symmetry

MDPI

Article

Extended Approach to the Asymptotic Behavior and Symmetric Solutions of Advanced Differential Equations

Omar Bazighifan [1,†], Ali Hasan Ali [2,†], Fatemah Mofarreh [3,†] and Youssef N. Raffoul [4,*,†]

1 Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy; o.bazighifan@gmail.com
2 Doctoral School of Mathematical and Computational Sciences, University of Debrecen, H-4002 Debrecen, Hungary; ali.hasan@science.unideb.hu
3 Mathematical Science Department, Faculty of Science, Princess Nourah Bint Abdulrahman University, Riyadh 11546, Saudi Arabia; fyalmofarrah@pnu.edu.sa
4 Department of Mathematics, University of Dayton, 300 College Park, Dayton, OH 45469, USA
* Correspondence: yraffoul1@udayton.edu
† These authors contributed equally to this work.

Abstract: We studied the asymptotic behavior of fourth-order advanced differential equations of the form $\left(a(v)(w'''(v))^{\beta}\right)' + q(v)g(w(\delta(v))) = 0$. New results are presented for the oscillatory behavior of these equations in the form of Philos-type and Hille–Nehari oscillation criteria. Some illustrative examples are presented.

Keywords: Philos-type oscillation criteria; Hille–Nehari-type oscillation criteria; asymptotic behavior; fourth-order equations; advanced differential equations

MSC: 34K11

check for updates

Citation: Bazighifan, O.; Ali, A.H.; Mofarreh, F.; Raffoul, Y.N. Extended Approach to the Asymptotic Behavior and Symmetric Solutions of Advanced Differential Equations. *Symmetry* **2022**, *14*, 686. https://doi.org/10.3390/sym14040686

Academic Editors: Wei-Shih Du and Alexander Zaslavski

Received: 28 February 2022
Accepted: 24 March 2022
Published: 26 March 2022

1. Introduction

As is well known, differential equations have many real-world applications [1]. Advanced differential equations, in particular, find applications in dynamical systems, mathematics of networks, optimization, and in the mathematical modeling of engineering processes, such as those found in electrical power systems, materials, and energy [2]. In the last decade, one important area of active research is the study of the qualitative oscillation behavior of differential equations [3–18]. In this paper, we investigate the oscillation of fourth-order nonlinear advanced differential equations of the following form:

$$\left(a(v)(w'''(v))^{\beta}\right)' + q(v)g(w(\delta(v))) = 0, \tag{1}$$

where $v \geq v_0$. Our main aim is to complement and improve the results in [19–21]. To motivate that, we briefly review and put into context those and related results.

In [17,18], Zhang et al. obtain, under mild assumptions and with the help of the comparison method with first-order equations, an oscillation criterion ensuring that every solution w of equation

$$\left(\left(w^{(\kappa-1)}(v)\right)^{\beta}\right)' + q(v)w^{\alpha}(\delta(v)) = 0 \tag{2}$$

with $\delta(v) \leq v$, $\alpha \leq \beta$, κ even, and α, β ratios of odd positive integers, is either oscillatory or satisfies $\lim_{v\to\infty} w(v) = 0$. For the special case when $\beta = \alpha$, Zhang et al. [22] obtain—under similar assumptions of those in [17,18], but now using the comparison method with second-order equations—some results on the asymptotic behavior of (2) in the case of $\kappa = 4$.

Agarwal and Grace [19] and Agarwal et al. [20] study canonical even-order nonlinear advanced differential equations, as follows:

$$\left(\left(w^{(\kappa-1)}(v)\right)^{\beta}\right)' + q(v)w^{\beta}(\delta(v)) = 0 \tag{3}$$

by means of the Riccati transformation technique, establishing some oscillation criteria for (3) when $\delta(v) \geq v$, κ is even, and β is a ratio of odd positive integers. As a special case, when $\beta = 1$, Equation (3) becomes

$$w^{(\kappa)}(v) + q(v)w(\delta(v)) = 0. \tag{4}$$

Grace and Lalli [21] study the oscillation of (4), in the case when κ is even, under the following condition:

$$\int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} \mathrm{d}s = \infty. \tag{5}$$

To prove their results, they apply previous mentioned results to the following equation:

$$w^{(\kappa)}(v) + \frac{q_0}{v^{\kappa}} w(\lambda v) = 0, \quad v \geq 1. \tag{6}$$

If we set $\kappa = 4$ and $\lambda = 2$, then, by applying the conditions in [19–21] to Equation (6), we find the results in [20], which improve those of [21]. Moreover, the results in [19] improve the ones of [20,21]. Thus, our motivation here is to complement and improve the results of [19–21]. From them, we obtain new criteria for the oscillation of Equation (1).

The paper is organized as follows. We begin with Section 2 of preliminaries, fixing our assumptions and notations and recalling necessary definitions and results from the literature. Our results are then given in Section 3: we prove conditions assuring that every solution w of (1) is either oscillatory or satisfies $\lim_{v\to\infty} w(v) = 0$ (see Theorems 1 and 2 and Corollary 1). In Section 4, we give two simple examples for which previous results of the literature do not apply, while our Hille–Nehari-type oscillation criterion holds. We end with Section 5—conclusions and future works—posing an interesting and challenging open question.

2. Hypotheses and Preliminaries

Throughout the work, we assume the following assumptions to (1):

A_1: β is a quotient of odd positive integers;

A_2: $a \in C^1([v_0,\infty),\mathbb{R})$, $a(v) > 0$, and $a'(v) \geq 0$ with

$$\int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} \mathrm{d}s < \infty; \tag{7}$$

A_3: $q \in C([v_0,\infty),\mathbb{R})$ with $q(v) \geq 0$;

A_4: $\delta \in C([v_0,\infty),\mathbb{R})$ with $\delta(v) \geq v$ and $\lim_{v\to\infty} \delta(v) = \infty$;

A_5: $g \in C(\mathbb{R},\mathbb{R})$ such that $g(x)/x^{\beta} \geq k > 0$ for $x \neq 0$.

By a solution of (1), we mean the function $w \in C^3[v_w,\infty)$, $v_w \geq v_0$, which has the property $a(v)(w'''(v))^{\beta} \in C^1[v_w,\infty)$ and satisfies (1) on $[v_w,\infty)$. We consider only those solutions w of (1) that satisfy $\sup\{|w(v)| : v \geq v_w\} > 0$. To prove our results, we make use of the following methods: (i) an integral averaging technique; (ii) Riccati transformation techniques; (iii) a comparison method with second-order differential equations.

Definition 1 (See [23]). *We say that the differential equation*

$$\left[a(v)\left(w'(v)\right)^{\beta}\right]' + q(v)w^{\beta}(g(v)) = 0, \quad v \geq v_0, \tag{8}$$

where $\beta > 0$ is a ratio of odd positive integers and $a, q \in C([v_0, \infty), \mathbb{R}^+)$, is nonoscillatory, if there exists a number $v \geq v_0$ and a function $\varsigma \in C^1([v_0, \infty), \mathbb{R})$ satisfying the inequality

$$\varsigma'(v) + \gamma a^{-1/\beta}(v)(\varsigma(v))^{(1+\beta)/\beta} + q(v) \leq 0$$

on $[v_0, \infty)$.

Definition 2 (See [24])**.** *Let $D = \{(v,s) \in \mathbb{R}^2 : v \geq s \geq v_0\}$ and $D_0 = \{(v,s) \in \mathbb{R}^2 : v > s \geq v_0\}$. A kernel function $H_i \in C(D, \mathbb{R})$, $(i = 1, 2, \ldots, n)$ is said to belong to the set $\Im$, written by $H \in \Im$, if, for $i = 1, 2$, one has:*

(i) *$H_i(v,s) = 0$ for $v \geq v_0$ and $H_i(v,s) > 0$ for $(v,s) \in D_0$;*
(ii) *$H_i(v,s)$ has a continuous and non-positive partial derivative $\partial H_i/\partial s$ on D_0, and there exist functions $\eta, \vartheta \in C^1([v_0, \infty), (0, \infty))$ and $h_i \in C(D_0, \mathbb{R})$, such that*

$$\frac{\partial}{\partial s} H_1(v,s) + \frac{\eta'(s)}{\eta(s)} H_1(v,s) = h_1(v,s) H_1^{\beta/(\beta+1)}(v,s) \tag{9}$$

and

$$\frac{\partial}{\partial s} H_2(v,s) + \frac{\vartheta'(s)}{\vartheta(s)} H_2(v,s) = h_2(v,s) \sqrt{H_2(v,s)}. \tag{10}$$

Notation 1. *For convenience, we denote*

$$\begin{aligned}
\zeta(s) &:= \int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} \mathrm{d}s, \\
\pi(s,v) &:= \frac{h_1^{\beta+1}(v,s) H_1^{\beta}(v,s)}{(\beta+1)^{\beta+1}} \frac{2^{\beta} \eta(s) a(s)}{(\theta s^2)^{\beta}}, \\
\tilde{\pi}(s,v) &:= \frac{\beta^{\beta+1} H_3(v,s)}{(\beta+1)^{\beta+1}} \frac{1}{a^{1/\beta}(s) \zeta(s)}
\end{aligned}$$

where $\theta \in (0,1)$, and

$$\varpi(v) := \int_{v}^{\infty} \left(\frac{k}{a(\varsigma)} \int_{\varsigma}^{\infty} q(s) \mathrm{d}s \right)^{1/\beta} \mathrm{d}\varsigma.$$

We shall employ the following four lemmas:

Lemma 1 (See [18])**.** *Suppose that $w \in C^{\kappa}([v_0, \infty), (0, \infty))$, $w^{(\kappa)}$ is of a fixed sign on $[v_0, \infty)$, $w^{(\kappa)}$ is not identically zero, and there exists a $v_1 \geq v_0$, such that*

$$w^{(\kappa-1)}(v) w^{(\kappa)}(v) \leq 0$$

for all $v \geq v_1$. If we have $\lim_{v \to \infty} w(v) \neq 0$, then there exists $v_{\theta} \geq v_1$, such that

$$w(v) \geq \frac{\theta}{(\kappa-1)!} v^{\kappa-1} \left| w^{(\kappa-1)}(v) \right|$$

for every $\theta \in (0,1)$ and $v \geq v_{\theta}$.

Lemma 2 (See [5])**.** *If $w^{(i)}(v) > 0$, $i = 0, 1, \ldots, \kappa$, and $w^{(\kappa+1)}(v) < 0$, then*

$$\frac{w(v)}{v^{\kappa}/\kappa!} \geq \frac{w'(v)}{v^{\kappa-1}/(\kappa-1)!}.$$

Lemma 3 (See [4]). *Let β be a ratio of two odd numbers and $V > 0$ and U be two constants. Then,*

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^\beta}.$$

Lemma 4 (See [25]). *If w is a positive solution of (1), then there exist three possible situations for $v \geq v_1$, where $v_1 \geq v_0$ is sufficiently large:*

$(\mathbf{S}_1)$ $w(v) > 0, w'(v) > 0, w''(v) > 0, w'''(v) > 0, w^{(4)}(v) < 0,$

$(\mathbf{S}_2)$ $w(v) > 0, w'(v) > 0, w''(v) < 0, w'''(v) > 0, w^{(4)}(v) < 0,$

$(\mathbf{S}_3)$ $w(v) > 0,\ w''(v) > 0,\ \ w'''(v) < 0.$

We are now in a position to formulate and prove our original results.

3. Main Results

In our first theorem, we employ an integral averaging technique to establish a Philos-type oscillation criterion.

Theorem 1 (Philos-type oscillation criterion for (1)). *Under assumptions A_1–A_5, if there exist positive functions $\eta, \vartheta \in C^1([v_0,\infty),\mathbb{R})$, such that*

$$\limsup_{v\to\infty} \frac{1}{H_1(v,v_1)} \int_{v_1}^{v} (H_1(v,s)k\eta(s)q(s) - \pi(s,v))\mathrm{d}s = \infty \tag{11}$$

for all $\theta \in (0,1)$, if

$$\limsup_{v\to\infty} \frac{1}{H_2(v,v_1)} \int_{v_1}^{v} \left(H_2(v,s)\vartheta(s)\varpi(s) - \frac{\vartheta(s)h_2^2(v,s)}{4} \right)\mathrm{d}s = \infty \tag{12}$$

and, for every $\theta_1 \in (0,1)$,

$$\limsup_{v\to\infty} \frac{1}{H_3(v,v_1)} \int_{v_1}^{v} \left(H_3(v,s)kq(s)\left(\frac{\theta_1\delta^2(s)}{2}\right)^\beta \zeta^\beta(\delta(s)) - \tilde{\pi}(s,v) \right)\mathrm{d}s = \infty,$$

then every solution of (1) is either oscillatory or satisfies $\lim\limits_{v\to\infty} w(v) = 0$.

Proof. Assume the contrary, that w is a positive solution of (1). Then, we can suppose that $w(v)$ and $w(\delta(v))$ are positive for all $v \geq v_1$ and are sufficiently large. From Lemma 4, we have three possible situations $(\mathbf{S}_1)$, $(\mathbf{S}_2)$ or $(\mathbf{S}_3)$. Assume that $(\mathbf{S}_1)$ holds. Using Lemma 1, we find that

$$w'(v) \geq \frac{\theta}{2}v^2 w'''(v) \tag{13}$$

for every $\theta \in (0,1)$ and for all large v. Define

$$\varphi(v) := \eta(v)\left(\frac{a(v)(w'''(v))^\beta}{w^\beta(v)}\right). \tag{14}$$

We see that $\varphi(v) > 0$ for $v \geq v_1$, where $\eta \in C^1([v_0,\infty),(0,\infty))$ and

$$\varphi'(v) = \eta'(v)\frac{a(v)(w'''(v))^\beta}{w^\beta(v)} + \eta(v)\frac{\left(a(w''')^\beta\right)'(v)}{w^\beta(v)} - \beta\eta(v)\frac{w^{\beta-1}(v)w'(v)a(v)(w'''(v))^\beta}{w^{2\beta}(v)}.$$

Using (13) and (14), we obtain that

$$\begin{aligned}\varphi'(v) &\leq \frac{\eta'(v)}{\eta(v)}\varphi(v) + \eta(v)\frac{\left(a(v)(w'''(v))^{\beta}\right)'}{w^{\beta}(v)} - \beta\eta(v)\frac{\theta}{2!}v^2\frac{a(v)(w'''(v))^{\beta+1}}{w^{\beta+1}(v)} \\ &\leq \frac{\eta'(v)}{\eta(v)}\varphi(v) + \eta(v)\frac{\left(a(v)(w'''(v))^{\beta}\right)'}{w^{\beta}(v)} - \frac{\beta\theta v^2}{2(\eta(v)a(v))^{\frac{1}{\beta}}}\varphi(v)^{\frac{\beta+1}{\beta}}.\end{aligned} \tag{15}$$

From (1) and (15), it follows that

$$\varphi'(v) \leq \frac{\eta'(v)}{\eta(v)}\varphi(v) - k\eta(v)\frac{q(v)w^{\beta}(\delta(v))}{w^{\beta}(v)} - \frac{\beta\theta v^2}{2(\eta(v)a(v))^{\frac{1}{\beta}}}\varphi(v)^{\frac{\beta+1}{\beta}}.$$

Note that $w'(v) > 0$ and $\delta(v) \geq v$. Thus, we find that

$$\varphi'(v) \leq \frac{\eta'(v)}{\eta(v)}\varphi(v) - k\eta(v)q(v) - \frac{\beta\theta v^2}{2(\eta(v)a(v))^{\frac{1}{\beta}}}\varphi(v)^{\frac{\beta+1}{\beta}}. \tag{16}$$

Multiplying (16) by $H_1(v,s)$ and integrating the resulting inequality from v_1 to v, we find that

$$\begin{aligned}\int_{v_1}^{v} H_1(v,s)k\eta(s)q(s)\mathrm{d}s \quad &\leq \quad \varphi(v_1)H_1(v,v_1) + \int_{v_1}^{v}\left(\frac{\partial}{\partial s}H_1(v,s) + \frac{\eta'(s)}{\eta(s)}H_1(v,s)\right)\varphi(s)\mathrm{d}s \\ &\quad - \int_{v_1}^{v}\frac{\beta\theta s^2}{2!(\eta(s)a(s))^{\frac{1}{\beta}}}H_1(v,s)\varphi^{\frac{\beta+1}{\beta}}(s)\mathrm{d}s.\end{aligned}$$

From (9), we obtain

$$\begin{aligned}\int_{v_1}^{v} H_1(v,s)k\eta(s)q(s)\mathrm{d}s \quad &\leq \quad \varphi(v_1)H_1(v,v_1) + \int_{v_1}^{v} h_1(v,s)H_1^{\beta/(\beta+1)}(v,s)\varphi(s)\mathrm{d}s \\ &\quad - \int_{v_1}^{v}\frac{\beta\theta s^2}{2(\eta(s)a(s))^{\frac{1}{\beta}}}H_1(v,s)\varphi^{\frac{\beta+1}{\beta}}(s)\mathrm{d}s.\end{aligned} \tag{17}$$

Lemma 3 with $V = \beta\theta s^2/\left(2(\eta(s)a(s))^{\frac{1}{\beta}}\right)H_1(v,s),\ U = h_1(v,s)H_1^{\beta/(\beta+1)}(v,s)$, and $x = \varphi(s)$, tell us that

$$h_1(v,s)H_1^{\beta/(\beta+1)}(v,s)\varphi(s) - \frac{\beta\theta s^2}{2(\eta(s)a(s))^{\frac{1}{\beta}}}H_1(v,s)\varphi^{\frac{\beta+1}{\beta}}(s) \leq \frac{h_1^{\beta+1}(v,s)H_1^{\beta}(v,s)}{(\beta+1)^{\beta+1}}\frac{2^{\beta}\eta(s)a(s)}{(\theta s^2)^{\beta}},$$

which, with (17), gives

$$\frac{1}{H_1(v,v_1)}\int_{v_1}^{v}(H_1(v,s)k\eta(s)q(s) - \pi(s,v))\mathrm{d}s \leq \varphi(v_1),$$

contradicting (11). Now, let us assume that $(\mathbf{S}_2)$ holds. Define

$$\psi(v) := \vartheta(v)\frac{w'(v)}{w(v)}.$$

We see that $\psi(v) > 0$ for $v \geq v_1$, where $\vartheta \in C^1([v_0,\infty),(0,\infty))$. Differentiating $\psi(v)$, we find

$$\psi'(v) = \frac{\vartheta'(v)}{\vartheta(v)}\psi(v) + \vartheta(v)\frac{w''(v)}{w(v)} - \frac{1}{\vartheta(v)}\psi(v)^2. \tag{18}$$

Integrating (1) from v to m and using the fact that $w'(v) > 0$, we find that

$$a(m)\left(w'''(m)\right)^{\beta} - a(v)\left(w'''(v)\right)^{\beta} = -\int_{v}^{m} q(s)g(w(\delta(s)))ds.$$

By virtue of $w'(v) > 0$ and $\delta(v) \geq v$, we obtain

$$a(m)\left(w'''(m)\right)^{\beta} - a(v)\left(w'''(v)\right)^{\beta} \leq -kw^{\beta}(v)\int_{v}^{u} q(s)ds.$$

Letting $m \to \infty$, we see that

$$a(v)\left(w'''(v)\right)^{\beta} \geq kw^{\beta}(v)\int_{v}^{\infty} q(s)\mathrm{d}s$$

and so

$$w'''(v) \geq w(v)\left(\frac{k}{a(v)}\int_{v}^{\infty} q(s)\mathrm{d}s\right)^{1/\beta}.$$

Integrating again from v to ∞, we obtain

$$w''(v) + w(v)\int_{v}^{\infty}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty} q(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma \leq 0. \tag{19}$$

From (18) and (19), we obtain that

$$\psi'(v) \leq \frac{\vartheta'(v)}{\vartheta(v)}\psi(v) - \vartheta(v)\varpi(s) - \frac{1}{\vartheta(v)}\psi(v)^2. \tag{20}$$

Multiplying (20) by $H_2(v,s)$ and integrating the resulting inequality from v_1 to v, it follows that

$$\begin{aligned}\int_{v_1}^{v} H_2(v,s)\vartheta(s)\varpi(s)\mathrm{d}s &\leq \psi(v_1)H_2(v,v_1) + \int_{v_1}^{v}\left(\frac{\partial}{\partial s}H_2(v,s) + \frac{\vartheta'(s)}{\vartheta(s)}H_2(v,s)\right)\psi(s)\mathrm{d}s \\ &- \int_{v_1}^{v}\frac{1}{\vartheta(s)}H_2(v,s)\psi^2(s)\mathrm{d}s.\end{aligned}$$

Thus, from (10), we obtain

$$\begin{aligned}\int_{v_1}^{v} H_2(v,s)\vartheta(s)\varpi(s)\mathrm{d}s &\leq \psi(v_1)H_2(v,v_1) + \int_{v_1}^{v} h_2(v,s)\sqrt{H_2(v,s)}\psi(s)\mathrm{d}s \\ &- \int_{v_1}^{v}\frac{1}{\vartheta(s)}H_2(v,s)\psi^2(s)\mathrm{d}s \leq \psi(v_1)H_2(v,v_1) + \int_{v_1}^{v}\frac{\vartheta(s)h_2^2(v,s)}{4}\mathrm{d}s\end{aligned}$$

and so

$$\frac{1}{H_2(v,v_1)}\int_{v_1}^{v}\left(H_2(v,s)\vartheta(s)\varpi(s) - \frac{\vartheta(s)h_2^2(v,s)}{4}\right)\mathrm{d}s \leq \psi(v_1),$$

which contradicts (12). Finally, assume that $(\mathbf{S}_3)$ holds and $\lim_{v\to\infty} w(v) \neq 0$. Recalling that $a(v)(w'''(v))^{\beta}$ is non-increasing, we obtain that

$$a^{1/\beta}(s)w'''(s) \leq a^{1/\beta}(v)w'''(v), \quad s \geq v \geq v_1.$$

Dividing the latter inequality by $a^{1/\beta}(s)$ and integrating the resulting inequality from v to u, we find

$$w''(u) \leq w''(v) + a^{1/\beta}(v)w'''(v)\int_{v}^{u} a^{-1/\beta}(s)\mathrm{d}s.$$

Letting $u \to \infty$, we obtain

$$0 \leq w''(v) + a^{1/\beta}(v)w'''(v)\zeta(v).$$

Thus,

$$\frac{-a^{1/\beta}(v)w'''(v)\zeta(v)}{w''(v)} \leq 1. \tag{21}$$

Furthermore, due to (21), we obtain that

$$\left(\frac{w''(v)}{\zeta(v)}\right)' \geq 0. \tag{22}$$

Now define

$$\phi(v) := \frac{a(v)(w'''(v))^{\beta}}{(w''(v))^{\beta}}. \tag{23}$$

We see that $\phi(v) < 0$ for $v \geq v_1$ and

$$\phi'(v) = \frac{\left(a(v)(w'''(v))^{\beta}\right)'}{(w''(v))^{\beta}} - \frac{\beta a(v)(w'''(v))^{\beta+1}}{(w''(v))^{\beta+1}}.$$

It follows from (1) and (21) that

$$\phi'(v) = \frac{-kq(v)w^{\beta}(\delta(v))}{(w''(v))^{\beta}} - \frac{\beta\phi^{\beta/\beta+1}(v)}{a^{1/\beta}(v)}.$$

From Lemma 1, we find

$$w(v) \geq \frac{\theta_1}{2}v^2 w''(v). \tag{24}$$

Thus, we have

$$\phi'(v) = \frac{-kq(v)w^{\beta}(\delta(v))}{(w''(\delta(v)))^{\beta}} \frac{(w''(\delta(v)))^{\beta}}{(w''(v))^{\beta}} - \frac{\beta\phi^{\beta/\beta+1}(v)}{a^{1/\beta}(v)}.$$

From (24), we obtain

$$\phi'(v) \leq -kq(v)\left(\frac{\theta_1\delta^2(v)}{2}\right)^{\beta}\left(\frac{\zeta(\delta(v))}{\zeta(v)}\right)^{\beta} - \frac{\beta\phi^{\beta/\beta+1}(v)}{a^{1/\beta}(v)}. \tag{25}$$

Using (21) and (23), we see, due to (26), that

$$-\phi(v)\zeta^{\beta}(v) \leq 1. \tag{26}$$

Multiplying (26) by $\zeta^{\beta}(v)$ and integrating the resulting inequality from v_1 to v, we obtain

$$\begin{aligned}
&\zeta^{\beta}(v)\phi(v) - \zeta^{\beta}(v_1)\phi(v_1) + \beta\int_{v_1}^{v} a^{-1/\beta}(s)\zeta^{\beta-1}(s)\phi(s)ds \\
&\qquad \leq -\int_{v_1}^{v} kq(s)\left(\frac{\theta_1\delta^2(s)}{2}\right)^{\beta}\zeta^{\beta}(\delta(s))ds - \beta\int_{v_1}^{v}\frac{\phi^{\beta/\beta+1}(s)}{a^{1/\beta}(s)}\zeta^{\beta}(s)ds.
\end{aligned} \tag{27}$$

Multiplying (27) by $H_3(v,s)$, we find that

$$\begin{aligned}\int_{v_1}^{v} H_3(v,s)kq(s)\left(\frac{\theta_1\delta^2(s)}{2}\right)^{\beta}\zeta^{\beta}(\delta(s))\mathrm{d}s \quad \leq \quad & \zeta^{\beta}(v_1)\phi(v_1)H_3(v,v_1)-\zeta^{\beta}(v)\phi(v)H_3(v,v_1)\\ &+\int_{v_1}^{v}\beta a^{-1/\beta}(s)\zeta^{\beta-1}(s)\phi(s)H_3(v,s)\mathrm{d}s\\ &-\int_{v_1}^{v}\frac{\beta\phi^{\beta/\beta+1}(s)}{a^{1/\beta}(s)}\zeta^{\beta}(s)H_3(v,s)\mathrm{d}s.\end{aligned}$$

Using Lemma 3 with $V=\zeta^{\beta}(s)H_3(v,s)/a^{1/\beta}(s)$, $U=a^{-1/\beta}(s)\zeta^{\beta-1}(s)H_3(v,s)$, and $x=\phi(s)$, we obtain

$$\beta a^{-1/\beta}(s)\zeta^{\beta-1}(s)\phi(s)H_3(v,s)-\frac{\beta\phi^{\beta/\beta+1}(s)}{a^{1/\beta}(s)}\zeta^{\beta}(s)H_3(v,s)\leq\frac{\beta^{\beta+1}H_3(v,s)}{(\beta+1)^{\beta+1}}\frac{1}{a^{1/\beta}(s)\zeta(s)}$$

and easily find, due to (26), that

$$\frac{1}{H_3(v,v_1)}\int_{v_1}^{v}\left(H_3(v,s)kq(s)\left(\frac{\theta_1\delta^2(s)}{2}\right)^{\beta}\zeta^{\beta}(\delta(s))-\tilde{\pi}(s,v)\right)\mathrm{d}s\leq\zeta^{\beta}(v_1)\phi(v_1)+1,$$

which contradicts (11). This completes the proof. □

With the help of our Theorem 1, we now prove a generalized Hille–Nehari-type oscillation criterion (cf. Corollary 1).

Theorem 2 (Generalized Hille–Nehari-type oscillation criterion for (1))**.** *Under assumptions A_1–A_5, if the differential equations*

$$\left(\frac{2a^{\frac{1}{\beta}}(v)}{(\theta v^2)^{\beta}}\left(w'(v)\right)^{\beta}\right)'+kq(v)w^{\beta}(v)=0, \tag{28}$$

for every $\theta\in(0,1)$,

$$w''(v)+w(v)\int_{v}^{\infty}\left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{\infty}q(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma=0 \tag{29}$$

and

$$\left(a(v)\left(w'(v)\right)^{\beta}\right)'+w^{\beta}(v)kq(v)\left(\frac{\zeta(\delta(v))}{\zeta(v)}\right)^{\beta}\left(\frac{\theta_1}{2}\delta^2(v)\right)^{\beta}=0, \tag{30}$$

for every $\theta_1\in(0,1)$, are oscillatory, then every solution of (1) is either oscillatory or satisfies $\lim_{v\to\infty}w(v)=0$.

Proof. Assume the contrary, that w is a positive solution of (1). Then, we can suppose that $w(v)$ and $w(\delta(v))$ are positive for all $v\geq v_1$ sufficiently large. From Lemma 4, we have three possible cases $(\mathbf{S}_1)$, $(\mathbf{S}_2)$, or $(\mathbf{S}_3)$. Let situation $(\mathbf{S}_1)$ hold. From Theorem 1, we obtain that (16) holds. If we set $\eta(v)=k=1$ in (16), then we find

$$\varphi'(v)+\frac{\beta\theta v^2}{2a^{\frac{1}{\beta}}(v)}\varphi(v)^{\frac{\beta+1}{\beta}}+q(v)\leq 0.$$

Thus, we can see that Equation (28) is nonoscillatory, which is a contradiction. Let case $(\mathbf{S}_2)$ holds. From Theorem 1, we obtain that (20) holds. If we now set $\vartheta(v)=k=1$ in (20), then we obtain

$$\psi'(v)+\psi^2(v)+\varpi(s)\varsigma\leq 0.$$

Hence, we see that Equation (29) is nonoscillatory, which is a contradiction. Let case ($\mathbf{S}_3$) hold and $\lim_{v\to\infty} w(v) \neq 0$. From Theorem 1, we obtain that (25) holds. Thus, we see that

$$\phi'(v) + \frac{\beta\phi^{\beta/\beta+1}(v)}{a^{1/\beta}(v)} + kq(v)\left(\frac{\theta_1\delta^2(v)}{2}\right)^{\beta}\left(\frac{\zeta(\delta(v))}{\zeta(v)}\right)^{\beta} \leq 0$$

and it follows that Equation (30) is nonoscillatory, which is a contradiction. Theorem 2 is proved. □

Let us now restrict ourselves to the case when $\beta = 1$. Note that, if

$$\int_{v_0}^{\infty} \frac{1}{a(v)}\mathrm{d}v < \infty$$

and

$$\liminf_{v\to\infty}\left(\int_{v_0}^{v}\frac{1}{a(s)}\mathrm{d}s\right)^{-1}\int_{v}^{\infty}\left(\int_{v_0}^{v}\frac{1}{a(s)}\mathrm{d}s\right)^{2}q(s)\mathrm{d}s > \frac{1}{4},$$

then Equation (8) with $\beta = 1$ is oscillatory [14]. For $\beta = 1$ our Theorem 2 gives a Hille–Nehari-type oscillation criterion.

Corollary 1 (Hille–Nehari-type oscillation criterion)**.** *Let $\beta = 1$. Under assumptions A_1–A_5 with $k = 1$, if*

$$\int_{v_0}^{\infty} \frac{\theta v^2}{2a(v)}\mathrm{d}v < \infty$$

and

$$\liminf_{v\to\infty}\left(\int_{v_0}^{v}\frac{\theta s^2}{2a(s)}\mathrm{d}s\right)^{-1}\int_{v}^{\infty}\left(\int_{v_0}^{v}\frac{\theta s^2}{2a(s)}\mathrm{d}s\right)^{2}q(s)\mathrm{d}s > \frac{1}{4} \tag{31}$$

for every constant $\theta \in (0,1)$,

$$\liminf_{v\to\infty} v\int_{v_0}^{v}\int_{v}^{v}\left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{v}q(s)\mathrm{d}s\right)\mathrm{d}\varsigma\mathrm{d}v > \frac{1}{4}, \tag{32}$$

and

$$\liminf_{v\to\infty}\left(\int_{v_0}^{v}\frac{1}{a(s)}\mathrm{d}s\right)^{-1}\int_{v}^{\infty}\left(\int_{v_0}^{v}\frac{1}{a(s)}\mathrm{d}s\right)^{2}\frac{\theta_1\zeta(\delta(s))\delta^2(s)q(s)}{2\zeta(s)}\mathrm{d}s > \frac{1}{4} \tag{33}$$

for every constant $\theta_1 \in (0,1)$, then every solution of (1) is either oscillatory or satisfies $\lim_{v\to\infty} w(v) = 0$.

The next section shows that our results are new even in the very special situation covered by Corollary 1.

4. Illustrative Examples

We give two examples for which all results of [17,18] cannot be applied, since $\delta(v) = v + 1 > v$, while our Hille–Nehari-type oscillation criterion holds.

Example 1. *Let us consider the following equation:*

$$(\mathrm{e}^{v}w'''(v))' + \frac{1}{16}\mathrm{e}^{v+\frac{1}{2}}w(v+1) = 0, \quad v \geq 1, \tag{34}$$

where $\alpha > 0$ is a constant. Note that $\beta = 1$, $a(v) = \mathrm{e}^{v}$, $q(v) = \mathrm{e}^{v+\frac{1}{2}}/16$ and $\delta(v) = v + 1$. It is easy to see that all conditions of our Corollary 1 are satisfied. Hence, all solutions of (34) are either oscillatory or satisfy $\lim_{v\to\infty} w(v) = 0$.

Following is a second example, where previous results in the literature cannot be applied, while our Corollary 1 is conclusive.

Example 2. *Consider the following equation:*

$$\left(v^4 w'''(v)\right)' + \alpha w(v+1) = 0, \quad v \geq 1, \tag{35}$$

where $\alpha > 0$ is a constant. Note that $\beta = 1$, $a(v) = v^4$, $q(v) = \alpha$, and $\delta(v) = v + 1$. If we set $k = 1$, then condition (31) becomes

$$\begin{aligned}
&\liminf_{v\to\infty}\left(\int_{v_0}^{v} \frac{\theta s^2}{2a(s)}\mathrm{d}s\right)^{-1} \int_{v}^{\infty}\left(\int_{v_0}^{v} \frac{\theta s^2}{2a(s)}\mathrm{d}s\right)^{2} q(s)\mathrm{d}s \\
&\quad = \liminf_{v\to\infty}(2v)\int_{v}^{\infty}\frac{\alpha}{4s^2}ds = \liminf_{v\to\infty}(2v)\left(\frac{\alpha}{4v}\right) \\
&\quad = \frac{\alpha}{2} > \frac{1}{4},
\end{aligned}$$

condition (32) becomes

$$\begin{aligned}
\liminf_{v\to\infty} v \int_{v_0}^{v}\int_{v}^{v}\left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{v} q(s)\mathrm{d}s\right)\mathrm{d}\varsigma\mathrm{d}v &= \liminf_{v\to\infty} v\left(\frac{\alpha}{2v}\right) \\
&= \frac{\alpha}{2} > \frac{1}{4},
\end{aligned}$$

and (33) is satisfied. Therefore, from Corollary 1, any solution of Equation (35) is either oscillatory, if $\alpha > 0.5$, or satisfies $\lim_{v\to\infty} w(v) = 0$.

5. Conclusions and Future Work

In this work, we obtained new Philos-type and Hille–Nehari-type oscillation criteria for equations of form (1). Our results are easy to generalize for the following equations:

$$\left(a(v)\left(w'''(v)\right)^{\beta}\right)' + \sum_{i=1}^{j} q_i(v) w^{\alpha}(\delta_i(v)) = 0, \quad j \geq 1, \tag{36}$$

where $v \geq v_0$, $\delta_i(v) \leq v$, $\alpha \leq \beta$, and α and β are ratios of odd positive integers. However, it is not easy to find analogous results for Equation (36) in the case $\alpha > \beta$. We leave this as an interesting open question.

Author Contributions: Conceptualization, O.B., A.H.A., F.M. and Y.N.R.; methodology, O.B., A.H.A., F.M. and Y.N.R.; software, O.B., A.H.A., F.M. and Y.N.R.; validation, O.B., A.H.A., F.M. and Y.N.R.; formal analysis, O.B., A.H.A., F.M. and Y.N.R.; investigation, O.B., A.H.A., F.M. and Y.N.R.; re-sources, O.B., A.H.A., F.M. and Y.N.R.; data curation, O.B., A.H.A., F.M. and Y.N.R.; writing—original draft preparation, O.B., A.H.A., F.M. and Y.N.R.; writing—review and editing, O.B., A.H.A., F.M. and Y.N.R.; visualization, O.B., A.H.A., F.M. and Y.N.R.; funding acquisition, O.B., A.H.A., F.M. and Y.N.R. All authors have read and agreed to the published version of the manuscript.

Funding: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2022R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The research is supported by Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. Researchers Supporting Project number: PNURSP2022R27.

Conflicts of Interest: The authors declear no conflict of interest.

References

1. Choquet, C.; Martins, N.; Sidi Ammi, M.R.; Tilioua, M.; Torres, D.F.M. Preface [Special issue on optimization and differential equations]. *Discret. Contin. Dyn. Syst. Ser. S* **2018**, *11*, 1–2. [CrossRef]
2. Hale, J. *Theory of Functional Differential Equations*, 2nd ed.; Springer: New York, NY, USA; Heidelberg, Germany, 1977.
3. Bazighifan, O.; Cesarano, C. Some New Oscillation Criteria for Second Order Neutral Differential Equations with Delayed Arguments. *Mathematics* **2019**, *7*, 619. [CrossRef]
4. Bazighifan, O.; Elabbasy, E.M.; Moaaz, O. Oscillation of higher-order differential equations with distributed delay. *J. Inequal. Appl.* **2019**, *2019*, 55. [CrossRef]
5. Chatzarakis, G.E.; Elabbasy, E.M.; Bazighifan, O. An oscillation criterion in 4th-order neutral differential equations with a continuously distributed delay. *Adv. Differ. Equ.* **2019**, *2019*, 426. [CrossRef]
6. Cesarano, C.; Pinelas, S.; Al-Showaikh, F.; Bazighifan, O. Asymptotic properties of solutions of fourth-order delay differential equations. *Symmetry* **2019**, *11*, 628. [CrossRef]
7. Cesarano, C.; Bazighifan, O. Oscillation of fourth-order functional differential equations with distributed delay. *Axioms* **2019**, *8*, 61. [CrossRef]
8. Cesarano, C.; Bazighifan, O. Qualitative behavior of solutions of second order differential equations. *Symmetry* **2019**, *11*, 777. [CrossRef]
9. Grace, S.R.; Džurina, J.; Jadlovská, I.; Li, T. On the oscillation of fourth-order delay differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 118. [CrossRef]
10. Győri, I.; Ladas, G. *Oscillation Theory of Delay Differential Equations*; Oxford Mathematical Monographs: New York, NY, USA, 1991.
11. Li, T.; Baculíková, B.; Džurina, J.; Zhang, C. Oscillation of fourth-order neutral differential equations with p-Laplacian like operators. *Bound. Value Probl.* **2014**, *2014*, 56. [CrossRef]
12. Moaaz, O.; Elabbasy, E.M.; Bazighifan, O. On the asymptotic behavior of fourth-order functional differential equations. *Adv. Differ. Equ.* **2017**, *2017*, 261. [CrossRef]
13. Moaaz, O.; Elabbasy, E.M.; Muhib, A. Oscillation criteria for even-order neutral differential equations with distributed deviating arguments. *Adv. Differ. Equ.* **2019**, *2019*, 297. [CrossRef]
14. Nehari, Z. Oscillation criteria for second-order linear differential equations. *Trans. Amer. Math. Soc.* **1957**, *85*, 428–445. [CrossRef]
15. Philos, C.G. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Arch. Math.* **1981**, *36*, 168–178. [CrossRef]
16. Řehák, P. How the constants in Hille-Nehari theorems depend on time scales. *Adv. Differ. Equ.* **2006**, *2006*, 64534. [CrossRef]
17. Zhang, C.; Agarwal, R.P.; Bohner, M.; Li, T. New results for oscillatory behavior of even-order half-linear delay differential equations. *Appl. Math. Lett.* **2013**, *26*, 179–183. [CrossRef]
18. Zhang, C.; Li, T.; Sun, B.; Thandapani, E. On the oscillation of higher-order half-linear delay differential equations. *Appl. Math. Lett.* **2011**, *24*, 1618–1621. [CrossRef]
19. Agarwal, R.P.; Grace, S.R. Oscillation theorems for certain functional differential equations of higher order. *Math. Comput. Model.* **2004**, *39*, 1185–1194. [CrossRef]
20. Agarwal, R.P.; Grace, S.R.; O'Regan, D. Oscillation criteria for certain nth order differential equations with deviating arguments. *J. Math. Anal. Appl.* **2001**, *262*, 601–622. [CrossRef]
21. Grace, S.R.; Lalli, B.S. Oscillation theorems for nth order nonlinear differential equations with deviating arguments. *Proc. Am. Math. Soc.* **1984**, *90*, 65–70. [CrossRef]
22. Zhang, C.; Li, T.; Saker, S.H. Oscillation of fourth-order delay differential equations. *Nelīnīīnī Koliv.* **2013**, *16*, 322–335. [CrossRef]
23. Agarwal, R.P.; Shieh, S.L.; Yeh, C.C. Oscillation criteria for second-order retarded differential equations. *Math. Comput. Model.* **1997**, *26*, 1–11. [CrossRef]
24. Liu, S.; Zhang, Q.; Yu, Y. Oscillation of even-order half-linear functional differential equations with damping. *Comput. Math. Appl.* **2011**, *61*, 2191–2196. [CrossRef]
25. Zhang, C.; Li, T.; Saker, S.H. Oscillation of fourth-order delay differential equations. *J. Math. Sci.* **2014**, *201*, 296–309. [CrossRef]