 := [0, \pi/2] \times [0, 1]$. *Consider the following integral equation of the form*

$$e^{x+y}u(x,y) = x^2y \int_0^{\pi/2} \int_0^1 (x+y+s+t)u(s,t)dtds + f(x,y), (x,y) \in D_2, \tag{17}$$

where,

$$f(x,y) = (e-1)(1-e^{\pi/2})(x^3y + x^2y^2) + \left[(e-1)\left(e^{\pi/2} - \frac{e^{\pi/2}}{2}\pi - 1\right) + (1-e^{\pi/2})\right]x^2y + e^{2x+2y},$$

and the exact solution is $u(x,y) = e^{x+y}$ *[13].*

Remark 1. *Regularized equally- spaced collocation points* $x_i, i = 0, 1, \ldots, m$ *and* $y_j, j = 0, 1, \ldots, n$ *for Example 2 are taken as*

$$x_i = \begin{cases} \varepsilon + \frac{\pi i}{2m}, for\ i \neq m \\ \frac{\pi}{2} - \varepsilon, for\ i = m \end{cases} \quad and\ y_j = \begin{cases} \varepsilon + \frac{j}{n}, for\ j \neq n \\ 1 - \varepsilon, for\ j = n \end{cases}, \tag{18}$$

and Chebyshev collocation points are taken as,

$$x_i = \begin{cases} \frac{1}{2}(\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{i\pi}{m}) + \varepsilon, for\ i \neq m, i \neq 0 \\ \frac{\pi}{2} - \varepsilon, for\ i = m \end{cases} \quad and\ y_j = \begin{cases} \frac{1-\cos\frac{i\pi}{n}}{2} + \varepsilon, for\ j \neq n, j \neq 0 \\ 1 - \varepsilon, for\ j = n \end{cases}. \tag{19}$$

Table 1. RMSE, maximum errors and $cond(A)$ obtained by Bernstein polynomial approximation with regularized-equal collocation points for the Example 1.

	RMSE	e	$cond(A)$
$m = 4, n = 4$	8.38179×10^{-4}	1.06334×10^{-3}	57.8964
$m = 6, n = 6$	1.58843×10^{-6}	2.00316×10^{-6}	2094.98
$m = 8, n = 8$	2.12976×10^{-9}	2.67504×10^{-9}	83,023.7
$m = 10, n = 10$	2.03771×10^{-12}	2.55296×10^{-12}	3.51706×10^{6}
$m = 12, n = 12$	6.94695×10^{-15}	9.21485×10^{-15}	1.57479×10^{8}

Table 2. RMSE, maximum errors and $cond(A)$ obtained by Bernstein polynomial approximation with Chebyshev collocation points for the Example 1.

	RMSE	e	$cond(A)$
$m = 4, n = 4$	4.83519×10^{-4}	6.20168×10^{-4}	45.0771
$m = 6, n = 6$	1.47242×10^{-7}	1.88455×10^{-7}	825.869
$m = 8, n = 8$	5.78929×10^{-11}	7.39292×10^{-11}	14,001.9
$m = 10, n = 10$	2.49619×10^{-14}	3.25295×10^{-14}	231,568
$m = 12, n = 12$	5.89465×10^{-15}	7.77156×10^{-15}	3.82524×10^{6}

The Table 3 presents the errors and $cond(A)$ of (11) for Example 2 with the given method for regularized-equal collocation points in (8). The Table 4 presents the errors and $cond(A)$ of (11) for Example 2 with the given method for Chebyshev collocation points in (9).

Table 3. RMSE, maximum errors and $cond(A)$ matrix obtained by Bernstein polynomial approximation with regularized-equal collocation points for the Example 2.

	RMSE	e	$cond(A)$
$m = 4, n = 4$	3.5208×10^{-3}	8.13074×10^{-3}	319.544
$m = 8, n = 8$	4.33182×10^{-8}	1.05168×10^{-7}	370,059
$m = 10, n = 10$	9.91076×10^{-11}	2.42851×10^{-10}	1.52628×10^{7}
$m = 12, n = 12$	1.7907×10^{-13}	4.3876×10^{-13}	6.67461×10^{8}
$m = 15, n = 15$	5.44544×10^{-14}	1.35003×10^{-13}	2.14727×10^{11}

Usually, solving the integral Equation (17) by Bernstein polynomial approximation with Chebyshev collocation points in (9) give more accurate solutions than with regularized-equal collocation points. $cond(A)$ of (10) is increasing according to m and n values. For this example ill-conditioned matrix is obtained when the m and n values are increased. Errors $|e_{p,q}|$ of Berstein polynomial approximation method for regularized-equal collocation points and for Chebyshev collocation points when $n = 4$ and $m = 4$ can be seen in Figures 3 and 4, respectively.

Table 4. RMSE, maximum errors and $cond(A)$ matrix obtained by Bernstein polynomial approximation with Chebyshev collocation points for the Example 2.

	RMSE	e	$cond(A)$
$m = 4, n = 4$	1.92655×10^{-3}	4.33359×10^{-3}	313.093
$m = 8, n = 8$	1.24258×10^{-9}	2.85366×10^{-9}	81,233.1
$m = 10, n = 10$	9.61004×10^{-13}	2.21334×10^{-12}	1.30198×10^{6}
$m = 12, n = 12$	7.95505×10^{-15}	2.66454×10^{-14}	2.08914×10^{7}
$m = 15, n = 15$	8.71513×10^{-15}	2.4869×10^{-14}	1.33228×10^{9}

Example 3. *Let $D := [0,1] \times [0,1]$. We consider the following integral equation of the form*

$$u(x,y) = \int_0^1 \int_0^1 (e^{x+y} + s + t)u(s,t)dtds + f(x,y), (x,y) \in D, \tag{20}$$

where, $f(x,y) = -\frac{46}{63} - \frac{4e^{x+y}}{7} + x^{5/2} + y^{5/2}$, has the exact solution $u(x,y) = x^{5/2} + y^{5/2}$. The Table 5 presents the errors and $cond(A)$ of (11) of the linear system (10) of Example 3 with the Bernstein polynomial approximation method for regularized-equal collocation points. The errors and $cond(A)$ of (11) of the linear system of Example 3 by the Bernstein polynomial approximation method with Chebyshev collocation points (9) are given in Table 6.

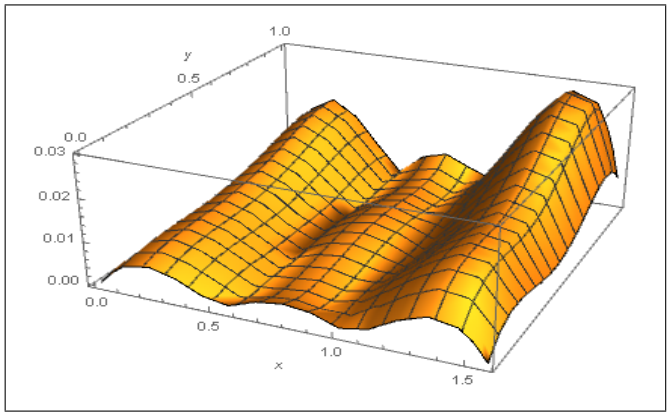

Figure 3. Error $|e_{p,q}|$ the Bernstein method with regularized-equal collocation points for Example 2.

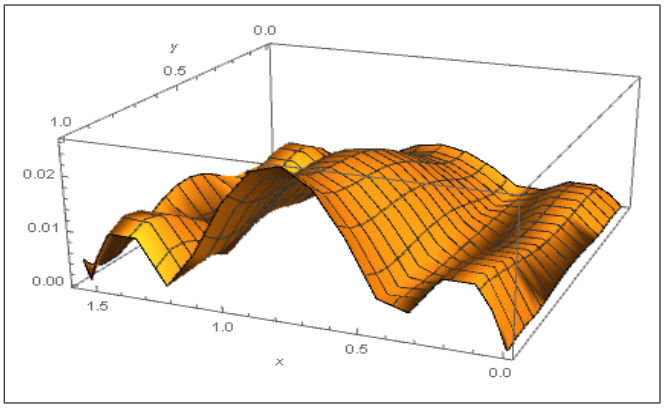

Figure 4. Error $|e_{p,q}|$ for the Bernstein method with Chebyshev collocation points for Example 2.

Table 5. RMSE, maximum errors and $cond(A)$ obtained by Bernstein polynomial approximation with regularized equal collocation points for the Example 3.

	RMSE	e	$cond(A)$
$m = 4, n = 4$	1.53888×10^{-3}	2.67654×10^{-3}	31.4408
$m = 8, n = 8$	1.34335×10^{-5}	2.23442×10^{-5}	405,765
$m = 10, n = 10$	3.77166×10^{-6}	6.17914×10^{-6}	1.73458×10^{7}
$m = 12, n = 12$	1.40399×10^{-6}	2.27531×10^{-6}	7.74265×10^{8}
$m = 13, n = 13$	7.40942×10^{-7}	1.09405×10^{-6}	5.5125×10^{9}
$m = 14, n = 14$	6.25117×10^{-7}	1.00499×10^{-6}	3.56596×10^{10}
$m = 15, n = 15$	3.67724×10^{-7}	5.48086×10^{-7}	2.54559×10^{11}

Table 6. RMSE, maximum errors and $cond(A)$ matrix obtained by Bernstein polynomial approximation with Chebyshev collocation points for the Example 3.

	RMSE	e	$cond(A)$
$m = 4, n = 4$	7.4401×10^{-4}	1.38474×10^{-3}	208.172
$m = 8, n = 8$	7.71115×10^{-7}	1.21018×10^{-6}	68,431.8
$m = 10, n = 10$	1.2929×10^{-7}	1.99855×10^{-7}	3.71914×10^{6}
$m = 12, n = 12$	3.25731×10^{-8}	4.99379×10^{-8}	1.88073×10^{7}
$m = 13, n = 13$	1.62656×10^{-8}	2.38047×10^{-8}	8.26725×10^{7}
$m = 14, n = 14$	1.06507×10^{-8}	1.6249×10^{-8}	3.07663×10^{8}
$m = 15, n = 15$	6.04148×10^{-9}	8.91422×10^{-9}	1.3338×10^{9}

Similarly, solving integral Equation (20) by Bernstein polynomial approximation method with Chebyshev collocation points give more accurate solutions than with regularized equally spaced collocatiton points. Errors $|e_{p,q}|$ of Berstein polynomial approximation method for regularized equal collocation points and for Chebyshev collocation points when $n = 5$ and $m = 5$ can be seen in Figures 5 and 6, respectively.

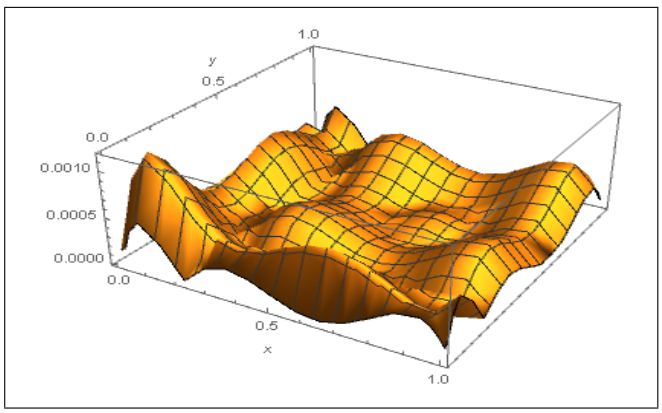

Figure 5. Error $|e_{p,q}|$ for the Bernstein method with regularized-equal collocation points for Example 3.

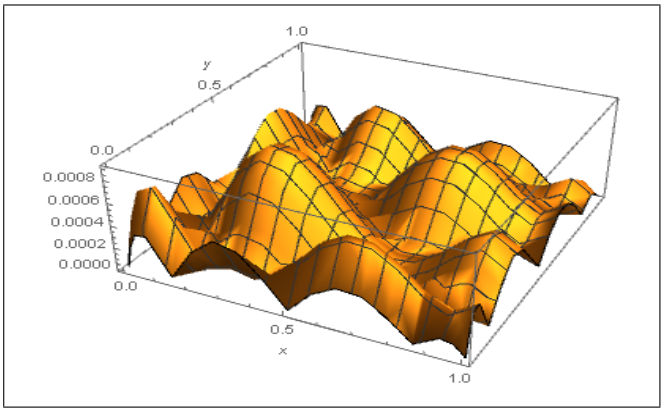

Figure 6. Error $|e_{p,q}|$ for the Bernstein method with Chebyshev collocation points for Example 3.

4. Conclusions

In this paper, two dimensional linear Fredholm integral equations of the second kind are solved by means of Bernstein polynomial approximation method. The considered approximation method with regularized equal collocation points and Chebyshev collocation points transforms the equations into a linear form of equations and numerical solutions are obtained. The numerical results indicate that the Bernstein polynomial approximation method is an accurate technique and can be applied to solve Fredholm integral equations of the second kind. Numerical results show that; when the collocation points are chosen as Chebyshev points, more stable results are obtained. The stability analysis of the Bernstein operators for solving other type of integral equations such as integro-differential equations and fractional integro-differential equations by using Chebyshev collocation points can be given in the further studies.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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