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Radius of $\mathtt{k}$-Parabolic Starlikeness for Some Entire Functions

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Abstract: This article considers three types of analytic functions based on their infinite product representation. The radius of the $\mathtt{k}$-parabolic starlikeness of the functions of these classes is studied. The optimal parameter values for $\mathtt{k}$-parabolic starlike functions are determined in the unit disk. Several examples are provided that include special functions such as Bessel, Struve, Lommel, and q-Bessel functions.

Keywords: parabolic starlike; entire functions; Bessel functions; Struve functions; Lommel functions; Wright functions

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1. Introduction

This article deals with the class $\mathtt{S}$ of functions f that are analytic and univalent in the unit disc $\mathtt{D}$, and normalized by $f(0) = f'(0) - 1$. A well-known, starlike class of order α, $\mathtt{S}^*(\alpha)$, consist of $f \in \mathtt{S}$ with the characterization

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D} \quad 0 \leq \alpha < 1. \tag{1}$$

The Inequality (1) is associated with the right half-plane. Rønning [1] defined the class $\mathtt{S}_p^*$ to be associated with a parabolic region $|w-1| = \mathrm{Re}\,(w)$ in the right half-plane and $f \in \mathtt{S}_p^*$ satisfies the relationship

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathbb{D}. \tag{2}$$

The class $\mathtt{S}_p^*$ is generalized in two ways. The very first generalization, $\mathtt{S}_p^*(\alpha)$, $\alpha < 1$, is given in [2] by searching functions $f \in \mathcal{A}$ satisfying

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha, \quad z \in \mathbb{D}. \tag{3}$$

It is shown that (3) also defines a parabola with vertex $w = (1+\alpha)/2$. The parabola became narrow with a growing α value and degenerate at $\alpha = 1$. It gives the previous class $\mathtt{S}_p^*$ for $\alpha = 0$ and notably $\mathtt{S}_p^*(\alpha) \subset \mathtt{S}^*(0)$ for $-1 \leq \alpha < 1$. However, for $\alpha < -1$, the class contains non-univalent functions.

The second form of the generalization $\mathtt{PS}^*(\rho)$ is given in [3] and consists of functions $\mathtt{f}$ such that $z\mathtt{f}'(z)/\mathtt{f}(z) \in \Omega_\rho$ for $z \in \mathtt{D}$, where

$$\begin{aligned}\Omega_\rho &= \left\{w = u + iv : v^2 < 4(1-\rho)(u-\rho)\right\} \\ &= \left\{w : |w-1| < 1 - 2\rho + \mathrm{Re}(w)\right\}.\end{aligned}$$

In this case, $f \in \mathtt{PS}^*(\rho)$ if, and only if,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| + 1 - 2\rho, \quad z \in \mathbb{D}, \quad 0 \leq \rho < 1.$$

More details and related properties of the class $\mathtt{S}_p^*(\alpha)$ and $\mathtt{PS}^*(\rho)$ can be found in previous articles [1–6].

The idea of parabolic starlikeness is expanded to include starlikeness in a conic region in [7], and the concern class is denoted by $\mathcal{ST}(k,\alpha)$. A function $f \in \mathcal{ST}(k,\alpha)$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha, \tag{4}$$

for $k \geq 0$, $0 \leq \alpha < 1$, and $z \in \mathbb{D}$. The class is also known as k-starlikeness of order α. The $\mathcal{ST}(k,\alpha)$-radius of a function f is the real number

$$\mathrm{r}^{s}_{k,\alpha}(f) = \sup_{z\in\mathbb{D}_r}\left\{r > 0 : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha, \quad z \in \mathbb{D}_r\right\}. \tag{5}$$

The concept of parabolic starlikeness is inspired from the idea of uniform convexity and unifrom starlikeness [8–10]. The concept of uniform convexity ($\mathcal{UCV}$) has undergone various generalizations, and there is a class known as k-uniform convex functions of order α denoted by $\mathcal{UCV}(k,\alpha)$. A function $f \in \mathcal{UCV}(k,\alpha)$ if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| + \alpha, \tag{6}$$

for $k \geq 0$, $0 \leq \alpha < 1$, and $z \in \mathbb{D}$. The class is also known as k-uniform convex of order α. The $\mathcal{UCV}(k,\alpha)$-radius of a function f is the real number

$$\mathrm{r}^{c}_{k,\alpha}(f) = \sup_{z\in\mathbb{D}_r}\left\{r > 0 : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| + \alpha, \quad z \in \mathbb{D}_r\right\}. \tag{7}$$

The $\mathcal{ST}(k,\alpha)$-radius of analytic functions defined by infinite product representation, including real zeroes, is discussed in the following section. The results can be applied to a variety of well-known special functions, such as the Bessel, Struve, and Lommel functions, generalized Mittag–Leffler functions, generalized Wright functions, and some trigonometric functions, which are given in Section 3. The concept of this study was inspired from various geometric properties of special function studied in [11–22].

The following results proved in [23] are also useful.

Lemma 1. *[23] If $|z| \leq r < a < b$, and $\lambda \in [0,1]$, then*

$$\left|\frac{z}{b-z} - \lambda\frac{z}{a-z}\right| \leq \frac{r}{b-r} - \lambda\frac{r}{a-r}. \tag{8}$$

As a consequence, it follows that

$$\operatorname{Re}\left(\frac{z}{b-z} - \lambda\frac{z}{a-z}\right) \leq \frac{r}{b-r} - \lambda\frac{r}{a-r} \tag{9}$$

and

$$\operatorname{Re}\left(\frac{z}{b-z}\right) \leq \left|\frac{z}{b-z}\right| \leq \frac{r}{b-r}. \tag{10}$$

2. $\mathcal{ST}(k,\alpha)$-Radius

In this section, three sub-classes of $\mathcal{A}$ are considered that consist of functions with a parameter $\nu \in \mathbb{R}$ and are classified by its infinite factorization involving zeros; they are denoted by $\mathbb{G}_1(\beta)$, $\mathbb{G}_2$, and $\mathbb{G}_3$, respectively. These classes can be represented analytically as follows:

$$\mathbb{G}_1(\beta) := \left\{ f_\nu \in \mathcal{A} \,\middle|\, f_\nu(z) := z\exp(\beta z)\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n^2(\nu)}\right), \quad \beta \geq 0, \quad \nu \in \mathbb{R} \right\}, \tag{11}$$

$$\mathbb{G}_2 := \left\{ \mathrm{g}_\nu \in \mathcal{A} \,\middle|\, \mathrm{g}_\nu(z) := z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{b_n^2(\nu)}\right), \nu \in \mathbb{R} \right\}, \tag{12}$$

$$\mathbb{G}_3 := \left\{ \mathrm{h}_\nu \in \mathcal{A} \,\middle|\, \mathrm{h}_\nu(z) := z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{d_n^2(\nu)}\right)^{\mu(\nu)}, \quad \nu \in \mathbb{R}\setminus\{0\} \right\}. \tag{13}$$

Here, μ is a positive real function of ν. The sequences $\{a_n(\nu)\}$, $\{b_n(\nu)\}$, and $\{d_n(\nu)\}$ respectively denote the n-th zero of the functions f_ν, g_ν, and h_ν; for all three functions, the nature of the zeros depends on ν in such a way that each of the infinite products mentioned above are uniformly convergent to each compact subset of $\mathbb{C}$. Several examples in Section 3 indicate that the classes $\mathbb{G}_1(\beta)$, $\mathbb{G}_2$, and $\mathbb{G}_3$ are non-empty.

Our first theorem gives the $\mathcal{ST}(k,\alpha)$-radius of the functions f_ν, g_ν, and h_ν.

Theorem 1. *Let $0 \leq \alpha < 1$ and $k \geq 0$. If for some $\nu \in \mathbb{R}$, the functions f_ν, g_ν, and h_ν have real positive zeros, then the following statements are true:*

1. *The $\mathcal{ST}(k,\alpha)$-radius of $f_\nu \in \mathbb{G}_1(\beta)$ is R_f, where R_f is the smallest positive root of the equation $(1+k)rf_\nu'(r) - (k+\alpha+\beta+r\beta)f_\nu(r) = 0$ in $(0, a_1^2(\nu))$ for $0 \leq \beta < 1-\alpha$;*
2. *The $\mathcal{ST}(k,\alpha)$-radius of $\mathrm{g}_\nu \in \mathbb{G}_2$ is R_g, where R_g is the smallest positive root of the equation $(1+k)r\mathrm{g}_\nu'(r) - (k+\alpha)\mathrm{g}_\nu(r) = 0$ in $(0, b_1(\nu))$;*
3. *The $\mathcal{ST}(k,\alpha)$-radius of $\mathrm{h}_\nu \in \mathbb{G}_3$ is R_h, where R_h is the smallest positive root of the equation $(1+k)r\mathrm{h}_\nu'(r) - (k+\alpha)\mathrm{h}_\nu(r) = 0$ in $(0, d_1(\nu))$.*

Proof. A logarithmic differentiation of $f_\nu \in \mathbb{G}_1(\beta)$ yields

$$\frac{zf_\nu'(z)}{f_\nu(z)} = 1 + \beta z - \sum_{n=1}^{\infty}\frac{z}{a_n^2(\nu) - z}. \tag{14}$$

Similarly, from (12) and (13), we get

$$\frac{z\mathrm{g}_\nu'(z)}{\mathrm{g}_\nu(z)} = 1 - \sum_{n=1}^{\infty}\frac{2z^2}{b_n^2(\nu) - z^2},$$
$$\frac{z\mathrm{h}_\nu'(z)}{\mathrm{h}_\nu(z)} = 1 - \mu(\nu)\sum_{n=1}^{\infty}\frac{2z^2}{d_n^2(\nu) - z^2}.$$

Suppose that $|z| \leq r < a_1^2(\nu)$. Next, by using (10), it can be shown that

$$\begin{aligned}\operatorname{Re}\frac{zf_\nu'(z)}{f_\nu(z)} &= 1 + \beta\operatorname{Re}(z) - \operatorname{Re}\sum_{n=1}^{\infty}\frac{z}{a_n^2(\nu) - z}\\ &> 1 - \beta - \sum_{n=1}^{\infty}\frac{|z|}{a_n^2(\nu) - |z|} = \frac{rf_\nu'(r)}{f_\nu(r)} - (1+r)\beta.\end{aligned}$$

Using (9), we get

$$\left|\frac{zf_\nu'(z)}{f_\nu(z)} - 1\right| \leq \beta|z| + \left|\sum_{n=1}^{\infty}\frac{z}{a_n^2(\nu) - z}\right| \leq \beta|z| + \sum_{n=1}^{\infty}\frac{|z|}{a_n^2(\nu) - |z|} = 1 - \frac{rf_\nu'(r)}{f_\nu(r)}.$$

Hence, we have

$$\operatorname{Re}\frac{zf_\nu'(z)}{f_\nu(z)}-k\left|\frac{zf_\nu'(z)}{f_\nu(z)}-1\right|-\alpha\geq(1+k)\frac{rf_\nu'(r)}{f_\nu(r)}-(k+\alpha+(1+r)\beta). \tag{15}$$

Similarly, for the functions g_ν and $\mathtt{h}_\nu$, it follows that

$$\operatorname{Re}\frac{z\mathrm{g}_\nu'(z)}{\mathrm{g}_\nu(z)}-k\left|\frac{z\mathrm{g}_\nu'(z)}{\mathrm{g}_\nu(z)}-1\right|-\alpha\geq(1+k)\frac{r\mathrm{g}_\nu'(r)}{\mathrm{g}_\nu(r)}-(k+\alpha) \tag{16}$$

and

$$\operatorname{Re}\frac{z\mathtt{h}_\nu'(z)}{\mathtt{h}_\nu(z)}-k\left|\frac{z\mathtt{h}_\nu'(z)}{\mathtt{h}_\nu(z)}-1\right|-\alpha\geq(1+k)\frac{r\mathtt{h}_\nu'(r)}{\mathtt{h}_\nu(r)}-(k+\alpha). \tag{17}$$

In all three cases, the equality holds when $z=|z|=r$. Finally, by the minimum principal of harmonic functions, the following inequalities

$$\operatorname{Re}\frac{zf_\nu'(z)}{f_\nu(z)}-k\left|\frac{zf_\nu'(z)}{f_\nu(z)}-1\right|-\alpha>0,\ \operatorname{Re}\frac{z\mathrm{g}_\nu'(z)}{\mathrm{g}_\nu(z)}-k\left|\frac{z\mathrm{g}_\nu'(z)}{\mathrm{g}_\nu(z)}-1\right|-\alpha>0\quad\text{, and}$$
$$\operatorname{Re}\frac{z\mathtt{h}_\nu'(z)}{\mathtt{h}_\nu(z)}-k\left|\frac{z\mathtt{h}_\nu'(z)}{\mathtt{h}_\nu(z)}-1\right|-\alpha>0$$

are valid if, and only if, $|z|<\mathtt{R}_f$, $|z|<\mathtt{R}_\mathrm{g}$, and $|z|<\mathtt{R}_\mathtt{h}$, respectively. Here, $\mathtt{R}_f$, $\mathtt{R}_\mathrm{g}$, and $\mathtt{R}_\mathtt{h}$ are the smallest positive roots of three equations

$$\begin{aligned}
&(1+k)\frac{rf_\nu'(r)}{f_\nu(r)}-(k+\alpha+(1+r)\beta)=0,\\
&(1+k)\frac{r\mathrm{g}_\nu'(r)}{\mathrm{g}_\nu(r)}-(k+\alpha)=0,\\
&(1+k)\frac{r\mathtt{h}_\nu'(r)}{\mathtt{h}_\nu(r)}-(k+\alpha)=0.
\end{aligned}$$

Note that for $(\alpha+\beta)<1$, we have

$$\begin{aligned}
&\lim_{r\to0^+}\left((1+k)\frac{rf_\nu'(r)}{f_\nu(r)}-(k+\alpha+(1+r)\beta\right)\\
&\quad=(1+k)\lim_{r\to0^+}\left(1+\beta r-\sum_{n=1}^{\infty}\frac{r}{a_n^2(\nu)-r}\right)-(k+\alpha+\beta)\\
&\quad=1-(\alpha+\beta)>0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{r\to a_1^2(\nu)^-}\left((1+k)\frac{rf_\nu'(r)}{f_\nu(r)}-(k+\alpha+(1+r)\beta\right)\\
&\quad=(1+k)\lim_{r\to a_1^2(\nu)^-}\left(1+\beta r-\sum_{n=1}^{\infty}\frac{r}{a_n^2(\nu)-r}\right)-(k+\alpha+\beta+\beta a_1^2(\nu))=-\infty.
\end{aligned}$$

This implies that $r(1+k)f_\nu'(r)-(k+\alpha+(1+r)\beta)f_\nu(r)=0$ has a root in $\left(0,a_1^2(\nu)\right)$. Similarly, it can be shown that

$$r(1+k)\mathrm{g}_\nu'(r)-(k+\alpha)\mathrm{g}_\nu(r)=0$$

and

$$r(1+k)\mathtt{h}_\nu'(r)-(k+\alpha)\mathtt{h}_\nu(r)=0,$$

have respective roots in $(0, b_1(\nu))$ and $(0, d_1(\nu))$ $\square$

Our next result gives the condition of ν for which the functions f_ν, $\mathtt{g}_\nu$, and $\mathtt{h}_\nu$ are in the class $\mathcal{ST}(k, \alpha)$.

Theorem 2. *Suppose that the function $\nu \to \mathtt{U}(\nu) := f_\nu'(1)/f_\nu(1)$ is well defined on an open interval $I = (r_1, r_2) \subseteq \mathbb{R}$. For $k, \beta \geq 0$, and $0 \leq \alpha < 1$, the following cases are true:*

(I) Assume that $\mathtt{U}(\nu)$ is increasing on I. The function $f_\nu \in \mathbb{G}_1(\beta)$ is in the class $\mathtt{ST}(k, \alpha)$

(a) for $\nu > \nu(f_\nu)$, where $\nu(f_\nu)$ is the unique solution of the equation

$$(1+k)f_\nu'(1) - (k+\alpha+2\beta)f_\nu(1) = 0,$$

in $I = (r_1, r_2)$, provided

$$\left.\begin{array}{l} \lim\limits_{\nu \to r_1^+} \big((1+k)\mathtt{U}(\nu) - (k+\alpha+2\beta)\big) < 0 \\ \text{and} \\ \lim\limits_{\nu \to r_2^-} \big((1+k)\mathtt{U}(\nu) - (k+\alpha+2\beta)\big) > 0 \end{array}\right\}, \tag{18}$$

(b) for $\nu > r_1$, provided $\lim\limits_{\nu \to r_1^+} \big((1+k)\mathtt{U}(\nu) - (k+\alpha+2\beta)\big) > 0$.

(II) Assume that $\mathtt{U}(\nu)$ is decreasing on I. The function $f_\nu \in \mathbb{G}_1(\beta)$ is in the class $\mathtt{ST}(k, \alpha)$

(a) for $r_1 < \nu < \nu(f_\nu)$, where $\nu(f_\nu)$ is the unique solution of the equation

$$(1+k)f_\nu'(1) - (k+\alpha+2\beta)f_\nu(1) = 0,$$

in $I = (r_1, r_2)$, provided

$$\left.\begin{array}{l} \lim\limits_{\nu \to r_1^+} \big((1+k)\mathtt{U}(\nu) - (k+\alpha+2\beta)\big) > 0 \\ \text{and} \\ \lim\limits_{\nu \to r_2^-} \big((1+k)\mathtt{U}(\nu) - (k+\alpha+2\beta)\big) < 0 \end{array}\right\}, \tag{19}$$

(b) for $\nu < r_2$, provided $\lim\limits_{\nu \to r_2^-} \big((1+k)\mathtt{U}(\nu) - (k+\alpha+2\beta)\big) > 0$.

(III) Other cases where the result holds, provided

$$\sum_{n=1}^{\infty} \frac{1}{a_n^2 - 1} \leq \frac{1-\alpha-\beta+\beta k}{1+k}. \tag{20}$$

Proof. Suppose that the function $\mathtt{U}(\nu) := f_\nu'(1)/f_\nu(1)$ is well defined on $\nu \in I \subseteq \mathbb{R}$. Then from (14), it follows that

$$\mathtt{U}(\nu) = \frac{f_\nu'(1)}{f_\nu(1)} = 1 + \beta - \sum_{n=1}^{\infty} \frac{1}{a_n^2(\nu) - 1}. \tag{21}$$

From the proof of Theorem 1, we have

$$\begin{aligned} \operatorname{Re} \frac{z f_\nu'(z)}{f_\nu(z)} - k\left|\frac{z f_\nu'(z)}{f_\nu(z)} - 1\right| - \alpha &\geq (1+k)\frac{r f_\nu'(r)}{f_\nu(r)} - (k+\alpha+(1+r)\beta) \\ &\geq (1+k)\mathtt{U}(\nu) - (k+\alpha+2\beta). \end{aligned} \tag{22}$$

The increasing property of $\mathtt{U}(\nu)$ in combination with (18) confirms the existence of the unique zero in (r_1, r_2). This implies $(1+k)\mathtt{U}(\nu) - (k+\alpha+2\beta) > 0$ if $\nu > \nu(f_\nu)$. In the case,

$\lim_{\nu\to r_1^+}\left((1+k)\mathtt{U}(\nu)-(k+\alpha+2\beta)\right)>0$, which clearly implies $(1+k)\mathtt{U}(\nu)-(k+\alpha+2\beta)>0$ if $\nu>r_1$. This completes the proof of part (I) using (22).

The decreasing property of $\mathtt{U}(\nu)$ in combination with (19) confirms the existence of the unique zero in (r_1,r_2). This implies $(1+k)\mathtt{U}(\nu)-(k+\alpha+2\beta)>0$ if $\nu<\nu(f_\nu)$. In the case, $\lim_{\nu\to r_2^-}\left((1+k)\mathtt{U}(\nu)-(k+\alpha+2\beta)\right)>0$, which clearly implies $(1+k)\mathtt{U}(\nu)-(k+\alpha+2\beta)>0$ if $\nu<r_2$. This completes the proof of part (II) using (22).

Now consider other cases, such as when $\mathtt{U}(\nu)$ is neither increasing nor decreasing on $\nu\in(r_1,r_2)$, or the limits mentioned in (I) and (II) do not appear as per requirements. Then, (21) and (22) together give

$$\operatorname{Re}\frac{zf_\nu'(z)}{f_\nu(z)}-k\left|\frac{zf_\nu'(z)}{f_\nu(z)}-1\right|-\alpha\geq(1+k)\left(1+\beta-\sum_{n=1}^{\infty}\frac{1}{a_n^2(\nu)-1}\right)-(k+\alpha+2\beta)>0, \tag{23}$$

provided

$$\sum_{n=1}^{\infty}\frac{1}{a_n^2(\nu)-1}\leq\frac{1-\alpha-\beta+\beta k}{1+k}.$$

This completes the proof. □

We state the following two similar results without proof.

Theorem 3. *Suppose the function $\nu\to\mathtt{w}(\nu):=\mathrm{g}_\nu'(1)/\mathrm{g}_\nu(1)$ is well defined on an open interval $I=(r_1,r_2)\subseteq\mathbb{R}$. For $k,\beta\geq 0$, and $0\leq\alpha<1$, the following cases are true:*

(I) Assume that $\mathtt{w}(\nu)$ is increasing on I. The function $g_\nu\in\mathbb{G}_2$ is in the class $\mathtt{ST}(k,\alpha)$

(a) for $\nu>\nu(g_\nu)$, where $\nu(g_\nu)$ is the unique solution of the equation

$$(1+k)\mathrm{g}_\nu'(1)-(k+\alpha)\mathrm{g}_\nu(1)=0,$$

in $I=(r_1,r_2)$, provided

$$\left.\begin{array}{l}\lim_{\nu\to r_1^+}\left((1+k)\mathtt{W}(\nu)-(k+\alpha+2\beta)\right)<0\\ \textit{and}\\ \lim_{\nu\to r_2^-}\left((1+k)\mathtt{W}(\nu)-(k+\alpha+2\beta)\right)>0\end{array}\right\}, \tag{24}$$

(b) for $\nu>r_1$, provided $\lim_{\nu\to r_1^+}\left((1+k)\mathtt{W}(\nu)-(k+\alpha+2\beta)\right)>0$.

(II) Assume that $\mathtt{W}(\nu)$ is decreasing on I. The function $g_\nu\in\mathbb{G}_2$ is in the class $\mathtt{ST}(k,\alpha)$

(a) for $r_1<\nu<\nu(g_\nu)$, where $\nu(g_\nu)$ is the unique solution of the equation

$$(1+k)g_\nu'(1)-(k+\alpha+2\beta)g_\nu(1)=0,$$

in $I=(r_1,r_2)$, provided

$$\left.\begin{array}{l}\lim_{\nu\to r_1^+}\left((1+k)\mathtt{W}(\nu)-(k+\alpha+2\beta)\right)>0\\ \textit{and}\\ \lim_{\nu\to r_2^-}\left((1+k)\mathtt{W}(\nu)-(k+\alpha+2\beta)\right)<0\end{array}\right\}, \tag{25}$$

(b) for $\nu<r_2$, provided $\lim_{\nu\to r_2^-}\left((1+k)\mathtt{W}(\nu)-(k+\alpha+2\beta)\right)>0$.

(III) *Other cases where the result holds, provided*

$$\sum_{n=1}^{\infty} \frac{1}{b_n^2(\nu)-1} < \frac{1-\alpha}{2(1+k)}. \tag{26}$$

Theorem 4. *Suppose that the function* $\nu \to \mathtt{T}(\nu) := \mathtt{h}_\nu'(1)/\mathtt{h}_\nu(1)$ *is well defined on an open interval* $I = (r_1, r_2) \subseteq \mathbb{R}$.

For $k, \beta \geq 0$, *and* $0 \leq \alpha < 1$, *the following cases are true:*

(I) *Assume that* $\mathtt{h}(\nu)$ *is increasing on I. The function* $h_\nu \in \mathbb{G}_3$ *is in the class* $\mathtt{ST}(k, \alpha)$

(a) *for* $\nu > \nu(h_\nu)$, *where* $\nu(h_\nu)$ *is the unique solution of the equation*

$$(1+k)\mathtt{h}_\nu'(1) - (k+\alpha)\mathtt{h}_\nu(1) = 0,$$

in $I = (r_1, r_2)$, *provided*

$$\left.\begin{array}{l} \lim\limits_{\nu \to r_1^+} \big((1+k)\mathtt{T}(\nu) - (k+\alpha)\big) < 0 \\ \textit{and} \\ \lim\limits_{\nu \to r_2^-} \big((1+k)\mathtt{T}(\nu) - (k+\alpha)\big) > 0 \end{array}\right\}, \tag{27}$$

(b) *for* $\nu > r_1$, *provided* $\lim\limits_{\nu \to r_1^+} \big((1+k)\mathtt{T}(\nu) - (k+\alpha)\big) > 0$.

(II) *Assume that* $\mathtt{T}(\nu)$ *is decreasing on I. The function* $g_\nu \in \mathbb{G}_3$ *is in the class* $\mathtt{ST}(k, \alpha)$

(a) *for* $r_1 < \nu < \nu(h_\nu)$, *where* $\nu(h_\nu)$ *is the unique solution of the equation*

$$(1+k)h_\nu'(1) - (k+\alpha)h_\nu(1) = 0,$$

in $I = (r_1, r_2)$, *provided*

$$\left.\begin{array}{l} \lim\limits_{\nu \to r_1^+} \big((1+k)\mathtt{T}(\nu) - (k+\alpha)\big) > 0 \\ \textit{and} \\ \lim\limits_{\nu \to r_2^-} \big((1+k)\mathtt{T}(\nu) - (k+\alpha)\big) < 0 \end{array}\right\}, \tag{28}$$

(b) *for* $\nu < r_2$, *provided* $\lim\limits_{\nu \to r_2^-} \big((1+k)\mathtt{T}(\nu) - (k+\alpha)\big) > 0$.

(III) *Other cases where the result holds, provided*

$$\sum_{n=1}^{\infty} \frac{1}{d_n^2(\nu)-1} < \frac{1-\alpha}{2\mu(1+k)}. \tag{29}$$

3. Application to Special Functions

In this section, we will provide several examples of special functions that are members of the class $\mathbb{G}_1(\beta)$, $\mathbb{G}_2$, and $\mathbb{G}_3$, and, hence, satisfy the results proved in the earlier sections.

3.1. Functions Involving Sin Functions

Example 1. *For* $\nu \neq 0$, *consider the function*

$$\begin{aligned} \mathcal{E}_\nu(z) :&= z\prod_{n=1}^{\infty}\left(1+\frac{z}{n^2\pi^2-\nu^2}\right) = \frac{z\nu\csc(\nu)\sin\left(\sqrt{\nu^2-z}\right)}{\sqrt{\nu^2-z}} \\ &= z + \frac{1-\nu\cot(\nu)}{2\nu^2}z + \frac{3-\nu^2+3\nu\cot(\nu)}{8\nu^4}z^2 + \ldots \end{aligned}$$

Now for $\beta \geq 0$, the normalized function $f_\nu(z) = ze^{\beta z}\mathcal{E}_\nu(z) \in \mathbb{G}_1(\beta)$ with zeros $a_n(\nu) = \sqrt{n^2\pi^2 - \nu^2}$, $n = 1, 2, 3, \ldots$, and $\nu \in (-\pi, \pi)$.

From Theorem 1, it follows that the $\mathcal{ST}(k, \alpha)$ radius of $f_\nu(z) = ze^{\beta z}\mathcal{E}_\nu(z)$ is $\mathtt{R}_f = \min\{1, \mathtt{R}_{f,0}\}$, where $\mathtt{R}_{f,0}$ is the unique solution of

$$(k+1)r\sqrt{\nu^2 - r}\cot\left(\sqrt{\nu^2 - r}\right) + 2\beta k r^2 - r\left(2\alpha + 2\beta + 2\beta k \nu^2 + k - 1\right) + 2\nu^2(\alpha + \beta - 1) = 0,$$

in $(0, \pi^2 - \nu^2)$ for $0 \leq \beta < 1 - \alpha$.

3.2. Functions Associated the Normalized Bessel Functions

It is well-known that the Bessel function, J_ν, of the first kind and order ν is the solution for the differential equation

$$z^2 y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = 0.$$

The zeros, $j_n(\nu)$, of J_ν are all real and appear in the increasing order $j_1(\nu) < j_2(\nu) < \ldots$ for $\nu \geq 0$. In addition, the Bessel function has the Weierstrass decomposition [24] of the form

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{j_n^2(\nu)}\right). \tag{30}$$

A logarithmic differentiation of (30) yields

$$\frac{zJ_\nu'(z)}{J_\nu(z)} = \nu - \sum_{n=1}^{\infty} \frac{2z^2}{j_n^2(\nu) - z^2} \tag{31}$$

The following three normalizations can be obtained from (30):

$$2^\nu\Gamma(\nu+1)z^{1-\frac{\nu}{2}}J_\nu(\sqrt{z}) = z\prod_{n=1}^{\infty}\left(1 - \frac{z}{j_n^2(\nu)}\right);$$
$$2^\nu\Gamma(\nu+1)z^{1-\nu}J_\nu(z) = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{j_n^2(\nu)}\right);$$
$$\left(2^\nu\Gamma(\nu+1)J_\nu(z)\right)^{\frac{1}{\nu}} = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{j_n^2(\nu)}\right)^{\frac{1}{\nu}},$$

for which it follows that the below example for $\mathbb{G}_1(\beta)$, $\mathbb{G}_2$, and $\mathbb{G}_3$.

Example 2 (The normalized Bessel function)**.** *Denote $j_n(\nu)$ as the n-th zero of the Bessel function $J_\nu(z)$. Then,*

(i) $\mathcal{B}_1(\nu, \beta, z) := 2^\nu\Gamma(\nu+1)z^{1-\frac{\nu}{2}}\exp(\beta z)J_\nu(\sqrt{z}) \in \mathbb{G}_1(\beta)$, *with* $\beta \geq 0$ *and* $a_n(\nu) = j_n(\nu)$;

(ii) $\mathcal{B}_2(\nu, z) = 2^\nu\Gamma(\nu+1)z^{1-\nu}J_\nu(z) \in \mathbb{G}_2$ *with* $b_n(\nu) = j_n(\nu)$;

(iii) $\mathcal{B}_3(\nu, z) = \left(2^\nu\Gamma(\nu+1)J_\nu(z)\right)^{\frac{1}{\nu}} = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{j_n^2(\nu)}\right)^{\frac{1}{\nu}} \in \mathbb{G}_3$ *with* $d_n(\nu) = j_n(\nu)$ *and* $\mu(\nu) := 1/\nu$.

We have the following results from Sections 2 and 3.

Theorem 5. *Suppose that $k, \nu \geq 0$, $0 \leq \alpha < 1$.*

(i) **Radius problem of $\mathcal{B}_1(\nu,\beta,z)$:** *For $0 \le \beta < 1-\alpha$, the $\mathcal{ST}(k,\alpha)$-radius of $\mathcal{B}_1(\nu,\beta,z)$ is the smallest positive root of the equation*

$$(1+k)\sqrt{r}J_\nu'(\sqrt{r}) + (2(1-\alpha) - \nu(1+k) + 2\beta(rk-1))J_\nu(\sqrt{r})) = 0,$$

in $(0, j_1^2(\nu))$.

(ii) **Radius problem of $\mathcal{B}_2(\nu,z)$:** *The $\mathcal{ST}(k,\alpha)$-radius of $\mathcal{B}_2(\nu,z)$ is the smallest positive root of the equation*

$$(1+k)rJ_\nu'(r) + ((1-\alpha) - \nu(1+k))J_\nu(r)) = 0$$

in $(0, j_1(\nu))$.

(iii) **Radius problem of $\mathcal{B}_3(\nu,z)$:** *The $\mathcal{ST}(k,\alpha)$-radius of $\mathcal{B}_3(\nu,z)$ is the smallest positive root of the equation*

$$(1+k)rJ_\nu'(r) - \nu(k+\alpha)J_\nu(r)) = 0$$

in $(0, j_1(\nu))$.

Proof. Following the notation of Theorem 1, let

$$f_\nu(z) = \mathcal{B}_1(\nu,\beta,z).$$

The logarithmic differentiation gives

$$\frac{zf_\nu'(z)}{f_\nu(z)} = 1 - \frac{\nu}{2} + \beta z + \frac{\sqrt{z}\,J_\nu'(\sqrt{z})}{2J_\nu(\sqrt{z})}. \tag{32}$$

Thus,

$$\begin{aligned}
&(1+k)rf_\nu'(r) - (k+\alpha+\beta+\beta r)f_\nu(r) = 0,\\
\Longrightarrow\ &(1+k)\big((2-\nu+2\beta r)J_\nu(\sqrt{r}+\sqrt{r}\,J_\nu'(\sqrt{r})\big) - 2(k+\alpha+\beta+\beta r)J_\nu(\sqrt{r}) = 0,\\
\Longrightarrow\ &(1+k)\sqrt{r}J_\nu'(\sqrt{r}) + (2(1-\alpha) - \nu(1+k) + 2\beta(rk-1))J_\nu(\sqrt{r})) = 0.
\end{aligned}$$

The conclusion follows from Theorem 1. Similarly, the $\mathcal{ST}(k,\alpha)$-radius of $\mathcal{B}_2(\nu,z)$ and $\mathcal{B}_3(\nu,z)$ follows from Theorem 1 by considering $\mathtt{g}_\nu(z) = \mathcal{B}_2(\nu,z)$ and $\mathtt{h}_\nu(z) = \mathcal{B}_3(\nu,z)$, respectively. □

The next result provides conditions on which the functions $\mathcal{B}_1(\nu,\beta,z)$, $\mathcal{B}_2(\nu,z)$, and $\mathcal{B}_3(\nu,z)$ are in the class $\mathcal{ST}(k,\alpha)$. For this purpose, let's recall some relevant properties of the ratio of Bessel functions, $J_\nu(z)$, with respect to ν and the fixed z. It is proved in [25] (Theorem 3, p. 193), that for all real z and ν

$$\frac{\partial}{\partial\nu}\left(\frac{zJ_\nu'(z)}{J_\nu(z)}\right) \ge 1, \tag{33}$$

away from the singularities of $zJ_\nu'(z)/J_\nu(z)$. Now for fixed $z=1$, it follows that

$$\frac{\partial}{\partial\nu}\left(\frac{J_\nu'(1)}{J_\nu(1)}\right) \ge 1, \tag{34}$$

in the interval between two ν, say ν_1 and ν_2 for which $J_{\nu_1}(1) = 0 = J_{\nu_2}(1)$, that is between any two singularities of $J_\nu'(1)/J_\nu(1)$. As per the requirement in this article, the zero of $J_\nu(1)$ for $\nu > -1$ is at $\nu_0 = -0.7745645128439621\ldots$, and

$$\lim_{\nu\to\nu_0}\frac{J_\nu'(1)}{J_\nu(1)} = -\infty.$$

The above information, along with Theorem 2, gives the following results.

Theorem 6. *For $k, \beta \geq 0$ and $0 \leq \alpha < 1$, the function $\mathcal{B}_1(\nu, \beta, z)$ is in the class $\mathcal{ST}(k, \alpha)$ for $\nu \geq \nu(\mathcal{B}_1)$, where $\nu(\mathcal{B}_1)$ is the unique solution of the differential equation*

$$(1+k)J'_\nu(1) - (2(1-\alpha) - \beta(1-k) - \nu(1+k))J_\nu(1) = 0$$

in (ν_0, ∞).

Proof. Denoting $f_\nu(z) = \mathcal{B}_1(\nu, \beta, z)$, it follows from (31) and (32) that

$$U(\nu) := \frac{f'_\nu(1)}{f_\nu(1)} = 1 - \frac{\nu}{2} + \beta + \frac{J'_\nu(1)}{2J_\nu(1)}.$$

Clearly, U is defined for $\nu \in (\nu_0, \infty)$, and together with (34), it follows that

$$U'(\nu) := -\frac{1}{2} + \frac{1}{2}\frac{\partial}{\partial \nu}\left(\frac{J'_\nu(1)}{J_\nu(1)}\right) \geq 0.$$

Thus, U is an increasing function of ν in (ν_0, ∞). Finally,

$$\begin{aligned}&\lim_{\nu \to \nu_0}\Big((1+k)U(\nu) - (k+\alpha+2\beta)\Big)\\ &= \lim_{\nu \to \nu_0}\left((1+k)\left(1 - \tfrac{\nu}{2} + \beta + \tfrac{J'_\nu(1)}{2J_\nu(1)}\right) - (k+\alpha+2\beta)\right) = -\infty\end{aligned}$$

and

$$\begin{aligned}&\lim_{\nu \to \infty}((1+k)U(\nu) - (k+\alpha+2\beta))\\ &= \lim_{\nu \to \infty}\left((1+k)\left(1 + \tfrac{\nu}{2} + \beta - \sum_{n=1}^{\infty}\tfrac{1}{j_n^2(\nu) - 1}\right) - (k+\alpha+2\beta)\right) = \infty.\end{aligned}$$

The results follow from Theorem 2. □

Next, a table is created that lists all values of $\nu(\mathcal{B}_1)$ for judicious choices of α, β, and k. Table 1 demonstrates the values of $\nu(\mathcal{B}_1)$ for fixed $\alpha = 0$, and for different values of β and k. We consider only the integer value of β and k. It can be observed from Table 1 that $\nu(\mathcal{B}_1)$ is

(i) decreasing with the increase of β for $k = 0$ and vice-versa;
(ii) independent of β for $k = 1$, and independent of k for $\beta = 1$. In either case, $\nu(\mathcal{B}_1) = 0.659908$;
(iii) increasing with the increase of β for $k \geq 2$, and increasing with the increase of k for $\beta \geq 2$.

Table 1. The values of $\nu(\mathcal{B}_1)$ for which $\mathcal{B}_1(\nu, \beta, z) \in \mathcal{ST}(k, \alpha = 0)$.

$\alpha = 0$	$\beta = 0$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$	$\beta = 6$
$k = 0$	1.1225	0.659908	0.225435	−0.147971	−0.409221	−0.550661	−0.621721
$k = 1$	0.659908	0.659908	0.659908	0.659908	0.659908	0.659908	0.659908
$k = 2$	0.510849	0.659908	0.811941	0.9663	1.1225	1.28017	1.43904
$k = 3$	0.437697	0.659908	0.888864	1.1225	1.35947	1.59888	1.84012
$k = 4$	0.394321	0.659908	0.935268	1.21695	1.50287	1.79175	2.08276
$k = 5$	0.365637	0.659908	0.9663	1.28017	1.59888	1.92086	2.24515
$k = 6$	0.34527	0.659908	0.988511	1.32545	1.66764	2.01331	2.36139

Table 2, shown below for $\alpha = 0.5$, also represents the same fact as presented above for $\alpha = 0$, except the fact that in this case, $\nu(\mathcal{B}_1)$ is increasing with k for $\beta = 1$.

Table 2. The values of $\nu(\mathcal{B}_1)$ for which $\mathcal{B}_1(\nu,\beta,z) \in \mathcal{ST}(k,\alpha=0.5)$.

$\alpha = 0.5$	$\beta = 0$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$	$\beta = 6$
$k = 0$	0.659908	0.225435	−0.147971	-0.409221	−0.550661	−0.621721	−0.660647
$k = 1$	0.437697	0.437697	0.437697	0.437697	0.437697	0.437697	0.437697
$k = 2$	0.365637	0.510849	0.659908	0.811941	0.9663	1.1225	1.28017
$k = 3$	0.330064	0.547791	0.773692	1.00519	1.24063	1.47891	1.7193
$k = 4$	0.308878	0.570065	0.842644	1.1225	1.40718	1.69518	1.98556
$k = 5$	0.294824	0.584958	0.888864	1.20117	1.51885	1.84012	2.1639
$k = 6$	0.284819	0.595616	0.921991	1.25757	1.59888	1.94396	2.29162

Similarly, by considering $w(\nu) = \mathcal{B}_2'(\nu,1)/\mathcal{B}_2(\nu,1)$ and $T(\nu) = \mathcal{B}_3'(\nu,1)/\mathcal{B}_3(\nu,1)$, the following result can be obtained from Theorems 3 and 4.

Theorem 7. *[21] (Theorem 2.4(b)) For $k \geq 0$ and $0 \leq \alpha < 1$, the function $\mathcal{B}_2(\nu,z)$ is in the class $\mathcal{ST}(k,\alpha)$ for $\nu \geq \nu(\mathcal{B}_2)$, where $\nu(\mathcal{B}_2)$ is the unique solution of the differential equation*

$$(1+k)J_\nu'(1) - (1-\alpha-\nu(1+k))J_\nu(1) = 0$$

in $(0,\infty)$.

Theorem 8. *[21] (Theorem 2.4(a)) For $k \geq 0$ and $0 \leq \alpha < 1$, the function $\mathcal{B}_3(\nu,z)$ is in the class $\mathcal{ST}(k,\alpha)$ for $\nu \geq \nu(\mathcal{B}_3)$, where $\nu(\mathcal{B}_3)$ is the unique solution of the differential equation*

$$(1+k)J_\nu'(1) - \nu(\alpha+k)J_\nu(1) = 0$$

in $(0,\infty)$.

Remark 1. *For $\beta = 0$, the radius problem for three different normalized Bessel functions, as stated in Theorem 5, is obtained in [21] (Theorem 2.1). The particular choice of α, β, and k leads to various known results related to the radius problem of normalized Bessel, and associated special, functions.*

3.3. Functions Associated with Struve Functions

Two of the most well-known special functions that are solutions of the non-homogeneous Bessel differential equations

$$z^2y''(z) + zy'(z) + (z^2-\nu^2)y(z) = z^{\mu+1},$$

and

$$z^2y''(z) + zy'(z) + (z^2-\nu^2)y(z) = \frac{4}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1},$$

are respectively known as the Lommel function, $\mathtt{S}_{\nu,\mu}$, and the Struve function, $\mathtt{S}_\nu$.

If $\mathtt{h}_{\nu,n}$ denotes the nth positive zero of $\mathtt{S}_\nu$, then (see [17]) for $|\nu| \leq 1/2$, the function $\mathtt{S}_\nu$ can be expressed as

$$\mathtt{S}_\nu(z) = \frac{z^{\nu+1}}{2^\nu\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\mathtt{h}_{\nu,n}^2}\right). \tag{35}$$

From the article [18] (Thorem 1), it is useful to note here that $\mathtt{h}_{\nu,n} > \mathtt{h}_{\nu,1} > 1$ for $|\nu| < 1/2$.

From (35), the following three normalized forms of the Struve functions can be considered here:

$$\left(\sqrt{\pi}2^{\nu}\Gamma\left(\nu+\frac{3}{2}\right)\mathtt{S}_{\nu}(z)\right)^{\frac{1}{\nu+1}} = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\mathtt{h}_{\nu,n}^2}\right)^{\frac{1}{\nu+1}}; \tag{36}$$

$$\sqrt{\pi}2^{\nu}\Gamma\left(\nu+\frac{3}{2}\right)z^{1-\nu}\mathtt{S}_{\nu}(z) = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\mathtt{h}_{\nu,n}^2}\right); \tag{37}$$

$$\sqrt{\pi}2^{\nu}\Gamma\left(\nu+\frac{3}{2}\right)z^{\frac{1-\nu}{2}}\mathtt{S}_{\nu}(\sqrt{z}) = z\prod_{n=1}^{\infty}\left(1-\frac{z}{\mathtt{h}_{\nu,n}^2}\right). \tag{38}$$

Further, this representation leads to

Example 3. *For $|\nu| \leq 1/2$,*

(i) $\mathcal{S}_1(\nu,\beta,z) = \exp(\beta z)\sqrt{\pi}2^{\nu}\Gamma(\nu+\frac{3}{2})z^{\frac{1-\nu}{2}}\mathtt{S}_{\nu}(\sqrt{z}) \in \mathbb{G}_1(\beta)$, *with* $\beta \geq 0$ *and* $a_n(\nu) = \mathtt{h}_{\nu,n}$;

(ii) $\mathcal{S}_2(\nu,z) = \sqrt{\pi}2^{\nu}\Gamma(\nu+\frac{3}{2})z^{1-\nu}\mathtt{S}_{\nu}(z) \in \mathbb{G}_2$ *with* $b_n(\nu) = \mathtt{h}_{\nu,n}$;

(iii) $\mathcal{S}_3(\nu,z) = \sqrt{\pi}2^{\nu}\Gamma(\nu+\frac{3}{2})(\mathtt{S}_{\nu}(z))^{\frac{1}{\nu+1}} \in \mathbb{G}_3$ *with* $d_n(\nu) = \mathtt{h}_{\nu,n}$ *and* $\mu(\nu) := 1/(\nu+1)$.

We have the following results form Sections 2 and 3.

Theorem 9. *Suppose that $k \geq 0$, $|\nu| \leq 1/2$, and $-1 \leq \alpha < 1$.*

(i) **Radius problem of $\mathcal{S}_1(\nu,\beta,z)$:** *For $0 \leq \beta < 1-\alpha$, the $\mathcal{ST}(k,\alpha)$-radius of $\mathcal{S}_1(\nu,\beta,z)$ is the smallest positive root of the equation*

$$(1+k)\sqrt{r}\mathtt{S}_{\nu}'(\sqrt{r}) + (2(1-\alpha)-(1+\nu)(1+k)+2\beta(rk-1))\mathtt{S}_{\nu}(\sqrt{r})) = 0,$$

in $(0, h_{\nu,1}^2)$.

(ii) **Radius problem of $\mathcal{S}_2(\nu,z)$:** *The $\mathcal{ST}(k,\alpha)$-radius of $\mathcal{S}_2(\nu,z)$ is the smallest positive root of the equation*

$$(1+k)r\mathtt{S}_{\nu}'(r) - ((1-\alpha)-\nu(1+k))\mathtt{S}_{\nu}(r)) = 0$$

in $(0, h_{\nu,1})$.

(iii) **Radius problem of $\mathcal{S}_3(\nu,z)$:** *The $\mathcal{ST}(k,\alpha)$-radius of $\mathcal{S}_3(\nu,z)$ is the smallest positive root of the equation*

$$(1+k)r\mathtt{S}_{\nu}'(r) - (\nu+1)(k+\alpha)\mathtt{S}_{\nu}(r) = 0$$

in $(0, h_{\nu,1})$.

Remark 2. *For $\beta = 0$, the radius problem for three different normalized Struve functions, as stated in Theorem 9, is obtained in [15] (Theorem 1, P.3). The particular choice of α, β, and k leads to various known results related to the radius problem of normalized Struve functions and related special functions. For $k = 0$, it gives a result on the radius of starlikeness for Struve functions [16] (Theorem 2).*

For the next result, lets denote (38) by $\mathtt{h}_{\nu}$. It was proved in [18] that $\mathtt{h}_{\nu}'(1) > 0$ for $\nu \geq -1/2$, and $\mathtt{h}_{\nu}(1) > 0$ for $|\nu| \leq 1/2$.

Theorem 10. *For $k,\beta \geq 0$, $0 \leq \alpha < 1$, the function $\mathcal{S}_1(\nu,\beta,z) \in \mathcal{ST}(k,\alpha)$ for all $|\nu| \leq 1/2$ provided*

$$\beta(k-1) - \alpha > k.$$

Proof. Clearly

$$\mathcal{S}_1(\nu,\beta,z) = \exp(\beta z)\mathtt{h}_{\nu}(z). \tag{39}$$

Then, by following the notation of Theorem 2, it follows that

$$\mathtt{U}(\nu) = \frac{\mathcal{S}_1'(\nu,\beta,1)}{\mathcal{S}_1(\nu,\beta,1)} = \beta + \frac{\mathtt{h}_\nu'(1)}{\mathtt{h}_\nu(1)}.$$

This gives $\mathtt{U}(\nu) \geq \beta$ for $|\nu| \leq 1/2$. On the other hand, using (13), it can be shown that

$$\mathtt{U}(\nu) = \beta + 1 - \sum_{n\geq 1} \frac{1}{\mathtt{h}_{\nu,n}^2 - 1} \geq \beta \implies \sum_{n\geq 1} \frac{1}{\mathtt{h}_{\nu,n}^2 - 1} \leq 1.$$

Finally, the result follows from Theorem 2 if

$$1 < \frac{1-\alpha-\beta+\beta k}{1+k} \implies 1-\alpha+\beta(k-1) > 1+k.$$

This completes the proof. □

Clearly, the above result does not hold for $k \leq 1$ and $\beta = 0$. To prove Theorem 10, we used the 3rd part of Theorem 2 because the increasing or decreasing property of $\mathtt{U}(\nu)$ is not known, and we are unable to prove it theoretically.

However, the graphical analysis (Figure 1) shows that $\mathtt{h}_\nu'(1)/h_\nu(1)$ is increasing and positive for all $\nu \in [-1/2, 1/2]$, and

$$\min_{\nu=-1/2} \frac{\mathtt{h}_\nu'(1)}{h_\nu(1)} = 1.64209.$$

Thus, for all $k,\beta \geq 0$, $0 \leq \alpha < 1$, and $|\nu| \leq 1/2$, the function $\mathcal{S}_1(\nu,\beta,z) \in \mathcal{ST}(k,\alpha)$ as

$$\begin{aligned}\lim_{\nu\to -1/2^+} \Big((1+k)\mathtt{U}(\nu) - (k+\alpha+2\beta) \Big) &= (1+k)(\beta+1.64209) - (k+\alpha+2\beta) \\ &= (k-1)\beta + (1.64209-\alpha) + 0.64209k > 0.\end{aligned}$$

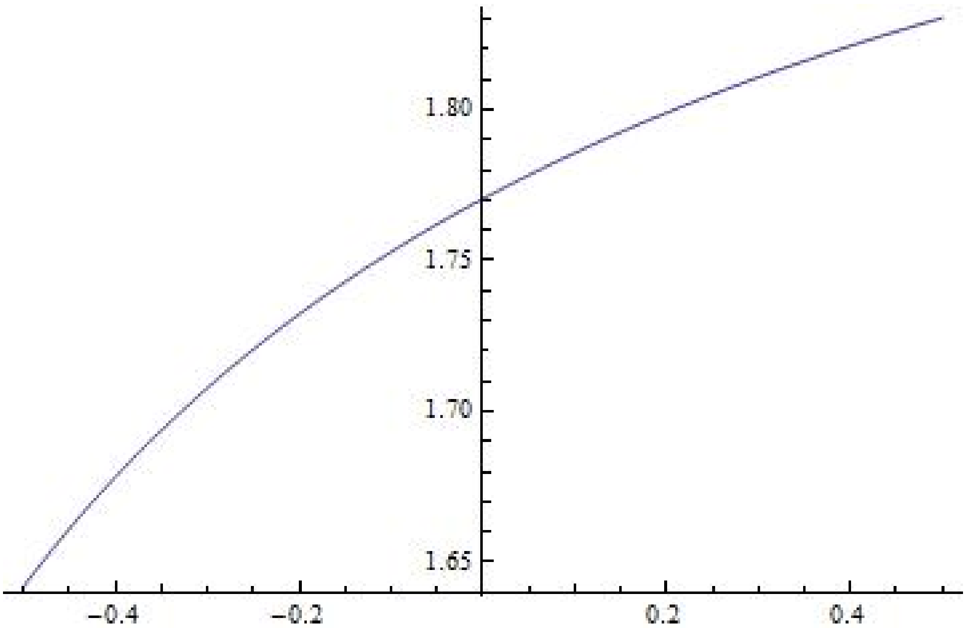

Figure 1. Graph of $\mathtt{h}_\nu'(1)/h_\nu(1)$ for $|\nu| \leq 1/2$.

By using the above fact, Theorem 10 may be improved as follows:

Theorem 11. *For all $k, \beta \geq 0$, $0 \leq \alpha < 1$, and $|\nu| \leq 1/2$, the function $\mathcal{S}_1(\nu, \beta, z) \in \mathcal{ST}(k, \alpha)$.*

3.4. Functions Associated with Wright Functions

In relation with the asymptotic theory of partitions, Wright [26] introduced the function

$$\phi(\rho, \beta, z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(n\rho + \delta)} \tag{40}$$

for $\rho > 0$ and $\delta, z \in \mathbb{C}$. The function is known as the Wright function. Note that the Wright function is also valid for $\rho > -1$ and, more significantly, it is an entire function of z for $\rho > -1$. More details about this function can be found in [26,27]. Various geometric properties of the Wright function are discussed in [19,28–33]. Denote, the nth positive zero of $\lambda_{\rho,\delta}(z) = \phi(\rho, \delta, -z^2)$ by $\lambda_{\rho,\delta,n}$, and the nth positive zero of $\Psi'_{\rho,\delta}$, where $\Psi(z) = z^\delta \phi(\rho, \delta, -z^2)$ by $\eta'_{\rho,\delta,n}$.

For $\rho > 0$ and $\delta > 0$, it is proved in [19] that the function $z \mapsto \lambda_{\rho,\delta} = \phi(\rho, \delta, -z^2)$ has infinitely many real zeros and have the Weierstrass decomposition

$$\Gamma(\delta)\phi(\rho, \delta, -z^2) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda^2_{\rho,\delta,n}}\right).$$

The product is convergent uniformly on the compact subset of the complex plane. The positive zeros of $\lambda_{\rho,\delta}$ are interlaced with zeros of $\Psi_{\rho,\delta}$. That means the zeros satisfy the chain of inequalities

$$\eta'_{\rho,\delta,1} < \lambda_{\rho,\delta,1} < \eta'_{\rho,\delta,2} < \lambda_{\rho,\delta,2} < \ldots.$$

We have following example related to $\phi(\rho, \delta, .)$.

Example 4 (The normalized Wright function)**.** *For $\rho, \delta, \beta > 0$, denote $\lambda_{\rho,\delta,n}$ as the n-th zero of the Bessel functions $\lambda_{\rho,\delta}(z)$. Then,*

$$\mathcal{W}_1(\rho, \delta, \beta, z) = z\exp(\beta z)\Gamma(\delta)\lambda_{\rho,\delta}(\sqrt{z}) = z\exp(\beta z)\prod_{n=1}^{\infty}\left(1 - \frac{z}{\lambda^2_{\rho,\delta,n}}\right) \in \mathbb{G}_1(\beta) \tag{41}$$

$$\mathcal{W}_2(\rho, \delta, z) = z\Gamma(\delta)\lambda_{\rho,\delta}(z) = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda^2_{\rho,\delta,n}}\right) \in \mathbb{G}_2 \tag{42}$$

$$\mathcal{W}_3(\rho, \delta, z) = \left(z^\delta\Gamma(\delta)\lambda_{\rho,\delta}(z)\right)^{\frac{1}{\delta}} = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda^2_{\rho,\delta,n}}\right)^{\frac{1}{\delta}} \in \mathbb{G}_3. \tag{43}$$

Theorem 12. *Suppose that $\rho, \delta \geq 0$ and $-1 < \alpha < 1$.*

(i) **Radius problem of $\mathcal{W}_1(\rho, \delta, \beta, z)$:** *For $0 \leq \beta < 1 - \alpha$, the $\mathcal{ST}(k, \alpha)$-radius of $\mathcal{W}_1(\rho, \delta, \beta, z)$ is the smallest positive root of the equation*

$$(1 + k)\sqrt{r}\lambda'_{\rho,\delta}(\sqrt{r}) + (2(1 - \alpha) + 2\beta(k - r))\lambda_{\rho,\delta}(\sqrt{r}) = 0$$

in $(0, \lambda_{\rho,\delta,1})$

(ii) **Radius problem of $\mathcal{W}_2(\rho, \delta, z)$:** *The $\mathcal{ST}(k, \alpha)$-radius of $\mathcal{W}_2(\rho, \delta, z)$ is the smallest positive root of the equation*

$$(1 + k)r\lambda'_{\rho,\delta}(r) - (1 - \alpha)\lambda_{\rho,\delta}(r)) = 0$$

in $(0, \lambda_{\rho,\delta,1})$.

(iii) **Radius problem of $\mathcal{W}_3(\rho, \delta, z)$:** *The $\mathcal{ST}(k, \alpha)$-radius of $\mathcal{W}_3(\rho, \delta, z)$ is the smallest positive root of the equation*

$$(1 + k)r\lambda'_{\rho,\delta}(r) - \delta(k + \alpha)\lambda_{\rho,\delta}(r) = 0$$

in $(0, \lambda_{\rho,\delta,1})$.

3.5. Functions Involving q-Bessel Functions

This section considers the Jackson and Hahn–Exton q-Bessel functions, respectively denoted by $\mathrm{J}_\nu^{(2)}(z;q)$ and $\mathrm{J}_\nu^{(3)}(z;q)$. For $z \in \mathbb{C}$, $\nu > -1$, and $q \in (0,1)$, both functions are defined by the series

$$\mathrm{J}_\nu^{(2)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{(q;q)_\infty (q^{\nu+1};q)_n} q^{n(n+\nu)} \tag{44}$$

$$\mathrm{J}_\nu^{(3)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n z^{2n+\nu}}{(q;q)_\infty (q^{\nu+1};q)_n} q^{\frac{n(n+1)}{2}}. \tag{45}$$

Here,

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n} \left(1 - aq^{k-1}\right), \textit{and} \quad (a;q)_\infty = \prod_{k\geq 1} \left(1 - aq^{k-1}\right)$$

are known as the q-Pochhammer symbol. For a fixed z and $q \to 1$, both of the above q-Bessel functions relate to the classical Bessel function J_ν as $\mathrm{J}_\nu^{(2)}((1-z)q;q) \to J_\nu(z)$ and $\mathrm{J}_\nu^{(3)}((1-z)q;q) \to J_\nu(2z)$. The q-extension of Bessel functions has been studied by several authors, notably [34–39] and various references therein. The geometric properties of q-Bessel is discussed in [11,20]. It is worth noting here that there are abundant results available in the literature related to the q extension of Bessel functions, but we limit ourselves with the requirement of this article. Here, we are going to explain that the result obtained in Section 2 complements results in [20] that are associated with the radius of starlikeness. For this purpose, lets recall the Hadamard factorization for the normalized q-Bessel functions

$$z \to \mathcal{J}_\nu^{(2)}(z;q) = 2^\nu c_\nu(q) z^{-\nu} \mathrm{J}_\nu^{(2)}(z;q) \quad \text{and} \quad z \to \mathcal{J}_\nu^{(3)}(z;q) = c_\nu(q) z^{-\nu} \mathrm{J}_\nu^{(3)}(z;q),$$

where $c_\nu(q) = (q;q)_\infty / \left(q^{\nu+1};q\right)_\infty$.

Lemma 2. *[20] For $\nu > -1$, the functions $z \to \mathcal{J}_\nu^{(2)}(z;q)$ and $z \to \mathcal{J}_\nu^{(3)}(z;q)$ are entire functions of order* zero *and pose the Hadamard factorization for the form*

$$\mathcal{J}_\nu^{(2)}(z;q) = \prod_{n\geq 1} \left(1 - \frac{z^2}{j_{\nu,n}^2(q)}\right), \quad \mathcal{J}_\nu^{(3)}(z;q) = \prod_{n\geq 1} \left(1 - \frac{z^2}{l_{\nu,n}^2(q)}\right) \tag{46}$$

where $j_{\nu,n}(q)$ and $l_{\nu,n}(q)$ are the n^{th} positive zero of the functions $\mathcal{J}_\nu^{(2)}(.;q)$ and $\mathcal{J}_\nu^{(3)}(.;q)$, respectively.

Now, we are ready to set our example for the $\mathbb{G}_1(\beta)$, $\mathbb{G}_2$, and $\mathbb{G}_3$ classes involving q-Bessel functions.

Example 5. *For $\nu > -1$, $q \in (0,1)$, denote $j_{\nu,n}(q)$ as the n-th zero of the q-Bessel functions $\mathrm{J}_\nu^{(2)}(z;q)$. Then,*

$$ {}_2f_{\nu,q}(z) = z\exp(\beta z)\mathcal{J}_\nu^{(2)}(\sqrt{z};q) = z\exp(\beta z)\prod_{n\geq 1}\left(1-\frac{z}{j_{\nu,n}^2(q)}\right) \in \mathbb{G}_1(\beta) \tag{47} $$

$$ {}_2g_{\nu,q}(z) = z\mathcal{J}_\nu^{(2)}(z;q) = z\prod_{n\geq 1}\left(1-\frac{z^2}{j_{\nu,n}^2(q)}\right) \in \mathbb{G}_2 \tag{48} $$

$$ {}_2h_{\nu,q}(z) = \left(z^\nu\mathcal{J}_\nu^{(2)}(z;q)\right)^{\frac{1}{\nu}} = z\prod_{n\geq 1}\left(1-\frac{z^2}{j_{\nu,n}^2(q)}\right)^{\frac{1}{\nu}} \in \mathbb{G}_3 \tag{49} $$

Example 6. *For $\nu > -1$, $q \in (0,1)$, denote $l_{\nu,n}(q)$ as the n-th zero of the q-Bessel functions $\mathrm{J}_\nu^{(3)}(z;q)$. Then,*

$$ {}_3f_{\nu,q}(z) = z\exp(\beta z)\mathcal{J}_\nu^{(3)}(\sqrt{z};q) \in \mathbb{G}_1(\beta) \tag{50} $$

$$ {}_3g_{\nu,q}(z) = z\mathcal{J}_\nu^{(3)}(z;q) \in \mathbb{G}_2 \tag{51} $$

$$ {}_3h_{\nu,q}(z) = \left(z^\nu\mathcal{J}_\nu^{(3)}(z;q)\right)^{\frac{1}{\nu}} \in \mathbb{G}_3 \tag{52} $$

Theorem 13. *Suppose that $s = \{1,2\}$, $-1 < \alpha < 1$, and $\nu > -1$.*

(i) ***Radius Problem of $_sf_{\nu,q}$:** For $0 \leq \beta \leq 1-\alpha$, the $\mathcal{ST}(k,\alpha)$ radius of $_sf_{\nu,q}$ is the smallest positive root of the equation*

$$ (1+k)\sqrt{r}\,\frac{d}{dr}\left(\mathrm{J}_\nu^{(s)}(\sqrt{r};q)\right) + (2-2\alpha-2\beta-(1+k)\nu+2k\beta r)\,\mathrm{J}_\nu^{(s)}(r;q) = 0 $$

(ii) ***Radius problem for $_sg_{\nu,q}$:** The $\mathcal{ST}(k,\alpha)$ radius of $_sg_{\nu,q}$ is the smallest positive root of the equation*

$$ (1+k)r\,\frac{d}{dr}\left(\mathrm{J}_\nu^{(s)}(r;q)\right) + (1-\alpha-2\beta-(1+k)\nu+\beta(1r))\,\mathrm{J}_\nu^{(s)}(r;q) = 0 $$

(iii) ***Radius problem for $_sh_{\nu,q}$:** The $\mathcal{ST}(k,\alpha)$ radius of $_sh_{\nu,q}$ is the smallest positive root of the equation*

$$ (1+k)r\,\frac{d}{dr}\left(\mathrm{J}_\nu^{(s)}(r;q)\right) - \nu(k+\alpha+\beta+\beta r)\,\mathrm{J}_\nu^{(s)}(r;q) = 0 $$

It is proved in [20] (Theorem 3) that the function $_2f_{\nu,q}$ is starlike and all of its derivatives are close-to-convex in $\mathbb{D}$ if, and only if, $\nu \geq \max\{\nu_0(q), \nu^*(q)\}$, where $\nu_0(q)$ is the unique root of $_2f'_{\nu,q}(1) = 0$ and $\nu^*(q)$ is the unique root of $j_{\nu,1} = 0$. Further, it is shown graphically that the inequality $\nu \geq \max\{\nu_0(q), \nu^*(q)\}$ can be replaced by $\nu \geq \nu_0(q)$.

In the next result, we partially complement the above result by covering the starlike portion. It is shown in [37] that $\nu \to j_{\nu,n}(q)$ is increasing for $\nu > -1$ and for fixed $0 < q < 1$. This implies

$$ \mathrm{u}(\nu) := \frac{_2f'_{\nu,q}(1)}{_2f_{\nu,q}(1)} = 1 - \sum_{n\geq 1}\frac{1}{j_{\nu,n}^2(q)-1} $$

is also increasing for $\nu > -1$ as

$$ \mathrm{u}'(\nu) = \sum_{n\geq 1}\frac{2j_{\nu,n}(q)\frac{d}{d\nu}j_{\nu,n}(q)}{\left(j_{\nu,n}^2(q)-1\right)^2} > 0 $$

Theorem 14. *For $\nu > -1$ and $q \in (0,1)$, the function ${}_2f_{\nu,q} \in \mathcal{ST}(k,\alpha)$ if*

$$\sum_{n\geq 1} \frac{1}{j_{\nu,n}^2(q) - 1} \leq \frac{1-\alpha-\beta+\beta k}{1+k}. \tag{53}$$

We recall from [20] that the q extension of the first Rayleigh sum for Bessel functions of the first kind

$$\sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu+1)},$$

is

$$\sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2(q)} = \frac{q^{\nu+1}}{4(q-1)(q^{\nu+1}-1)}. \tag{54}$$

This gives

$$\sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2(q) - 1} > \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2(q)} = \frac{q^{\nu+1}}{4(q-1)(q^{\nu+1}-1)} \tag{55}$$

By a calculation, it follows that

$$\nu \to \frac{q^{\nu+1}}{4(q-1)(q^{\nu+1}-1)}$$

is decreasing and tends to zero when $\nu \to \infty$. Thus, for all $\nu > -1$

$$\sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2(q) - 1} > 0.$$

This fact gives us the lower bound of the sum in (53). However, we are unable to find specific range of ν for which the the upper bound in (53) holds.

4. Discussion

The results obtained in this article complement the results related to the radius of starlikeness of entire functions. The functions having a Hadamard factorization are contained in any of the normalized classes of $\mathbb{G}_1(\beta)$, $\mathbb{G}_2$, and $\mathbb{G}_3$. Thus, the radius of starlikeness follows from Theorems 1 and 2.

Using a similar concept, there are results related to the radius of convexity of special functions, such as Bessel, Struve, and their q-analogs. One can expect similar generalized concepts and theorems like this work except for the convexity, which have an independent interest and are, hence, not included in this article.

It is also evident that, unlike Bessel functions, the results related to q-Bessel functions are not straight forward, as evidenced in the results of Section 2. This is due to the fact that there are several open aspects about the properties of zero q-Bessel functions.

The Lommel functions of a certain order also have factorization consisting of zeros, and the radius problems discussed in [16] can be complemented by using the results in Section 2. The proof and details of the results are similar to that discussed in Section 3. We avoid more details to limit the length of the article.

5. Conclusions

The concept introduced in this article can be further used for the study of the $\mathcal{UCV}(k,\alpha)$-radius of an analytic function f involving special functions such as Bessel, Struve, etc.

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References

1. Rønning, F. Uniformly convex functions and a corresponding class of starlike functions. *Proc. Am. Math. Soc.* **1993**, *118*, 189–196. [CrossRef]
2. Rønning, F. On starlike functions associated with parabolic regions. *Ann. Univ. Mar. Curie-Skłodowska Sect. A* **1991**, *45*, 117–122.
3. Ali, R.M.; Singh, V. Coefficients of parabolic starlike functions of order ρ, Computational Methods and Function Theory 1994 (Penang). *Ser. Approx. Decompos.* **1995**, *5*, 23–36.
4. Aghalary, R.; Kulkarni, S.R. Certain properties of parabolic starlike and convex functions of order ρ. *Bull. Malays. Math. Sci. Soc.* **2003**, *26*, 153–162.
5. Ali, R.M. Starlikeness associated with parabolic regions. *Int. J. Math. Math. Sci.* **2005**, *2005*, 561–570. [CrossRef]
6. Ma, W.C.; Minda, D. Uniformly convex functions. *Ann. Polon. Math.* **1992**, *57*, 165–175. [CrossRef]
7. Kanas, S.; Wiśniowska, A. Conic domains and starlike functions. *Rev. Roumaine Math. Pures Appl.* **2000**, *45*, 647–657.
8. Goodman, A.W. On uniformly starlike functions. *J. Math. Anal. Appl.* **1991**, *155*, 364–370 [CrossRef]
9. Goodman, A.W. On uniformly convex functions. *Ann. Polon. Math.* **1991**, *56*, 87–92. [CrossRef]
10. Kanas, S.; Wisniowska, A. Conic regions and k-uniform convexity. *J. Comput. Appl. Math.* **1999**, *105*, 327–336. [CrossRef]
11. Aktaş, I.; Baricz, A. Bounds for radii of starlikeness of some q-Bessel functions. *Results Math.* **2017**, *72*, 947–963. [CrossRef]
12. Aktaş, I.; Baricz, A.; Orhan, H. Bounds for radii of starlikeness and convexity of some special functions. *Turkish J. Math.* **2018**, *42*, 211–226. [CrossRef]
13. Aktaş, I.; Baricz, A.; Yağmur, N. Bounds for the radii of univalence of some special functions. *Math. Inequal. Appl.* **2017**, *20*, 825–843.
14. Aktaş, I.; Orhan, H. Bounds for radii of convexity of some q-Bessel functions. *Bull. Korean Math. Soc.* **2020**, *57*, 355–369.
15. Aktaş, I.; Toklu, E.; Orhan, H. Radii of k-starlikeness of order α of Struve and Lommel functions. *Miskolc Math. Notes* **2021**, *22*, 5–15. [CrossRef]
16. Baricz, A.; Dimitrov, D.K.; Orhan, H.; Yagmur, N. Radii of starlikeness of some special functions. *Proc. Am. Math. Soc.* **2016**, *144*, 3355–3367. [CrossRef]
17. Baricz, A.; Ponnusamy, S.; Singh, S. Turán type inequalities for Struve functions. *J. Math. Anal. Appl.* **2017**, *445*, 971–984. [CrossRef]
18. Baricz, A.; Szász, R. Close-to-convexity of some special functions and their derivatives. *Bull. Malays. Math. Sci. Soc.* **2016**, *39* 427–437. [CrossRef]
19. Baricz, A.; Toklu, E.; Kadioğlu, E. Radii of starlikeness and convexity of Wright functions. *Math. Commun.* **2018**, *23*, 97–117.
20. Baricz, A.; Dimitrov, D.K.; Mező, I. Radii of starlikeness and convexity of some q-Bessel functions. *J. Math. Anal. Appl.* **2016**, *435*, 968–985. [CrossRef]
21. Bohra, N.; Ravichandran, V. Radii problems for normalized Bessel functions of first kind. *Comput. Methods Funct. Theory* **2018**, *18*, 99–123. [CrossRef]
22. Toklu, E. Radii of starlikeness and convexity of q-Mittag-Leffler functions. *Turkish J. Math.* **2019**, *43*, 2610–2630. [CrossRef]
23. Deniz, E.; Szász, R. The radius of uniform convexity of Bessel functions. *J. Math. Anal. Appl.* **2017**, *453*, 572–588. [CrossRef]
24. Watson, G.N. *A Treatise on the Theory of Bessel Functions*; Cambridge University Press: Cambridge, UK, 1944.
25. Landau, L.J. Ratios of Bessel functions and roots of $\alpha J_\nu(x) + xJ'_\nu(x) = 0$. *J. Math. Anal. Appl.* **1999**, *240*, 174–204. [CrossRef]
26. Wright, E.M. On the Coefficients of Power Series Having Exponential Singularities. *J. Lond. Math. Soc.* **1933**, *S1-8*, 71–79. [CrossRef]
27. Gorenflo, R.; Luchko, Y.; Mainardi, F. Analytical properties and applications of the Wright function. *Fract. Calc. Appl. Anal.* **1999**, *2*, 383–414.
28. Altinkaya, Ş. On the inclusion properties for ϑ-spirallike functions involving both Mittag-Leffler and Wright function. *Turk. J. Math.* **2022**, *46*, 1119–1131.
29. Das, S.; Mehrez, K. On geometric properties of the Mittag-Leffler and Wright functions. *J. Korean Math. Soc.* **2021**, *58*, 949–965.
30. Mustafa, N. Geometric properties of normalized Wright functions. *Math. Comput. Appl.* **2016**, *21*, 14. [CrossRef]
31. Prajapat, J.K. Certain geometric properties of the Wright function. *Integral Transforms Spec. Funct.* **2015**, *26*, 203–212. [CrossRef]
32. Prajapat, J.K. Geometric properties of the Wright functions. *J. Rajasthan Acad. Phys. Sci.* **2016**, *15*, 63–71.
33. Raza, M.; Din, M.U.; Malik, S.N. Certain geometric properties of normalized Wright functions. *J. Funct. Spaces* **2016**, *2016*, 1896154. [CrossRef]
34. Abreu, L.D. A q-sampling theorem related to the q-Hankel transform. *Proc. Am. Math. Soc.* **2005**, *133*, 1197–1203. [CrossRef]

35. Annaby, M.H.; Mansour, Z.S.; Ashour, O.A. Sampling theorems associated with biorthogonal q-Bessel functions. *J. Phys. A* **2010**, *43*, 295204. [CrossRef]
36. Ismail, M.E.H. The zeros of basic Bessel functions, the functions $J_{\nu+ax}(x)$, and associated orthogonal polynomials. *J. Math. Anal. Appl.* **1982**, *86*, 1–19. [CrossRef]
37. Ismail, M.E.H.; Muldoon, M.E. On the variation with respect to a parameter of zeros of Bessel and q-Bessel functions. *J. Math. Anal. Appl.* **1988**, *135*, 187–207. [CrossRef]
38. Koelink, H.T.; Swarttouw, R.F. On the zeros of the Hahn-Exton q-Bessel function and associated q-Lommel polynomials. *J. Math. Anal. Appl.* **1994**, *186*, 690–710. [CrossRef]
39. Koornwinder, T.H.; Swarttouw, R.F. On q-analogues of the Fourier and Hankel transforms. *Trans. Am. Math. Soc.* **1992**, *333*, 445–461. [CrossRef]