symmetry

MDPI

Article

Stability of the Equation of q-Wright Affine Functions in Non-Archimedean (n, β)-Banach Spaces

El-Sayed El-Hady [1,2,*,†] and Iz-iddine El-Fassi [3,†]

1 Mathematics Department, College of Science, Jouf University, Sakaka P.O. Box 2014, Saudi Arabia
2 Basic Science Department, Faculty of Computers and Informatics, Suez Canal University, Ismailia 41522, Egypt
3 Department of Mathematics, Faculty of Sciences and Techniques, S. M. Ben Abdellah University, Fez B.P. 2202, Morocco; izidd-math@hotmail.fr
* Correspondence: eaelhady@ju.edu.sa
† These authors contributed equally to this work.

Abstract: In this article, we employ a version of some fixed point theory (FPT) to obtain stability results for the symmetric functional equation (FE) of q-Wright affine functions in non-Archimedean (n, β)-Banach spaces (nArch(n, β)-BS). Furthermore, we give some interesting consequences of our results. In this way, we generalize several earlier outcomes.

Keywords: ulam stability; q-Wright affine functions; fixed point theorem; functional equation; non-Archimedean (n, β)-Banach spaces

MSC: 39B82; 39B05

check for updates

Citation: El-Hady, E.-S.; El-Fassi, I.-i. Stability of the Equation of q-Wright Affine Functions in Non-Archimedean (n, β)-Banach Spaces. *Symmetry* **2022**, *14*, 633. https://doi.org/10.3390/sym14040633

Academic Editor: Ioan Rașa

Received: 10 March 2022
Accepted: 20 March 2022
Published: 22 March 2022

1. Introduction

Stability of FEs in the sense of Ulam (see, e.g., [1–13]) plays an essential role in many applications. It provides close to exact solutions for many kinds of equations where the exact solutions are unreachable. An equation is called *stable*, in the sense of *Ulam* or *Ulam-Hyers*, provided, roughly speaking, that each function satisfying our equation approximately (in some sense) is near (in some way) to its exact solution.

The concept of the nearness of two functions can be obviously understood in various ways. Some of such ways are 2-norm, nArch norm, and n-norm. S. Gähler [14] in the mid 1960s seems to be among the first who developed the theory of 2-normed spaces. That of 2-BS was studied later by S. Gähler [15], see also [16,17]. nArch spaces have many important applications (see, e.g., [18–20]). In particular, they have applications in biology, economics, physics, and engineering (see [21] for more details).

The stability problem emerged as a consequence of the famous question asked by Ulam at a conference in Wisconsin University in the fall of 1940 (see [8]). The stability problem of Ulam can be rewritten as follows.

Given two groups G^* and a metric group (G^{**}, η), is it true that for $\varepsilon > 0$, there exists $\delta > 0$, such that, if $\mathcal{T} : G^* \to G^{**}$ satisfies

$$\eta(\mathcal{T}(t_1 t_2), \mathcal{T}(t_1)\mathcal{T}(t_2)) < \delta, \quad \text{for every } t_1, t_2 \in G^*,$$

then a homomorphism $\mathcal{M} : G^* \to G^{**}$ exists such that

$$\eta(\mathcal{T}(t_1), \mathcal{M}(t_1)) < \varepsilon$$

for all $t_1, t_2 \in G^*$?

In 1941, D.H. Hyers provided a positive answer to Ulam's question in case of BS. Since then, the stability problem has been known as the *Ulam-Hyers* or *Hyers-Ulam* stability problem. In 1950, Aoki (see [1]) generalized the result of Hyers for approximate additive mapping. In 1978, Rassias (see [10]) introduced a general form of the result of Hyers by investigating the stability in case of unbounded Cauchy differences. The famous result of Rassias can be rewritten as follows (see [10]):

Theorem 1. *Assume BS B_1, B_2, and a continuous mapping $\mathcal{F} : B_1 \to B_2$ from $\mathbb{R}$ into B_2. Suppose that there exists $c \geq 0$, and $\vartheta_1 \in [0,1)$, such that*

$$\|\mathcal{F}(b_1+b_2) - \mathcal{F}(b_1) - \mathcal{F}(b_2)\| \leq c(\|b_1\|^{\vartheta_1} + \|b_2\|^{\vartheta_1}),\ b_1, b_2 \in B_1 \setminus \{0\}. \tag{1}$$

Then, a unique solution $\mathcal{T} : B_1 \to B_2$ of the Cauchy FE exists with

$$\|\mathcal{F}(b_1) - \mathcal{T}(b_1)\| \leq \frac{2c\|b_1\|^{\vartheta_1}}{|2-2^{\vartheta_1}|},\ b_1 \in B_1 \setminus \{0\}. \tag{2}$$

The theorem above is known as the *Ulam-Hyers-Rassias* or the *Hyers-Ulam-Rassias* stability. Further interesting recent results in stability can be found in the following [22–25].

Fix a real number $0 < q < 1$. A function $H : I \to \mathbb{R}$ (with I some real nonempty interval) is $q-$Wright convex provided (see, e.g., [26–28])

$$H(q\nu_1 + (1-q)\nu_2) + H((1-q)\nu_1 + q\nu_2) \leq H(\nu_1) + H(\nu_2),\ \nu_1, \nu_2 \in I.$$

If H satisfies the FE

$$H(q\nu_1 + (1-q)\nu_2) + H((1-q)\nu_1 + q\nu_2) = H(\nu_1) + H(\nu_2), \tag{3}$$

then we say that it is q-Wright affine. Solutions of (3) are called the q-Wright affine functions, which are both q-Wright convex and concave see, e.g., [26–30]. Equation (3) is interesting because of the following.

- Equation (3) becomes the Jensen's FE (when $q = 1/2$)

$$H\left(\frac{\nu_1+\nu_2}{2}\right) = \frac{H(\nu_1) + H(\nu_2)}{2}.$$

- Equation (3) takes the form (when $q = 1/3$)

$$H(2\nu_1 + \nu_2) + H(\nu_1 + 2\nu_2) = H(3\nu_1) + H(3\nu_2),$$

  which has been studied in [31].

It should be remarked that the cases of more arbitrary q were studied in [27] (see also [32]). The stability of some classes of generalizing Equation (3) have been investigated in [33]. Note also that some interesting hyperstability results have been obtained by the first author in [34]. The first author investigated (3) in 2-BS (see [35]) and also in $(2,\alpha)$-BS (see [36]). As far as we know, stability results for (3) in nArch(n,β)-BS do not exist; so, the current article fills this gap. Moreover, the current results can be seen as a generalization of the results obtained in [35,36] in (n,β)-nArch spaces. Furthermore, our results are improvements and generalizations of the results obtained in [26] on nArch (n,β)-BS.

Throughout the paper, we denote the sets of all positive integers by $\mathbb{N}$, $\mathbb{R}$ the real numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = [0,\infty)$, and $\mathcal{B}^{\mathcal{A}}$ the family of all functions from a nonempty set $\mathcal{A}$ into a nonempty set $\mathcal{B}$. The article is organized as follows: In Section 2, we recall some basic notions and the main tool which is a version of an FPT. In Section 3, we apply the FPT to investigate the stability of (3) in nArch (n,β)-BS, and in Section 4 we introduce some consequences of our analysis.

2. Preliminaries

Here, we recall some basic notions concerning the nArch (n, β)-normed space. We start with the notion of an nArch field.

Definition 1. *By an nArch field, we mean a field $\mathcal{F}$ equipped with a function (valuation) $|\cdot|_* : \mathcal{F} \to [0, \infty)$: for all $k_1, k_2 \in \mathcal{F}$, the following hold:*

(C1) $|k_1|_* = 0 \Leftrightarrow k_1 = 0$;
(C2) $|k_1 k_2|_* = |k_1|_* |k_2|_*$;
(C3) $|k_1 + k_2|_* \le \max(|k_1|_*, |k_2|_*)$ *for every* $k_1, k_2 \in \mathcal{F}$.

The function $|\cdot|_*$ is called the trivial valuation, if $|k_1|_* = 1$, for all $k_1 \in \mathcal{F}$, $k_1 \neq 0$, and $|0|_* = 0$. Now, we recall the concept of an nArch space.

Definition 2. *Suppose that $\mathcal{V}$ is a vector space acting on some field $\mathcal{F}$ with an nArch nontrivial valuation $|\cdot|_*$. A function $\|\cdot\|_* : \mathcal{V} \to \mathbb{R}_+$ is called an nArch norm if:*

1. $\|v_1\|_* = 0 \Leftrightarrow v_1 = 0$;
2. $\|a v_1\|_* = |a|_* \|v_1\|_*$, *for all* $a \in \mathcal{F}$ *and* $v_1 \in \mathcal{V}$;
3. $\|v_1 + v_2\|_* \le \max(\|v_1\|_*, \|v_2\|_*)$, *for all* $v_1, v_2 \in \mathcal{V}$.

Then, $(\mathcal{V}, \|\cdot\|_*)$ is called an nArch space (or an ultrametric normed space). A $\{x_n\}$ is Cauchy in an nArch normed space, if and only if $\{x_{n+1} - x_n\}$ converges to zero in the space. In a complete nArch space, every Cauchy sequence is convergent.

For some examples of an nArch norm, the reader is advised to see, e.g., [9,37]. The following is the definition of the nArch (n, β)-norm.

Definition 3. *Assume a real vector space $\mathcal{X}$ with dimension at least n over some scalar field $\mathbb{F}$ with an nArch nontrivial valuation $|\cdot|_*$, $n \in \mathbb{N}$, $\beta \in (0, 1]$ is some fixed number. A function $\|\cdot, \dots, \cdot\|_{*,\beta} : \mathcal{X}^n \to \mathbb{R}_+$ is said to be an nArch (n, β)-norm on $\mathcal{X}$ $\Leftrightarrow$ it satisfies:*

(1) $\|\nu_1, u_2, \dots, u_n\|_{*,\beta} = 0 \Leftrightarrow \nu_1, u_2, \dots, u_n$ *are linearly dependent;*
(2) $\|\nu_1, u_2, \dots, u_n\|_{*,\beta}$ *is unchanged under permutations of* $\nu_1, u_2, \dots, u_n$;
(3) $\|\lambda \nu_1, u_2, \dots, u_n\|_{*\beta} = |\lambda|_*^\beta \|\nu_1, u_2, \dots, u_n\|_{*,\beta}$;
(4) $\|\nu_1 + \nu_2, u_2, \dots, u_n\|_{*,\beta} \le \max\{\|\nu_1, u_2, \dots, u_n\|_{*,\beta}, \|\nu_2, u_2, \dots, u_n\|_{*\beta}\}$

for all $\nu_1, \nu_2, u_2, \dots, u_n \in \mathcal{X}$, $\lambda \in \mathbb{K}$. Then, $(\mathcal{X}, \|\cdot, \dots, \cdot\|_{*,\beta})$ is called an nArch (n, β)-normed space.

A famous example of an nArch $(2, \beta)$-norm is given as follows.

Example 1. *Assume $\mathbb{K}$ is an nArch field with a valuation $|\cdot|_*$ that is nontrivial. For $n = 2$, $\lambda \in \mathbb{K}$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in X = \mathbb{K}^2$ with $x + y = (x_1 + y_1, x_2 + y_2)$, $\lambda x = (\lambda x_1, \lambda x_2)$, the nArch $(2, \beta)$-norm on X is defined by*

$$\|x, y\|_{*,\beta} = |x_1 y_2 - x_2 y_1|_*^\beta,$$

and $\beta \in (0, 1]$ is some fixed number. In general, it is not easy to find examples for any $n > 2$, see, e.g., [37,38].

It is now clear from the above definition that the nArch (n, β)-normed space is an nArch n-normed space if $\beta = 1$ and is an nArch β-normed space if $n = 1$, respectively. The following is an essential lemma (see [39]).

Lemma 1. *Let $(\mathcal{X}_1, \|\cdot, \dots, \cdot\|_{*,q})$ be an nArch (n, κ)-normed space, such that $n \ge 2$, $0 < \kappa \le 1$. Then,*

(L1) *if* $\nu_1 \in \mathcal{X}_1$, $\|\nu_1, u_2, \dots, u_n\|_{*,\kappa} = 0$ *for every* $u_2, \dots, u_n \in \mathcal{X}_1$, *then* $\nu_1 = 0$;

(L2) a sequence $\{x_k\}$ in an nArch (n,κ)-normed space $\mathcal{X}_1$ is a Cauchy sequence $\Leftrightarrow$ $\{x_{k+1}-x_k\}$ converges to zero in $\mathcal{X}_1$.

Proof. See [39]. □

Definition 4. *(a) The sequence $\{x_i\}, i \in \mathbb{N}$ in an nArch (n,r)-normed space $\mathcal{X}$ is said to be a convergent sequence if there exists an element $v_1 \in \mathcal{X}$: $\lim_{i\to\infty} \|x_i - v_1, u_2, \ldots, u_n\|_{*,r} = 0$ for all $u_2, \ldots, u_n \in \mathcal{X}$. In this case, we write $\lim_{i\to\infty} x_i := v_1$,*

$$\lim_{i\to\infty} \|x_i, u_2, \ldots, u_n\|_{*,r} = \|\lim_{i\to\infty} x_i, u_2, \ldots, u_n\|_{*,r}$$

for all $u_2, \ldots, u_n \in \mathcal{X}$.

(b) If every Cauchy sequence in the nArch (n,r)-normed space $\mathcal{X}$ converges, then $\mathcal{X}$ is called an nArch (n,r)-BS.

The following theorem is the basic tool in our analysis. It is a version of an FPT introduced by Brzdęk and Ciepliński in nArch metric spaces (see ([3], Theorem 1)).

Fixed Point Theorem

We use this section to recall the FPT ([3], Theorem 1) in nArch (n,β)-BS, see also other fixed point results [40,41]. For this purpose, we need the following hypothesis.

(A1) W is a nonempty set, $j, n \in \mathbb{N}$, X is a nArch (n,β)-BS.

(A2) $f_1, \ldots, f_j : W \to W$ and $K_1, \ldots, K_j : W \times X^{n-1} \to \mathbb{R}_+$ are given maps.

(A3) $\Lambda : {\mathbb{R}_+}^{W\times X^{n-1}} \to \mathbb{R}_+^{W\times X^{n-1}}$ is a non-decreasing operator defined by

$$(\Lambda\delta)(x, u_2, \ldots, u_n) := \max_{1\le i\le j} K_i(x, u_2, \ldots, u_n)\delta(f_i(x), u_2, \ldots, u_n)$$

for all $\delta \in \mathbb{R}_+^{W\times X^{n-1}}$, $(x, u_2, \ldots, u_n) \in W \times X^{n-1}$.

(A4) $\Gamma : X^W \to X^W$ is an operator that satisfies

$$\|\Gamma\xi(x) - \Gamma\mu(x), u_2, \ldots, u_n\|_{*,\beta} \le \max_{1\le i\le j}\{K_i(x, u_2, \ldots, u_n) \\ \|\xi(f_i(x)) - \mu(f_i(x)), u_2, \ldots, u_n\|_{*,\beta}\}$$

for all $\xi, \mu \in X^W$ and $(x, u_2, \ldots, u_n) \in W \times X^{n-1}$.

From the definition of the stability, one can deduce that stability implies the existence, and the converse is not always true. This means that stability guarantees the existence of a solution. The basic tool in our analysis is the following FPT (see [39]).

Theorem 2. *Let assumptions* **(A1)–(A4)** be satisfied. Consider functions $\varepsilon : W \times X^{n-1} \to \mathbb{R}_+$, $\varphi : W \to X$:

$$\left\|\Gamma\varphi(x) - \varphi(x), u_2, \ldots, u_n\right\|_{*,\beta} \le \varepsilon(x, u_2, \ldots, u_n), \ \ (x, u_2, \ldots, u_n) \in W \times X^{n-1}, \tag{4}$$

and

$$\lim_{m\to\infty} \Lambda^m \varepsilon(x, u_2, \ldots, u_n) = 0, \ \ (x, u_2, \ldots, u_n) \in W \times X^{n-1}. \tag{5}$$

Then, for all $x \in W$, the limit $\psi(x) := \lim_{m\to\infty}(\Gamma^m\varphi)(x)$ exists, the function $\psi \in X^W$ is a fixed point of Γ with

$$\|\varphi(x) - \psi(x), u_2, \ldots, u_n\|_{*,\beta} \le \sup_{m\in\mathbb{N}_0} (\Lambda^m\varepsilon)(x, u_2, \ldots, u_n) \tag{6}$$

for all $(x, u_2, \ldots, u_n) \in W \times X^{n-1}$. Moreover, if

$$\Lambda\big(\sup_{m\in\mathbb{N}_0}(\Lambda^m\varepsilon)\big)(x, u_2, \ldots, u_n) \leq \sup_{m\in\mathbb{N}_0}(\Lambda^{m+1}\varepsilon)(x, u_2, \ldots, u_n)$$

for all $(x, u_2, \ldots, u_n) \in W \times X^{n-1}$, then, ψ is the unique fixed point of Γ satisfying (6).

Proof. The proof is illustrated in [39]. □

Now, we present the stability of (3) in nArch (n, β)-BS.

3. Main Results

We assume that Y is a normed space over some field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and X is an nArch (n, β)-Banach space. The theorem below is our main theorem by which we show that, under certain conditions, functions that satisfy (3) approximately (in some sense) are close (in some way) to the exact solutions of (3) in nArch (n, β)-BS.

Theorem 3. *Let $H : Y \to X$, $\theta : Y \times Y \times X^{n-1} \to \mathbb{R}_+$ and $c : (0,1) \to \mathbb{R}_+$ be functions satisfying:*

$$\mathcal{M} := \{q \in (0,1) | a_q := \max\{c(q), c(1-q)\} < 1\} \neq \emptyset, \tag{7}$$

$$\theta\big(tv_1, tv_2, u_2, \ldots, u_n\big) \leq c(t)\theta\big(v_1, v_2, u_2, \ldots, u_n\big),\ t \in \{q, 1-q\}, n \in \mathbb{N} \tag{8}$$

and

$$\begin{aligned}\|H(qv_1 + (1-q)v_2) + H((1-q)v_1 + qv_2) - H(v_1) - H(v_2), u_2, \ldots, u_n\|_{*,\beta} \\ \leq \theta\big(v_1, v_2, u_2, \ldots, u_n\big)\end{aligned} \tag{9}$$

for all $v_1, v_2 \in Y$, $u_2, \ldots, u_n \in X$. Then, there exists a unique solution $F : Y \to X$ of (3):

$$\begin{aligned}\|H(v_1) - F(v_1), u_2, \ldots, u_n\|_{*,\beta} \leq \min\{\theta(v_1, 0, u_2, \ldots, u_n), \theta(0, v_1, u_2, \ldots, u_n)\}, \\ v_1 \in Y,\ \ u_2, \ldots, u_n \in X.\end{aligned} \tag{10}$$

Moreover, F is the unique solution of (3): there exists a constant $\mathcal{C} \in (0, \infty)$ with

$$\begin{aligned}\|H(v_1) - F(v_1), u_2, \ldots, u_n\|_{*,\beta} \leq \mathcal{C}\min\{\theta(v_1, 0, u_2, \ldots, u_n), \theta(0, v_1, u_2, \ldots, u_n)\}, \\ v_1 \in Y,\ \ u_2, \ldots, u_n \in X.\end{aligned} \tag{11}$$

Proof. Plugging into (9) first $v_2 = 0$ and next $v_1 = 0$, we obtain

$$\begin{cases}\|H(qv_1) + H((1-q)v_1) - H(v_1) - H(0), u_2, \ldots, u_n\|_{*,\beta} \leq \theta\big(v_1, 0, u_2, \ldots, u_n\big) \\ \|H((1-q)v_1) + H(qv_1) - H(0) - H(v_1), u_2, \ldots, u_n\|_{*,\beta} \leq \theta\big(0, v_1, u_2, \ldots, u_n\big)\end{cases} \tag{12}$$

for every $v_1 \in Y$, $u_2, u_3, \cdots, u_n \in X$. Then,

$$\begin{aligned}&\|H(qv_1) + H((1-q)v_1) - H(v_1) - H(0), u_2, \ldots, u_n\|_{*,\beta} \\ &\leq \min\{\theta\big(v_1, 0, u_2, \ldots, u_n\big), \theta\big(0, v_1, u_2, \ldots, u_n\big)\} \\ &=: \varepsilon(v_1, u_2, \ldots, u_n)\end{aligned} \tag{13}$$

for all $v_1 \in Y$, $u_2, \ldots, u_n \in X$. Introducing

$$h(v_1) = H(v_1) - H(0),\ v_1 \in Y;$$

then, (13) takes the form

$$\|h(qv_1) + h((1-q)v_1) - h(v_1), u_2, \ldots, u_n\|_{*,\beta} \leq \varepsilon(v_1, u_2, \ldots, u_n). \tag{14}$$

Now, a basic role in the proof is played by the operator $\Gamma : X^Y \to X^Y$ defined by

$$\Gamma\xi(\nu_1) := \xi(q\nu_1) + \xi((1-q)\nu_1), \quad \xi \in X^Y, \quad \nu_1 \in Y.$$

Then, inequality (13) takes the form

$$\|\Gamma h(\nu_1) - h(\nu_1), u_2, \ldots, u_n\|_{*,\beta} \leq \varepsilon(\nu_1, u_2, \ldots, u_n)$$

for all $\nu_1 \in Y$, $u_2, \ldots, u_n \in X$.

Now, let $\Lambda_q : \mathbb{R}_+^{Y\times X^{n-1}} \to \mathbb{R}_+^{Y\times X^{n-1}}$ be an operator of the form

$$\Lambda_q\delta(\nu_1, u_2, \ldots, u_n) = \max\{\delta(q\nu_1, u_2, \ldots, u_n), \delta((1-q)\nu_1, u_2, \ldots, u_n)\},$$

for all $\delta \in \mathbb{R}_+^{Y\times X^{n-1}}$, $\nu_1 \in Y$ and $u_2, \ldots, u_n \in X$; then, it is easily seen that, for each $q \in (0,1)$, the operator $\Lambda := \Lambda_q$ has the form (**A3**), with $j = 2$, $W = Y$, and

$$f_1(\nu_1) = q\nu_1, \qquad f_2(\nu_1) = (1-q)\nu_1, \qquad K_1(\nu_1, u_2, \ldots, u_n) = K_2(\nu_1, u_2, \ldots, u_n) = 1$$

for all $\nu_1 \in Y$ and $u_2, \ldots, u_n \in X$.

Moreover, for all $\xi, \mu \in X^Y$, $\nu_1 \in Y$, $u_2, \ldots, u_n \in X$, we obtain

$$\begin{aligned}
&\left\|\Gamma\xi(x) - \Gamma\mu(\nu_1), u_2, \ldots, u_n\right\|_{*,\beta} \\
&\quad = \|\xi(q\nu_1) - \xi((1-q)\nu_1) - \mu(q\nu_1) + \mu((1-q)\nu_1), u_2, \ldots, u_n\|_{*,\beta} \\
&\quad \leq \max\Big\{\left\|\xi(f_1(\nu_1)) - \mu(f_1(\nu_1)), u_2, \ldots, u_n\right\|_{*,\beta}, \left\|\xi(f_2(\nu_1)) - \mu(f_2(\nu_1)), u_2, \ldots, u_n\right\|_{*,\beta}\Big\} \\
&\quad = \max_{1\leq i\leq 2} K_i(\nu_1, u_2, \ldots, u_n)\left\|(\xi - \mu)(f_i(\nu_1)), u_2, \ldots, u_n\right\|_{*,\beta}
\end{aligned}$$

where $(\xi - \mu)(\nu_1) \equiv \xi(\nu_1) - \mu(x)$. So, **(A4)** is valid for Γ. Note that, from (8), and employing the definition of Λ_q we obtain

$$\begin{aligned}
\Lambda_q\varepsilon(\nu_1, u_2, \ldots, u_n) &= \max\{\varepsilon(q\nu_1, u_2, \ldots, u_n), \varepsilon((1-q)\nu_1, u_2, \ldots, u_n)\} \\
&= \max\{\min\{\theta(q\nu_1, 0, u_2, \cdots, u_n), \theta(0, q\nu_1, u_2, \cdots, u_n)\}, \\
&\quad \min\{\theta((1-q)\nu_1, 0, u_2, \cdots, u_n), \theta(0, (1-q)\nu_1, u_2, \cdots, u_n)\}\} \\
&\quad \leq \max\{c(q)\varepsilon(\nu_1, u_2, \ldots, u_n); c(1-q)\varepsilon(\nu_1, u_2, \ldots, u_n)\} \\
&\qquad = a_q\varepsilon(\nu_1, u_2, \ldots, u_n), \nu_1 \in Y, u_2, u_3, \cdots, u_n \in X.
\end{aligned} \tag{15}$$

By using induction, we will prove that for all $\nu_1 \in Y$, $u_2, \ldots, u_n \in X$, we have

$$\Lambda_q^\ell\varepsilon(q\nu_1, u_2, \ldots, u_n) \leq a_q^\ell\varepsilon(\nu_1, u_2, \ldots, u_n) \tag{16}$$

for all $\ell \in \mathbb{N}$, $q \in \mathcal{M}$. From (15), (16) holds when $\ell = 1$. Next, assume that (16) holds for $\ell = r$, $r \in \mathbb{N}$. Then, by the definition of Λ_q and (15), we have

$$\begin{aligned}
\Lambda_q^{r+1}\varepsilon(\nu_1, u_2, \ldots, u_n) =& \Lambda_q\left(\Lambda_q^r\varepsilon(\nu_1, u_2, \ldots, u_n)\right) \\
&= \max\Big\{\Lambda_q^r\varepsilon(q\nu_1, u_2, \ldots, u_n), \Lambda_q^r\varepsilon((1-q)\nu_1, u_2, \ldots, u_n)\Big\} \\
&\leq a_q^r \max\{\varepsilon(q\nu_1, u_2, \ldots, u_n), \varepsilon((1-q)\nu_1, u_2, \ldots, u_n)\} \\
&= a_q^r\Lambda_q\varepsilon(q\nu_1, u_2, \ldots, u_n). \\
&\quad \leq a_q^{r+1}\varepsilon(\nu_1, u_2, \ldots, u_n), \nu_1 \in Y, u_2, \cdots, u_n \in X.
\end{aligned}$$

This proves (16) when $\ell = r + 1$.

Now, we can conclude that (16) holds for all $\ell \in \mathbb{N}$. Therefore, by (16), we obtain

$$\lim_{\ell\to\infty} \Lambda_q^{\ell}\varepsilon(v_1, u_2, \dots, u_n) = 0$$

for all $v_1 \in Y,\ u_2, \cdots, u_n \in X$ and for some $q \in \mathcal{M}$. Further, for every $\ell \in \mathbb{N}_0$, $q \in \mathcal{M}$, $v_1 \in Y$ and $u_2, \dots, u_n \in X$, we have

$$\sup_{\ell\in\mathbb{N}_0} \Lambda_q^{\ell}\varepsilon(v_1, u_2, \dots, u_n) = \varepsilon(v_1, u_2, \dots, u_n),$$

and

$$\sup_{\ell\in\mathbb{N}_0} \Lambda_q^{\ell+1}\varepsilon(v_1, u_2, \dots, u_n) = \Lambda_q\varepsilon(v_1, u_2, \dots, u_n).$$

In view of Theorem 2 (with $W = Y$ and $\varphi = f$), for every $q \in \mathcal{M}$, the mapping $B : Y \to X$, given by $B(v_1) = \lim_{\ell\to\infty} \Gamma^{\ell} g(v_1)$ for $v_1 \in Y$, is a fixed point of Γ, i.e.,

$$B(v_1) = B(qv_1) + B((1-q)v_1), \quad v_1 \in Y.$$

Moreover,

$$\left\|h(v_1) - B(v_1), u_2, \dots, u_n\right\|_{*,\beta} \le \sup_{\ell\in\mathbb{N}_0} \Lambda^{\ell}\varepsilon(v_1, u_2, \dots, u_n)$$

for all $v_1 \in Y$, $u_2, \dots, u_n \in X$.

Next, we prove

$$\begin{aligned}\Big\|\Gamma^{\ell}h(qv_1 + (1-q)v_2) + \Gamma^{\ell}h((1-q)v_1 + qv_2) - \Gamma^{\ell}h(v_1) \ - \Gamma^{\ell}h(v_2), u_2, \dots, u_n\Big\|_{*,\beta} \\ \le a_q^{\ell}\theta(v_1, v_2, u_2, \dots, u_n)\end{aligned} \tag{17}$$

for every $v_1, v_2 \in Y$, $u_2, \dots, u_n \in X$, $\ell \in \mathbb{N}_0$, and $q \in \mathcal{M}$.

Clearly, if $\ell = 0$, then (17) is simply (9). So, fix $\ell \in \mathbb{N}_0$, and suppose that (17) holds for n, every $v_1, y \in Y$, and $u_2, \dots, u_n \in X$. Then, for every $v_1, v_2 \in Y$ and $u_2, \dots, u_n \in X$,

$$\begin{aligned}&\Big\|\Gamma^{\ell+1}h(qv_1 + (1-q)v_2) + \Gamma^{\ell+1}h((1-q)v_1 + qv_2) - \Gamma^{\ell+1}h(v_1) - \Gamma^{\ell+1}h(v_2), u_2, \dots, u_n\Big\|_{*,\beta} \\ &= \Big\|\Gamma^{\ell}h\Big(q(qv_1 + (1-q)v_2)\Big) + \Gamma^{\ell}h\Big((1-q)(qv_1 + (1-q)v_2)\Big) \\ &+ \Gamma^{\ell}g\Big(q((1-q)v_1 + qv_2)\Big) + \Gamma^{\ell}h\Big((1-q)((1-q)v_1 + qv_2)\Big) \\ &\quad + \Gamma^{\ell}h(qv_1) - \Gamma^{\ell}h((1-q)v_1) - \Gamma^{\ell}h(qv_2) - \Gamma^{\ell}h(1-q)v_2), u_2, \dots, u_n\Big\|_{*,\beta} \\ &\le \max\Big\{\Big\|\Gamma^{\ell}h\Big(q(qv_1 + (1-q)v_2)\Big) + \Gamma^{\ell}h\Big(q((1-q)v_1 + qv_2)\Big) - \Gamma^{\ell}h\Big(qv_1\Big) \\ &\qquad - \Gamma^{\ell}h\Big(qv_2\Big), u_2, \dots, u_n\Big\|_{*,\beta}, \Big\|\Gamma^{\ell}h\Big((1-q)(qv_1 + (1-q)v_2)\Big) \\ &\quad - \Gamma^{\ell}h\Big((1-q)(qv_1 + (1-q)v_2\Big) - \Gamma^{\ell}h((1-q)v_1) - \Gamma^{\ell}h((1-q)v_2), u_2, \dots, u_n\Big\|_{*,\beta}\Big\} \\ &\le \max\Big\{a_q^{\ell}\theta(qv_1, qv_2, u_2, \dots, u_n), a_q^{\ell}\theta((1-q)v_1, (1-q)v_2, u_2, \dots, u_n)\Big\} \\ &\le a_q^{\ell+1}\theta(v_1, v_2, u_2, \dots, u_n).\end{aligned}$$

Thus, by induction, we prove that (17) holds for every $v_1, v_2 \in Y$, $u_2, \dots, u_n \in X$, for all $\ell \in \mathbb{N}_0$. Now, letting $\ell \to \infty$ in (17), we obtain

$$B(qv_1 + (1-q)v_2) + B((1-q)v_1 + qv_2) = B(v_1) + B(v_2), \ \ v_1, v_2 \in Y. \tag{18}$$

So, we have proved that the existence of $F : Y \to X$ satisfies (3) for $\nu_1, \nu_2 \in Y$, such that

$$\|h(\nu_1) - B(\nu_1), u_2, \ldots, u_n\|_{*,\beta} \le \sup_{\ell \in \mathbb{N}_0} \Lambda^{\ell} \varepsilon(\nu_1, u_2, \ldots, u_n) = \varepsilon(\nu_1, u_2, \ldots, u_n) \tag{19}$$

for all $\nu_1 \in Y, u_2, \ldots, u_n \in X$.

Write $F(\nu_1) := B(\nu_1) + H(0)$. Then (10) holds, and

$$F(q\nu_1 + (1-q)\nu_2) + F((1-q)\nu_1 + q\nu_2) = F(\nu_1) + F(\nu_2), \tag{20}$$

for all $\nu_1, \nu_2 \in Y$. It remains to prove the uniqueness of F. So, let $\mathcal{C} \in (0, \infty)$, and $F' : Y \to X$ be a solution of (3) with

$$\begin{aligned} &\|f(\nu_1) - F'(\nu_1), u_2, \ldots, u_n\|_{*,\beta} \le \mathcal{C}\varepsilon(\nu_1, u_2, \ldots, u_n) \\ &\nu_1 \in Y, u_2, \cdots, u_n \in X. \end{aligned} \tag{21}$$

Then,

$$\begin{cases} F(q\nu_1) + F((1-q)\nu_1) = F(\nu_1) + F(0), \ \nu_1 \in Y \\ F'(q\nu_1) + F'((1-q)\nu_1) = F'(\nu_1) + F'(0), \ \nu_1 \in Y, \end{cases} \tag{22}$$

and by (10),

$$\begin{aligned} &\|F(\nu_1) - F'(\nu_1), u_2, \ldots, u_n\|_{*,\beta} \\ &\le \max\{\|F(\nu_1) - f(\nu_1), u_2, \ldots, u_n\|_{*,\beta}; \|f(\nu_1) - F'(\nu_1), u_2, \ldots, u_n\|_{*,\beta}\} \\ &\le \max\{\varepsilon(\nu_1, u_2, \ldots, u_n), \mathcal{C}\varepsilon(\nu_1, u_2, \ldots, u_n)\} \\ &= \max\{1, \mathcal{C}\}\varepsilon(\nu_1, u_2, \ldots, u_n), \end{aligned} \tag{23}$$

for all $\nu_1 \in Y$, and $u_2, \ldots, u_n \in X$. Further, by (8), $\theta(0, 0, u_2, \ldots, u_n) = 0$ for all $u_2, \ldots, u_n \in X$; so, $\varepsilon(0, u_2, \ldots, u_n) = 0$ for all $u_2, \ldots, u_n \in X$. This and (23) yield

$$F(0) = F'(0) = H(0). \tag{24}$$

Now, we show that, for each $j \in \mathbb{N}_0$,

$$\|F(\nu_1) - F'(\nu_1), u_2, \ldots, u_n\|_{*,\beta} \le \max\{1, \mathcal{C}\}\varepsilon(\nu_1, u_2, \ldots, u_n) \sup_{l \ge j}(a_q^l), \tag{25}$$

for all $\nu_1 \in Y, u_2, \ldots, u_n \in X$ and for some $q \in \mathcal{M}$. The case $j = 0$ is just (23). So fix $m \in \mathbb{N}_0$, and assume (25) holds for $j = m$. Then, from (22), (24), and (15), we obtain

$$\begin{aligned} &\|F(\nu_1) - F'(\nu_1), u_2, \ldots, u_n\|_{*,\beta} \\ &= \|F(q\nu_1) + F((1-q)\nu_1) - F(0) - F'(q\nu_1) - F'((1-q)\nu_1) - F'(0), u_2, \ldots, u_n\|_{*,\beta} \\ &\le \max\{\|F(q\nu_1) - F'(q\nu_1), u_2, \ldots, u_n\|_{*,\beta}, \|F((1-q)\nu_1) - F'((1-q)\nu_1), u_2, \ldots, u_n\|_{*,\beta}\} \\ &\le \max\{\max\{1, \mathcal{C}\}\varepsilon(q\nu_1, u_2, \ldots, u_n) \sup_{l \ge m}(a_q^l), \max\{1, \mathcal{C}\}\varepsilon((1-q)\nu_1, u_2, \ldots, u_n) \sup_{l \ge m}(a_q^l)\} \\ &= \max\{1, \mathcal{C}\} \sup_{l \ge m}(a_q^l) \max\{\varepsilon(q\nu_1, u_2, \ldots, u_n), \varepsilon((1-q)\nu_1, u_2, \ldots, u_n)\} \\ &\le \max\{1, \mathcal{C}\} \sup_{l \ge m}(a_q^l) a_q \, \varepsilon(\nu_1, u_2, \ldots, u_n), \ \nu_1 \in Y, u_2, \cdots, u_n \in X \\ &= \max\{1, \mathcal{C}\} \sup_{l \ge m+1}(a_q^l) \, \varepsilon(\nu_1, u_2, \ldots, u_n), \ \nu_1 \in Y, u_2, \cdots, u_n \in X. \end{aligned}$$

Thus, we have shown (25). So, letting $j \to \infty$ in (25), we obtain $F = F'$. □

4. Some Consequences

Using Theorem 3, we provide four natural examples of functions θ, c satisfying (7) and (8). Namely,

(i) $\theta\big(\nu_1,\nu_2,u_2,\ldots,u_n\big) := \epsilon\|\nu_1\|^r\|\nu_2\|^s\|z,u_2,\cdots,u_n\|_{*,\beta}$,

(ii) $\theta\big(\nu_1,\nu_2,u_2,\ldots,u_n\big) := \epsilon\Big(\max\{\|\nu_1\|^r,\|\nu_2\|^s\}\Big)^w\|z,u_2,\cdots,u_n\|_{*,\beta}$,

(iii) $\theta\big(\nu_1,\nu_2,u_2,\ldots,u_n\big) := \epsilon\Big(\|\nu_1\|^r\|\nu_2\|^s+\|\nu_1\|^{r+s}+\|\nu_2\|^{r+s}\Big)\times\|z,u_2,\cdots,u_n\|_{*,\beta}$,

(iv) $\theta\big(\nu_1,\nu_2,u_2,\ldots,u_n\big) := \epsilon\Big(\alpha_1\|\nu_1\|^r+\alpha_2\|\nu_2\|^s\Big)^w\times\|z,u_2,\cdots,u_n\|_{*,\beta}$,

for every $\nu_1,\nu_2\in Y$, $u_2,\cdots,u_n\in X$, for some $z\in X$, and $\alpha_i,\epsilon,r,s,w\in(0,\infty)$, $i=1,2$.

Corollary 1. *Assume a normed space Y, an nArch $(n,\beta)-BS$ X, and let $\epsilon,r,s\in(0,\infty)$, $q\in(0,1)$. If $H:Y\to X$ satisfies*

$$\|H(q\nu_1+(1-q)\nu_2)+H((1-q)\nu_1+q\nu_2)-H(\nu_1)-H(\nu_2)\|\le\epsilon\|\nu_1\|^r\|\nu_2\|^s\|z,u_2,\cdots,u_n\|_{*,\beta},$$

for every $\nu_1,\nu_2\in Y$, $u_2,\cdots,u_n\in X$, for some arbitrary element $z\in X$; then, H is a solution of (3) on Y.

Proof. Assume that

$$\theta(\nu_1,\nu_2,u_2,\ldots,u_n) := \epsilon\|\nu_1\|^r\|\nu_2\|^s\|u_2,\ldots,u_n\|_{*,\beta},$$

and $c(t)=t^{r+s}$ in Theorem 3 for every $\nu_1,\nu_2\in Y$, $u_2,\ldots,u_n\in X$ and for some $z\in X$, where $t\in(0,1)$, and $\epsilon,r,s\in(0,\infty)$. So, conditions (7) and (8) are valid, i.e., $a_q=\max\{q^{r+s},(1-q)^{r+s}\}<1$, and $\theta(t\nu_1,t\nu_2,u_2,\ldots,u_n)=c(t)\theta(\nu_1,\nu_2,u_2,\ldots,u_n)$ for every $\nu_1,\nu_2\in Y$, and $u_2,\ldots,u_n\in X$, where $t\in\{q,1-q\}$. Furthermore, we have

$$\min\{\theta(0,\nu_1,u_2,\ldots,u_n),\theta(\nu_1,0,u_2,\ldots,u_n)\}=0.$$

So that, by Theorem 3, we obtain $H(\nu_1)=F(\nu_1)$ for every $\nu_1\in Y$, i.e., H is a solution of (3) on Y. □

Corollary 2. *Suppose a normed space Y, an nArch $(n,\beta)-BS$ X, and let $\epsilon,r,s\in(0,\infty)$, and $q\in(0,1)$. If $H:Y\to X$ satisfies*

$$\begin{aligned}&\|H(q\nu_1+(1-q)\nu_2)+H((1-q)\nu_1+q\nu_2)-H(\nu_1)-H(\nu_2)\|_{*,\beta}\\&\qquad\le\epsilon\Big(\|\nu_1\|^r\|\nu_2\|^s+\|\nu_1\|^{r+s}+\|\nu_2\|^{r+s}\Big)\|z,u_2,\ldots,u_n\|_{*,\beta},\end{aligned}$$

for every $\nu_1,\nu_2\in Y$, $u_2,\ldots,u_n\in X$ and for some $z\in X$; then, there exists a unique solution $F:Y\to X$ of (3):

$$\|H(\nu_1)-F(\nu_1),u_2,\ldots,u_n\|_{*,\beta}\le\epsilon\|\nu_1\|^{r+s}\|z,u_2,\cdots,u_n\|_{*,\beta}$$

for every $\nu_1\in Y$, $u_2,\ldots,u_n\in X$ and for some $z\in X$.

Proof. Let

$$\theta(\nu_1,\nu_2,u_2,\ldots,u_n) := \epsilon\Big(\|\nu_1\|^r\|\nu_2\|^s+\|\nu_1\|^{r+s}+\|\nu_2\|^{r+s}\Big)\|z,u_2,\ldots,u_n\|_{*,\beta}$$

and $c(t)=t^{r+s}$ in Theorem 3 for every $\nu_1,\nu_2\in Y$, $u_2,\ldots,u_n\in X$ and for some $z\in X$, where $t\in(0,1)$ and $\epsilon,r,s\in(0,\infty)$. Then, (7) and (8) are valid, i.e., $a_q=\max\{q^{r+s},(1-q)^{r+s}\}<1$, and $\theta(tv_1, tv_2, u_2, \ldots, u_n) = c(t)\theta(v_1, v_2, u_2, \cdots, u_n)$ for all $v_1, v_2 \in Y$ and $u_2, \ldots, u_n \in X$, where $t \in \{q, 1-q\}$. In addition, we have

$$\min\{\theta(0, v_1, u_2, \ldots, u_n), \theta(v_1, 0, u_2, \ldots, u_n)\} = \epsilon \|v_1\|^{r+s} \|z, u_2, \ldots, u_n\|_{*,\beta};$$

so, by Theorem 3, we obtain the desired results. □

Corollary 3. *Suppose a normed space Y, X an nArch $(n, \beta)-BS$, and let $\epsilon, r, s, w \in (0, \infty)$, $q \in (0, 1)$. If $H : Y \to X$ satisfies*

$$\begin{aligned} \|H(qv_1 + (1-q)v_2) \ + H((1-q)v_1 + qv_2) - H(v_1) - H(v_2)\|_{*,\beta} \leq \\ \epsilon \left(\max\{\|v_1\|^r; \|v_2\|^s\} \right)^w \times \|z, u_2, \ldots, u_n\|_{*,\beta}, \end{aligned}$$

for all $v_1, v_2 \in Y$, $u_2, \ldots, u_n \in X$ and for some $z \in X$. Then, there exists a unique solution $F : Y \to X$ of (3):

$$\|H(v_1) - F(v_1), u_2, \cdots, u_n\|_{*,\beta} \leq \epsilon \Big(\min\{\|v_1\|^r; \|v_2\|^s\} \Big)^w \|z, u_2, \ldots, u_n\|_{*,\beta}$$

for every $v_1 \in Y$, $u_2, \ldots, u_n \in X$ and for some $z \in X$.

Proof. Let

$$\theta(v_1, v_2, u_2, \ldots, u_n) := \epsilon \Big(\max\{\|v_1\|^r; \|v_2\|^s\} \Big)^w \|z, u_2, \ldots, u_n\|_{*,\beta},$$

and $c(t) = \Big(\max\{t^r, t^s\} \Big)^w$ in Theorem 3 for all $v_1, v_2 \in Y$, $u_2, \ldots, u_n \in X$ and for some $z \in X$, where $t \in (0, 1)$ and $\epsilon, r, s, w \in (0, \infty)$. Therefore, conditions (7) and (8) are satisfied. Thus, by Theorem 3, we obtain the desired results. □

Corollary 4. *Assume a normed space Y, an nArch $(n, \beta)-BS$ X, and let $\epsilon, r, s, w, \alpha_i \in (0, \infty)$ for $i = 1, 2$ and $q \in (0, 1)$. If $H : Y \to X$ satisfies:*

$$\begin{aligned} \|H(qv_1 + (1-q)v_2) \ + H((1-q)v_1 + qv_2) - H(v_1) - H(v_2)\|_{*,\beta} \leq \\ \epsilon \left(\alpha_1 \|v_1\|^r + \alpha_2 \|v_2\|^s \right)^w \times \|z, u_2, \ldots, u_n\|_{*,\beta}, \end{aligned}$$

for every $v_1, v_2 \in Y$, $u_2, \ldots, u_n \in X$ and for some $z \in X$. Then, there exists a unique solution $F : Y \to X$ of (3), such that

$$\|H(v_1) - F(v_1), u_2, \ldots, u_n\|_{*,\beta} \leq \epsilon \Big(\min\{\alpha_1 \|v_1\|^r, \alpha_2 \|v_2\|^s\} \Big)^w \|z, u_2, \ldots, u_n\|_{*,\beta}$$

for every $v_1 \in Y$, $u_2, \ldots, u_n \in X$ and for some $z \in X$.

Proof. Let

$$\theta(v_1, v_2, u_2, \cdots, u_n) := \epsilon \Big(\alpha_1 \|v_1\|^r + \alpha_2 \|v_2\|^s \Big)^w \times \|z, u_2, \cdots, u_n\|_{*,\beta},$$

and $c(t) = t^{w \min\{r,s\}}$ in Theorem 3 for every $v_1, v_2 \in Y$, $u_2, \cdots, u_n \in X$ and for some $z \in X$, where $t \in (0, 1)$, $\epsilon, r, s, w, \alpha_i \in (0, \infty)$, for $i = 1, 2$. So, conditions (7) and (8) are satisfied. Hence, by Theorem 3, we obtain the desired results. □

5. Conclusions

We studied the stability of the FE of the q-Wright affine functions in nArch (n, β)-BS by using some recent FPT. In other words, using a version of an FPT and based on some assumptions, we obtain functions that satisfy the given FE approximately in nArch

(n, β)-BS. The results obtained are useful, because it means that we obtain estimates for the difference between the exact and approximate solutions of the equation of interest. Our results bridged the gap that exists in the literature concerning the stability results of the equation of interest in nArch (n, β)-BS. We also presented some important consequences of our results. In this way, we improve several earlier outcomes. Potential future work could be to investigate the stability of the given FE in some other spaces such as dq-metric spaces.

Author Contributions: Conceptualization, E.-S.E.-H., I.-i.E.-F.; methodology, E.-S.E.-H., I.-i.E.-F.; software, E.-S.E.-H., I.-i.E.-F.; validation, E.-S.E.-H., I.-i.E.-F.; formal analysis, E.-S.E.-H., I.-i.E.-F.; investigation, E.-S.E.-H., I.-i.E.-F.; resources, E.-S.E.-H., I.-i.E.-F.; data curation, E.-S.E.-H., I.-i.E.-F.; writing—original draft preparation, E.-S.E.-H.; writing—review and editing, E.-S.E.-H., I.-i.E.-F.; visualization, E.-S.E.-H., I.-i.E.-F.; supervision, E.-S.E.-H., I.-i.E.-F.; project administration, E.-S.E.-H., I.-i.E.-F. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful to their universities for their support.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Aoki, T. On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Jpn.* **1950**, *2*, 64–66. [CrossRef]
2. Bourgin, D.G. Classes of transformations and bordering transformations. *Bull. Amer. Math. Soc.* **1951**, *57*, 223–237. [CrossRef]
3. Brzdęk, J.; Ciepliński, K. A fixed point approach to the stability of functional equations in non-Archimedean metric spaces. *Nonlinear Anal.* **2011**, *74*, 6861–6867. [CrossRef]
4. Brzdęk, J.; Fechner, W.; Moslehian, M.S.; Sikorska, J. Recent developments of the conditional stability of the homomorphism equation. *Banach J. Math. Anal.* **2015**, *9*, 278–327. [CrossRef]
5. Chu, H.-Y.; Kim, A.; Park, J. On the Hyers-Ulam stabilities of functional equations on n-Banach spaces. *Math. Nachr.* **2016**, *289*, 1177–1188. [CrossRef]
6. Ebadian, A.; Aghalary, R.; Abolfathi, M.A. On approximate dectic mappings in non-Archimedean spaces: A fixed point approach. *Int. J. Nonlinear Anal. Appl.* **2014**, *5*, 111–122.
7. Găvruta, P. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **1994**, *184*, 431–436. [CrossRef]
8. Hyers, D.H. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef]
9. Moslehian, M.S.; Rassias, T.M. Stability of functional equations in non-Archimedean spaces. *Appl. Anal. Discret. Math.* **2007**, *1*, 325–334. [CrossRef]
10. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.* **1978**, *72*, 297–300. [CrossRef]
11. Rassias, T.M. On a modified Hyers-Ulam sequence. *J. Math. Anal. Appl.* **1991**, *158*, 106–113. [CrossRef]
12. Rassias, T.M.; Semrl, P. On the behavior of mappings which do not satisfy Hyers-Ulam stability. *Proc. Amer. Math. Soc.* **1992**, *114*, 989–993. [CrossRef]
13. Ulam, S.M. *Problems in Modern Mathematics*; Science Editions; Wiley: Hoboken, NJ, USA, 1960; Chapter IV.
14. Gähler, S. Lineare 2-normierte Räume. *Math. Nachr.* **1964**, *28*, 1–43. (In German) [CrossRef]
15. Gähler, S. Über 2-Banach-Räume. *Math. Nachr.* **1969**, *42*, 335–347. (In German) [CrossRef]
16. Hensel, K. Über eine neue Begründung der Theorie der algebraischen Zahlen. *Jahresber. Dtsch. Math.-Ver.* **1897**, *6*, 83–88. [CrossRef]
17. White, A. 2-Banach spaces. *Math. Nachr.* **1969**, *42*, 43–60. [CrossRef]
18. Beg, I.; Ahmed, M.; Nafadi, A. (JCLR) property and fixed point in non-Archimedean fuzzy metric spaces. *Int. J. Nonlinear Anal. Appl.* **2018**, *9*, 195–201.
19. Katsaras, A.K.; Beoyiannis, A. Tensor products of non-Archimedean weighted spaces of continuous functions. *Georgian Math. J.* **1999**, *6*, 33–44. [CrossRef]
20. Khrennikov, A. Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models. In *Mathematics and Its Applications*; Kluwer Academic: Dordrecht, The Netherlands, 1997; Volume 427.
21. Zúñiga-Galindo, W.A. *Advances in Non-Archimedean Analysis and Applications: The p-adic Methodology in STEAM-H*; Springer Nature: Singapore, 2021.
22. Brzdęk, J. Banach limit, fixed points and Ulam stability. *Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. Mat.* **2022**, *116*, 1–16. [CrossRef]

23. Badora, R.; Brzdęk, J.; Ciepliński, K. Applications of Banach limit in Ulam stability. *Symmetry* **2021**, *13*, 841. [CrossRef]
24. Bahyrycz, A.; Brzdęk, J.; El-hady, E.; Lesniak, Z. On Ulam Stability of Functional Equations in 2-Normed Spaces—A Survey. *Symmetry* **2021**, *13*, 2200. [CrossRef]
25. Brzdęk, J.; Eghbali, N.; Kalvandi, V. On Ulam Stability of a Generalized Delayed Differential Equation of Fractional Order. *Results Math.* **2022**, *77*, 1–21. [CrossRef]
26. Brzdęk, J. Stability of the equation of the p-Wright affine functions. *Aequ. Math.* **2013**, *85*, 497–503. [CrossRef]
27. Daróczy, Z.; Lajkó, K.; Lovas, R.; Maksa, G.; Páles, Z. Functional equations involving means. *Acta Math. Hung.* **2007**, *116*, 79–87. [CrossRef]
28. Wright, E.M. An inequality for convex functions. *Amer. Math. Mon.* **1954**, *61*, 620–622. [CrossRef]
29. Bahyrycz, A.; Brzdęk, J.; Piszczek, M. On approximately p-Wright affine functions in ultrametric spaces. *J. Funct. Spaces Appl.* **2013**, *2013*, 723545. [CrossRef]
30. Gilányi, A.; Páles, Z. On Dinghas-type derivatives and convex functions of higher order. *Real Anal. Exch.* **2002**, *27*, 485–493. [CrossRef]
31. Najati, A.; Park, C. Stability of homomorphisms and generalized derivations on Banach algebras. *J. Inequal. Appl.* **2009**, *2009*, 595439. [CrossRef]
32. Jarczyk, W.; Sablik, M. Duplicating the cube and functional equations. *Results Math.* **1994**, *26*, 324–335. [CrossRef]
33. Brzdęk, J.; Liviu, C. Stability for a family of equations generalizing the equation of p-Wright affine functions. *Appl. Math. Comput.* **2016**, *276*, 158–171. [CrossRef]
34. EL-Fassi, I. On a New Type of Hyperstability for Radical Cubic Functional Equation in Non-Archimedean Metric Spaces. *Results Math.* **2017**, *72*, 991–1005. [CrossRef]
35. El-hady, E. *On Stability of the Functional Equation of p-Wright Affine Functions in 2-Banach Spaces, Ulam Type Stability*; Springer: Cham, Switzerland, 2019; pp. 131–141.
36. El-hady, E. On stability of the functional equation of p-Wright affine functions in (2, α)-Banach spaces. *J. Egypt. Math. Soc.* **2019**, *27*, 1–9. [CrossRef]
37. Diagana, T.; François, R. *Non-Archimedean Operator Theory*; Springer: Cham, Switzerland, 2016.
38. Yang, X.; Chang, L.; Liu, G.; Shen, G. Stability of functional equations in (n,β)-normed spaces. *J. Inequalities Appl.* **2015**, *2015*, 18. [CrossRef]
39. EL-Fassi, I.; Elqorachi, E.; Khodaei, H. A Fixed Point Approach to Stability of k-th Radical Functional Equation in Non-Archimedean (n,β)-Banach Spaces. *Bull. Iran. Math. Soc.* **2021**, *47*, 487–504. [CrossRef]
40. Rasham, T.; Marino, G.; Shahzad, A.; Park, C.; Shoaib, A. Fixed point results for a pair of fuzzy mappings and related applications in b-metric like spaces. *Adv. Differ. Equ.* **2021**, *1*, 1–18. [CrossRef]
41. Debnath, P.; Nabanita, K.; Radenovic, S. *Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Sciences*; Springer: Singapore, 2021.