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On Generating Functions for Parametrically Generalized Polynomials Involving Combinatorial, Bernoulli and Euler Polynomials and Numbers

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Abstract: The aim of this paper is to give generating functions for parametrically generalized polynomials that are related to the combinatorial numbers, the Bernoulli polynomials and numbers, the Euler polynomials and numbers, the cosine-Bernoulli polynomials, the sine-Bernoulli polynomials, the cosine-Euler polynomials, and the sine-Euler polynomials. We investigate some properties of these generating functions. By applying Euler's formula to these generating functions, we derive many new and interesting formulas and relations related to these special polynomials and numbers mentioned as above. Some special cases of the results obtained in this article are examined. With this special case, detailed comments and comparisons with previously available results are also provided. Furthermore, we come up with open questions about interpolation functions for these polynomials. The main results of this paper highlight the existing symmetry between numbers and polynomials in a more general framework. These include Bernouilli, Euler, and Catalan polynomials.

Keywords: Bernoulli and Euler numbers and polynomials; cosine-type Bernoulli and Euler polynomials; sine-type Bernoulli and Euler polynomials; Stirling numbers; generating functions; special numbers and polynomials

MSC: 05A15; 11B68; 11B73; 11B83; 33B10

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1. Motivation and Preliminaries

1.1. Motivation

For $k \in \mathbb{Z} \backslash \{-2\}$ and $a \in \mathbb{C}$, in [1], Simsek introduced the functions

$$F_{\mathcal{Y}}(w,k,a) = \frac{aw}{e^{(k+1)w} - e^{-(k+1)w} + e^{w} - e^{-w}} = \sum_{n=0}^{\infty} \mathcal{Y}_n(k,a) \frac{w^n}{n!}, \tag{1}$$

$$K_{\mathcal{Y}}(w,x,k,a) = e^{wx} F_{\mathcal{Y}}(w,k,a) = \sum_{n=0}^{\infty} Q_n(x,k,a) \frac{w^n}{n!}. \tag{2}$$

Letting $x = 0$ in (2), we have

$$Q_n(0,k,a) = \mathcal{Y}_n(k,a).$$

Another interesting example is given by

$$B_n(x) = 2^{-n} Q_n(2x-1,-1,2)$$

which are the well-known Bernoulli polynomials. Moreover, we have

$$B_n = 2^{-n} \sum_{v=0}^{n} (-1)^v \binom{n}{v} \mathcal{Y}_{n-v}(-1, 2)$$

where B_n are the Bernoulli numbers.

Bernoulli numbers and polynomials are very important and fundamental in many areas. Bernoulli numbers sit in the center of a number of mathematical fields. Among other things, we can mention, for example that

- Bernoulli numbers are rational numbers;
- Their numerators are very important for differential topology via the Kervaire–Milnor Formula;
- Their denominators are very important for homotopy theory;
- Bernoulli number are central in Number theory and are special values of zeta functions on integers;
- Interpolation theory connects Bernoulli Numbers and of Eisenstein series, modular forms, and complex analysis;
- Homotopy theory and number theory and the special values of zetas functions on the integers.

For more details, see [2–22].

Thus, the Bernoulli numbers and polynomials have received much considerable attention throughout the mathematical literature.

Because of the limited attention given to the numbers $\mathcal{Y}_n(k, a)$ and the polynomials $Q_n(x, k, a)$, there is no question of a standard. Among other things, in this article, we present a systematic treatment of these polynomials and numbers. This will give new insight into the subject.

By using (1) and (2), it is easy to see that the polynomials $Q_n(x, k, a)$ and the numbers $\mathcal{Y}_n(k, a)$ are linked by the following relationship:

$$Q_n(x, k, a) = \sum_{v=0}^{n} \binom{n}{v} \mathcal{Y}_{n-v}(k, a) x^v \tag{3}$$

(cf. [1]).

We now introduce the following generating functions involving $\sin(yw)$ and $\cos(yw)$ for the following parametrically generalized polynomials: the polynomials $Q_n^{(S)}(x, y, k, a)$ and $Q_n^{(C)}(x, y, k, a)$, respectively:

$$H(w, x, y, a, k) = \frac{e^{xw} \sin(yw) aw}{e^{(k+1)w} - e^{-(k+1)w} + e^w - e^{-w}} = \sum_{n=0}^{\infty} Q_n^{(S)}(x, y, k, a) \frac{w^n}{n!} \tag{4}$$

and

$$G(w, x, y, a, k) = \frac{e^{xw} \cos(yw) aw}{e^{(k+1)w} - e^{-(k+1)w} + e^w - e^{-w}} = \sum_{n=0}^{\infty} Q_n^{(C)}(x, y, k, a) \frac{w^n}{n!}, \tag{5}$$

where $k \in \mathbb{Z} \backslash \{-2\}$ and $a \in \mathbb{C}$.

We investigate and study these generating functions. In Section 2, by using these generating functions and their functional equations, we give relations among the polynomials $Q_n^{(S)}(x, y, k, a)$, the polynomials $Q_n^{(C)}(x, y, k, a)$, the polynomials $Q_n(x, k, a)$, the numbers $\mathcal{Y}_{n-v}(k, a)$, the Bernoulli polynomials and numbers, the Euler polynomials and numbers, the cosine-Bernoulli polynomials, the sine-Bernoulli polynomials, the cosine-Euler polynomials, and the sine-Euler polynomials.

We now give in more detail the contents of this paper. In Section 2, by using generating functions for parametrically generalized polynomials and numbers, we obtain interesting identities and relations including the polynomials $Q_n^{(S)}(x, y, k, a)$, the polynomials

$Q_n^{(C)}(x,y,k,a)$, the polynomials $Q_n(x,k,a)$, the numbers $\mathcal{Y}_{n-v}(k,a)$, the Bernoulli polynomials and numbers, the Euler polynomials and numbers, the cosine-Bernoulli polynomials, the sine-Bernoulli polynomials, the cosine-Euler polynomials, the sine-Euler polynomials, the numbers $\beta_n(k)$, and the Stirling numbers of the second kind.

In Section 3, we give open questions related to the interpolation functions for the polynomials $Q_n(x,k,a)$, $Q_n^{(C)}(x,y,k,a)$, and $Q_n^{(S)}(x,y,k,a)$.

In Section 4, we provide a concluding statement.

1.2. Preliminaries

In order to give the results of this paper, we need the following standard notation, definitions, and relations. Throughout this paper, we use the following notations:

$$\mathbb{N}=\{1,2,\ldots\}, \quad \mathbb{N}_0=\{0,1,2,\ldots\}=\mathbb{N}\cup\{0\}.$$

As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$c and $\mathbb{C}$ denote the set of integers, the set of rational numbers, the set of real numbers, and the set of complex numbers, respectively. Moreover, the falling factorial is given by

$$(\lambda)_n=\begin{cases}\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1) & \text{if} \quad n\in\mathbb{N},\\ 1 & \text{if} \quad n=0,\end{cases}$$

where $(\lambda)_0=1$ and $\lambda\in\mathbb{C}$.

Let $\alpha\in\mathbb{R}$ (or $\mathbb{C}$). The Bernoulli numbers and polynomials of higher order are defined by means of the following generating functions:

$$F_B(w,\alpha)=\left(\frac{w}{e^w-1}\right)^\alpha=\sum_{n=0}^{\infty}B_n^{(\alpha)}\frac{w^n}{n!} \tag{6}$$

and

$$G_B(w,x,\alpha)=F_B(w,\alpha)e^{wx}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{w^n}{n!}, \tag{7}$$

respectively (cf. [18,23,24]).

By substituting $\alpha=1$ into (6) and (7), we obtain the classical Bernoulli numbers and polynomials

$$B_n^{(1)}=B_n \quad \text{and} \quad B_n^{(1)}(x)=B_n(x).$$

For $\alpha=0$, we have

$$B_n^{(0)}(x)=x^n$$

and

$$B_n^{(0)}=\begin{cases}1 & \text{if} \quad n=0,\\ 0 & \text{if} \quad n>0.\end{cases}$$

By using (6) and (7), we have

$$B_n^{(\alpha)}(x)=\sum_{v=0}^{n}\binom{n}{v}x^vB_{n-v}^{(\alpha)},$$

where $n\in\mathbb{N}_0$ (cf. [18,23,24]).

Let $\alpha\in\mathbb{R}$ (or $\mathbb{C}$). The Euler numbers and polynomials of a higher order are defined by means of the following generating functions:

$$F_E(w,\alpha)=\left(\frac{2}{e^w+1}\right)^\alpha=\sum_{n=0}^{\infty}E_n^{(\alpha)}\frac{w^n}{n!} \tag{8}$$

and

$$G_E(w,x,\alpha) = F_E(w,\alpha)e^{wx} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{w^n}{n!}, \quad (9)$$

respectively (cf. [18,23,24]).

By substituting $\alpha = 1$ into (8) and (9), the classical Euler numbers and polynomials are derived as follows:

$$E_n^{(1)} = E_n \quad \text{and} \quad E_n^{(1)}(x) = E_n(x).$$

For $\alpha = 0$, we have

$$E_n^{(0)}(x) = x^n$$

and

$$E_n^{(0)} = \begin{cases} 1 & \text{if} \quad n = 0, \\ 0 & \text{if} \quad n > 0. \end{cases}$$

By using (8) and (9), we have

$$E_n^{(\alpha)}(x) = \sum_{v=0}^{n} \binom{n}{v} x^v E_{n-v}^{(\alpha)},$$

where $n \in \mathbb{N}_0$ (cf. [18,23,24]).

By using (1), (6), and (9), we have the following identity:

$$\mathcal{Y}_n(k,a) = \frac{a}{2(k+2)} \sum_{v=0}^{n} \binom{n}{v} k^{n-v}(k+2)^v B_v E_{n-v}\left(\frac{k+1}{k}\right), \quad (10)$$

where $n \in \mathbb{N}_0$ (cf. [1]).

By using (1), (7), and (8), we have the following identity:

$$\mathcal{Y}_n(k,a) = \frac{a}{2(k+2)} \sum_{v=0}^{n} \binom{n}{v} k^{n-v}(k+2)^v E_{n-v} B_v\left(\frac{k+1}{k+2}\right), \quad (11)$$

where $n \in \mathbb{N}_0$ (cf. [1]).

The Stirling numbers of the second kind, $S_2(n,k)$, are defined by means of the following generating function:

$$F_S(w,k) = \frac{(e^w - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{w^n}{n!} \quad (12)$$

and

$$y^n = \sum_{v=0}^{n} S_2(n,v)(y)_v, \quad (13)$$

where $k \in \mathbb{N}_0$ (cf. [18,23,24]).

By using (12), we have

$$S_2(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n,$$

where $n,k \in \mathbb{N}_0$.

By using (8) and (12), we get the following identity:

$$S_2(n,k) = \frac{2^{k-n}}{k!}\sum_{m=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{n}{m}\binom{k}{j} j^m E_{n-m}^{(-k)} \quad (14)$$

(cf. [23]).

The polynomials $C_n(x,y)$ and $S_n(x,y)$ are defined by means of the following generating functions:

$$K_C(w,x,y) = e^{xw}\cos(yw) = \sum_{n=0}^{\infty} C_n(x,y)\frac{w^n}{n!} \tag{15}$$

and

$$K_S(w,x,y) = e^{xw}\sin(yw) = \sum_{n=0}^{\infty} S_n(x,y)\frac{w^n}{n!}, \tag{16}$$

(cf. [25–31]).

The polynomials $C_n(x,y)$ and $S_n(x,y)$ are computed by the following formulas

$$C_n(x,y) = \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j}$$

and

$$S_n(x,y) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} (-1)^j \binom{n}{2j+1} x^{n-2j-1} y^{2j+1},$$

respectively (cf. [25–31]).

The cosine-Bernoulli polynomials $B_n^{(C)}(x,y)$ and the sine-Bernoulli polynomials $B_n^{(S)}(x,y)$ are defined by means of the following generating functions:

$$g_C(w,x,y) = \frac{w\cos(yw)}{e^w - 1}e^{xw} = \sum_{n=0}^{\infty} B_n^{(C)}(x,y)\frac{w^n}{n!} \tag{17}$$

and

$$g_S(w,x,y) = \frac{w\sin(yw)}{e^w - 1}e^{xw} = \sum_{n=0}^{\infty} B_n^{(S)}(x,y)\frac{w^n}{n!} \tag{18}$$

(cf. [30,31]; see also [26–29]).

The cosine-Euler polynomials $E_n^{(C)}(x,y)$ and the sine-Euler polynomials $E_n^{(S)}(x,y)$ are defined by means of the following generating functions:

$$h_C(w,x,y) = \frac{2\cos(yw)}{e^w + 1}e^{xw} = \sum_{n=0}^{\infty} E_n^{(C)}(x,y)\frac{w^n}{n!} \tag{19}$$

and

$$h_S(w,x,y) = \frac{2\sin(yw)}{e^w + 1}e^{xw} = \sum_{n=0}^{\infty} E_n^{(S)}(x,y)\frac{w^n}{n!} \tag{20}$$

(cf. [30,31]; see also [26–29]).

Kucukoglu and Simsek [32] defined a new sequence of special numbers $\beta_n(k)$ by means of the following generating function:

$$\left(1 - \frac{z}{2}\right)^k = \sum_{n=0}^{\infty} \beta_n(k)\frac{z^n}{n!}, \tag{21}$$

where $k \in \mathbb{N}_0$, $z \in \mathbb{C}$ with $|z| < 2$.

By using (21), we have

$$\beta_n(k) = \frac{(-1)^n n!}{2^n}\binom{k}{n}, \tag{22}$$

where $n, k \in \mathbb{N}_0$ (cf. [32] (Equation (4.9))).

2. Generating Functions for New Classes of Parametric Kinds of Special Polynomials

In this section, we investigate some properties of Equations (4) and (5). By using these generating functions, we derive some new identities and relations involving the polynomials $Q_n^{(S)}(x,y,k,a)$, the polynomials $Q_n^{(C)}(x,y,k,a)$, the polynomials $Q_n(x,k,a)$, the numbers $\mathcal{Y}_{n-v}(k,a)$, the Bernoulli polynomials and numbers, the Euler polynomials and numbers, the cosine-Bernoulli polynomials, the sine-Bernoulli polynomials, the cosine-Euler polynomials, and the sine-Euler polynomials.

By substituting $y=0$ and $x=0$ into the Equations (4) and (5), we have the following identities:

$$Q_n^{(C)}(x,0,k,a) = Q_n(x,k,a),$$

$$Q_n^{(C)}(0,0,k,a) = \mathcal{Y}_n(k,a),$$

and

$$Q_n^{(S)}(x,0,k,a) = 0.$$

In [1], using Equation (1), Simsek gave

$$F_{\mathcal{Y}}(w,k,a) = \frac{awe^{(k+1)w}}{\left(e^{(k+2)w}-1\right)\left(e^{kw}+1\right)}.$$

Combining the above equation with (7), (9), (17), (18), (19), and (20), we obtain the following functional equations:

$$\begin{aligned} H(w,x,y,2,k) &= \frac{1}{k+2} g_S\left((k+2)w, \frac{k+1+x}{k+2}, \frac{y}{k+2}\right) F_E(kw,1) \\ H(w,x,y,2,k) &= \frac{1}{k+2} G_B((k+2)w,0,1) h_S\left(kw, \frac{k+1+x}{k}, \frac{y}{k}\right), \end{aligned}$$

and

$$\begin{aligned} G(w,x,y,2,k) &= \frac{1}{k+2} g_C\left((k+2)w, \frac{k+1+x}{k+2}, \frac{y}{k+2}\right) F_E(kw,1) \\ G(w,x,y,2,k) &= \frac{1}{k+2} G_B((k+2)w,0,1) h_S\left(kw, \frac{k+1+x}{k}, \frac{y}{k}\right). \end{aligned}$$

Using similar functional equations to the above equations, we give some novel formulas and relations including the polynomials $Q_n^{(S)}(x,y,k,a)$, the polynomials $Q_n^{(C)}(x,y,k,a)$, the polynomials $Q_n(x,k,a)$, the numbers $\mathcal{Y}_{n-v}(k,a)$, the Bernoulli polynomials and numbers, the Euler polynomials and numbers, the cosine-Bernoulli polynomials, the sine-Bernoulli polynomials, the cosine-Euler polynomials, and the sine-Euler polynomials.

Theorem 1. *Let $n \in \mathbb{N}_0$. Then, we have*

$$Q_n^{(S)}(x,y,k,a) = \frac{a}{2}\sum_{j=0}^{n}\binom{n}{j} E_j^{(S)}\left(\frac{x}{k},\frac{y}{k}\right) B_{n-j}\left(\frac{k+1}{k+2}\right) k^j (k+2)^{n-j-1}.$$

Proof. By using (4), we obtain the following functional equation:

$$\frac{e^{xw}\sin(yw)}{e^{kw}+1}\left(\frac{aw}{e^{(k+2)w}-1}\right)e^{(k+1)w} = \sum_{n=0}^{\infty} Q_n^{(S)}(x,y,k,a)\frac{w^n}{n!}.$$

By combining the above equation with (7) and (20), we have

$$\begin{aligned}&\frac{a}{2(k+2)}\sum_{n=0}^{\infty}E_n^{(S)}\left(\frac{x}{k},\frac{y}{k}\right)k^n\frac{w^n}{n!}\sum_{n=0}^{\infty}B_n\left(\frac{k+1}{k+2}\right)(k+2)^n\frac{w^n}{n!}\\=&\sum_{n=0}^{\infty}Q_n^{(S)}(x,y,k,a)\frac{w^n}{n!}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{a}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}E_j^{(S)}\left(\frac{x}{k},\frac{y}{k}\right)B_{n-j}\left(\frac{k+1}{k+2}\right)k^j(k+2)^{n-j-1}\frac{w^n}{n!}\\=&\sum_{n=0}^{\infty}Q_n^{(S)}(x,y,k,a)\frac{w^n}{n!}.\end{aligned}$$

By comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we get the desired result. □

Theorem 2. *Let $n \in \mathbb{N}_0$. Then, we have*

$$Q_n^{(C)}(x,y,k,a)=\frac{a}{2}\sum_{j=0}^{n}\binom{n}{j}E_j^{(C)}\left(\frac{x}{k},\frac{y}{k}\right)B_{n-j}\left(\frac{k+1}{k+2}\right)k^j(k+2)^{n-j-1}.$$

Proof. By using (5), we obtain the following functional equation:

$$\frac{e^{xw}\cos(yw)}{e^{kw}+1}\left(\frac{aw}{e^{(k+2)w}-1}\right)e^{(k+1)w}=\sum_{n=0}^{\infty}Q_n^{(C)}(x,y,k,a)\frac{w^n}{n!}.$$

By combining the above equation with (7) and (19), we have

$$\begin{aligned}&\frac{a}{2(k+2)}\sum_{n=0}^{\infty}E_n^{(C)}\left(\frac{x}{k},\frac{y}{k}\right)k^n\frac{w^n}{n!}\sum_{n=0}^{\infty}B_n\left(\frac{k+1}{k+2}\right)(k+2)^n\frac{w^n}{n!}\\=&\sum_{n=0}^{\infty}Q_n^{(C)}(x,y,k,a)\frac{w^n}{n!}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{a}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}E_j^{(C)}\left(\frac{x}{k},\frac{y}{k}\right)B_{n-j}\left(\frac{k+1}{k+2}\right)k^j(k+2)^{n-j-1}\frac{w^n}{n!}\\=&\sum_{n=0}^{\infty}Q_n^{(C)}(x,y,k,a)\frac{w^n}{n!}.\end{aligned}$$

By comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we obtain the desired result. □

Theorem 3. *Let $n \in \mathbb{N}$. Then we have*

$$Q_n^{(S)}(x,y,k,a)=\sum_{j=0}^{\left[\frac{n-1}{2}\right]}(-1)^j\binom{n}{2j+1}y^{2j+1}Q_{n-1-2j}(x,k,a).$$

Proof. By using (2) and (4), we obtain

$$\sum_{n=0}^{\infty}Q_n^{(S)}(x,y,k,a)\frac{w^n}{n!}=\sum_{n=0}^{\infty}(-1)^n\frac{(yw)^{2n+1}}{(2n+1)!}\sum_{n=0}^{\infty}Q_n(x,k,a)\frac{w^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} Q_n^{(S)}(x,y,k,a)\frac{w^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n-1}{2}\right]}(-1)^j\binom{n}{2j+1}y^{2j}Q_{n-1-2j}(x,k,a)\frac{w^n}{n!}.$$

By comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we achieve the desired result. □

Theorem 4. *Let $n \in \mathbb{N}_0$. Then, we have*

$$Q_n^{(C)}(x,y,k,a) = \sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{n}{2j}y^{2j}Q_{n-2j}(x,k,a).$$

Proof. By using (2) and (4), we obtain

$$\sum_{n=0}^{\infty} Q_n^{(C)}(x,y,k,a)\frac{w^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{(yw)^{2n}}{(2n)!}\sum_{n=0}^{\infty} Q_n(x,k,a)\frac{w^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} Q_n^{(C)}(x,y,k,a)\frac{w^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{n}{2j}y^{2j}Q_{n-2j}(x,k,a)\frac{w^n}{n!}.$$

By comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we achieve the desired result. □

Theorem 5. *Let $n \in \mathbb{N}_0$. Then, we have*

$$Q_n^{(S)}(x,y,k,a) = \sum_{j=0}^{n}\binom{n}{j}S_j(x,y)\mathcal{Y}_{n-j}(k,a).$$

Proof. By using (1), (4), and (16), we obtain

$$\sum_{n=0}^{\infty} Q_n^{(S)}(x,y,k,a)\frac{w^n}{n!} = \sum_{n=0}^{\infty} S_n(x,y)\frac{w^n}{n!}\sum_{n=0}^{\infty}\mathcal{Y}_n(k,a)\frac{w^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} Q_n^{(S)}(x,y,k,a)\frac{w^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}S_j(x,y)\mathcal{Y}_{n-j}(k,a)\frac{w^n}{n!}.$$

Byy omparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we achieve the desired result. □

Theorem 6. *Let $n \in \mathbb{N}_0$. Then, we have*

$$Q_n^{(C)}(x,y,k,a) = \sum_{j=0}^{n}\binom{n}{j}C_j(x,y)\mathcal{Y}_{n-j}(k,a).$$

Proof. By using (1), (4), and (15), we have

$$\sum_{n=0}^{\infty} Q_n^{(C)}(x,y,k,a)\frac{w^n}{n!} = \sum_{n=0}^{\infty} C_n(x,y)\frac{w^n}{n!}\sum_{n=0}^{\infty}\mathcal{Y}_n(k,a)\frac{w^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} Q_n^{(C)}(x,y,k,a)\frac{w^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}C_j(x,y)\mathcal{Y}_{n-j}(k,a)\frac{w^n}{n!}.$$

By comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we achieve the desired result. □

Theorem 7. *Let $n \in \mathbb{N}_0$. Then, we have*

$$Q_n(x+iy,k,a) = Q_n^{(C)}(x,y,k,a) + iQ_n^{(S)}(x,y,k,a).$$

Proof. By using (4) and (5), we have

$$\sum_{n=0}^{\infty}\left(Q_n^{(C)}(x,y,k,a) + iQ_n^{(S)}(x,y,k,a)\right)\frac{w^n}{n!} = \frac{e^{xw+iyw}aw}{e^{(k+1)w} - e^{-(k+1)w} + e^{w} - e^{-w}}.$$

By using the the above equation and the Euler's formula, we obtain

$$\sum_{n=0}^{\infty}\left(Q_n^{(C)}(x,y,k,a) + iQ_n^{(S)}(x,y,k,a)\right)\frac{w^n}{n!} = \sum_{n=0}^{\infty} Q_n(x+iy,k,a)\frac{w^n}{n!}.$$

By comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we get the desired result. □

Theorem 8. *Let $n \in \mathbb{N}_0$. Then, we have*

$$\begin{aligned}&\sum_{v=0}^{n}\binom{n}{v}\sum_{j=0}^{v}\binom{v}{j}Q_j^{(C)}(x,y,k,2)Q_{v-j}^{(S)}(x,y,k,2)k^{n-v}E_{n-v}^{(-2)}\\ =\ &\frac{(k+2)^{n-2}}{2}B_n^{(C,2)}\left(\frac{2(k+1+x)}{k+2},\frac{2y}{k+2}\right),\end{aligned}$$

where

$$B_n^{(C,2)}\left(\frac{2(k+1+x)}{k+2},\frac{2y}{k+2}\right) = \sum_{v=0}^{n}\binom{n}{v}B_v^{(2)}S_{n-v}\left(\frac{2(k+1+x)}{k+2},\frac{2y}{k+2}\right).$$

Proof. By using (4) and (5), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}Q_n^{(C)}(x,y,k,2)\frac{w^n}{n!}\sum_{n=0}^{\infty}Q_n^{(S)}(x,y,k,2)\frac{w^n}{n!}\\ =\ &\frac{1}{2}\left(\frac{w^2e^{2(k+1+x)w}\sin(2yw)}{\left(e^{(k+2)w}-1\right)^2}\right)\left(\frac{2}{e^{kw}+1}\right)^2.\end{aligned}$$

By combining the above equation with (8) and (17), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}Q_j^{(C)}(x,y,k,2)Q_{n-j}^{(S)}(x,y,k,2)\frac{w^n}{n!}\sum_{n=0}^{\infty}E_n^{(-2)}\frac{(kw)^n}{n!}\\ =\ &\frac{1}{2(k+2)^2}\sum_{n=0}^{\infty}B_n^{(C,2)}\left(\frac{2(k+1+x)}{k+2},\frac{2y}{k+2}\right)(k+2)^n\frac{w^n}{n!}.\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty}\sum_{v=0}^{n}\binom{n}{v}\sum_{j=0}^{v}\binom{v}{j}Q_j^{(C)}(x,y,k,2)Q_{v-j}^{(S)}(x,y,k,2)k^{n-v}E_{n-v}^{(-2)}\frac{w^n}{n!}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}B_n^{(C,2)}\left(\frac{2(k+1+x)}{k+2},\frac{2y}{k+2}\right)(k+2)^{n-2}\frac{w^n}{n!}.$$

By comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we achieve the desired result. □

In the following we state interesting identities related to the Stirling numbers, the polynomials $C_m(x,y)$ and $S_m(x,y)$.

By combining (12) with (15) and (16), we get the following functional equations:

$$\cos(yw)\sum_{n=0}^{\infty}(x)_n F_S(w,n) = K_C(w,x,y) \tag{23}$$

and

$$\sin(yw)\sum_{n=0}^{\infty}(x)_n F_S(w,n) = K_S(w,x,y). \tag{24}$$

Theorem 9. *Let $m \in \mathbb{N}_0$. Then we have*

$$C_m(x,y) = \sum_{j=0}^{\left[\frac{m}{2}\right]}(-1)^j y^{2j}\binom{m}{2j}\sum_{n=0}^{m-2j}(x)_n S_2(m-2j,n). \tag{25}$$

Proof. By using (23), we obtain

$$\sum_{m=0}^{\infty}(-1)^m\frac{(yw)^{2m}}{(2m)!}\sum_{m=0}^{\infty}\sum_{n=0}^{m}(x)_n S_2(m,n)\frac{w^m}{m!} = \sum_{m=0}^{\infty}C_m(x,y)\frac{w^m}{m!}.$$

Therefore

$$\sum_{m=0}^{\infty}C_m(x,y)\frac{w^m}{m!} = \sum_{m=0}^{\infty}\sum_{j=0}^{\left[\frac{m}{2}\right]}(-1)^j y^{2j}\binom{m}{2j}\sum_{n=0}^{m-2j}(x)_n S_2(m-2j,n)\frac{w^m}{m!}.$$

By comparing the coefficients of $\frac{w^m}{m!}$ on both sides of the above equation, we obtain the desired result. □

By combining (25) with (14), we arrive at the following theorem:

Theorem 10. *Let $m \in \mathbb{N}_0$. Then, we have*

$$C_m(x,y) = \sum_{j=0}^{\left[\frac{m}{2}\right]}(-1)^j y^{2j}\binom{m}{2j}\sum_{n=0}^{m-2j}\frac{(x)_n 2^{n-m+2j}}{n!}$$
$$\times\sum_{v=0}^{m-2j}\sum_{d=0}^{n}(-1)^{n-d}\binom{m-2j}{v}\binom{n}{d}d^v E_{m-2j-v}^{(-n)}.$$

By combining (25) with (22), we arrive at the following theorem:

Theorem 11. *Let $m \in \mathbb{N}_0$. Then, we have*

$$C_m(x,y) = \sum_{j=0}^{\left[\frac{m}{2}\right]} (-1)^j \frac{(2y)^{2j}\beta_{2j}(m)}{(2j)!} \sum_{n=0}^{m-2j} (x)_n S_2(m-2j,n).$$

Theorem 12. *Let $m \in \mathbb{N}$. Then, we have*

$$S_m(x,y) = \sum_{j=0}^{\left[\frac{m-1}{2}\right]} (-1)^j \binom{m}{2j+1} y^{2j+1} \sum_{n=0}^{m-1-2j} (x)_n S_2(m-1-2j,n). \tag{26}$$

Proof. By using (24), we obtain

$$\sum_{m=0}^{\infty} S_m(x,y)\frac{w^m}{m!} = \sum_{n=0}^{\infty} (-1)^n \frac{(yw)^{2n+1}}{(2n+1)!} \sum_{m=0}^{\infty} \sum_{n=0}^{m} (x)_n S_2(m,n) \frac{w^m}{m!}.$$

Therefore

$$\begin{aligned} \sum_{m=0}^{\infty} S_m(x,y)\frac{w^m}{m!} &= \sum_{m=0}^{\infty} m \sum_{j=0}^{\left[\frac{m-1}{2}\right]} (-1)^j \binom{m-1}{2j} \frac{y^{2j+1}}{2j+1} \\ &\quad \times \sum_{n=0}^{m-1-2j} (x)_n S_2(m-1-2j,n) \frac{w^m}{m!}. \end{aligned}$$

By comparing the coefficients of $\frac{w^m}{m!}$ on both sides of the above equation, we obtain the desired result. □

Remark 1. *By combining* (13) *with* (15) *and* (16), *we also arrive at Formulas* (25) *and* (26). *For these and similar formulas, the above formula may be used.*

3. Questions

In [33], Kim, and Simsek gave interpolation functions involving the Hurwitz zeta function and the alternating Hurwitz zeta function for the numbers $\mathcal{Y}_n(k,a)$.

How can we define interpolation functions for the following polynomials:

$$Q_n(x,k,a), Q_n^{(C)}(x,y,k,a), \text{and } Q_n^{(S)}(x,y,k,a)?$$

4. Conclusions

Applications of generating functions are used in a remarkably wide range of areas, and we used them to define new classes of parametric kinds of special polynomials. By using the method of generating functions and Euler's formula, we investigated properties of these new parametric kinds of special polynomials. We also provide new identities and relations involving these classes of special polynomials, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the cosine-Bernoulli polynomials, the sine-Bernoulli polynomials, the cosine-Euler polynomials, the sine-Euler polynomials, and known special polynomials.

In the near future, with the help of the results given in this article, solutions of the open questions put forward will be investigated.

In general, these results have the potential to be used many branches of mathematics, probability, statistics, mathematical physics, and engineering.

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