Article

Fourth Hankel Determinant Problem Based on Certain Analytic Functions

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Abstract: In recent years, the Hankel determinant bounds for different subclasses of analytic, starlike and symmetric starlike functions have been discussed and studied by the many well-known authors. In this paper, we first consider a new subclass of analytic function and then we derive the fourth Hankel determinant bound for this class.

1. Introduction

We need to present some basic Geometric Function Theory literature for a better understanding of the topic discussed in this article. In this regard, the letters $\mathfrak{A}$ and $\mathcal{S}$ are used to represent the classes of normalized analytic and univalent functions, respectively. The following set-builder form is used to define these classes:

$$\mathfrak{A} := \left\{ f \in \mathcal{H}(\mathfrak{D}) : f(z) = \sum_{j=1}^{\infty} a_j z^j \quad (a_1 = 1) \right\} \tag{1}$$

and:

$$\mathcal{S} := \{ f \in \mathfrak{A} : f \text{ is univalent in } \mathfrak{D} \},$$

where $\mathcal{H}(\mathfrak{D})$ stands for the set of analytic functions in the region $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$. Although function theory was started in 1851, it emerged as a good area of new research in 1916, due to the conjecture $|a_n| \leq n$, which was proved by De-Branges in 1985 and many scholars attempted to prove or disprove this conjecture as a result they discovered multiple subfamilies of a class $\mathcal{S}$ of univalent functions that are associated with different image domains. The most basic of these families are the families of star-like, convex, and close-to-convex functions which are defined by:

$$\mathcal{S}^* = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in \mathfrak{D} \right\},$$

$$\mathfrak{C} = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \qquad z \in \mathfrak{D} \right\},$$

$$\mathfrak{K} = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \ g(z) \in \mathcal{S}^* \ z \in \mathfrak{D} \right\}.$$

Each of the functions classes described above has a distinct symmetry. We denote by $\mathcal{P}$, the class of analytic functions p normalized by:

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{2}$$

and:

$$\Re(p(z)) > 0, \ (z \in \mathfrak{D}). \tag{3}$$

Assume that f and g are two analytic functions in $\mathfrak{D}$. Then, we say that the function f is subordinate to the function g, and we can write:

$$f(z) \prec g(z), \ (z \in \mathfrak{D}), \tag{4}$$

if there exists a Schwarz function $w(z)$ with the following conditions:

$$w(0) = 0 \text{ and } |w(z)| < 1, \ (z \in \mathfrak{D}),$$

such that:

$$f(z) = g(w(z)), \ (z \in \mathfrak{D}). \tag{5}$$

Now, take the non-vanishing analytic functions $q_1(z)$ and $q_2(z)$ in $\mathfrak{D}$ that satisfy the following condition:

$$q_1(0) = q_2(0) = 1.$$

In this paper, we define a class of functions $f(z) \in \mathfrak{A}$ that satisfy the following condition:

$$\frac{f'(z)}{q_1(z)} \prec q_2(z).$$

Instead of $q_2(z)$, we will now select a specific function. Additionally, $q_1(z)$ should be subordinated to another function. These options are:

$$q_1(z) \prec e^z \text{ and } q_2(z) = 1 + \sin z.$$

Using the above-mentioned concept, we now consider the following class:

$$p^* = \left\{ f \in \mathfrak{A} : \frac{f'(z)}{q_1(z)} \prec 1 + \sin z \ \& \ q_1(z) \prec e^z, \ z \in \mathfrak{D} \right\}. \tag{6}$$

To show the functions class p^* is nonempty. For this, let $f_1, q_1 \to \mathbb{C}$ be given by:

$$q_1(z) = e^z$$

and:

$$f_1(z) = \frac{e^z}{2}(\sin(z) - \cos(z) + 2).$$

Then:

$$\frac{f_1'(z)}{q_1(z)} \prec 1 + \sin z.$$

The problem of determining coefficient bounds offers information on a complex-valued function's geometry. In particular, the second coefficient provides information about the growth and distortion theorems for functions in class $\mathcal{S}$. Similarly, in the study of singularities and power series with integral coefficients, the Hankel determinants are particularly useful. In 1976, Noonan and Thomas [1] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ of functions f as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}, \quad (a_1 = 1).$$

For some special choices of n and q we have the following selections.

1. For $q = 2$, $n = 1$:

$$H_{2,1}(f) = \Delta_1, \text{ where } \Delta_1 = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \; a_1 = 1,$$

is the famed Fekete-Szegő functional.

2. For $q = 2$, $n = 2$:

$$H_{2,2}(f) = \Delta_2, \text{ where } \Delta_2 = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_3 - a_3^2,$$

is the second Hankel determinant. Janteng et al. [2] (see also [3]) investigated the sharp boundaries of $H_2(2)$ for the class of $\mathcal{S}^*$, $\mathcal{C}$, and $\mathfrak{R}$, which are listed below:

$$|H_{2,2}(f)| \leq \begin{cases} 1 & f \in \mathcal{S}^*, \\ \frac{1}{8} & f \in \mathcal{C}, \\ \frac{4}{9} & f \in \mathfrak{K}. \end{cases}$$

Krishna [4] derived a precise estimate of $H_{2,2}(f)$ for the class of Bazilevič functions. On the other hand the sharp bound of $H_{2,2}(f)$ for the class of close-to-convex functions remains unknown.

3. For $q = 3$, $n = 1$:

$$H_{3,1}(f) = \Delta_3, \text{ where } \Delta_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix},$$

is the third Hankel determinant. Babalola [5] obtained the upper bound of $|H_{3,1}(f)|$ for $\mathcal{S}^*$, $\mathcal{C}$, and $\mathfrak{K}$. Later, other writers calculated the bounds of $|H_{3,1}(f)|$ for different subclasses of analytic and univalent functions. In 2016, Zaprawa [6] enhanced Babalola's results and demonstrated that:

$$|H_{3,1}(f)| \leqq \begin{cases} 1 & f \in \mathcal{S}^*, \\ \frac{49}{540} & f \in \mathcal{C}, \\ \frac{41}{60} & f \in \mathfrak{K}. \end{cases}$$

He also thought that the bounds were still not sharp. Later, in 2018, Kwon improved the Zaprawa inequality for $f \in \mathcal{S}^*$ by achieving $|H_{3,1}(f)| \leqq \frac{8}{9}$, and in 2021, Zaprawa refined this bound even further by establishing that $|H_{3,1}(f)| \leqq \frac{5}{9}$ for $f \in \mathcal{S}^*$. In the papers [7,8], the non-sharp bounds of this determinant for the sets $\mathcal{S}^*_{\sin}$ and $\mathcal{S}^*_{car}$, respectively, were also computed. They succeeded in achieving:

$$|H_{3,1}(f)| \leqq \begin{cases} 0.51856, & \text{for } f \in \mathcal{S}^*_{\sin}, \\ 1.1989, & \text{for } f \in \mathcal{S}^*_{car}. \end{cases}$$

Many specialists have attempted to find the determinant's sharp bounds, but none has been successful. Finally, in 2018, Kowalczyk et al. [9] and Lecko et al. [10] achieved the following sharp bounds of $|H_{3,1}(f)|$ for the sets $\mathcal{C}$ and $\mathcal{S}^*\left(\frac{1}{2}\right)$, respectively:

$$|H_{3,1}(f)| \leqq \begin{cases} \frac{4}{135}, & \text{for} \quad f \in \mathcal{C}, \\ \frac{1}{9}, & \text{for} \quad f \in \mathcal{S}^*\left(\frac{1}{2}\right). \end{cases}$$

For more information on this topic, the reader should look at the works of Srivastava et al. [11], and Wang et al. [12].

4. For $q = 4$, $n = 1$:

$$H_{4,1}(f) = \Delta_4, \text{ where } \Delta_4 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix},$$

is the fourth Hankel determinant. Since $f \in \mathcal{S}$ and $a_1 = 1$, thus:

$$\begin{aligned} H_4(1) = &\; a_7\{a_3 I_1 - a_4 I_2 + a_5 I_3\} \\ &- a_6\{a_3 I_4 - a_4 I_5 + a_6 I_3\} \\ &+ a_5\{a_3 I_6 - a_5 I_5 + a_6 I_2\} \\ &- a_4\{a_4 I_6 - a_5 I_4 + a_6 I_2\}, \end{aligned}$$

where:

$$\begin{aligned} I_1 &= a_2 a_4 - a_3^2, \; I_2 = a_4 - a_2 a_3, \; I_3 = a_3 - a_2^2, \\ I_4 &= a_2 a_5 - a_3 a_4, \; I_5 = a_5 - a_2 a_4, \; I_6 = a_3 a_5 - a_4^2. \end{aligned}$$

Many articles have been published in the last few years looking for upper bounds for the second-order Hankel determinant $H_2(2)$, the third-order Hankel determinant $H_3(1)$ and the fourth hankel determinant $H_4(1)$, see for example [13,14]. Arif et al. [15] recently researched the problem of the fourth Hankel determinant for the class of bounded turning functions for the first time and successfully achieved the bound of $H_{4,1}(f)$. Khan et al. [16] examined a range of bounded-turning functions that are connected to sine functions and found upper bounds for the third- and fourth-order Hankel determinants. As far as we know, there is minimal work related with the fourth Hankel determinant in the literature. The major objective of this work is to define a new subclass of analytic function using a new technique, we then find the fourth Hankel determinant for the our newly defined functions class.

2. A Set of Lemmas

In order to prove our desired results, we shall require the following Lemmas:

Lemma 1 (see [17]). *If $p(z) \in \mathcal{P}$, then there exist some x, z with $|x| \leq 1$, $|z| \leq 1$, such that:*

$$2c_2 = c_1^2 + \left(4 - c_1^2\right),$$

$$4c_3 = c_1^3 + 2c_1 x\left(4 - c_1^2\right) - c_1 x^2\left(4 - c_1^2\right) + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z.$$

Lemma 2 (see [18]). *Let $p(z) \in \mathcal{P}$, then:*

$$\left| c_4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4 \right| \leq 2,$$

$$\left| c_1^5 + 3c_1c_2^2 + 3c_1^2c_3 - 4c_1^3c_2 - 2c_1c_4 - 2c_2c_3 + c_5 \right| \leq 2,$$

$$\left| \begin{array}{c} c_1^6 + 6c_1^2c_2^2 + 4c_1^3c_3 + 2c_1c_5 + 2c_2c_4 + c_3^2 - c_2^3 - 5c_1^4c_2 \\ -3c_1^2c_4 - 6c_1c_2c_3 - c_6 \end{array} \right| \leq 2,$$

$$|c_n| \leq 2, n = 1, 2, 3, \ldots$$

(7)

Lemma 3 (see [19]). *Let $p(z) \in \mathcal{P}$, then:*

$$\left| c_2 - \frac{|c_1^2|}{2} \right| \leq 2 - \frac{|c_1^2|}{2},$$

$$|c_{n+k} - \mu c_n c_k| < 2 \quad 0 \leq \mu \leq 1,$$

$$\left| c_{n+2k} - \mu c_n c_k^2 \right| \leq 2(1 + 2\mu).$$

(8)

3. Main Results

We now state and prove the main results of our present investigation. The first result is about to find the bounds for the first seven initial coefficients for our defined functions class p^*. The proceeding results shall be used in order to prove the major result (the fourth Hankel Determinant) for this define functions class.

Theorem 1. *If the function $f(z) \in p^*$ and is of the form (1), then:*

$$\begin{aligned} |a_2| &\leq 1, \\ |a_3| &\leq 0.666\,67, \\ |a_4| &\leq 0.5481125224, \\ |a_5| &\leq 0.608, \\ |a_6| &\leq 0.5878016243, \\ |a_7| &\leq 0.42857. \end{aligned}$$

(9)

Proof. Since $f(z) \in p^*$, according to the definition of subordination, then there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that:

$$\frac{f'(z)}{q_1(z)} = 1 + \sin w(z),$$

(10)

where:

$$q_1(z) \prec e^z$$

and we define a function:

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots .$$

(11)

It is easy to see that $p(z) \in \mathcal{P}$ and:

$$w(z) = \frac{p(z) + 1}{p(z) - 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}.$$

On the other hand:

$$1 + \sin(w(z)) = 1 + \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(-\frac{1}{2}c_1 c_2 + \frac{1}{2}c_3 + \frac{5}{48}c_1^3\right)z^3$$
$$+ \left(\frac{5}{16}c_2 c_1^2 - \frac{1}{32}c_1^4 - \frac{1}{4}c_2^2 - \frac{1}{2}c_1 c_3 + \frac{1}{2}c_4\right)z^4 \dots \tag{12}$$

and:

$$e^{w(z)} = 1 + \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{8}c_1^2\right)z^2 + \left(-\frac{1}{4}c_1 c_2 + \frac{1}{2}c_3 + \frac{1}{48}c_1^3\right)z^3$$
$$+ \left(\frac{1}{16}c_2 c_1^2 + \frac{1}{384}c_1^4 - \frac{1}{8}c_2^2 - \frac{1}{4}c_1 c_3 + \frac{1}{2}c_4\right)z^4 \dots \tag{13}$$

Using (12) and (13) we achieve:

$$(1 + \sin w(z))e^{w(z)} = c_1 z + \left(c_2 - \frac{c_1^2}{8}\right)z^2 + \left(c_3 - \frac{c_1 c_2}{4} - \frac{c_1^3}{16}\right)z^3$$
$$+ \left(c_4 - \frac{3c_2 c_1^2}{16} - \frac{c_1 c_3}{4} - \frac{1c_2^2}{8} + \frac{25c_1^4}{384}\right)z^4$$
$$+ \left(c_5 - \frac{c_2 c_3}{4} - \frac{-c_1 c_4}{4} + \frac{25c_1^3 c_2}{96} - \frac{3c_2 c_1^2}{16} - \frac{143c_1^5}{3840} - \frac{3c_1^2 c_3}{16}\right)z^5 \tag{14}$$
$$+ \left(c_6 - \frac{3c_4 c_1^2}{16} - \frac{c_2^3 c_3}{16} - \frac{-c_1 c_5}{4} + \frac{25c_1^3 c_3}{96} + \frac{25c_1^2 c_2^2}{64}\right.$$
$$\left. - \frac{143c_2 c_1^4}{768} - \frac{743c_1^6 c_3}{46080} - \frac{c_3^2}{8} - \frac{3c_1 c_2 c_3}{8} - \frac{c_2 c_4}{4}\right)z^6 + \cdots .$$

Additionally:

$$f'(z) = z + \sum_{n=2}^{\infty} n a_n z^{n-1} = 1 + 2a_2 z^1 + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots . \tag{15}$$

When the coefficients of z, z^2, z^3 are compared between the Equations (14) and (15), we get:

$$a_2 = \frac{c_1}{2}, \ a_3 = \frac{c_2}{3} - \frac{c_1^2}{24}, \ a_4 = \frac{c_1 c_2}{16} - \frac{c_1^3}{64} + \frac{c_3}{4}, \tag{16}$$

$$a_5 = -\frac{3c_1^2 c_2}{80} - \frac{c_1 c_3}{20} - \frac{c_2^2}{40} + \frac{c_4}{5} + \frac{5c_1^4}{384}, \tag{17}$$

$$a_6 = \frac{-c_2 c_3}{24} - \frac{c_1 c_4}{24} + \frac{25c_2 c_1^3}{576} + \frac{c_5}{6} - \frac{3c_1 c_2^2}{96} - \frac{143c_1^5}{23040} - \frac{3c_1^2 c_3}{96}, \tag{18}$$

$$a_7 = \frac{-3c_1^2 c_4}{112} - \frac{c_2^3}{112} - \frac{c_1 c_5}{28} + \frac{25c_3 c_1^3}{672} + \frac{25c_1^2 c_2^2}{448} - \frac{143c_2 c_1^4}{5376}$$
$$+ \frac{743c_1^6}{322560} + \frac{c_6}{7} - \frac{c_3^2}{56} - \frac{3c_1 c_2 c_3}{56} - \frac{c_2 c_4}{28}. \tag{19}$$

Using Lemma 2, we are easily able to obtain:

$$|a_2| \leq 1,$$

$$|a_3| = \left|\frac{c_2}{3} - \frac{c_1^2}{24}\right|.$$

Using Lemma 1, we get:

$$|a_3| = \left|\frac{c_1^2 + x\left(4 - c_1^2\right)}{6} - \frac{c_1^2}{24}\right| = \left|\frac{c_1^2}{8} + \frac{x\left(4 - c_1^2\right)}{6}\right|. \tag{20}$$

We suppose that $|x| = t \in [0,1]$, $c_1 = c \in [0,2]$. Additionally, if we apply the triangle inequality to the equation above, we get:

$$|a_3| \leqslant \frac{c^2}{8} + \frac{t(4 - c^2)}{6}.$$

Assume that:

$$F(c,t) = \frac{c^2}{8} + \frac{t(4 - c^2)}{6}.$$

Then there is what we achieved:

$$\frac{\partial F}{\partial t} = \frac{(4 - c^2)}{6} \geq 0,$$

$F(c,t)$ is clearly increasing on $[0,1]$. As a result, at $t = 1$, the function $F(c,t)$ can obtain the maximum value:

$$\max F(c,t) = F(c,1) = \frac{c^2}{8} + \frac{(4 - c^2)}{6}.$$

Let:

$$G(c) = \frac{2}{3} - \frac{c^2}{24}, \ G'(c) = -\frac{c}{12} \leq 0.$$

As a result, $G(c)$ has a maximum value at $c = 0$, as seen below:

$$|a_3| \leq G(0) = \frac{2}{3},$$

$$|a_4| = \left| \frac{c_1 c_2}{16} - \frac{c_1^3}{64} + \frac{c_3}{4} \right| = \left| \frac{1}{4} \left[c_3 - \frac{c_1 c_2}{8} \right] + \frac{c_1}{32} \left[c_2 - \frac{c_1^2}{2} \right] \right|.$$

Let $c_1 = c, c \in [0,2]$; by using Lemma 3, we get:

$$|a_4| = \left| \frac{1}{4} \left[c_3 - \frac{c_1 c_2}{8} \right] + \frac{c}{32} \left[c_2 - \frac{c_1^2}{2} \right] \right| \leq \frac{1}{2} + \frac{c}{32} \left[2 - \frac{c^2}{2} \right].$$

Now, suppose:

$$F(c) = \frac{1}{2} + \frac{c}{16} - \frac{c^3}{64}.$$

Obviously, we come across:

$$F'(c) = \frac{1}{16} - \frac{3c^2}{64},$$

the critical points of the function $F(c)$ are $c = \pm\frac{2\sqrt{3}}{3}$, and we have:

$$F''(c) = F''\left(\frac{2\sqrt{3}}{3} \right) = -0.1082531755 < 0.$$

Hence, the maximum value of $F(c)$ is given by:

$$|a_4| \leq F\left(\frac{2\sqrt{3}}{3} \right) = \frac{1}{2} + \frac{c}{16} - \frac{c^3}{64} = 0.5481125224,$$

$$|a_5| = \left| -\frac{3c_1^2 c_2}{80} - \frac{c_1 c_3}{20} - \frac{c_2^2}{40} + \frac{c_4}{5} + \frac{5c_1^4}{384} \right|$$

$$= \left| \frac{1}{5} \left[c_4 - \frac{c_1 c_3}{4} \right] - \frac{5c_1^2}{192} \left[c_2 - \frac{c_1^2}{2} \right] - \frac{c_2}{40} \left[c_2 - \frac{c_1^2}{2} \right] - \frac{11c_1^2 c_2}{960} \right|.$$

Let $c_1 = c, c \in [0,2]$ according to Lemma 3:

$$|a_5| \leq \frac{2}{5} + \frac{5c^2}{192}\left[2 - \frac{c^2}{2}\right] + \frac{1}{20}\left[2 - \frac{c^2}{2}\right] + \frac{23c^2}{480}$$
$$= \frac{1}{2} + \frac{3c^2}{40} - \frac{5c^4}{384}.$$

Assume that:

$$F(c) = \frac{1}{2} + \frac{3c^2}{40} - \frac{5c^4}{384}.$$

Obviously, we come across:

$$F'(c) = \frac{3c}{20} - \frac{5c^3}{96}.$$

Setting $F'(c) = 0$, we get:

$$c = \frac{6\sqrt{2}}{5}, c = 0.$$

So, for $c = \frac{6\sqrt{2}}{5}$, we achieved:

$$F''\left(\frac{6\sqrt{2}}{5}\right) = -\frac{3}{10} < 0.$$

As a result, at $c = \frac{6\sqrt{2}}{5}$, the function $F(c)$ can obtain the maximum value:

$$|a_5| \leq F\left(\frac{6\sqrt{2}}{5}\right) = \frac{76}{125} = 0.608.$$

$$|a_6| = \left| \frac{-c_2 c_3}{24} - \frac{c_1 c_4}{24} + \frac{25 c_2 c_1^3}{576} + \frac{c_5}{6} - \frac{3c_1 c_2^2}{96} - \frac{143 c_1^5}{23040} - \frac{3c_1^2 c_3}{96} \right|$$
$$= \left| \frac{1}{8}\left[c_5 - \frac{c_1 c_4}{3}\right] + \frac{1}{24}[c_5 - c_1 c_3] + \frac{143 c_1^3}{11520}\left[c_2 - \frac{c_1^2}{2}\right] \right.$$
$$\left. - \frac{3c_1 c_2}{96}\left[c_2 - \frac{c_1^2}{2}\right] + \frac{59 c_1^3 c_2}{3840} - \frac{3c_1^2 c_3}{96} \right|.$$

Let $c_1 = c, c \in [0,2]$, by using Lemma 3 we get:

$$|a_6| \leq \frac{2}{8} + \frac{2}{24} + \frac{143c^3}{11520}\left[2 - \frac{c^2}{2}\right] + \frac{1}{8}\left[2 - \frac{c^2}{2}\right] + \frac{59c^3}{1920}$$
$$= \frac{7}{12} - \frac{c^2}{16} + \frac{c^3}{18} - \frac{143c^5}{23040} + \frac{3c^2}{48}.$$

Assume:

$$F(c) = \frac{7}{12} + \frac{c^3}{18} - \frac{143c^5}{23040}.$$

Obviously, we come across:

$$F'(c) = \frac{c^2}{6} - \frac{143c^4}{4608}.$$

Setting $F'(c) = 0$, we get $c = 0$ is only one root lies in $[0,2]$. So for $c = 0$, the function $F(c)$ can obtain the maximum value:

$$|a_6| \leq F(0) = \frac{7}{12} = 0.5833.$$

$$|a_7| = \left| \frac{-3c_1^2 c_4}{112} - \frac{c_2^3}{112} - \frac{c_1 c_5}{28} + \frac{25 c_3 c_1^3}{672} + \frac{25 c_1^2 c_2^2}{448} - \frac{143 c_2 c_1^4}{5376} \right.$$

$$\left. + \frac{743 c_1^6}{322560} + \frac{c_6}{7} - \frac{c_3^2}{56} - \frac{3 c_1 c_2 c_3}{56} - \frac{c_2 c_4}{28} \right|$$

$$= \left| \frac{-3 c_1^2 [c_4 - c_1^2]}{112} - \frac{3 c_1 c_2 [c_3 - c_1 c_2]}{56} + \frac{25 c_1^3 [c_3 - c_1 c_3]}{672} \right.$$

$$+ \frac{19 c_1^4}{1792} \left[c_2 - \frac{c_1^2}{2} \right] - \frac{c_2^2}{112} \left[c_2 - \frac{c_1^2}{2} \right]$$

$$\left. - \frac{1}{7} \left[c_6 - \frac{c_2 c_4}{4} \right] + \frac{2453 c_1^6}{322560} + \frac{11 c_1^2 c_2^2}{64} - \frac{c_3^2}{56} - \frac{c_1 c_5}{28} \right|.$$

By taking $c_1 = c, c \in [0, 2]$, along with the use of Lemma 3, we obtained:

$$|a_7| \leq \frac{5}{14} + \frac{2c}{7} - \frac{81 c^2}{56} + \frac{25 c^3}{336} + \frac{19 c^4}{896} + \frac{743 c^6}{322560}.$$

Assume:

$$F(c) = \frac{5}{14} + \frac{2c}{7} - \frac{81 c^2}{112} + \frac{25 c^3}{336} + \frac{19 c^4}{896} + \frac{743 c^6}{322560}.$$

Obviously, we come across:

$$F'(c) = \frac{2}{7} - \frac{81 c}{56} + \frac{25 c^2}{112} + \frac{19 c^3}{224} + \frac{743 c^5}{53760}.$$

When we set $F'(c) = 0$, we get $c = 0.20449$, which is the only root of $F'(c) = 0$, belonging to the interval $[0, 2]$, obviously, we find:

$$F''(c) = -\frac{81}{56} + \frac{25 c}{56} + \frac{57 c^2}{224} + \frac{743 c^4}{10752},$$

$$F''(c = 0.20449) = -1.3439.$$

As a result, at $c = 0.20449$, $F(c)$ reaches its maximum value:

$$|a_7| \leq F(0.20449) = 0.386000091.$$

Hence the proof is completed. □

Theorem 2. *If the function $f(z) \in p^*$ and is of the form (1), then we have:*

$$\left| a_3 - a_2^2 \right| \leq \frac{2}{3}. \tag{21}$$

Proof. From (16), we have:

$$\left| a_3 - a_2^2 \right| = \left| \frac{c_2}{3} - \frac{7 c_1^2}{24} \right|.$$

Using Lemma 1, we get:

$$\left| a_3 - a_2^2 \right| = \left| \frac{c_1^2 + x(4 - c_1^2)}{6} - \frac{7 c_1^2}{24} \right|. \tag{22}$$

We suppose that $|x| = t \in [0,1]$, $c_1 = c \in [0,2]$. Additionally, if we apply the triangle inequality to the equation above, we get:

$$\left| a_3 - a_2^2 \right| \leq \frac{t(4-c^2)}{6} - \frac{c^2}{8}.$$

Assume that:

$$F(c,t) = \left| a_3 - a_2^2 \right| \leq \frac{t(4-c^2)}{6} - \frac{c^2}{8}.$$

Obviously, we find:

$$\frac{\partial F}{\partial t} = \frac{(4-c^2)}{6} \geq 0,$$

$F(c,t)$ is clearly increasing on $[0,1]$. As a result, at $t = 1$, the function $F(c,t)$ can obtain the maximum value:

$$\max F(c,t) = F(c,1) = \frac{(4-c^2)}{6} - \frac{c^2}{8} = \frac{2}{3} - \frac{c^2}{24}.$$

Let:

$$\begin{aligned} G(c) &= \frac{2}{3} - \frac{c^2}{24} \\ G'(c) &= -\frac{c}{12} \leq 0, \end{aligned}$$

$G(c)$ is clearly decreasing on $[0,2]$. As a result, at $c = 0$, the function $G(c)$ can obtain the maximum value:

$$\left| a_3 - a_2^2 \right| \leq G(0) = \frac{2}{3}.$$

Hence, proving Theorem 2. $\square$

Theorem 3. *If the function* $f(z) \in p^*$ *and of the form* (1), *then we have:*

$$|a_2 a_3 - a_4| \leq 0.540256083. \tag{23}$$

Proof. From (16), we have:

$$\begin{aligned} |a_2 a_3 - a_4| &= \left| \frac{c_1 c_2}{6} - \frac{c_1^3}{48} - \frac{c_1 c_2}{16} + \frac{c_1^3}{64} - \frac{c_3}{4} \right| \\ &= \left| \frac{5 c_1 c_2}{48} - \frac{c_1^3}{192} - \frac{c_3}{4} \right|. \end{aligned}$$

We can deduce from the Lemma 1 that:

$$|a_2 a_3 - a_4| = \left| \frac{-c_1^3}{64} + \frac{7 c_1 x (4 - c_1^2)}{96} + \frac{c_1 x^2 (4 - c_1^2)}{16} - \frac{(4 - c_1^2)(1 - |x|^2) z}{8} \right|.$$

We suppose that $|x| = t \in [0,1]$, $c_1 = c \in [0,2]$. Additionally, if we apply the triangle inequality to the equation above, we get:

$$|a_2 a_3 - a_4| \leq \frac{c^3}{64} + \frac{7ct(4-c^2)}{96} + \frac{ct^2(4-c^2)}{16} + \frac{(4-c^2)}{8}.$$

Suppose that:

$$F(c,t) = \frac{c^3}{64} + \frac{7ct(4-c^2)}{96} + \frac{ct^2(4-c^2)}{16} + \frac{(4-c^2)}{8}.$$

Then, we obtain:

$$\frac{\partial F}{\partial t} = \frac{7c(4-c^2)}{96} + \frac{ct(4-c^2)}{8} \geq 0.$$

As a result, $F(c,t)$ is an increasing function about t on the closed interval $[0,1]$. This means that $F(c,t)$, reaches its maximum value at $t = 1$, which is:

$$\max F(c,t) = F(c,1) = \frac{c^3}{64} + \frac{7c(4-c^2)}{96} + \frac{c(4-c^2)}{16} + \frac{(4-c^2)}{8}.$$

Now, define:

$$G(c) = \frac{c^3}{64} + \frac{7c(4-c^2)}{96} + \frac{c(4-c^2)}{16} + \frac{(4-c^2)}{8}$$

$$= \frac{1}{4} + \frac{13c}{24} - \frac{c^2}{8} - \frac{23c^3}{192}.$$

Obviously, we find:

$$G'(c) = \frac{13}{24} - \frac{c}{4} \frac{-23c^2}{64}.$$

When we set $G'(c) = 0$, we get $c = \frac{-24+\sqrt{29280}}{138}$, obviously, we find:

$$G''(c) = -\frac{1}{4} - \frac{23c}{32},$$

$$G''(c) = G''\left(\frac{-24+\sqrt{29280}}{138}\right) = -1.066043457.$$

As a result, the function $G(c)$ reaches its greatest value at c $c = r = \frac{-24+\sqrt{29280}}{138}$, which is also:

$$|a_2 a_3 - a_4| = G(c) = G\left(\frac{-24+\sqrt{29280}}{138}\right) \leq 0.540256083.$$

The proof of Theorem 3 is completed. □

Theorem 4. *If the function $f(z) \in p^*$ and is of the form (1), then we have:*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{17}{18}. \tag{24}$$

Proof. From (16), we have:

$$\left|a_2 a_4 - a_3^2\right| = \left|\frac{c_1^2 c_2}{32} - \frac{c_1^4}{128} + \frac{c_1 c_3}{8} - \left(\frac{c_2}{3} - \frac{c_1^2}{24}\right)^2\right|$$

$$= \left|\frac{17c_1^2 c_2}{288} - \frac{11c_1^4}{64} + \frac{c_1 c_3}{8} - \frac{c_2^2}{9}\right|.$$

As a result of Lemma 1, we obtain:

$$\left| a_2 a_4 - a_3^2 \right| = \left| \frac{17 c_1^2 c_2}{288} - \frac{11 c_1^4}{64} + \frac{c_1 c_3}{8} - \frac{c_2^2}{9} \right|$$

$$= \left| \frac{7 c_1^2 x \left(4 - c_1^2 \right)}{192} - \frac{c_1^2 x^2 \left(4 - c_1^2 \right)}{32} + \frac{c_1 \left(4 - c_1^2 \right) \left(1 - |x|^2 \right) z}{16} \right.$$

$$\left. + \frac{x^2 \left(4 - c_1^2 \right)^2}{36} + \frac{3 c_1^4}{128} \right|.$$

We suppose that $|x| = t \in [0, 1]$, $c_1 = c \in [0, 2]$. Additionally, if we apply the triangle inequality to the equation above, we get:

$$\left| a_2 a_4 - a_3^2 \right| \le \frac{7 t c^2 \left(4 - c^2 \right)}{192} + \frac{c^2 t^2 \left(4 - c^2 \right)}{32} + \frac{t^2 \left(4 - c^2 \right)^2}{36} + \frac{\left(4 - c^2 \right)}{2} + \frac{3 c^4}{128}.$$

Suppose that:

$$F(c, t) = \frac{7 t c^2 \left(4 - c^2 \right)}{192} + \frac{c^2 t^2 \left(4 - c^2 \right)}{32} + \frac{t^2 \left(4 - c^2 \right)^2}{36} + \frac{\left(4 - c^2 \right)}{2} + \frac{3 c^4}{128}.$$

Obviously, we find:

$$\frac{\partial F}{\partial t} = \frac{7 c^2 \left(4 - c^2 \right)}{192} + \frac{t c^2 \left(4 - c^2 \right)}{16} + \frac{t \left(4 - c^2 \right)^2}{18} \ge 0.$$

As a result, $F(c, t)$ is an increasing function about t on the closed interval $[0, 1]$. This means that $F(c, t)$, reaches its maximum value at $t = 1$, which is:

$$\max F(c, t) = F(c, 1) = \frac{7 c^2 \left(4 - c^2 \right)}{192} + \frac{c^2 \left(4 - c^2 \right)}{32} + \frac{\left(4 - c^2 \right)^2}{36} + \frac{\left(4 - c^2 \right)}{2} + \frac{3 c^4}{128}.$$

Now, define:

$$G(c) = \frac{7 c^2 \left(4 - c^2 \right)}{192} + \frac{c^2 \left(4 - c^2 \right)}{32} + \frac{\left(4 - c^2 \right)^2}{36} + \frac{\left(4 - c^2 \right)}{2} + \frac{3 c^4}{128}$$

$$= \frac{17}{18} - \frac{11 c^2}{144} - \frac{19 c^4}{1152}.$$

Then:

$$G'(c) = -\frac{11 c}{72} - \frac{19 c^3}{288} \le 0.$$

This means that at $c = 0$, the function $G(c)$ can reach its maximum value:

$$\left| a_2 a_4 - a_3^2 \right| = G(c) = G(0) \le \frac{17}{18}.$$

We complete the proof of Theorem 4. □

Theorem 5. *If the function $f(z) \in p^*$ and is of the form* (1)*, then we have:*

$$\left| a_2 a_5 - a_3 a_4 \right| \le \frac{1}{3}. \tag{25}$$

Proof. From (16) and (17), we have:

$$|a_2 a_5 - a_3 a_4| = \left| \frac{9c_1^3 c_2}{640} - \frac{7c_1^2 c_3}{480} + \frac{c_1 c_2^2}{30} + \frac{c_1 c_4}{10} - \frac{3c_1^5}{512} - \frac{c_2 c_3}{12} \right|$$

$$= \left| \frac{9c_1^3 \left[c_2 - \frac{c_1^2}{2} \right]}{640} - \frac{c_3 \left[c_2 - \frac{c_1^2}{2} \right]}{12} - \frac{9c_1 [c_4 - c_1 c_3]}{160} \right.$$

$$\left. + \frac{7c_1 \left[c_4 - \frac{16c_2^2}{21} \right]}{160} + \frac{33c_1^5}{2560} \right|.$$

Using Lemma 3, we obtain:

$$|a_2 a_5 - a_3 a_4| \leq \frac{9c^3 \left[2 - \frac{c^2}{2} \right]}{640} + \frac{\left[2 - \frac{c^2}{2} \right]}{6} + \frac{c}{5} + \frac{33c^5}{2560}$$

$$= \frac{1}{3} + \frac{c}{5} - \frac{c^2}{12} + \frac{9c^3}{320} + \frac{3c^5}{512}.$$

Assume that:

$$F(c) = \frac{1}{3} + \frac{c}{5} - \frac{c^2}{12} + \frac{9c^3}{320} + \frac{3c^5}{512}.$$

Obviously, we find:

$$F'(c) = \frac{1}{5} - \frac{c}{6} + \frac{27c^3}{320} + \frac{15c^4}{512}.$$

When we set $F'(c) = 0$, we get $c = 0$, $c = 1.2678$. Consequently, we find:

$$F''(c) = -\frac{1}{6} + \frac{27c}{160} + \frac{15c^3}{512},$$

$$F''(0) = -\frac{1}{6} < 0.$$

Consequently, at $c = 0$, $F(c)$ reaches its maximum value, which is:

$$|a_2 a_5 - a_3 a_4| \leq F(0) \leq \frac{1}{3}.$$

We complete the proof of Theorem 5. □

Theorem 6. *If the function $f(z) \in p^*$ and is of the form* (1), *then we have:*

$$|a_5 - a_2 a_4| \leq \frac{2147}{3680}. \tag{26}$$

Proof. From (16) and (17), we have:

$$|a_5 - a_2 a_4| = \left| \frac{c_1^4}{48} - \frac{7c_1 c_3}{40} - \frac{3c_1 c_2^2}{160} - \frac{c_2^2}{40} + \frac{c_4}{5} \right|$$

$$= \left| \frac{\left[c_1^4 + c_2^2 + 2c_1 c_3 - 3c_1^2 c_2 - c_4 \right]}{40} + \frac{3c_1^2 [c_2 - c_1^2 / 2]}{32} \right.$$

$$\left. - \frac{c_1^4}{960} + \frac{7[c_4 - 40c_1 c_3 / 56]}{40} \right|.$$

Letting $|x| = t \in [0, 1]$, $c_1 = c \in [0, 2]$ and using Lemmas 2 and 3, we obtain:

$$|a_5 - a_2 a_4| \leq \frac{2}{40} + \frac{3c^2 \left[2 - c^2/2 \right]}{32} + \frac{7}{20} + \frac{c^4}{960}.$$

Suppose that:

$$F(c) = \frac{2}{5} + \frac{3c^2}{16} - \frac{23c^4}{480}.$$

Obviously, we find:

$$F'(c) = \frac{3c}{8} - \frac{23c^3}{120}.$$

When we set $F'(c) = 0$, we get $c = 0$, $c = \pm \frac{3\sqrt{115}}{23}$. Consequently, we find:

$$F''(c) = \frac{3}{8} - \frac{23c^2}{40}, \quad F''\left(\frac{3\sqrt{115}}{23} \right) = -0.7499999.$$

This means that $F(c)$ reaches its maximum value at $c = \frac{3\sqrt{115}}{23}$, which is:

$$|a_5 - a_2 a_4| \leq F\left(\frac{3\sqrt{115}}{23} \right) = \frac{2147}{3680}.$$

The proof of the Theorem 6 is now complete. $\square$

Theorem 7. *If the function $f(z) \in p^*$ and is of the form* (1), *then we have:*

$$\left| a_5 a_3 - a_4^2 \right| \leq \frac{7}{12}. \tag{27}$$

Proof. From (16) and (17), we have:

$$\left| a_5 a_3 - a_4^2 \right| = \left| \frac{9c_1^2 c_2^2}{768} - \frac{23 c_1 c_2 c_3}{480} - \frac{c_2^3}{120} + \frac{c_2 c_4}{15} + \frac{133 c_1^4 c_2}{23040} \right.$$

$$\left. + \frac{19 c_1^3 c_3}{1920} - \frac{c_1^2 c_4}{120} - \frac{29 c_1^6}{36864} - \frac{c_3^2}{16} \right|$$

$$= \left| \frac{c_2}{15} \left[c_4 - \frac{23 c_1 c_3}{2} \right] - \frac{c_3}{16} \left[c_3 - \frac{23 c_1 c_2}{2} \right] - \frac{c_2^2 \left[c_2 - \frac{c_1^2}{2} \right]}{120} \right.$$

$$\left. - \frac{11 c_1^2 c_2 \left[c_2 - \frac{c_1^2}{2} \right]}{3840} + \frac{19 c_1^3}{1920} \left[c_3 - \frac{9600 c_1 c_2}{21888} \right] - \frac{c_1^2 c_4}{120} - \frac{29 c_1^6}{36864} \right|.$$

After that, use Lemmas 2 and 3, we obtain:

$$\left| a_5 a_3 - a_4^2 \right| \leq \frac{4}{15} + \frac{1}{4} + \frac{1}{30} \left[2 - \frac{c^2}{2} \right] + \frac{11 c^2}{1920} \left[2 - \frac{c^2}{2} \right]$$

$$+ \frac{19 c^3}{960} + \frac{c_1^2 c_4}{120} - \frac{29 c^6}{36864}.$$

Assume that:

$$F(c) = \frac{4}{15} + \frac{1}{4} + \frac{1}{30} \left[2 - \frac{c^2}{2} \right] + \frac{11 c^2}{1920} \left[2 - \frac{c^2}{2} \right] + \frac{19 c^3}{960} + \frac{c^2 c_4}{120} - \frac{29 c^6}{36864}$$

$$= \frac{7}{12} + \frac{11 c^2}{960} + \frac{19 c^3}{960} - \frac{11 c^4}{3840} + \frac{29 c^6}{36834}.$$

Then for all $c \in [0, 2]$ we have:

$$F'(c) = \frac{11c}{480} + \frac{19c^2}{320} - \frac{11c^3}{960} + \frac{29c^5}{6144}.$$

When we set $F'(c) = 0$, we get $c = 0$, which is the only root of $F'(c) = 0$, belonging to the $[0, 2]$, obviously, we find:

$$F''(c) = -\frac{7}{160} + \frac{19c}{160} - \frac{11c^2}{320} + \frac{145c^4}{6144},$$

$$F''(0) = -\frac{7}{160}.$$

This means that $F(c)$ reaches its maximum value at $c = 0$, which is:

$$\left| a_5 a_3 - a_4^2 \right| \leq \frac{7}{12}. \tag{28}$$

We complete the proof of Theorem 7. □

We now state and prove the result related to fourth Hankel Determinant. We will use all the above results in order to obtain the bound for $H_4(1)$.

Theorem 8. *If the function $f(z) \in p^*$ and is of the form (1), then we have:*

$$|H_4(1)| \leq 1.15 \tag{29}$$

Proof.

$$\begin{aligned}
H_4(1) = a_7 &\left\{ a_3 \left(a_2 a_4 - a_3^2 \right) - a_4 (a_4 - a_2 a_3) + a_5 \left(a_3 - a_2^2 \right) \right\} \\
- a_6 &\left\{ a_3 (a_2 a_5 - a_3 a_4) - a_4 (a_5 - a_2 a_4) + a_6 \left(a_3 - a_2^2 \right) \right\} \\
&\left\{ -a_6 a_3 (a_2 a_5 - a_3 a_4) - a_4 (a_5 - a_2 a_4) + \left(a_6 a_3 - a_2^2 \right) \right\} \\
+ a_5 &\left\{ a_3 \left(a_3 a_5 - a_4^2 \right) - a_5 (a_5 - a_2 a_4) + a_6 (a_4 - a_2 a_3) \right\} \\
- a_4 &\left\{ a_4 \left(a_3 a_5 - a_4^2 \right) - a_5 (a_2 a_5 - a_3 a_4) + a_6 (a_4 - a_2 a_3) \right\},
\end{aligned}$$

so, by applying the triangle inequality, we obtain:

$$\begin{aligned}
H_4(1) = &|a_7||a_3|\left| a_2 a_4 - a_3^2 \right| + |a_4||a_4 - a_2 a_3| + |a_5|\left| a_3 - a_2^2 \right| \\
&+ |a_6||a_3||a_2 a_5 - a_3 a_4| + |a_4||a_5 - a_2 a_4| + |a_6|\left| a_3 - a_2^2 \right| \\
&+ |a_6||a_3||a_2 a_5 - a_3 a_4| + |a_4||a_5 - a_2 a_4| + |a_6|\left| a_3 - a_2^2 \right| \\
&+ |a_5||a_3|\left| a_3 a_5 - a_4^2 \right| + |a_5||(a_5 - a_2 a_4)| + |a_6||a_4 - a_2 a_3| \\
&+ |a_4|^2\left| a_3 a_5 - a_4^2 \right| - |a_5||a_2 a_5 - a_3 a_4| + |a_6||a_4 - a_2 a_3|. \tag{30}
\end{aligned}$$

Next, substituting Equations (9), (21), (23)–(28) into (30), we easily get the desired assertion as given in (29). □

4. Conclusions

In Geometric Function Theory, many authors have studied and investigated the third Hankel determinant problems for different subclasses of analytic functions as described in the introduction section. Recently, the investigation of the third and fourth Hankel determinant got attractions of many well-known mathematicians see for example [20–27]. We have essentially motivated by the recent research going on, in this paper, we have first

considered the class of normalized holomorphic functions f in such way that the ratio $\frac{f'(z)}{q_1(z)}$ is subordinate to $q_2(z)$, where $q_1(z)$ and $q_2(z)$ are non-vanishing holomorphic functions in the open unit disc. We have then derived the fourth Hankel determinant bound for our defined functions class.

In concluding our present investigation, one may attempt to produce the similar bounds for different subclasses of analytic functions. The current results presented in this article can be derive by means of certain q-difference operators.

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