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q-Analogue of a New Subclass of Harmonic Univalent Functions Associated with Subordination

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Abstract: In this article, we introduce and investigate the q-analogue of a new subclass of harmonic univalent functions defined by subordination. We first obtain a coefficient characterization of these functions. We give compactness and extreme points, distortion bounds, necessary and sufficient convolution conditions for this subclass of harmonic univalent functions with negative coefficients. The symmetry properties and other properties of the q-analogue subclass of functions presented in this paper shed light on future studies.

Keywords: harmonic functions; univalent functions; star-like functions; q-analogue; subordination

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1. Introduction

The theory of geometric functions is a branch of mathematical science that has been studied since the late 1800s and early 1900s, which remains popular today. We can say that the theory of geometric functions is a multidisciplinary field consisting of a combination of geometry and analysis disciplines. Refs. [1–3] can be cited as references for the origins of the geometric function theory. Of course, studies in this area are not limited to these articles. We know from ancient studies that this field is important in engineering and closely related fields [4]. This theory, which is also used in other subjects, such as electricity and magnetism [5], has a wide application area in mathematical physics [6]. Actually, new developments in the productive process to linear and nonlinear boundary-value and initial-value problems utilizing spectral analysis [7] are likely to lead to a role for geometric function theory in solving a wide range of partial differential equations (PDEs). The theory of geometric functions is also widely used in fluid mechanics, which is popular in engineering [8]. It is essential that the theory of geometric functions, which has a wide range of applications in all these popular scientific disciplines, is still up-to-date today. Many articles have been published in this field to date.

This study examines harmonic univalent functions in the geometric function theory. We will examine the q-analogue subclass of the harmonic univalent function class. The foundations of q-analogue functions are based on q-gamma and q-beta functions. The first studies on these functions started in the early 1980s (for example, [9–11]) and have been used and developed in many different disciplines. Recent studies have also examined the symmetric properties of functions in subclasses in the theory of geometric functions. Subclass studies are also among the crucial problems that are frequently studied today [12–16].

Let $\mathcal{H}$ indicate the class of complex-valued continuous harmonic functions which are harmonic in $\mathcal{U} = \{\varsigma : \varsigma \in \mathbb{C} \text{ and } |\varsigma| < 1\}$ (the open unit disk). Let us denote the subclasses of $\mathcal{H}$ consisting of functions which are analytic in $\mathcal{U}$ with $\mathcal{A}$. In $\mathcal{U}$, a harmonic function can be written in the form $\wp = \ell + \bar{\jmath}$, where ℓ and $\jmath$ are analytic in $\mathcal{U}$. Function ℓ is called the analytic part and $\jmath$ is called co-analytic part of $\wp$. A necessary and sufficient condition for $\wp$ to be locally univalent and sense-preserving in $\mathcal{U}$ is that $|\jmath'(\varsigma)| < |\ell'(\varsigma)|$(see [17]). Thus, without loss of the generality, we can write

$$\ell(\varsigma) = \varsigma + \sum_{\varphi=2}^{\infty} a_\varphi \varsigma^\varphi, \quad \jmath(\varsigma) = \sum_{\varphi=1}^{\infty} b_\varphi \varsigma^\varphi. \tag{1}$$

Let us denote the subclass of $\wp = \ell + \bar{\jmath}$ which is univalent, harmonic and sense-preserving in $\mathcal{U}$ for which $\ell(0) = \ell'(0) - 1 = 0 = \jmath(0)$ with $\mathcal{SH}$. Obviously, the sense-preserving feature means $|b_1| < 1$.

Clunie and Sheil-Small [17] explored the class $\mathcal{SH}$ and its geometric subclasses. They found some coefficient bounds. Later, many researchers published a large number of articles on the $\mathcal{SH}$ and its subclasses.

We recollect here the q-difference operator that was used in geometric function theory and in several areas of science. We give basic definitions and properties about the q-difference operator that are used in this study (for details, see [18,19]). For $0 < q < 1$, we define the q-integer $[\varphi]_q$ by

$$[\varphi]_q = \frac{1 - q^\varphi}{1 - q}, \quad (\varphi = 1, 2, 3, \cdots).$$

Notice that if $q \to 1^-$, then $[\varphi]_q \to \varphi$.

In 1990, İsmail et al. [20] used q-calculus in analytic univalent function theory by describing a subclass of complex-valued functions which are analytic in $\mathcal{U}$ with the normalizations $\wp(0) = 0$, $\wp'(0) = 1$, and $|\wp(q\varsigma)| \leq |\wp(\varsigma)|$ on $\mathcal{U}$ for every q, $q \in (0,1)$. The theory of analytic univalent functions and q-calculus have been explored by some researchers referencing these authors, for example, see [21,22]. ℓ and $\jmath$ are, without loss of generality, analytic functions given by (1) and can be written with q-difference operator as [19]

$$D_q \ell(\varsigma) = \begin{cases} \frac{\ell(\varsigma) - \ell(q\varsigma)}{(1-q)\varsigma} & ; \varsigma \neq 0 \\ \ell'(0) & ; \varsigma = 0 \end{cases} \text{ and } D_q \jmath(\varsigma) = \begin{cases} \frac{\jmath(\varsigma) - \jmath(q\varsigma)}{(1-q)\varsigma} & ; \varsigma \neq 0 \\ \jmath'(0) & ; \varsigma = 0 \end{cases}.$$

Thus, for the function ℓ and $\jmath$ to be defined as in (1), we obtain

$$D_q \ell(\varsigma) = 1 + \sum_{\varphi=2}^{\infty} [\varphi]_q a_\varphi \varsigma^{\varphi-1} \text{ and } D_q \jmath(\varsigma) = \sum_{\varphi=1}^{\infty} [\varphi]_q b_\varphi \varsigma^{\varphi-1}. \tag{2}$$

Let $\wp = \ell + \bar{\jmath}$ be a harmonic function which is defined as in (1). We call these functions $q-$harmonic, locally univalent and sense-preserving in $\mathcal{U}$ and denoted by $\mathcal{SH}q$, if and only if the second dilatation $\mathfrak{w}_q$ fulfills the following condition

$$|\mathfrak{w}_q(\varsigma)| = \left| \frac{D_q \jmath(\varsigma)}{D_q \ell(\varsigma)} \right| < 1 \tag{3}$$

where $\varsigma \in \mathcal{U}$ and $0 < q < 1$. It is obvious that from $q \to 1^-$, class $\mathcal{SH}_q$ becomes class $\mathcal{SH}$ (see [23,24]).

Let $\mathcal{TSH}_q$ be a subclass which consists of harmonic univalent functions $\wp = \ell + \bar{\jmath}$ in $\mathcal{SH}_q$. Here, ℓ and $\jmath$ are defined as in the following equation

$$\ell(\varsigma) = \varsigma - \sum_{\varphi=2}^{\infty} |a_\varphi| \varsigma^\varphi \text{ and } \jmath(\varsigma) = \sum_{\varphi=1}^{\infty} |b_\varphi| \varsigma^\varphi. \tag{4}$$

We say that an analytic function $\wp$ is subordinate to an analytic function $\mathfrak{F}$ and write $\wp \prec \mathfrak{F}$, if there exists a complex-valued function $\mathfrak{w}$ which maps $\mathcal{U}$ onto itself with $\mathfrak{w}(0) = 0$, such that $\wp(\varsigma) = \mathfrak{F}(\mathfrak{w}(\varsigma)) \quad (\varsigma \in \mathcal{U})$.

Furthermore, if the function $\mathfrak{F}$ is univalent in $\mathcal{U}$, then we have the following equivalence:

$$\wp(\varsigma) \prec \mathfrak{F}(\varsigma) \Leftrightarrow \wp(0) = \mathfrak{F}(0) \text{ and } \wp(\mathcal{U}) \subset \mathfrak{F}(\mathcal{U}).$$

Denote by $\mathcal{SH}_q(\gamma,\mu,\nu)$, the subclass of $\mathcal{SH}_q$ consists of $\wp$ functions as in (1) which fulfill the following condition

$$\frac{\varsigma D_q\ell(\varsigma)-\overline{\varsigma D_q\jmath(\varsigma)}}{\gamma(\varsigma D_q\ell(\varsigma)-\overline{\varsigma D_q\jmath(\varsigma)})+(1-\gamma)(\ell(\varsigma)+\overline{\jmath(\varsigma)})}\prec\frac{1+\mu\varsigma}{1+\nu\varsigma}, \tag{5}$$

where $-1\le\nu\le 0<\mu\le 1$ and $0\le\gamma<1$.

Finally, we let $\mathcal{TSH}_q(\gamma,\mu,\nu)\equiv\mathcal{SH}_q(\gamma,\mu,\nu)\cap\mathcal{TSH}_q$. If we choose γ,μ and ν specifically, the classes $\mathcal{SH}_q(\mu,\nu)$ reduce to the miscellaneous subclasses as follows:

(i) $\mathcal{SH}_q(\gamma,\mu,\nu)=\mathcal{S}^*_{\mathcal{H}}(\gamma,\mu,\nu)$ for $q\to 1^-$[25];
(ii) $\mathcal{SH}_q(\gamma,2\alpha-1,1)=\mathcal{S}^*_{\mathcal{H}}(\gamma,\alpha)$ for $q\to 1^-$ and $0\le\alpha<1$ [26];
(iii) $\mathcal{SH}_q(0,\mu,\nu)=\mathcal{SH}_q(\mu,\nu)$ [27];
(iv) $\mathcal{SH}_q(0,\mu,\nu)=\mathcal{SH}^*(\mu,\nu)$ for $q\to 1^-$ [28];
(v) $\mathcal{SH}_q(0,-1,1)=\mathcal{H}^0_q(0)$ [24,29];
(vi) $\mathcal{SH}_q(0,2\alpha-1,1)=\mathcal{H}^0_q(\alpha)$ for $0\le\alpha<1$ [24,29];
(vii) $\mathcal{SH}_q(0,-1,1)=S^*_H$ for $q\to 1^-$ [30–32];
(viii) $\mathcal{SH}_q(0,2\alpha-1,1)=S^*_H(\alpha)$ for $q\to 1^-$ and $0\le\alpha<1$ [30,33].

By using the method and technique that Dziok (see [28,34,35]) and Dziok et al. (see [36,37]) used, in this article, we find necessary and sufficient conditions for class $\mathcal{TSH}_q(\mu,\nu)$. Moreover, we will determine distortion bounds, radii of star-likeness and convexity, compactness and extreme points for the above defined class $\mathcal{TSH}_q(\mu,\nu)$. In this paper, we find necessary and sufficient conditions to be in the $\mathcal{TSH}_q(\gamma,\mu,\nu)$ class. Therewith, we determine distortion bounds, extreme points for the above defined class $\mathcal{TSH}_q(\gamma,\mu,\nu)$.

2. Main Results

For functions $\wp_1$ and $\wp_2\in\mathcal{SH}_q$ of the form

$$\wp_k(\varsigma)=\varsigma+\sum_{\varphi=2}^{\infty}a_{k,\varphi}\varsigma^{\varphi}+\sum_{\varphi=1}^{\infty}\overline{b_{k,\varphi}\varsigma^{\varphi}},\qquad(\varsigma\in\mathcal{U},\,k=1,2), \tag{6}$$

we define the Hadamard product of $\wp_1$ and $\wp_2$ by

$$(\wp_1*\wp_2)(\varsigma)=\varsigma+\sum_{\varphi=2}^{\infty}a_{1,\varphi}\,a_{2,\varphi}\varsigma^{\varphi}+\sum_{\varphi=1}^{\infty}\overline{b_{1,\varphi}\,b_{2,\varphi}\varsigma^{\varphi}}\qquad(\varsigma\in\mathcal{U}).$$

With the following theorem, we find out under which conditions a function $\wp$ is in the $\mathcal{SH}_q(\gamma,\mu,\nu)$ class.

Theorem 1. *Let $\wp\in\mathcal{SH}_q$. Then, $\wp\in\mathcal{SH}_q(\gamma,\mu,\nu)$ if and only if*

$$\wp(\varsigma)*\Phi(\varsigma;\xi)\neq 0,\qquad(\xi\in\mathbb{C},\,|\xi|=1,\,\varsigma\in\mathcal{U}\backslash\{0\}),$$

where

$$\Phi_q(\varsigma;\xi)=\frac{(\mu-\nu)\xi\varsigma-(1+\mu\xi)(1-\gamma)q\varsigma^2}{(1-\varsigma)(1-q\varsigma)}+\frac{[2+(\mu+\nu)\xi-2(1+\mu\xi)\gamma]\overline{\varsigma}-(1+\mu\xi)(1-\gamma)q\overline{\varsigma}^2}{(1-\overline{\varsigma})(1-q\overline{\varsigma})}.$$

Proof. Let $\wp\in\mathcal{SH}_q$ be as in (1). In that case, $\wp\in\mathcal{SH}_q(\gamma,\mu,\nu)$ if and only if it satisfies (5) or we can also use the following inequality instead

$$\frac{\varsigma D_q\ell(\varsigma)-\overline{\varsigma D_q\jmath(\varsigma)}}{\gamma(\varsigma D_q\ell(\varsigma)-\overline{\varsigma D_q\jmath(\varsigma)})+(1-\gamma)(\ell(\varsigma)+\overline{\jmath(\varsigma)})}\neq\frac{1+\mu\xi}{1+\nu\xi}, \tag{7}$$

where $\xi \in \mathbb{C}$, $|\xi| = 1$ and $\varsigma \in \mathcal{U}\backslash\{0\}$. Since

$$\ell(\varsigma) = \ell(\varsigma) * \frac{\varsigma}{1-\varsigma},\ \jmath(\varsigma) = \jmath(\varsigma) * \frac{\varsigma}{1-\varsigma},$$

and

$$\varsigma D_q \ell(\varsigma) = \ell(\varsigma) * \frac{\varsigma}{(1-\varsigma)(1-q\varsigma)},\ \varsigma D_q \jmath(\varsigma) = \jmath(\varsigma) * \frac{\varsigma}{(1-\varsigma)(1-q\varsigma)}\ ,$$

the inequality (7) yields

$$\begin{aligned}
& (1+\mu\xi)\Big[\gamma(\varsigma D_q \ell(\varsigma) - \overline{\varsigma D_q \jmath(\varsigma)}) + (1-\gamma)(\ell(\varsigma) + \overline{\jmath(\varsigma)})\Big] \\
& -(1+\nu\xi)\Big[\varsigma D_q \ell(\varsigma) - \overline{\varsigma D_q \jmath(\varsigma)}\Big] \\
= \ & \ell(\varsigma) * \left\{[(1+\mu\xi)\gamma - (1+\nu\xi)]\frac{\varsigma}{(1-\varsigma)(1-q\varsigma)} + (1+\mu\xi)(1-\gamma)\frac{\varsigma}{1-\varsigma}\right\} \\
& + \overline{\jmath(\varsigma)} * \left\{[(1+\nu\xi) - (1+\mu\xi)\gamma]\frac{\bar{\varsigma}}{(1-\bar{\varsigma})(1-q\bar{\varsigma})} + (1+\mu\xi)(1-\gamma)\frac{\bar{\varsigma}}{1-\bar{\varsigma}}\right\} \\
= \ & \ell(\varsigma) * \frac{(\mu-\nu)\xi\varsigma - (1+\mu\xi)(1-\gamma)q\varsigma^2}{(1-\varsigma)(1-q\varsigma)} \\
& + \overline{\jmath(\varsigma)} * \frac{[2+(\mu+\nu)\xi - 2(1+\mu\xi)\gamma]\overline{\varsigma} - (1+\mu\xi)(1-\gamma)q\overline{\varsigma}^2}{(1-\overline{\varsigma})(1-q\overline{\varsigma})} \\
= \ & \wp(\varsigma) * \Phi(\varsigma;\xi) \neq 0.
\end{aligned}$$

□

Now, we will give a sufficient coefficient bound for functions in the $\mathcal{SH}_q(\gamma, \mu, \nu)$ class and we prove this theorem with using a special inequality technique.

Theorem 2. *Let $\wp$ be of the form (1). If $-1 \le \nu \le 0 < \mu \le 1, 0 \le \gamma < 1$ and*

$$\sum_{\varphi=1}^{\infty} \left(\Xi_\varphi |a_\varphi| + \Psi_\varphi |b_\varphi|\right) \le 2(\mu - \nu), \tag{8}$$

where

$$\Xi_\varphi = (\mu\gamma - \nu)[\varphi]_q + (1-\gamma)([\varphi]_q - 1 + \mu) \tag{9}$$

and

$$\Psi_\varphi = (\mu\gamma - \nu)[\varphi]_q + (1-\gamma)([\varphi]_q + 1 - \mu) \tag{10}$$

then $\wp$ is harmonic, sense preserving, univalent in $\mathcal{U}$, and $\wp \in \mathcal{SH}_q(\gamma, \mu, \nu)$.

Proof. Since

$$\begin{aligned}
|D_q\ell(\varsigma)| &\geq 1-\sum_{\varphi=2}^{\infty}[\varphi]_q|a_\varphi||z|^{\varphi-1} \\
&> 1-\sum_{\varphi=2}^{\infty}\frac{(\mu\gamma-\nu)[\varphi]_q+(1-\gamma)([\varphi]_q-1+\mu)}{\nu-\mu}|a_\varphi| \\
&\geq \sum_{\varphi=1}^{\infty}\frac{(\mu\gamma-\nu)[\varphi]_q+(1-\gamma)([\varphi]_q+1-\mu)}{\nu-\mu}|b_\varphi| \\
&> \sum_{\varphi=1}^{\infty}\frac{(\mu\gamma-\nu)[\varphi]_q+(1-\gamma)([\varphi]_q+1-\mu)}{\nu-\mu}|b_\varphi||\varsigma|^{\varphi-1} \\
&\geq \sum_{\varphi=1}^{\infty}[\varphi]_q|b_\varphi||\varsigma|^{\varphi-1}\geq |D_q\jmath(\varsigma)|,
\end{aligned}$$

it follows that $\wp \in S\mathcal{H}_q$. Otherwise, $\wp \in \mathcal{SH}_q(\gamma,\mu,\nu)$ if and only if there exists a complex-valued function that satisfies $\mathfrak{w}$ with $\mathfrak{w}(0)=0$, $|\mathfrak{w}(\varsigma)|<1$ $(\varsigma\in\mathcal{U})$ such that

$$\frac{\varsigma D_q\ell(\varsigma)-\overline{\varsigma D_q\jmath(\varsigma)}}{\gamma(\varsigma D_q\ell(\varsigma)-\overline{\varsigma D_q\jmath(\varsigma)})+(1-\gamma)(\ell(\varsigma)+\overline{\jmath(\varsigma)})}=\frac{1+\mu\mathfrak{w}(\varsigma)}{1+\nu\mathfrak{w}(\varsigma)}$$

or, equivalently,

$$\left|\frac{(1-\gamma)(\varsigma D_q\ell(\varsigma)-\overline{\varsigma D_q\jmath(\varsigma)}-\ell(\varsigma)-\overline{\jmath(\varsigma)})}{(\mu\gamma-\nu)(\varsigma D_q\ell(\varsigma)-\overline{\varsigma D_q\jmath(\varsigma)})+\mu(1-\gamma)(\ell(\varsigma)+\overline{\jmath(\varsigma)})}\right|<1. \tag{11}$$

After making the basic mathematical operations in inequality (11), if we substitute the $D_q\ell(\varsigma)$ and $D_q\jmath(\varsigma)$ equations which we find from (2), we obtain the following result:

$$\begin{aligned}
&\left|(1-\gamma)(\varsigma D_q\ell(\varsigma)-\overline{\varsigma D_q\jmath(\varsigma)}-\ell(\varsigma)-\overline{\jmath(\varsigma)})\right| \\
&-\left|(\mu\gamma-\nu)(\varsigma D_q\ell(\varsigma)-\overline{\varsigma D_q\jmath(\varsigma)})+\mu(1-\gamma)(\ell(\varsigma)+\overline{\jmath(\varsigma)})\right| \\
= &\left|\sum_{\varphi=2}^{\infty}(1-\gamma)([\varphi]_q-1)a_\varphi\varsigma^\varphi-\sum_{\varphi=1}^{\infty}(1-\gamma)([\varphi]_q+1)\overline{b_\varphi\varsigma^\varphi}\right| \\
&-\left|(\mu-\nu)\varsigma+\sum_{\varphi=2}^{\infty}\left[(\mu\gamma-\nu)[\varphi]_q+\mu(1-\gamma)\right]a_\varphi\varsigma^\varphi\right. \\
&\left.-\sum_{\varphi=1}^{\infty}\left[(\mu\gamma-\nu)[\varphi]_q-\mu(1-\gamma)\right]\overline{b_\varphi\varsigma^\varphi}\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{\varphi=2}^{\infty}(1-\gamma)([\varphi]_q-1)|a_\varphi||\varsigma|^\varphi \\
&\quad+\sum_{\varphi=1}^{\infty}(1-\gamma)([\varphi]_q+1)|b_\varphi||\varsigma|^\varphi-(\mu-\nu)|\varsigma| \\
&\quad+\sum_{\varphi=2}^{\infty}\left[(\mu\gamma-\nu)[\varphi]_q+\mu(1-\gamma)\right]|a_\varphi||\varsigma|^\varphi \\
&\quad+\sum_{\varphi=1}^{\infty}\left[(\mu\gamma-\nu)[\varphi]_q-\mu(1-\gamma)\right]|b_\varphi||\varsigma|^\varphi \\
&\le |\varsigma|\left\{\sum_{\varphi=2}^{\infty}[(\mu\gamma-\nu)[\varphi]_q+(1-\gamma)([\varphi]_q-1+\mu)]|a_\varphi|\right. \\
&\qquad\left.+\sum_{\varphi=1}^{\infty}[(\mu\gamma-\nu)[\varphi]_q+(1-\gamma)([\varphi]_q+1-\mu)]|b_\varphi|-(\mu-\nu)\right\} \\
&< 0,
\end{aligned}
$$

the harmonic univalent function

$$\wp(\varsigma)=\varsigma+\sum_{\varphi=2}^{\infty}\frac{(\mu-\nu)x_\varphi}{\Xi_\varphi}\varsigma^\varphi+\sum_{\varphi=1}^{\infty}\frac{(\mu-\nu)y_\varphi}{\Psi_\varphi}\overline{\varsigma^\varphi} \tag{12}$$

where

$$\sum_{\varphi=1}^{\infty}|x_\varphi|+\sum_{\varphi=1}^{\infty}|y_\varphi|=1$$

shows that coefficient bound given by (8) is sharp. The functions of the form (12) are in $\mathcal{SH}_q(\gamma,\mu,\nu)$ because

$$\sum_{\varphi=1}^{\infty}\left(\frac{\Xi_\varphi}{2(\mu-\nu)}|a_\varphi|+\frac{\Psi_\varphi}{2(\mu-\nu)}|b_\varphi|\right)=\sum_{\varphi=1}^{\infty}(|x_\varphi|+|y_\varphi|)=1$$

by (8). □

Next, we define and prove that the bound (8) is also necessary for the $\mathcal{TSH}_q(\gamma,\mu,\nu)$ class.

Theorem 3. *Let $\wp=\ell+\bar{\jmath}$ with ℓ and $\jmath$ of the form (4). Then, $\wp\in\mathcal{TSH}_q(\gamma,\mu,\nu)$ if and only if the condition (8) holds.*

Proof. According to Theorem 2, we just need to prove that $\wp\notin\mathcal{TSH}_q(\gamma,\mu,\nu)$ if stipulation (8) is not satisfied. We note that a necessary and sufficient condition for $\wp=\ell+\bar{\jmath}$ given by (4) to be in $\mathcal{TSH}_q(\gamma,\mu,\nu)$ is that the coefficient bound (8) is satisfied. Equivalently, we must have

$$\left|\frac{-\sum_{\varphi=2}^{\infty}([\varphi]_q-1)(1-\gamma)|a_\varphi|\varsigma^\varphi-\sum_{\varphi=1}^{\infty}([\varphi]_q+1)(1-\gamma)|b_\varphi|\bar{\varsigma}^\varphi}{(\mu-\nu)\varsigma-\sum_{\varphi=2}^{\infty}\left[(\mu\gamma-\nu)[\varphi]_q+\mu(1-\gamma)\right]|a_\varphi|\varsigma^\varphi-\sum_{\varphi=1}^{\infty}\left[(\mu\gamma-\nu)[\varphi]_q-\mu(1-\gamma)\right]|b_\varphi|\bar{\varsigma}^\varphi}\right|<1.$$

For $\varsigma=r<1$, we obtain

$$\frac{\sum_{\varphi=2}^{\infty}([\varphi]_q-1)(1-\gamma)|a_\varphi|r^{\varphi-1}+\sum_{\varphi=1}^{\infty}([\varphi]_q+1)(1-\gamma)|b_\varphi|r^{\varphi-1}}{(\mu-\nu)-\sum_{\varphi=2}^{\infty}\left[(\mu\gamma-\nu)[\varphi]_q+\mu(1-\gamma)\right]|a_\varphi|r^{\varphi-1}-\sum_{\varphi=1}^{\infty}\left[(\mu\gamma-\nu)[\varphi]_q-\mu(1-\gamma)\right]|b_\varphi|r^{\varphi-1}}<1. \tag{13}$$

If the inequality (8) is not satisfied when $r \to 1$, the inequality (13) is not satisfied either. Thus, we can find at least one $\varsigma_0 = r_0$ in the range $(0,1)$ for which the quotient (13) is greater than 1. This contradicts the required condition for $\wp \in \mathcal{TSH}_q(\gamma,\mu,\nu)$ and so the proof is complete. □

Theorem 4. *Let $\wp \in \mathcal{TSH}_q(\gamma,\mu,\nu)$. In this case, for $|\varsigma| = r < 1$, we have*

$$|\wp(\varsigma)| \leq (1+|b_1|)r + \left(\frac{\mu-\nu}{[2]_q(\mu\gamma-\nu)+(1-\gamma)(q+\mu)} - \frac{\mu\gamma-\nu+(1-\gamma)(2-\mu)}{[2]_q(\mu\gamma-\nu)+(1-\gamma)(q+\mu)}|b_1|\right)r^2,$$

and

$$|\wp(\varsigma)| \geq (1-|b_1|)r - \left(\frac{\mu-\nu}{[2]_q(\mu\gamma-\nu)+(1-\gamma)(q+\mu)} - \frac{\mu\gamma-\nu+(1-\gamma)(2-\mu)}{[2]_q(\mu\gamma-\nu)+(1-\gamma)(q+\mu)}|b_1|\right)r^2.$$

Proof. In this proof, we prove the right-hand inequality, since the left-hand inequality can be similarly derived. Let $\wp \in \mathcal{TSH}_q(\gamma,\mu,\nu)$. Taking the absolute value of $\wp$, we have

$$\begin{aligned}
|\wp(\varsigma)| &\leq (1+|b_1|)r + \sum_{\varphi=2}^{\infty}\left(|a_\varphi| + |b_\varphi|\right)r^\varphi \\
&\leq (1+|b_1|)r + \frac{r^2}{[2]_q(\mu\gamma-\nu)+(1-\gamma)(q+\mu)}\sum_{\varphi=2}^{\infty}\left(\Xi_\varphi|a_\varphi| + \Psi_\varphi|b_\varphi|\right) \\
&\leq (1+|b_1|)r + \frac{\mu-\nu-[(\mu\gamma-\nu)+(1-\gamma)(2-\mu)]|b_1|}{[2]_q(\mu\gamma-\nu)+(1-\gamma)(q+\mu)}r^2.
\end{aligned}$$

□

Using the left-hand inequality of Theorem 4, we get the following covering result.

Corollary 1. *Let $\wp = \ell + \bar{\jmath}$ with ℓ and $\jmath$ of the form (4). If $\wp \in \mathcal{TSH}_q(\gamma,\mu,\nu)$, then*

$$\left\{\mathfrak{w} : |\mathfrak{w}| < \frac{q[\gamma(\mu-1)-\nu+1]+[(2-q)[1+\gamma(\mu-1)]-2\mu+\nu q]|b_1|}{\mu-\nu+q[\gamma(\mu-1)-\nu+1]}\right\} \subset f(\mathcal{U}).$$

Theorem 5. *Set*

$$\ell_1(\varsigma) = \varsigma,\ \ell_\varphi(\varsigma) = \varsigma - \frac{\mu-\nu}{\Xi_\varphi}\varsigma^\varphi,\ (\varphi = 2,3,\cdots),$$

and

$$\jmath_\varphi(\varsigma) = \varsigma + \frac{\mu-\nu}{\Psi_\varphi}\bar{\varsigma}^\varphi,\ (\varphi = 1,2,\cdots).$$

Then, $\wp \in \mathcal{TSH}_q(\gamma,\mu,\nu)$ if and only if it can be expressed as

$$\wp(\varsigma) = \sum_{\varphi=1}^{\infty}\left(x_\varphi\ell_\varphi(\varsigma) + y_\varphi\jmath_\varphi(\varsigma)\right)$$

where $x_\varphi \geq 0$, $y_\varphi \geq 0$ and $\sum_{\varphi=1}^{\infty}(x_\varphi + y_\varphi) = 1$. In particular, the extreme points of $\mathcal{TSH}_q(\gamma,\mu,\nu)$ are $\{\ell_\varphi\}$ and $\{\jmath_\varphi\}$.

Proof. Suppose

$$\begin{aligned}
\wp(\varsigma) &= \sum_{\varphi=1}^{\infty}\left(x_\varphi\ell_\varphi(\varsigma) + y_\varphi\jmath_\varphi(\varsigma)\right) \\
&= \sum_{\varphi=1}^{\infty}(x_\varphi + y_\varphi)\varsigma - \sum_{\varphi=2}^{\infty}\frac{\mu-\nu}{\Xi_\varphi}x_\varphi\varsigma^\varphi + \sum_{\varphi=1}^{\infty}\frac{\mu-\nu}{\Psi_\varphi}y_\varphi\bar{\varsigma}^\varphi.
\end{aligned}$$

Then,

$$\begin{aligned}\sum_{\varphi=2}^{\infty}\Xi_{\varphi}|a_{\varphi}|+\sum_{\varphi=1}^{\infty}\Psi_{\varphi}|b_{\varphi}| &= (\mu-\nu)\sum_{\varphi=2}^{\infty}x_{\varphi}+(\mu-\nu)\sum_{\varphi=1}^{\infty}y_{\varphi} \\ &= (\mu-\nu)(1-x_1)\le\mu-\nu\end{aligned}$$

and so $\wp\in\mathcal{TSH}_q(\gamma,\mu,\nu)$. Conversely, if $\wp\in\mathcal{TSH}_q(\gamma,\mu,\nu)$, then

$$|a_{\varphi}|\le\frac{\mu-\nu}{\Xi_{\varphi}}\text{ and }|b_{\varphi}|\le\frac{\mu-\nu}{\Psi_{\varphi}}.$$

Set

$$x_{\varphi}=\frac{\Xi_{\varphi}}{\mu-\nu}|a_{\varphi}|\ (\varphi=2,3,\cdots)\text{ and }y_{\varphi}=\frac{\Psi_{\varphi}}{\mu-\nu}|b_{\varphi}|\ (\varphi=1,2,\cdots).$$

Then, by Theorem 4, $0\le x_{\varphi}\le 1(\varphi=2,3,\cdots)$, $0\le y_{\varphi}\le 1\ (\varphi=1,2,\cdots)$. We define

$$x_1=1-\sum_{\varphi=2}^{\infty}x_{\varphi}-\sum_{\varphi=1}^{\infty}y_{\varphi}$$

and note that by Theorem 4, $x_1\ge 0$. Consequently, we obtain

$$\wp(\varsigma)=\sum_{\varphi=1}^{\infty}\left(x_{\varphi}\ell_{\varphi}(\varsigma)+y_{\varphi}\jmath_{\varphi}(\varsigma)\right)$$

as required. □

Finally, we define and prove the following theorem about the $\mathcal{TSH}_q(\gamma,\mu,\nu)$ class.

Theorem 6. *The class $\mathcal{TSH}_q(\gamma,\mu,\nu)$ is closed under convex combination.*

Proof. For $i=1,2,3,\cdots$ let $\wp_i\in\mathcal{TSH}_q(\gamma,\mu,\nu)$, where $\wp_i$ is given by

$$\wp_i(\varsigma)=\varsigma-\sum_{\varphi=2}^{\infty}|a_{\varphi_i}|\varsigma^{\varphi}+\sum_{\varphi=1}^{\infty}|b_{\varphi_i}|\bar{\varsigma}^{\varphi}.$$

Then, by (8),

$$\sum_{\varphi=1}^{\infty}\left(\Xi_{\varphi}|a_{\varphi_i}|+\Psi_{\varphi}|b_{\varphi_i}|\right)\le 2(\mu-\nu).$$

For $\sum_{i=1}^{\infty}\varkappa_i=1,\ 0\le\varkappa_i\le 1$, we can write the convex combination of $\wp_i$ as follows

$$\sum_{i=1}^{\infty}\varkappa_i\wp_i(\varsigma)=\varsigma-\sum_{\varphi=2}^{\infty}\left(\sum_{i=1}^{\infty}\varkappa_i|a_{\varphi_i}|\right)\varsigma^{\varphi}+\sum_{\varphi=1}^{\infty}\left(\sum_{i=1}^{\infty}\varkappa_i|b_{\varphi_i}|\right)\bar{\varsigma}^{\varphi}.$$

Then, by (8),

$$\begin{aligned}\sum_{\varphi=1}^{\infty}\left(\Xi_{\varphi}\sum_{i=1}^{\infty}\varkappa_i|a_{\varphi_i}|+\Psi_{\varphi}\sum_{i=1}^{\infty}\varkappa_i|b_{\varphi_i}|\right) &= \sum_{i=1}^{\infty}\varkappa_i\left(\sum_{\varphi=1}^{\infty}\left[\Xi_{\varphi}|a_{\varphi_i}|+\Psi_{\varphi}|b_{\varphi_i}|\right]\right) \\ &\le 2(\mu-\nu)\sum_{i=1}^{\infty}\varkappa_i=2(\mu-\nu).\end{aligned}$$

Thus, it shows that the condition in inequality (8) is satisfied, and so we can write

$$\sum_{i=1}^{\infty} \varkappa_i \wp_i(\varsigma) \in \mathcal{TSH}_q(\gamma, \mu, \nu).$$

□

3. Discussion

In this article, as mentioned in the introduction, we defined a subclass with a subordination technique on the important subject of harmonic functions. We obtained significant correlations for q-calculus. We defined the q-analogue operation as a new subclass of harmonic univalent function class with the subordination principle. We have defined necessary and sufficient conditions for this new subclass, which we have defined with the help of subordination. Moreover, we analyzed the distortion bounds and extreme points and obtained significant results. This study is both a guide for future articles and will shed light on new ideas. For example, regarding this article, the (p, q)-analogue function subclass can be defined with the same technique in the future. The motivation for this definition is that new results can be obtained with the aid of subordination of the (p, q)-analogue class.

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