symmetry

Article

New Generalized Class of Convex Functions and Some Related Integral Inequalities

Artion Kashuri [1], Ravi P. Agarwal [2], Pshtiwan Othman Mohammed [3], Kamsing Nonlaopon [4,*], Khadijah M. Abualnaja [5] and Yasser S. Hamed [5]

[1] Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", 9400 Vlora, Albania; artion.kashuri@univlora.edu.al
[2] Department of Mathematics, Texas A & M University-Kingsville, Kingsville, TX 78363, USA; Ravi.Agarwal@tamuk.edu
[3] Department of Mathematics, College of Education, University of Sulaimani, Sulaimani 46001, Kurdistan Region, Iraq; pshtiwan.muhammad@univsul.edu.iq
[4] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
[5] Department of Mathematics and Statistics, College of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia; Kh.abualnaja@tu.edu.sa (K.M.A.); yasersalah@tu.edu.sa (Y.S.H.)
* Correspondence: nkamsi@kku.ac.th; Tel.: +66-86642-1582

Abstract: There is a strong correlation between convexity and symmetry concepts. In this study, we investigated the new generic class of functions called the (n, m)–generalized convex and studied its basic algebraic properties. The Hermite–Hadamard inequality for the (n, m)–generalized convex function, for the products of two functions and of this type, were proven. Moreover, this class of functions was applied to several known identities; midpoint-type inequalities of Ostrowski and Simpson were derived. Our results are extensions of many previous contributions related to integral inequalities via different convexities.

Keywords: Hermite–Hadamard inequality; Ostrowski inequality; Simpson inequality; (n, m)–generalized convexity

MSC: 26A33; 26A51; 26D07; 26D10; 26D15; 26D20

Citation: Kashuri, A.; Agarwal, R.P.; Mohammed, P.O.; Nonlaopon, K.; Abualnaja, K.M.; Hamed, Y.S. New Generalized Class of Convex Functions and Some Related Integral Inequalities. *Symmetry* **2022**, *14*, 722. https://doi.org/10.3390/sym14040722

Academic Editor: Nicusor Minculete

Received: 12 March 2022
Accepted: 31 March 2022
Published: 2 April 2022

1. Introduction and Preliminaries

The twenty-first century began with the introduction and establishment of new tools used to solve linear and nonlinear differential and difference equations. In terms of the convexity theory, one important development involves defining a new class of convex functions, which is then tested on the well-known inequalities. "As it is known, inequalities aim to develop different mathematical methods. Nowadays, we need to seek accurate inequalities for proving the existence and uniqueness of the mathematical methods. In recent years, especially over the past two decades, several authors have been engaged in the study of inequalities, including various function classes (symmetric or asymmetric)", see [1]. Moreover, the modern convexity theory has motivated researchers to propose a new generalized class of convex functions and to investigate their special models, which could effectively be used in different fields, in particular, agriculture, medicine, reliability engineering, demography, actuarial study, survival analysis, and others. Kasamsetty et al. in [2] defined a new class of convex functions used to delay modeling and established an application to the transistor sizing problem. Awan et al. in [3] obtained new classes of convex functions and inequalities. Hudzik and Maligranda in [4] investigated the class of s-convex functions. Eftekhari in [5] derived new results using (s, m)–convexity in the second sense. Kadakal and İşcan in [6] established related inequalities via the exponential type convexity. Agarwal and Choi in [7] used fractional operators and found their image formulas. Rekhviashvili et al. in [8] described damped vibrations via a fractional oscillator model.

In much of the literature, we can see various Hermite–Hadamard (HH) inequality types, in which one of the known classes of convex functions is utilized (e.g., [9–11]). Moreover, some generalizations of the HH integral inequalities, such as HH–Fejér, AB HH, midpoint HH, mid-end-point HH, conformable HH, and HH–Mercer integral inequalities are found (e.g., [12–14]). In addition, different integral inequalities using those convexities are investigated. Ujević in [15] obtained sharp inequalities for Simpson and Ostrowski types. Liu et al. in [16], using the MT–convexity class derived Ostrowski fractional inequalities. Kaijser et al. in [17] established Hardy-type inequalities via convexity. Rashid et al. in [18], using generalized k–fractional integrals, found Grüss inequalities. For more recent published papers on HH, see [19,20].

Let us review some fundamental and preliminary results on convexity and inequality.

Definition 1. *Function $\Theta : \tau \subseteq \mathrm{R} \to \mathrm{R}$ is called convex, if*

$$\Theta(\varrho\chi_1 + (1-\varrho)\chi_2) \leq \varrho\Theta(\chi_1) + (1-\varrho)\Theta(\chi_2), \tag{1}$$

holds for all $\chi_1, \chi_2 \in \tau$ (τ is an interval with real numbers and R is the set of real numbers) and $\varrho \in [0,1]$. Moreover, Θ is concave if $(-\Theta)$ is convex.

Definition 2 ([4])**.** *Let $s \in (0,1]$ be a real number. A function $\Theta : \tau \subseteq \mathrm{R} \to \mathrm{R}$ is called s-convex (in the second sense), if*

$$\Theta(\varrho\chi_1 + (1-\varrho)\chi_2) \leq \varrho^s\Theta(\chi_1) + (1-\varrho)^s\Theta(\chi_2), \tag{2}$$

holds for all $\chi_1, \chi_2 \in \tau$, and $\varrho \in [0,1]$.

Definition 3 ([21])**.** *Let $\tau, \mathcal{J}$ be intervals in R, $(0,1) \subseteq \mathcal{J}$ and let $h : \mathcal{J} \to \mathrm{R}$ be a nonnegative function, and $h \neq 0$. A nonnegative function $\Theta : \tau \to \mathrm{R}$ is called h-convex, if*

$$\Theta(\varrho\chi_1 + (1-\varrho)\chi_2) \leq h(\varrho)\Theta(\chi_1) + h(1-\varrho)\Theta(\chi_2), \tag{3}$$

holds for all $\chi_1, \chi_2 \in \tau$, $\varrho \in (0,1)$.

Toply et al. [22] introduced the following class of convex functions:

Definition 4. *Let $n \in \mathbb{N}$. A function $\Theta : \tau \to \mathrm{R}$ is called n–polynomial convex, if*

$$\Theta(\varrho\chi_1 + (1-\varrho)\chi_2) \leq \frac{1}{n}\sum_{\ell_1=1}^{n}\left[1-(1-\varrho)^{\ell_1}\right]\Theta(\chi_1) + \frac{1}{n}\sum_{\ell_1=1}^{n}\left[1-\varrho^{\ell_1}\right]\Theta(\chi_2), \tag{4}$$

holds for every $\chi_1, \chi_2 \in \tau$, and $\varrho \in [0,1]$.

Recently, Rashid et al. [23] defined the following class of convex functions:

Definition 5 ([23])**.** *Assume that $s \in [0,1]$ and $n \in \mathbb{N}$. A function $\Theta : \tau \to \mathrm{R}$ is said to be n–polynomial s–type convex, if*

$$\Theta(\varrho\chi_1 + (1-\varrho)\chi_2) \leq \frac{1}{n}\sum_{\ell_1=1}^{n}\left[1-(s(1-\varrho))^{\ell_1}\right]\Theta(\chi_1) + \frac{1}{n}\sum_{\ell_1=1}^{n}\left[1-(s\varrho)^{\ell_1}\right]\Theta(\chi_2), \tag{5}$$

holds for every $\chi_1, \chi_2 \in \tau$, and $\varrho \in [0,1]$.

The following double inequality, namely the HH inequality, is remarkable, and it played an important role in the analysis.

Theorem 1 (HH inequality [24])**.** *Let $\Theta : \tau \subseteq \mathrm{R} \to \mathrm{R}$ be a convex function on τ for $\chi_1, \chi_2 \in \tau$ and $\chi_1 < \chi_2$, then*

$$\Theta\left(\frac{\chi_1 + \chi_2}{2}\right) \leq \frac{1}{\chi_2 - \chi_1} \int_{\chi_1}^{\chi_2} \Theta(\varrho) d\varrho \leq \frac{\Theta(\chi_1) + \Theta(\chi_2)}{2}. \tag{6}$$

The following well-known inequality is called the Ostrowski inequality:

Theorem 2 (Ostrowski inequality [16])**.** *Let $\Theta : \tau \subseteq \mathrm{R} \to \mathrm{R}$ a differentiable function in the interval τ and let $\chi_1, \chi_2 \in \tau$ with $\chi_1 < \chi_2$. If $|\Theta'(x)| \leq M$ for all $x \in [\chi_1, \chi_2]$, then*

$$\left|\Theta(x) - \frac{1}{\chi_2 - \chi_1} \int_{\chi_1}^{\chi_2} \Theta(\varrho) d\varrho\right| \leq M(\chi_2 - \chi_1)\left[\frac{1}{4} + \frac{\left(x - \frac{\chi_1+\chi_2}{2}\right)^2}{(\chi_2 - \chi_1)^2}\right]. \tag{7}$$

Another type of inequality is obtained by Dragomir et al. [25], which is as follows:

Theorem 3 (Simpson inequality [25])**.** *Assume that $\Theta : [\chi_1, \chi_2] \to \mathrm{R}$ is a four-time continuous and differentiable function on (χ_1, χ_2) such that $\|\Theta^{(4)}\|_\infty := \sup_{x \in (\chi_1, \chi_2)} |\Theta^{(4)}(x)| < \infty$ with $\chi_1 < \chi_2$, then*

$$\left|\frac{1}{6}\left[\Theta(\chi_1) + 4\Theta\left(\frac{\chi_1 + \chi_2}{2}\right) + \Theta(\chi_2)\right] - \frac{1}{\chi_2 - \chi_1} \int_{\chi_1}^{\chi_2} \Theta(x) dx\right| \leq \frac{1}{2880}(\chi_2 - \chi_1)^4 \|\Theta^{(4)}\|_\infty. \tag{8}$$

For brevity, we denote by $\mathcal{D} = \{h_1, h_2, \ldots, h_n, g_1, g_2, \ldots, g_m\}$ the convex set in the sequel.

Motivated by the above results, we introduce the following generic class of convex functions:

Definition 6. *Suppose that $1 \leq n \leq m$, where $n, m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0, 1] \to [0, +\infty)$ are continuous functions for all $\ell_1 = 1, 2, \ldots, n$ and $\ell_2 = 1, 2, \ldots, m$. A function $\Theta : \tau \to \mathrm{R}$, which is nonnegative, is said to be (n, m)–generalized convex with respect to $\mathcal{D}$, if*

$$\Theta(\varrho\chi_1 + (1 - \varrho)\chi_2) \leq \left(\frac{1}{n} \sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)\Theta(\chi_1) + \left(\frac{1}{m} \sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)\Theta(\chi_2), \tag{9}$$

holds for every $\chi_1, \chi_2 \in \tau$ and $\varrho \in [0, 1]$.

Remark 1. *From Definition 6, we can observe that:*

1. *If $n = m = 1$, $h_{\ell_1}(\varrho) = 1 - (1 - \varrho)^{\ell_1}$ and $g_{\ell_2}(\varrho) = 1 - \varrho^{\ell_2}$, then we have Definition 1.*
2. *If $n = m = 1$, $h_{\ell_1}(\varrho) = \varrho^s$ and $g_{\ell_2}(\varrho) = (1 - \varrho)^s$, then we obtain Definition 2.*
3. *If $n = m = 1$, $h_{\ell_1}(\varrho) = h(\varrho)$ and $g_{\ell_2}(\varrho) = h(1 - \varrho)$, then we obtain Definition 3.*
4. *If $n = m$, $h_{\ell_1}(\varrho) = 1 - (1 - \varrho)^{\ell_1}$ and $g_{\ell_2}(\varrho) = 1 - \varrho^{\ell_2}$, then we obtain Definition 4.*
5. *If $n = m$, $h_{\ell_1}(\varrho) = 1 - (s(1 - \varrho))^{\ell_1}$ and $g_{\ell_2}(\varrho) = 1 - (s\varrho)^{\ell_2}$, then we obtain Definition 5.*

Interested readers can derive many other known and unknown classes for suitable choices of the above functions h_{ℓ_1} and g_{ℓ_2}.

This article is divided into five sections: in Section 2, algebraic properties of the (n, m)–generalized convex function are presented. In Section 3, a new version of the HH inequality is presented; by using this definition, we will also derive the products of two functions of this type. In Section 4, we obtain general results by using the well-known identities of midpoint-type inequalities of Ostrowski and Simpson for our new defined convex functions; we obtain special cases from these. Section 5 concludes the article.

2. Algebraic Properties of the New Convex Function

This section deals with algebraic properties of our new definition.

Theorem 4. *Suppose that $1 \leq n \leq m$, where $n, m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ are continuous functions for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$, and Θ, Θ_1, $\Theta_2 : \tau \to \mathrm{R}$. If Θ, Θ_1, and Θ_2 are three nonnegative (n,m)–generalized convex functions with respect to $\mathcal{D}$, then*

1. *$\Theta_1 + \Theta_2$ is the (n,m)–generalized convex with respect to $\mathcal{D}$;*
2. *$c\Theta$ is the (n,m)–generalized convex with respect to $\mathcal{D}$ for any nonnegative real number c.*

Proof. The proof is evident, so we omit it. □

Theorem 5. *Suppose that $1 \leq n \leq m$, where $n, m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ are continuous functions for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$. Let $\Theta_1 : \tau \to \mathrm{R}$ be a convex function and $\Theta_2 : \mathrm{R} \to \mathrm{R}$ is a non-decreasing and nonnegative (n,m)–generalized convex function with respect to $\mathcal{D}$. Then the function $\Theta_2 \circ \Theta_1 : \tau \to \mathrm{R}$ is an (n,m)–generalized convex with respect to $\mathcal{D}$.*

Proof. For all $\chi_1, \chi_2 \in \tau$ and $\varrho \in [0,1]$, we have

$$\begin{aligned}
(\Theta_2 \circ \Theta_1)(\varrho\chi_1 + (1-\varrho)\chi_2) &= \Theta_2(\Theta_1(\varrho\chi_1 + (1-\varrho)\chi_2)) \\
&\leq \Theta_2(\varrho\Theta_1(\chi_1) + (1-\varrho)\Theta_1(\chi_2)) \\
&\leq \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)\Theta_2(\Theta_1(\chi_1)) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)\Theta_2(\Theta_1(\chi_2)) \\
&= \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)(\Theta_2 \circ \Theta_1)(\chi_1) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)(\Theta_2 \circ \Theta_1)(\chi_2),
\end{aligned}$$

which ends our proof. □

Theorem 6. *Suppose that $1 \leq n \leq m$, where $n, m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ are continuous functions for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$. Let $\Theta_k : [\chi_1, \chi_2] \to \mathrm{R}$ be a family of nonnegative (n,m)–generalized convex functions with respect to $\mathcal{D}$ and $\Theta(\chi) = \sup_k \Theta_k(\chi)$. Then Θ is an (n,m)–generalized convex function with respect to $\mathcal{D}$ and $\mathcal{U} = \{\chi \in [\chi_1, \chi_2] : \Theta(\chi) < +\infty\}$ is an interval.*

Proof. Let $\chi_1, \chi_2 \in \mathcal{U}$ and $\varrho \in [0,1]$, then

$$\begin{aligned}
\Theta(\varrho\chi_1 + (1-\varrho)\chi_2) &= \sup_k \Theta_k(\varrho\chi_1 + (1-\varrho)\chi_2) \\
&\leq \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)\sup_k \Theta_k(\chi_1) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)\sup_k \Theta_k(\chi_2) \\
&= \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)\Theta(\chi_1) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)\Theta(\chi_2) < +\infty,
\end{aligned}$$

which ends our proof. □

Theorem 7. *Suppose that $1 \leq n \leq m$, where $n, m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ are continuous functions for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$. If $\Theta : [\chi_1, \chi_2] \to \mathrm{R}$ is a nonnegative (n,m)–generalized convex function with respect to $\mathcal{D}$, then Θ is bounded on $[\chi_1, \chi_2]$.*

Proof. Let $K = \max\{\Theta(\chi_1), \Theta(\chi_2)\}$ and $x \in [\chi_1, \chi_2]$. Then, there exists $\varrho \in [0,1]$, such that $x = \varrho\chi_1 + (1-\varrho)\chi_2$. Moreover, since h_{ℓ_1}, g_{ℓ_2} are continuous functions on $[0,1]$ for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$, then we denote, respectively, $L_1 = \max\{h_1, h_2, \ldots, h_n\}$ and $L_2 = \max\{g_1, g_2, \ldots, g_m\}$. Hence,

$$\begin{aligned}
\Theta(x) = \Theta(\varrho\chi_1 + (1-\varrho)\chi_2) &\leq \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)\Theta(\chi_1) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)\Theta(\chi_2) \\
&\leq K\left[\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho) + \frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right] \\
&\leq K\left[\frac{1}{n}\sum_{\ell_1=1}^{n} L_1 + \frac{1}{m}\sum_{\ell_2=1}^{m} L_2\right] \\
&= K(L_1 + L_2) = M.
\end{aligned}$$

Moreover , for all $x \in [\chi_1, \chi_2]$, there exists $\xi \in \left[0, \frac{\chi_2 - \chi_1}{2}\right]$, such that $x = \frac{\chi_1+\chi_2}{2} + \xi$ or $x = \frac{\chi_1+\chi_2}{2} - \xi$. Let us suppose that $x = \frac{\chi_1+\chi_2}{2} + \xi$ without loss of generality. So, we have

$$\begin{aligned}
\Theta\left(\frac{\chi_1+\chi_2}{2}\right) &= \Theta\left(\frac{1}{2}\left[\frac{\chi_1+\chi_2}{2} + \xi\right] + \frac{1}{2}\left[\frac{\chi_1+\chi_2}{2} - \xi\right]\right) \\
&\leq \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}\left(\frac{1}{2}\right)\right)\Theta(x) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}\left(\frac{1}{2}\right)\right)\Theta\left(\frac{\chi_1+\chi_2}{2} - \xi\right) \\
&\leq L_1\Theta(x) + L_2\Theta\left(\frac{\chi_1+\chi_2}{2} - \xi\right).
\end{aligned}$$

By making use of M as the upper bound, we can deduce

$$\Theta(x) \geq \frac{1}{L_1}\Theta\left(\frac{\chi_1+\chi_2}{2}\right) - M = m,$$

which ends our proof. □

3. The HH Inequality for the New Convex Function

In this section, we will establish some integral inequalities of the HH-type pertaining to the (n,m)–generalized convex functions.

Theorem 8. *Assume that $1 \leq n \leq m$, where $n, m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ are continuous functions for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$. If $\Theta : [\chi_1, \chi_2] \to \mathrm{R}$ is a nonnegative (n,m)–generalized convex function with respect to $\mathcal{D}$, then we have*

$$\begin{aligned}
\frac{1}{\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}\left(\frac{1}{2}\right) + \frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}\left(\frac{1}{2}\right)}\Theta\left(\frac{\chi_1+\chi_2}{2}\right) &\leq \frac{1}{\chi_2 - \chi_1}\int_{\chi_1}^{\chi_2}\Theta(x)dx \\
&\leq \left(\frac{\Theta(\chi_1) + \Theta(\chi_2)}{2}\right)\left[\frac{1}{n}\sum_{\ell_1=1}^{n} H_{\ell_1} + \frac{1}{m}\sum_{\ell_2=1}^{m} G_{\ell_2}\right], \qquad (10)
\end{aligned}$$

where

$$H_{\ell_1} := \int_0^1 h_{\ell_1}(\varrho)d\varrho, \quad \forall\, \ell_1 = 1,2,\ldots,n \quad \text{and} \quad G_{\ell_2} := \int_0^1 g_{\ell_2}(\varrho)d\varrho, \quad \forall\, \ell_2 = 1,2,\ldots,m.$$

Proof. Let $w_1, w_2 \in [\chi_1, \chi_2]$. Applying the (n,m)–generalized convexity with respect to $\mathcal{D}$ of Θ on $[\chi_1, \chi_2]$, we have

$$\Theta\left(\frac{w_1+w_2}{2}\right) \leq \frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}\left(\frac{1}{2}\right)\Theta(w_1) + \frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}\left(\frac{1}{2}\right)\Theta(w_2). \tag{11}$$

Let us denote, respectively, $w_1 = \varrho\chi_2 + (1-\varrho)\chi_1$ and $w_2 = \varrho\chi_1 + (1-\varrho)\chi_2$. From inequality (11), we obtain

$$\Theta\left(\frac{\chi_1+\chi_2}{2}\right) \leq \frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}\left(\frac{1}{2}\right)\Theta(\varrho\chi_2 + (1-\varrho)\chi_1) + \frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}\left(\frac{1}{2}\right)\Theta(\varrho\chi_1 + (1-\varrho)\chi_2). \tag{12}$$

Integrating on both sides (12), with respect to ϱ from 0 to 1, we obtain

$$\begin{aligned}\Theta\left(\frac{\chi_1+\chi_2}{2}\right) &\leq \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}\left(\frac{1}{2}\right)\right)\int_0^1 \Theta(\varrho\chi_2 + (1-\varrho)\chi_1)d\varrho \\ &\quad + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}\left(\frac{1}{2}\right)\right)\int_0^1 \Theta(\varrho\chi_1 + (1-\varrho)\chi_2)d\varrho \\ &= \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}\left(\frac{1}{2}\right) + \frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}\left(\frac{1}{2}\right)\right)\frac{1}{\chi_2-\chi_1}\int_{\chi_1}^{\chi_2}\Theta(x)dx,\end{aligned}$$

which gives the proof of the left hand side of (10). For the right hand side of (10), we use the definition of (n,m)–generalized convexity with respect to $\mathcal{D}$ of Θ, where $\varrho \in [0,1]$. Hence,

$$\Theta(\varrho\chi_1 + (1-\varrho)\chi_2) \leq \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)\Theta(\chi_1) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)\Theta(\chi_2),$$

and

$$\Theta(\varrho\chi_2 + (1-\varrho)\chi_1) \leq \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)\Theta(\chi_2) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)\Theta(\chi_1).$$

Adding both of them, we have

$$\begin{aligned}\Theta(\varrho\chi_1 + (1-\varrho)\chi_2) + \Theta(\varrho\chi_2 + (1-\varrho)\chi_1) &\leq \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)\Theta(\chi_1) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)\Theta(\chi_2) \\ &\quad + \left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)\Theta(\chi_2) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)\Theta(\chi_1).\end{aligned} \tag{13}$$

Integrating on both sides (13) with respect to ϱ from 0 to 1, we obtain

$$\begin{aligned}&\int_0^1 \Theta(\varrho\chi_1 + (1-\varrho)\chi_2)d\varrho + \int_0^1 \Theta(\varrho\chi_2 + (1-\varrho)\chi_1)d\varrho \\ &\leq \int_0^1 \left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)\Theta(\chi_1) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)\Theta(\chi_2)\right]d\varrho \\ &\quad + \int_0^1 \left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)\Theta(\chi_2) + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)\Theta(\chi_1)\right]d\varrho,\end{aligned}$$

which leads to

$$\frac{1}{\chi_2-\chi_1}\int_{\chi_1}^{\chi_2}\Theta(x)dx \leq \left(\frac{\Theta(\chi_1)+\Theta(\chi_2)}{2}\right)\left[\frac{1}{n}\sum_{\ell_1=1}^{n} H_{\ell_1} + \frac{1}{m}\sum_{\ell_2=1}^{m} G_{\ell_2}\right],$$

which ends our proof. □

Remark 2. *We have particular cases from Theorem 8:*

- *If $h_{\ell_1}(\varrho) = \varrho$ and $g_{\ell_2}(\varrho) = \varrho$ for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$, we have Theorem 1.*
- *If $n = m$, $h_{\ell_1}(\varrho) = 1-(s(1-\varrho))^{\ell_1}$ and $g_{\ell_2}(\varrho) = 1-(s\varrho)^{\ell_2}$ for $s \in [0,1]$, $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$, we obtain ([23], Theorem 2.1).*
- *If $n = m$, $h_{\ell_1}(\varrho) = 1-(1-\varrho)^{\ell_1}$ and $g_{\ell_2}(\varrho) = 1-\varrho^{\ell_2}$ for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$, we obtain ([22], Theorem 4).*

Theorem 9. *Let $1 \le n_1 \le m_1$ and $1 \le n_2 \le m_2$ where $n_1, n_2, m_1, m_2 \in \mathbb{N}$. Assume that $h^{(1)}_{\ell_1}, g^{(1)}_{\ell_2}, h^{(2)}_k, g^{(2)}_l : [0,1] \to [0,+\infty)$ are continuous functions for all $\ell_1 = 1,\ldots,n_1$, $\ell_2 = 1,\ldots,m_1$, $k = 1,\ldots,n_2$ and $l = 1,\ldots,m_2$. If $\Theta, \psi : [\chi_1,\chi_2] \to \mathrm{R}$ are nonnegative (n_1,m_1) and (n_2,m_2)–generalized convex functions with respect to*
$\mathcal{D}^{(1)} = \left\{h^{(1)}_1, h^{(1)}_2, \ldots, h^{(1)}_{n_1}, g^{(1)}_1, g^{(1)}_2, \ldots, g^{(1)}_{m_1}\right\}$, and
$\mathcal{D}^{(2)} = \left\{h^{(2)}_1, h^{(2)}_2, \ldots, h^{(2)}_{n_2}, g^{(2)}_1, g^{(2)}_2, \ldots, g^{(2)}_{m_2}\right\}$, respectively, then we have

$$\begin{aligned}
&\frac{1}{\chi_2-\chi_1}\int_{\chi_1}^{\chi_2}\Theta(x)\psi(x)dx \\
&\quad\le \left(\frac{1}{n_1n_2}\sum_{\ell_1=1}^{n_1}\sum_{k=1}^{n_2}A_{\ell_1,k}\right)\Theta(\chi_1)\psi(\chi_1) + \left(\frac{1}{n_1m_2}\sum_{\ell_1=1}^{n_1}\sum_{l=1}^{m_2}B_{\ell_1,l}\right)\Theta(\chi_1)\psi(\chi_2) \\
&\quad + \left(\frac{1}{n_2m_1}\sum_{k=1}^{n_2}\sum_{\ell_2=1}^{m_1}C_{k,\ell_2}\right)\Theta(\chi_2)\psi(\chi_1) + \left(\frac{1}{m_1m_2}\sum_{\ell_2=1}^{m_1}\sum_{l=1}^{m_2}D_{\ell_2,l}\right)\Theta(\chi_2)\psi(\chi_2), \qquad (14)
\end{aligned}$$

where

$$A_{\ell_1,k} := \int_0^1 h^{(1)}_{\ell_1}(\varrho)h^{(2)}_k(\varrho)d\varrho, \quad \forall\,\ell_1 = 1,2,\ldots,n_1, \;\; \forall\,k = 1,2,\ldots,n_2,$$

$$B_{\ell_1,l} := \int_0^1 h^{(1)}_{\ell_1}(\varrho)g^{(2)}_l(\varrho)d\varrho, \quad \forall\,\ell_1 = 1,2,\ldots,n_1, \;\; \forall\,l = 1,2,\ldots,m_2,$$

$$C_{k,\ell_2} := \int_0^1 h^{(2)}_k(\varrho)g^{(1)}_{\ell_2}(\varrho)d\varrho, \quad \forall\,k = 1,2,\ldots,n_2, \;\; \forall\,\ell_2 = 1,2,\ldots,m_1,$$

and

$$D_{\ell_2,l} := \int_0^1 g^{(1)}_{\ell_2}(\varrho)g^{(2)}_l(\varrho)d\varrho, \quad \forall\,\ell_2 = 1,2,\ldots,m_1, \;\; \forall\,l = 1,2,\ldots,m_2.$$

Proof. Applying (n_1,m_1) and (n_2,m_2)–generalized convexity with respect to $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ of Θ, ψ on $[\chi_1,\chi_2]$, respectively, we have

$$\Theta(\varrho\chi_1+(1-\varrho)\chi_2) \le \left(\frac{1}{n_1}\sum_{\ell_1=1}^{n_1}h^{(1)}_{\ell_1}(\varrho)\right)\Theta(\chi_1) + \left(\frac{1}{m_1}\sum_{\ell_2=1}^{m_1}g^{(1)}_{\ell_2}(\varrho)\right)\Theta(\chi_2) \qquad (15)$$

and

$$\psi(\varrho\chi_1+(1-\varrho)\chi_2) \le \left(\frac{1}{n_2}\sum_{k=1}^{n_2}h^{(2)}_k(\varrho)\right)\psi(\chi_1) + \left(\frac{1}{m_2}\sum_{l=1}^{m_2}g^{(2)}_l(\varrho)\right)\psi(\chi_2). \qquad (16)$$

Multiplying inequalities (15) and (16) on both sides, we obtain

$$\begin{aligned}
&\Theta(\varrho\chi_1+(1-\varrho)\chi_2)\psi(\varrho\chi_1+(1-\varrho)\chi_2)\\
&\leq \left(\frac{1}{n_1}\sum_{\ell_1=1}^{n_1} h_{\ell_1}^{(1)}(\varrho)\right)\left(\frac{1}{n_2}\sum_{k=1}^{n_2} h_{k}^{(2)}(\varrho)\right)\Theta(\chi_1)\psi(\chi_1)\\
&+\left(\frac{1}{n_1}\sum_{\ell_1=1}^{n_1} h_{\ell_1}^{(1)}(\varrho)\right)\left(\frac{1}{m_2}\sum_{l=1}^{m_2} g_{l}^{(2)}(\varrho)\right)\Theta(\chi_1)\psi(\chi_2)\\
&+\left(\frac{1}{n_2}\sum_{k=1}^{n_2} h_{k}^{(2)}(\varrho)\right)\left(\frac{1}{m_1}\sum_{\ell_2=1}^{m_1} g_{\ell_2}^{(1)}(\varrho)\right)\Theta(\chi_2)\psi(\chi_1)\\
&+\left(\frac{1}{m_1}\sum_{\ell_2=1}^{m_1} g_{\ell_2}^{(1)}(\varrho)\right)\left(\frac{1}{m_2}\sum_{l=1}^{m_2} g_{l}^{(2)}(\varrho)\right)\Theta(\chi_2)\psi(\chi_2). \qquad (17)
\end{aligned}$$

Integrating inequality (17) with respect to ϱ from 0 to 1 on both sides, we obtain

$$\begin{aligned}
&\int_0^1 \Theta(\varrho\chi_1+(1-\varrho)\chi_2)\psi(\varrho\chi_1+(1-\varrho)\chi_2)d\varrho = \frac{1}{\chi_2-\chi_1}\int_{\chi_1}^{\chi_2}\Theta(x)\psi(x)dx\\
&\leq \left(\frac{1}{n_1n_2}\sum_{\ell_1=1}^{n_1}\sum_{k=1}^{n_2}\int_0^1 h_{\ell_1}^{(1)}(\varrho)h_{k}^{(2)}(\varrho)d\varrho\right)\Theta(\chi_1)\psi(\chi_1)\\
&+\left(\frac{1}{n_1m_2}\sum_{\ell_1=1}^{n_1}\sum_{l=1}^{m_2}\int_0^1 h_{\ell_1}^{(1)}(\varrho)g_{l}^{(2)}(\varrho)d\varrho\right)\Theta(\chi_1)\psi(\chi_2)\\
&+\left(\frac{1}{n_2m_1}\sum_{k=1}^{n_2}\sum_{\ell_2=1}^{m_1}\int_0^1 h_{k}^{(2)}(\varrho)g_{\ell_2}^{(1)}(\varrho)d\varrho\right)\Theta(\chi_2)\psi(\chi_1)\\
&+\left(\frac{1}{m_1m_2}\sum_{\ell_2=1}^{m_1}\sum_{l=1}^{m_2}\int_0^1 g_{\ell_2}^{(1)}(\varrho)g_{l}^{(2)}(\varrho)d\varrho\right)\Theta(\chi_2)\psi(\chi_2).\\
&=\left(\frac{1}{n_1n_2}\sum_{\ell_1=1}^{n_1}\sum_{k=1}^{n_2}A_{\ell_1,k}\right)\Theta(\chi_1)\psi(\chi_1)+\left(\frac{1}{n_1m_2}\sum_{\ell_1=1}^{n_1}\sum_{l=1}^{m_2}B_{\ell_1,l}\right)\Theta(\chi_1)\psi(\chi_2)\\
&+\left(\frac{1}{n_2m_1}\sum_{k=1}^{n_2}\sum_{\ell_2=1}^{m_1}C_{k,\ell_2}\right)\Theta(\chi_2)\psi(\chi_1)+\left(\frac{1}{m_1m_2}\sum_{\ell_2=1}^{m_1}\sum_{l=1}^{m_2}D_{\ell_2,l}\right)\Theta(\chi_2)\psi(\chi_2),
\end{aligned}$$

which ends our proof. □

4. Further Results

We denote by $\mathcal{L}[\chi_1,\chi_2]$ the set of all integrable functions on $[\chi_1,\chi_2]$. Let us recall the following lemmas in order to establish our following results.

Lemma 1 (Midpoint identity [26])**.** *Let $\Theta:\tau\subseteq \mathrm{R}\to\mathrm{R}$ be a differentiable function on τ and $\chi_1,\chi_2\in\tau$ with $\chi_1<\chi_2$. If $\Theta'\in\mathcal{L}[\chi_1,\chi_2]$, then*

$$\begin{aligned}
\mathcal{T}(\Theta;\chi_1,\chi_2) &:= \frac{1}{\chi_2-\chi_1}\int_{\chi_1}^{\chi_2}\Theta(\varrho)d\varrho-\Theta\left(\frac{\chi_1+\chi_2}{2}\right)\\
&=\frac{(\chi_2-\chi_1)}{4}\left\{\int_0^1 \varrho\Theta'\left(\frac{\varrho}{2}\chi_1+\frac{2-\varrho}{2}\chi_2\right)d\varrho-\int_0^1\varrho\Theta'\left(\frac{\varrho}{2}\chi_2+\frac{2-\varrho}{2}\chi_1\right)d\varrho\right\}. \qquad (18)
\end{aligned}$$

Lemma 2 (Ostrowski identity [27])**.** *Let $\Theta:\tau\subseteq \mathrm{R}\to\mathrm{R}$ be a differentiable function on τ and $\chi_1,\chi_2\in\tau$ with $\chi_1<\chi_2$. If $\Theta'\in\mathcal{L}[\chi_1,\chi_2]$, then*

$$\mathcal{T}_1(\Theta;x,\chi_1,\chi_2) := \Theta(x) - \frac{1}{\chi_2-\chi_1}\int_{\chi_1}^{\chi_2}\Theta(\varrho)d\varrho$$
$$= \frac{(x-\chi_1)^2}{\chi_2-\chi_1}\int_0^1 \varrho\Theta'(\varrho x+(1-\varrho)\chi_1)d\varrho - \frac{(\chi_2-x)^2}{\chi_2-\chi_1}\int_0^1 \varrho\Theta'(\varrho x+(1-\varrho)\chi_2)d\varrho. \quad (19)$$

Lemma 3 (Simpson identity [28]). *Let $\Theta : \tau \subseteq R \to R$ be a differentiable function on τ and $\chi_1, \chi_2 \in \tau$ with $\chi_1 < \chi_2$. If $\Theta' \in \mathcal{L}[\chi_1,\chi_2]$, then*

$$\mathcal{T}_2(\Theta;\chi_1,\chi_2) := \frac{1}{6}\left[\Theta(\chi_1)+4\Theta\left(\frac{\chi_1+\chi_2}{2}\right)+\Theta(\chi_2)\right] - \frac{1}{\chi_2-\chi_1}\int_{\chi_1}^{\chi_2}\Theta(\varrho)d\varrho$$
$$= (\chi_2-\chi_1)\left\{\int_0^{\frac{1}{2}}\left(\varrho-\frac{1}{6}\right)\Theta'(\varrho\chi_2+(1-\varrho)\chi_1)d\varrho + \int_{\frac{1}{2}}^1\left(\varrho-\frac{5}{6}\right)\Theta'(\varrho\chi_2+(1-\varrho)\chi_1)d\varrho\right\}. \quad (20)$$

Theorem 10. *Suppose that $1 \le n \le m$, where $n,m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ are continuous functions for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$, and $\Theta : [\chi_1,\chi_2] \to R$ be a differentiable function on (χ_1,χ_2) such that $\Theta' \in \mathcal{L}[\chi_1,\chi_2]$. If $|\Theta'|$ is an (n,m)–generalized convex function with respect to $\mathcal{D}$ on $[\chi_1,\chi_2]$, then we have*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4}(|\Theta'(\chi_1)|+|\Theta'(\chi_2)|)\left[\frac{1}{n}\sum_{\ell_1=1}^{n}U_{\ell_1}+\frac{1}{m}\sum_{\ell_2=1}^{m}V_{\ell_2}\right], \quad (21)$$

where

$$U_{\ell_1} := \int_0^1 \varrho h_{\ell_1}\left(\frac{\varrho}{2}\right)d\varrho, \quad \forall\, \ell_1 = 1,2,\ldots,n \quad \text{and} \quad V_{\ell_2} := \int_0^1 \varrho g_{\ell_2}\left(\frac{\varrho}{2}\right)d\varrho, \quad \forall\, \ell_2 = 1,2,\ldots,m.$$

Proof. By using Lemma 1 and the (n,m)–generalized convexity of $|\Theta'|$ with respect to $\mathcal{D}$, we have

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4}\left\{\int_0^1 \varrho\left|\Theta'\left(\frac{\varrho}{2}\chi_1+\frac{(2-\varrho)}{2}\chi_2\right)\right|d\varrho + \int_0^1 \varrho\left|\Theta'\left(\frac{\varrho}{2}\chi_2+\frac{(2-\varrho)}{2}\chi_1\right)\right|d\varrho\right\}$$
$$\le \frac{(\chi_2-\chi_1)}{4}\left\{\int_0^1 \varrho\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_1)| + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_2)|\right]d\varrho\right.$$
$$\left. + \int_0^1 \varrho\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_2)| + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_1)|\right]d\varrho\right\}$$
$$= \frac{(\chi_2-\chi_1)}{4}(|\Theta'(\chi_1)|+|\Theta'(\chi_2)|)\left[\frac{1}{n}\sum_{\ell_1=1}^{n}U_{\ell_1}+\frac{1}{m}\sum_{\ell_2=1}^{m}V_{\ell_2}\right],$$

which ends our proof. □

Corollary 1. *We have particular cases from Theorem 10:*

- *If $h_{\ell_1}(\varrho) = \varrho$ and $g_{\ell_2}(\varrho) = 1-\varrho$ for all $\ell_1 = 1,2,\ldots,n$, and $\ell_2 = 1,2,\ldots,m$, we obtain*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{8}[|\Theta'(\chi_1)|+|\Theta'(\chi_2)|].$$

- *If* $h_{\ell_1}(\varrho) = \varrho^{\ell_1}$ *and* $g_{\ell_2}(\varrho) = (1-\varrho)^{\ell_2}$ *for all* $\ell_1 = 1,2,\ldots,n$, *and* $\ell_2 = 1,2,\ldots,m$, *we obtain*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4}\left[|\Theta'(\chi_1)| + |\Theta'(\chi_2)|\right]$$
$$\times\left[\frac{1}{n}\sum_{\ell_1=1}^{n}\frac{1}{2^{\ell_1}(\ell_1+2)} + \frac{1}{m}\sum_{\ell_2=1}^{m}\frac{1}{2^{\ell_2}}\left(\frac{2}{\ell_2+1}\left(2^{\ell_2+1}-1\right) - \frac{1}{\ell_2+2}\left(2^{\ell_2+2}-1\right)\right)\right].$$

- *If* $h_{\ell_1}(\varrho) = \varrho^{s}$ *and* $g_{\ell_2}(\varrho) = (1-\varrho)^{s}$ *for all* $\ell_1 = 1,\ldots,n$, $\ell_2 = 1,\ldots,m$, *and* $s \in (0,1]$, *we obtain*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4}\left[|\Theta'(\chi_1)| + |\Theta'(\chi_2)|\right]$$
$$\times\left[\frac{1}{2^s(s+2)} + \frac{1}{2^s}\left(\frac{2}{s+1}\left(2^{s+1}-1\right) - \frac{1}{s+2}\left(2^{s+2}-1\right)\right)\right].$$

Theorem 11. *Suppose that* $1 \le n \le m$, *where* $n,m \in \mathbb{N}$, *and assume that* $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ *are continuous functions for all* $\ell_1 = 1,2,\ldots,n$ *and* $\ell_2 = 1,2,\ldots,m$, *and* $\Theta : [\chi_1,\chi_2] \to$ R *be a differentiable function on* (χ_1,χ_2), *such that* $\Theta' \in \mathcal{L}[\chi_1,\chi_2]$. *If* $|\Theta'|^q$ *is an* (n,m)*–generalized convex function with respect to* $\mathcal{D}$ *on* $[\chi_1,\chi_2]$, *then for* $q > 1$ *and* $\frac{1}{p}+\frac{1}{q} = 1$, *we have*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}M_{\ell_1}\right)|\Theta'(\chi_1)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}N_{\ell_2}\right)|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right.$$
$$\left. + \left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}M_{\ell_1}\right)|\Theta'(\chi_2)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}N_{\ell_2}\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right\}, \tag{22}$$

where

$$M_{\ell_1} := \int_0^1 h_{\ell_1}\left(\frac{\varrho}{2}\right)d\varrho, \quad \forall\, \ell_1 = 1,2,\ldots,n \quad \text{and} \quad N_{\ell_2} := \int_0^1 g_{\ell_2}\left(\frac{\varrho}{2}\right)d\varrho, \quad \forall\, \ell_2 = 1,2,\ldots,m.$$

Proof. By using Lemma 1, Hölder's inequality and the (n,m)–generalized convexity of $|\Theta'|^q$ with respect to $\mathcal{D}$, we have

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4}\left\{\int_0^1 \varrho\left|\Theta'\left(\frac{\varrho}{2}\chi_1 + \frac{(2-\varrho)}{2}\chi_2\right)\right|d\varrho + \int_0^1 \varrho\left|\Theta'\left(\frac{\varrho}{2}\chi_2 + \frac{(2-\varrho)}{2}\chi_1\right)\right|d\varrho\right\}$$
$$\le \frac{(\chi_2-\chi_1)}{4}\left(\int_0^1 \varrho^p d\varrho\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\int_0^1\left|\Theta'\left(\frac{\varrho}{2}\chi_1 + \frac{(2-\varrho)}{2}\chi_2\right)\right|^q d\varrho\right)^{\frac{1}{q}} + \left(\int_0^1\left|\Theta'\left(\frac{\varrho}{2}\chi_2 + \frac{(2-\varrho)}{2}\chi_1\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right\}$$
$$\le \frac{(\chi_2-\chi_1)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left[\int_0^1\left(\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_1)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_2)|^q\right)d\varrho\right]^{\frac{1}{q}}\right.$$
$$\left. + \left[\int_0^1\left(\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_2)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_1)|^q\right)d\varrho\right]^{\frac{1}{q}}\right\}$$

$$= \frac{(\chi_2-\chi_1)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}M_{\ell_1}\right)|\Theta'(\chi_1)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}N_{\ell_2}\right)|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}M_{\ell_1}\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}N_{\ell_2}\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right\},$$

which ends our proof. □

Corollary 2. *We have particular cases from Theorem 11:*

- *If $h_{\ell_1}(\varrho)=\varrho$ and $g_{\ell_2}(\varrho)=1-\varrho$ for all $\ell_1=1,2,\ldots,n$, and $\ell_2=1,2,\ldots,m$, we have*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4\sqrt[q]{4}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left[|\Theta'(\chi_1)|^q+3|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}+\left[3|\Theta'(\chi_1)|^q+|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right\}.$$

- *If $h_{\ell_1}(\varrho)=\varrho^{\ell_1}$ and $g_{\ell_2}(\varrho)=(1-\varrho)^{\ell_2}$ for all $\ell_1=1,2,\ldots,n$, and $\ell_2=1,2,\ldots,m$, we obtain*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\frac{1}{2^{\ell_1}(\ell_1+1)}\right)|\Theta'(\chi_1)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\frac{2^{\ell_2+1}-1}{2^{\ell_2}(\ell_2+1)}\right)|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\frac{1}{2^{\ell_1}(\ell_1+1)}\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\frac{2^{\ell_2+1}-1}{2^{\ell_2}(\ell_2+1)}\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right\}.$$

- *If $h_{\ell_1}(\varrho)=\varrho^s$ and $g_{\ell_2}(\varrho)=(1-\varrho)^s$ for all $\ell_1=1,\ldots,n$, $\ell_2=1,\ldots,m$, and $s\in(0,1]$, we obtain*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4\sqrt[q]{2^s(s+1)}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left[|\Theta'(\chi_1)|^q+(2^{s+1}-1)|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}+\left[(2^{s+1}-1)|\Theta'(\chi_1)|^q+|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right\}.$$

Theorem 12. *Suppose that $1\le n\le m$, where $n,m\in\mathbb{N}$, and assume that $h_{\ell_1},g_{\ell_2}:[0,1]\to[0,+\infty)$ are continuous functions for all $\ell_1=1,2,\ldots,n$ and $\ell_2=1,2,\ldots,m$, and $\Theta:[\chi_1,\chi_2]\to$ R be a differentiable function on (χ_1,χ_2), such that $\Theta'\in\mathcal{L}[\chi_1,\chi_2]$. If $|\Theta'|^q$ is an (n,m)–generalized convex function with respect to $\mathcal{D}$ on $[\chi_1,\chi_2]$, then for $q>1$, we have*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}U_{\ell_1}\right)|\Theta'(\chi_1)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}V_{\ell_2}\right)|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}U_{\ell_1}\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}V_{\ell_2}\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right\}, \tag{23}$$

where U_{ℓ_1} and V_{ℓ_2} are defined as in Theorem 10.

Proof. By using Lemma 1, the well-known power mean inequality and the (n,m)–generalized convexity of $|\Theta'|^q$ with respect to $\mathcal{D}$, we have

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4}\left\{\int_0^1\varrho\left|\Theta'\left(\frac{\varrho}{2}\chi_1+\frac{(2-\varrho)}{2}\chi_2\right)\right|d\varrho+\int_0^1\varrho\left|\Theta'\left(\frac{\varrho}{2}\chi_2+\frac{(2-\varrho)}{2}\chi_1\right)\right|d\varrho\right\}$$

$$\leq \frac{(\chi_2-\chi_1)}{4}\left(\int_0^1 \varrho d\varrho\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1 \varrho\left|\Theta'\left(\frac{\varrho}{2}\chi_1+\frac{(2-\varrho)}{2}\chi_2\right)\right|^q d\varrho\right)^{\frac{1}{q}}+\left(\int_0^1 \varrho\left|\Theta'\left(\frac{\varrho}{2}\chi_2+\frac{(2-\varrho)}{2}\chi_1\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right\}$$

$$\leq \frac{(\chi_2-\chi_1)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left[\int_0^1 \varrho\left(\left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_1)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_2)|^q\right)d\varrho\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\int_0^1 \varrho\left(\left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}\left(\frac{\varrho}{2}\right)\right)|\Theta'(\chi_1)|^q\right)d\varrho\right]^{\frac{1}{q}}\right\}$$

$$=\frac{(\chi_2-\chi_1)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n} U_{\ell_1}\right)|\Theta'(\chi_1)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m} V_{\ell_2}\right)|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n} U_{\ell_1}\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m} V_{\ell_2}\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right\},$$

which ends our proof. □

Corollary 3. *We have particular cases from Theorem 12:*

- *If $h_{\ell_1}(\varrho)=\varrho$ and $g_{\ell_2}(\varrho)=1-\varrho$ for all $\ell_1=1,2,\ldots,n$, and $\ell_2=1,2,\ldots,m$, we obtain*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)|\leq \frac{(\chi_2-\chi_1)}{8\sqrt[q]{3}}\left\{\left[|\Theta'(\chi_1)|^q+2|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}+\left[2|\Theta'(\chi_1)|^q+|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right\}.$$

- *If $h_{\ell_1}(\varrho)=\varrho^{\ell_1}$ and $g_{\ell_2}(\varrho)=(1-\varrho)^{\ell_2}$ for all $\ell_1=1,2,\ldots,n$, and $\ell_2=1,2,\ldots,m$, we obtain*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)|\leq \frac{(\chi_2-\chi_1)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\frac{1}{2^{\ell_1}(\ell_1+2)}\right)|\Theta'(\chi_1)|^q\right.\right.$$

$$\left.+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\frac{1}{2^{\ell_2}}\left(\frac{2}{\ell_2+1}\left(2^{\ell_2+1}-1\right)-\frac{1}{\ell_2+2}\left(2^{\ell_2+2}-1\right)\right)\right)|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\frac{1}{2^{\ell_1}(\ell_1+2)}\right)|\Theta'(\chi_2)|^q\right.$$

$$\left.\left.+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\frac{1}{2^{\ell_2}}\left(\frac{2}{\ell_2+1}\left(2^{\ell_2+1}-1\right)-\frac{1}{\ell_2+2}\left(2^{\ell_2+2}-1\right)\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right\}.$$

- *If $h_{\ell_1}(\varrho)=\varrho^s$ and $g_{\ell_2}(\varrho)=(1-\varrho)^s$ for all $\ell_1=1,\ldots,n$, $\ell_2=1,\ldots,m$, and $s\in(0,1]$, we obtain*

$$|\mathcal{T}(\Theta;\chi_1,\chi_2)| \le \frac{(\chi_2-\chi_1)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\frac{1}{2^s(s+2)}|\Theta'(\chi_1)|^q+\frac{1}{2^s}\left(\frac{2}{s+1}\left(2^{s+1}-1\right)-\frac{1}{s+2}\left(2^{s+2}-1\right)\right)|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{1}{2^s(s+2)}|\Theta'(\chi_2)|^q+\frac{1}{2^s}\left(\frac{2}{s+1}\left(2^{s+1}-1\right)-\frac{1}{s+2}\left(2^{s+2}-1\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right\}.$$

Theorem 13. *Suppose that $1 \le n \le m$, where $n,m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ are continuous functions for all $\ell_1 = 1,2,\dots,n$ and $\ell_2 = 1,2,\dots,m$, and $\Theta : [\chi_1,\chi_2] \to \mathrm{R}$ be a differentiable function on (χ_1,χ_2), such that $\Theta' \in \mathcal{L}[\chi_1,\chi_2]$. If $|\Theta'|$ is an (n,m)–generalized convex function with respect to $\mathcal{D}$ on $[\chi_1,\chi_2]$, then we have*

$$|\mathcal{T}_1(\Theta;x,\chi_1,\chi_2)| \le \frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}E_{\ell_1}\right)|\Theta'(x)|+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}F_{\ell_2}\right)|\Theta'(\chi_1)|\right]$$
$$+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}E_{\ell_1}\right)|\Theta'(x)|+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}F_{\ell_2}\right)|\Theta'(\chi_2)|\right], \quad (24)$$

where

$$E_{\ell_1} := \int_0^1 \varrho h_{\ell_1}(\varrho)d\varrho, \quad \forall\, \ell_1 = 1,2,\dots,n \quad \text{and} \quad F_{\ell_2} := \int_0^1 \varrho g_{\ell_2}(\varrho)d\varrho, \quad \forall\, \ell_2 = 1,2,\dots,m.$$

Proof. By using Lemma 2 and the (n,m)–generalized convexity of $|\Theta'|$ with respect to $\mathcal{D}$, we have

$$|\mathcal{T}_1(\Theta;x,\chi_1,\chi_2)| \le \frac{(x-\chi_1)^2}{\chi_2-\chi_1}\int_0^1 \varrho\left|\Theta'(\varrho x+(1-\varrho)\chi_1)\right|d\varrho$$
$$+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\int_0^1 \varrho\left|\Theta'(\varrho x+(1-\varrho)\chi_2)\right|d\varrho$$
$$\le \frac{(x-\chi_1)^2}{\chi_2-\chi_1}\int_0^1 \varrho\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}(\varrho)\right)|\Theta'(x)|+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}(\varrho)\right)|\Theta'(\chi_1)|\right]d\varrho$$
$$+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\int_0^1 \varrho\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}(\varrho)\right)|\Theta'(x)|+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}(\varrho)\right)|\Theta'(\chi_2)|\right]d\varrho$$
$$=\frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}E_{\ell_1}\right)|\Theta'(x)|+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}F_{\ell_2}\right)|\Theta'(\chi_1)|\right]$$
$$+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}E_{\ell_1}\right)|\Theta'(x)|+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}F_{\ell_2}\right)|\Theta'(\chi_2)|\right],$$

which ends our proof. □

Corollary 4. *We have particular cases from Theorem 13:*

- *If $h_{\ell_1}(\varrho) = \varrho$ and $g_{\ell_2}(\varrho) = 1-\varrho$ for all $\ell_1 = 1,2,\dots,n$, and $\ell_2 = 1,2,\dots,m$, we obtain*

$$|\mathcal{T}_1(\Theta;x,\chi_1,\chi_2)| \le \frac{(x-\chi_1)^2}{6(\chi_2-\chi_1)}\left[2|\Theta'(x)|+|\Theta'(\chi_1)|\right]+\frac{(\chi_2-x)^2}{6(\chi_2-\chi_1)}\left[2|\Theta'(x)|+|\Theta'(\chi_2)|\right].$$

- *If $h_{\ell_1}(\varrho) = \varrho^{\ell_1}$ and $g_{\ell_2}(\varrho) = (1-\varrho)^{\ell_2}$ for all $\ell_1 = 1,2,\dots,n$, and $\ell_2 = 1,2,\dots,m$, we obtain*

$$‘|\mathcal{T}_1(\Theta;x,\chi_1,\chi_2)| \leq \frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\frac{1}{\ell_1+2}\right)|\Theta'(x)| + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}\frac{1}{(\ell_2+1)(\ell_2+2)}\right)|\Theta'(\chi_1)|\right]$$
$$+ \frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\frac{1}{\ell_1+2}\right)|\Theta'(x)| + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}\frac{1}{(\ell_2+1)(\ell_2+2)}\right)|\Theta'(\chi_2)|\right].$$

- *If $h_{\ell_1}(\varrho) = \varrho^s$ and $g_{\ell_2}(\varrho) = (1-\varrho)^s$ for all $\ell_1 = 1,\dots,n$, $\ell_2 = 1,\dots,m$, and $s \in (0,1]$, we obtain*

$$|\mathcal{T}_1(\Theta;x,\chi_1,\chi_2)| \leq \frac{1}{(s+1)(s+2)}$$
$$\times\left\{\frac{(x-\chi_1)^2}{\chi_2-\chi_1}\big[(s+1)|\Theta'(x)| + |\Theta'(\chi_1)|\big] + \frac{(\chi_2-x)^2}{\chi_2-\chi_1}\big[(s+1)|\Theta'(x)| + |\Theta'(\chi_2)|\big]\right\}.$$

Theorem 14. *Suppose that $1 \leq n \leq m$, where $n,m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ are continuous functions for all $\ell_1 = 1,2,\dots,n$ and $\ell_2 = 1,2,\dots,m$, and $\Theta : [\chi_1,\chi_2] \to$ R be a differentiable function on (χ_1,χ_2) such that $\Theta' \in \mathcal{L}[\chi_1,\chi_2]$. If $|\Theta'|^q$ is (n,m)–generalized convex function with respect to $\mathcal{D}$ on $[\chi_1,\chi_2]$, then for $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, we obtain*

$$|\mathcal{T}_1(\Theta;x,\chi_1,\chi_2)| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}H_{\ell_1}\right)|\Theta'(x)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}G_{\ell_2}\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+ \frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}H_{\ell_1}\right)|\Theta'(x)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}G_{\ell_2}\right)|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right\}, \quad (25)$$

where H_{ℓ_1} and G_{ℓ_2} are defined as in Theorem 8.

Proof. By using Lemma 2, Hölder's inequality and the (n,m)–generalized convexity of $|\Theta'|^q$ with respect to $\mathcal{D}$, we obtain

$$|\mathcal{T}_1(\Theta;x,\chi_1,\chi_2)| \leq \frac{(x-\chi_1)^2}{\chi_2-\chi_1}\int_0^1 \varrho|\Theta'(\varrho x+(1-\varrho)\chi_1)|d\varrho$$
$$+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\int_0^1 \varrho|\Theta'(\varrho x+(1-\varrho)\chi_2)|d\varrho$$
$$\leq \frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left(\int_0^1 \varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1 |\Theta'(\varrho x+(1-\varrho)\chi_1)|^q d\varrho\right)^{\frac{1}{q}}$$
$$+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left(\int_0^1 \varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1 |\Theta'(\varrho x+(1-\varrho)\chi_2)|^q d\varrho\right)^{\frac{1}{q}}$$
$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left[\int_0^1\left(\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}(\varrho)\right)|\Theta'(x)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}(\varrho)\right)|\Theta'(\chi_1)|^q\right)d\varrho\right]^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left[\int_0^1\left(\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}(\varrho)\right)|\Theta'(x)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}(\varrho)\right)|\Theta'(\chi_2)|^q\right)d\varrho\right]^{\frac{1}{q}}\right\}$$

$$= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ \frac{(x-\chi_1)^2}{\chi_2-\chi_1} \left[\left(\frac{1}{n} \sum_{\ell_1=1}^{n} H_{\ell_1} \right) |\Theta'(x)|^q + \left(\frac{1}{m} \sum_{\ell_2=1}^{m} G_{\ell_2} \right) |\Theta'(\chi_1)|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + \frac{(\chi_2-x)^2}{\chi_2-\chi_1} \left[\left(\frac{1}{n} \sum_{\ell_1=1}^{n} H_{\ell_1} \right) |\Theta'(x)|^q + \left(\frac{1}{m} \sum_{\ell_2=1}^{m} G_{\ell_2} \right) |\Theta'(\chi_2)|^q \right]^{\frac{1}{q}} \right\},$$

which ends our proof. □

Corollary 5. *We have particular cases from Theorem 14:*

- *If $h_{\ell_1}(\varrho) = \varrho$ and $g_{\ell_2}(\varrho) = 1 - \varrho$ for all $\ell_1 = 1, 2, \ldots, n$, and $\ell_2 = 1, 2, \ldots, m$, we obtain*

$$|\mathcal{T}_1(\Theta; x, \chi_1, \chi_2)| \le \left(\frac{1}{2}\right)^{\frac{1}{q}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times \left\{ \frac{(x-\chi_1)^2}{\chi_2-\chi_1} \left[|\Theta'(x)|^q + |\Theta'(\chi_1)|^q \right]^{\frac{1}{q}} + \frac{(\chi_2-x)^2}{\chi_2-\chi_1} \left[|\Theta'(x)|^q + |\Theta'(\chi_2)|^q \right]^{\frac{1}{q}} \right\}.$$

- *If $h_{\ell_1}(\varrho) = \varrho^{\ell_1}$ and $g_{\ell_2}(\varrho) = (1-\varrho)^{\ell_2}$ for all $\ell_1 = 1, 2, \ldots, n$, and $\ell_2 = 1, 2, \ldots, m$, we obtain*

$$|\mathcal{T}_1(\Theta; x, \chi_1, \chi_2)| \le \left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times \left\{ \frac{(x-\chi_1)^2}{\chi_2-\chi_1} \left[\left(\frac{1}{n} \sum_{\ell_1=1}^{n} \frac{1}{\ell_1+1} \right) |\Theta'(x)|^q + \left(\frac{1}{m} \sum_{\ell_2=1}^{m} \frac{1}{\ell_2+1} \right) |\Theta'(\chi_1)|^q \right]^{\frac{1}{q}} \right.$$
$$\left. + \frac{(\chi_2-x)^2}{\chi_2-\chi_1} \left[\left(\frac{1}{n} \sum_{\ell_1=1}^{n} \frac{1}{\ell_1+1} \right) |\Theta'(x)|^q + \left(\frac{1}{m} \sum_{\ell_2=1}^{m} \frac{1}{\ell_2+1} \right) |\Theta'(\chi_2)|^q \right]^{\frac{1}{q}} \right\}.$$

- *If $h_{\ell_1}(\varrho) = \varrho^s$ and $g_{\ell_2}(\varrho) = (1-\varrho)^s$ for all $\ell_1 = 1, \ldots, n$, $\ell_2 = 1, \ldots, m$, and $s \in (0, 1]$, we obtain*

$$|\mathcal{T}_1(\Theta; x, \chi_1, \chi_2)| \le \left(\frac{1}{s+1}\right)^{\frac{1}{q}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times \left\{ \frac{(x-\chi_1)^2}{\chi_2-\chi_1} \left[|\Theta'(x)|^q + |\Theta'(\chi_1)|^q \right]^{\frac{1}{q}} + \frac{(\chi_2-x)^2}{\chi_2-\chi_1} \left[|\Theta'(x)|^q + |\Theta'(\chi_2)|^q \right]^{\frac{1}{q}} \right\}.$$

Theorem 15. *Suppose that $1 \le n \le m$, where $n, m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0, 1] \to [0, +\infty)$ are continuous functions for all $\ell_1 = 1, 2, \ldots, n$ and $\ell_2 = 1, 2, \ldots, m$, and $\Theta : [\chi_1, \chi_2] \to$ R be a differentiable function on (χ_1, χ_2), such that $\Theta' \in \mathcal{L}[\chi_1, \chi_2]$. If $|\Theta'|^q$ is the (n, m)–generalized convex function with respect to $\mathcal{D}$ on $[\chi_1, \chi_2]$, then for $q > 1$, we have*

$$|\mathcal{T}_1(\Theta; x, \chi_1, \chi_2)| \le \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ \frac{(x-\chi_1)^2}{\chi_2-\chi_1} \left[\left(\frac{1}{n} \sum_{\ell_1=1}^{n} E_{\ell_1} \right) |\Theta'(x)|^q + \left(\frac{1}{m} \sum_{\ell_2=1}^{m} F_{\ell_2} \right) |\Theta'(\chi_1)|^q \right]^{\frac{1}{q}} \right.$$
$$\left. + \frac{(\chi_2-x)^2}{\chi_2-\chi_1} \left[\left(\frac{1}{n} \sum_{\ell_1=1}^{n} E_{\ell_1} \right) |\Theta'(x)|^q + \left(\frac{1}{m} \sum_{\ell_2=1}^{m} F_{\ell_2} \right) |\Theta'(\chi_2)|^q \right]^{\frac{1}{q}} \right\}, \quad (26)$$

where E_{ℓ_1} and F_{ℓ_2} are defined as in Theorem 13.

Proof. By using Lemma 2, the well-known power mean inequality and the (n,m)–generalized convexity of $|\Theta'|^q$ with respect to $\mathcal{D}$, we have

$$\begin{aligned}
|\mathcal{T}_1(\Theta;x,\chi_1,\chi_2)| &\le \frac{(x-\chi_1)^2}{\chi_2-\chi_1}\int_0^1 \varrho|\Theta'(\varrho x+(1-\varrho)\chi_1)|d\varrho \\
&+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\int_0^1 \varrho|\Theta'(\varrho x+(1-\varrho)\chi_2)|d\varrho \\
&\le \frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left(\int_0^1 \varrho d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho|\Theta'(\varrho x+(1-\varrho)\chi_1)|^q d\varrho\right)^{\frac{1}{q}} \\
&+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left(\int_0^1 \varrho d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho|\Theta'(\varrho x+(1-\varrho)\chi_2)|^q d\varrho\right)^{\frac{1}{q}} \\
\le \left(\frac{1}{2}\right)^{1-\frac{1}{q}}&\left\{\frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left[\int_0^1 \varrho\left(\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}(\varrho)\right)|\Theta'(x)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}(\varrho)\right)|\Theta'(\chi_1)|^q\right)d\varrho\right]^{\frac{1}{q}}\right. \\
&\left.+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left[\int_0^1 \varrho\left(\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}(\varrho)\right)|\Theta'(x)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}(\varrho)\right)|\Theta'(\chi_2)|^q\right)d\varrho\right]^{\frac{1}{q}}\right\} \\
&=\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}E_{\ell_1}\right)|\Theta'(x)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}F_{\ell_2}\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}E_{\ell_1}\right)|\Theta'(x)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}F_{\ell_2}\right)|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

which ends our proof. □

Corollary 6. *We have particular cases from Theorem 15:*

- *If $h_{\ell_1}(\varrho)=\varrho$ and $g_{\ell_2}(\varrho)=1-\varrho$ for all $\ell_1=1,2,\dots,n$, and $\ell_2=1,2,\dots,m$, we obtain*

$$|\mathcal{T}_1(\Theta;x,\chi_1,\chi_2)|\le\frac{1}{2\sqrt[q]{3}}$$
$$\times\left\{\frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left[2|\Theta'(x)|^q+|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left[2|\Theta'(x)|^q+|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right\}.$$

- *If $h_{\ell_1}(\varrho)=\varrho^{\ell_1}$ and $g_{\ell_2}(\varrho)=(1-\varrho)^{\ell_2}$ for all $\ell_1=1,2,\dots,n$, and $\ell_2=1,2,\dots,m$, we obtain*

$$\begin{aligned}
&|\mathcal{T}_1(\Theta;x,\chi_1,\chi_2)|\le\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\times\left\{\frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\frac{1}{\ell_1+2}\right)|\Theta'(x)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\frac{1}{(\ell_2+1)(\ell_2+2)}\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\frac{1}{\ell_1+2}\right)|\Theta'(x)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\frac{1}{(\ell_2+1)(\ell_2+2)}\right)|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

- *If $h_{\ell_1}(\varrho)=\varrho^s$ and $g_{\ell_2}(\varrho)=(1-\varrho)^s$ for all $\ell_1=1,\dots,n$, $\ell_2=1,\dots,m$, and $s\in(0,1]$, we obtain*

$$|\mathcal{T}_1(\Theta;x,\chi_1,\chi_2)| \leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}}$$
$$\times\left\{\frac{(x-\chi_1)^2}{\chi_2-\chi_1}\left[(s+1)|\Theta'(x)|^q+|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}+\frac{(\chi_2-x)^2}{\chi_2-\chi_1}\left[(s+1)|\Theta'(x)|^q+|\Theta'(\chi_2)|^q\right]^{\frac{1}{q}}\right\}.$$

Theorem 16. *Suppose that $1 \leq n \leq m$, where $n,m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ are continuous functions for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$, and $\Theta : [\chi_1,\chi_2] \to$ R be a differentiable function on (χ_1,χ_2), such that $\Theta' \in \mathcal{L}[\chi_1,\chi_2]$. If $|\Theta'|$ is the (n,m)–generalized convex function with respect to $\mathcal{D}$ on $[\chi_1,\chi_2]$, then we have*

$$\begin{aligned}|\mathcal{T}_2(\Theta;\chi_1,\chi_2)| &\leq (\chi_2-\chi_1)\\ &\times\left\{\frac{1}{n}\sum_{\ell_1=1}^{n}\left[\left(A_{\ell_1}^{(1)}+A_{\ell_1}^{(3)}\right)-\left(A_{\ell_1}^{(2)}+A_{\ell_1}^{(4)}\right)+\left(B_{\ell_1}^{(2)}+B_{\ell_1}^{(4)}\right)-\left(B_{\ell_1}^{(1)}+B_{\ell_1}^{(3)}\right)\right]|\Theta'(\chi_1)|\right.\\ &\left.+\frac{1}{m}\sum_{\ell_2=1}^{m}\left[\left(C_{\ell_2}^{(1)}+C_{\ell_2}^{(3)}\right)-\left(C_{\ell_2}^{(2)}+C_{\ell_2}^{(4)}\right)+\left(D_{\ell_2}^{(2)}+D_{\ell_2}^{(4)}\right)-\left(D_{\ell_2}^{(1)}+D_{\ell_2}^{(3)}\right)\right]|\Theta'(\chi_2)|\right\},\end{aligned} \tag{27}$$

where

$$A_{\ell_1}^{(1)} := \int_0^{\frac{1}{6}} h_{\ell_1}(\varrho)d\varrho, \quad A_{\ell_1}^{(2)} := \int_{\frac{1}{6}}^{\frac{1}{2}} h_{\ell_1}(\varrho)d\varrho,$$
$$A_{\ell_1}^{(3)} := \int_{\frac{1}{2}}^{\frac{5}{6}} h_{\ell_1}(\varrho)d\varrho, \quad A_{\ell_1}^{(4)} := \int_{\frac{5}{6}}^{1} h_{\ell_1}(\varrho)d\varrho,$$
$$B_{\ell_1}^{(1)} := \int_0^{\frac{1}{6}} \varrho h_{\ell_1}(\varrho)d\varrho, \quad B_{\ell_1}^{(2)} := \int_{\frac{1}{6}}^{\frac{1}{2}} \varrho h_{\ell_1}(\varrho)d\varrho,$$
$$B_{\ell_1}^{(3)} := \int_{\frac{1}{2}}^{\frac{5}{6}} \varrho h_{\ell_1}(\varrho)d\varrho, \quad B_{\ell_1}^{(4)} := \int_{\frac{5}{6}}^{1} \varrho h_{\ell_1}(\varrho)d\varrho,$$

and

$$C_{\ell_2}^{(1)} := \int_0^{\frac{1}{6}} g_{\ell_2}(\varrho)d\varrho, \quad C_{\ell_2}^{(2)} := \int_{\frac{1}{6}}^{\frac{1}{2}} g_{\ell_2}(\varrho)d\varrho,$$
$$C_{\ell_2}^{(3)} := \int_{\frac{1}{2}}^{\frac{5}{6}} g_{\ell_2}(\varrho)d\varrho, \quad C_{\ell_2}^{(4)} := \int_{\frac{5}{6}}^{1} g_{\ell_2}(\varrho)d\varrho,$$
$$D_{\ell_2}^{(1)} := \int_0^{\frac{1}{6}} \varrho g_{\ell_2}(\varrho)d\varrho, \quad D_{\ell_2}^{(2)} := \int_{\frac{1}{6}}^{\frac{1}{2}} \varrho g_{\ell_2}(\varrho)d\varrho,$$
$$D_{\ell_2}^{(3)} := \int_{\frac{1}{2}}^{\frac{5}{6}} \varrho g_{\ell_2}(\varrho)d\varrho, \quad D_{\ell_2}^{(4)} := \int_{\frac{5}{6}}^{1} \varrho g_{\ell_2}(\varrho)d\varrho,$$

for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$.

Proof. By using Lemma 3 and the (n,m)–generalized convexity of $|\Theta'|$ with respect to $\mathcal{D}$, we have

$$\begin{aligned}|\mathcal{T}_2(\Theta;\chi_1,\chi_2)| \leq (\chi_2-\chi_1)\Bigg\{&\int_0^{\frac{1}{2}}\left|\varrho-\frac{1}{6}\right||\Theta'(\varrho\chi_2+(1-\varrho)\chi_1)|d\varrho\\ &+\int_{\frac{1}{2}}^{1}\left|\varrho-\frac{5}{6}\right||\Theta'(\varrho\chi_2+(1-\varrho)\chi_1)|d\varrho\Bigg\}\end{aligned}$$

$$\leq (\chi_2-\chi_1)\Bigg\{\int_0^{\frac{1}{2}}\left|\varrho-\frac{1}{6}\right|\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}(\varrho)\right)|\Theta'(\chi_2)|+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}(\varrho)\right)|\Theta'(\chi_1)|\right]d\varrho$$

$$+\int_{\frac{1}{2}}^{1}\left|\varrho-\frac{5}{6}\right|\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}(\varrho)\right)|\Theta'(\chi_2)|+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}(\varrho)\right)|\Theta'(\chi_1)|\right]d\varrho\Bigg\}$$

$$=(\chi_2-\chi_1)\Bigg\{\frac{1}{n}\sum_{\ell_1=1}^{n}\left[\left(A_{\ell_1}^{(1)}+A_{\ell_1}^{(3)}\right)-\left(A_{\ell_1}^{(2)}+A_{\ell_1}^{(4)}\right)+\left(B_{\ell_1}^{(2)}+B_{\ell_1}^{(4)}\right)-\left(B_{\ell_1}^{(1)}+B_{\ell_1}^{(3)}\right)\right]|\Theta'(\chi_1)|$$

$$+\frac{1}{m}\sum_{\ell_2=1}^{m}\left[\left(C_{\ell_2}^{(1)}+C_{\ell_2}^{(3)}\right)-\left(C_{\ell_2}^{(2)}+C_{\ell_2}^{(4)}\right)+\left(D_{\ell_2}^{(2)}+D_{\ell_2}^{(4)}\right)-\left(D_{\ell_2}^{(1)}+D_{\ell_2}^{(3)}\right)\right]|\Theta'(\chi_2)|\Bigg\},$$

which ends our proof. □

Corollary 7. *If we take $h_{\ell_1}(\varrho)=\varrho^{\ell_1}$, $\forall\,\ell_1=1,\dots,n$ and $g_{\ell_2}(\varrho)=(1-\varrho)^{\ell_2}$, $\forall\,\ell_2=1,\dots,m$ in Theorem 16, we obtain*

$$|\mathcal{T}_2(\Theta;\chi_1,\chi_2)|\leq(\chi_2-\chi_1)$$
$$\times\Bigg\{\frac{1}{n}\sum_{\ell_1=1}^{n}\left[\left(U_{\ell_1}^{(1)}+U_{\ell_1}^{(3)}\right)-\left(U_{\ell_1}^{(2)}+U_{\ell_1}^{(4)}\right)+\left(V_{\ell_1}^{(2)}+V_{\ell_1}^{(4)}\right)-\left(V_{\ell_1}^{(1)}+V_{\ell_1}^{(3)}\right)\right]|\Theta'(\chi_1)|$$
$$+\frac{1}{m}\sum_{\ell_2=1}^{m}\left[\left(P_{\ell_2}^{(1)}+P_{\ell_2}^{(3)}\right)-\left(P_{\ell_2}^{(2)}+P_{\ell_2}^{(4)}\right)+\left(Q_{\ell_2}^{(2)}+Q_{\ell_2}^{(4)}\right)-\left(Q_{\ell_2}^{(1)}+Q_{\ell_2}^{(3)}\right)\right]|\Theta'(\chi_2)|\Bigg\},$$

where

$$U_{\ell_1}^{(1)}:=\frac{1}{6^{\ell_1+1}(\ell_1+1)},\quad U_{\ell_1}^{(2)}:=\frac{1}{\ell_1+1}\left(\frac{1}{2^{\ell_1+1}}-\frac{1}{6^{\ell_1+1}}\right),$$

$$U_{\ell_1}^{(3)}:=\frac{1}{\ell_1+1}\left(\left(\frac{5}{6}\right)^{\ell_1+1}-\frac{1}{2^{\ell_1+1}}\right),\quad U_{\ell_1}^{(4)}:=\frac{1}{\ell_1+1}\left(1-\left(\frac{5}{6}\right)^{\ell_1+1}\right),$$

$$V_{\ell_1}^{(1)}:=\frac{1}{6^{\ell_1+2}(\ell_1+2)},\quad V_{\ell_1}^{(2)}:=\frac{1}{\ell_1+2}\left(\frac{1}{2^{\ell_1+2}}-\frac{1}{6^{\ell_1+2}}\right),$$

$$V_{\ell_1}^{(3)}:=\frac{1}{\ell_1+2}\left(\left(\frac{5}{6}\right)^{\ell_1+2}-\frac{1}{2^{\ell_1+2}}\right),\quad V_{\ell_1}^{(4)}:=\frac{1}{\ell_1+2}\left(1-\left(\frac{5}{6}\right)^{\ell_1+2}\right),$$

$$P_{\ell_2}^{(1)}:=\frac{1}{\ell_2+1}\left(1-\left(\frac{5}{6}\right)^{\ell_2+1}\right),\quad P_{\ell_2}^{(2)}:=\frac{1}{\ell_2+1}\left(\left(\frac{5}{6}\right)^{\ell_2+1}-\frac{1}{2^{\ell_2+1}}\right),$$

$$P_{\ell_2}^{(3)}:=\frac{1}{\ell_2+1}\left(\frac{1}{2^{\ell_2+1}}-\frac{1}{6^{\ell_2+1}}\right),\quad P_{\ell_2}^{(4)}:=\frac{1}{6^{\ell_2+1}(\ell_2+1)},$$

and

$$Q_{\ell_2}^{(1)}:=\frac{1}{(\ell_2+1)(\ell_2+2)}-\left(\frac{1}{\ell_2+1}\left(\frac{5}{6}\right)^{\ell_2+1}-\frac{1}{\ell_2+2}\left(\frac{5}{6}\right)^{\ell_2+2}\right),$$

$$Q_{\ell_2}^{(2)}:=\left(\frac{1}{\ell_2+1}\left(\frac{5}{6}\right)^{\ell_2+1}-\frac{1}{\ell_2+2}\left(\frac{5}{6}\right)^{\ell_2+2}\right)-\left(\frac{1}{\ell_2+1}\left(\frac{1}{2}\right)^{\ell_2+1}-\frac{1}{\ell_2+2}\left(\frac{1}{2}\right)^{\ell_2+2}\right),$$

$$Q_{\ell_2}^{(3)}:=\left(\frac{1}{\ell_2+1}\left(\frac{1}{2}\right)^{\ell_2+1}-\frac{1}{\ell_2+2}\left(\frac{1}{2}\right)^{\ell_2+2}\right)-\left(\frac{1}{\ell_2+1}\left(\frac{1}{6}\right)^{\ell_2+1}-\frac{1}{\ell_2+2}\left(\frac{1}{6}\right)^{\ell_2+2}\right),$$

$$Q_{\ell_2}^{(4)}:=\frac{1}{\ell_2+1}\left(\frac{1}{6}\right)^{\ell_2+1}-\frac{1}{\ell_2+2}\left(\frac{1}{6}\right)^{\ell_2+2}.$$

Theorem 17. *Suppose that $1 \leq n \leq m$, where $n, m \in \mathbb{N}$, and assume that $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ are continuous functions for all $\ell_1 = 1,2,\ldots,n$ and $\ell_2 = 1,2,\ldots,m$, and $\Theta : [\chi_1,\chi_2] \to$ R be a differentiable function on (χ_1,χ_2) such that $\Theta' \in \mathcal{L}[\chi_1,\chi_2]$. If $|\Theta'|^q$ is the (n,m)–generalized convex function with respect to $\mathcal{D}$ on $[\chi_1,\chi_2]$, then for $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, we have*

$$|\mathcal{T}_2(\Theta;\chi_1,\chi_2)| \leq (\chi_2-\chi_1)\left[\frac{2}{p+1}\left(\frac{2^{p+1}+1}{6^{p+1}}\right)\right]^{\frac{1}{p}} \times \left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n} H_{\ell_1}\right)|\Theta'(\chi_2)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} G_{\ell_2}\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}, \quad (28)$$

where H_{ℓ_1} and G_{ℓ_2} are as defined in Theorem 8.

Proof. By using Lemma 3, Hölder's inequality and the (n,m)–generalized convexity of $|\Theta'|^q$ with respect to $\mathcal{D}$, we have

$$\begin{aligned}|\mathcal{T}_2(\Theta;\chi_1,\chi_2)| &\leq (\chi_2-\chi_1)\left\{\int_0^{\frac{1}{2}}\left|\varrho-\frac{1}{6}\right||\Theta'(\varrho\chi_2+(1-\varrho)\chi_1)|d\varrho + \int_{\frac{1}{2}}^{1}\left|\varrho-\frac{5}{6}\right||\Theta'(\varrho\chi_2+(1-\varrho)\chi_1)|d\varrho\right\}\\
&\leq (\chi_2-\chi_1)\left[\left(\int_0^{\frac{1}{2}}\left|\varrho-\frac{1}{6}\right|^p d\varrho\right)^{\frac{1}{p}} + \left(\int_{\frac{1}{2}}^{1}\left|\varrho-\frac{5}{6}\right|^p d\varrho\right)^{\frac{1}{p}}\right]\left(\int_0^1|\Theta'(\varrho\chi_2+(1-\varrho)\chi_1)|^q d\varrho\right)^{\frac{1}{q}}\\
&\leq (\chi_2-\chi_1)\left[\left(\int_0^{\frac{1}{2}}\left|\varrho-\frac{1}{6}\right|^p d\varrho\right)^{\frac{1}{p}} + \left(\int_{\frac{1}{2}}^{1}\left|\varrho-\frac{5}{6}\right|^p d\varrho\right)^{\frac{1}{p}}\right]\\
&\quad\times\left[\int_0^1\left(\left(\frac{1}{n}\sum_{\ell_1=1}^{n} h_{\ell_1}(\varrho)\right)|\Theta'(\chi_2)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} g_{\ell_2}(\varrho)\right)|\Theta'(\chi_1)|^q\right)d\varrho\right]^{\frac{1}{q}}\\
&= (\chi_2-\chi_1)\left[\frac{2}{p+1}\left(\frac{2^{p+1}+1}{6^{p+1}}\right)\right]^{\frac{1}{p}}\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n} H_{\ell_1}\right)|\Theta'(\chi_2)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m} G_{\ell_2}\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}},\end{aligned}$$

which ends our proof. □

Corollary 8. *We have particular cases from Theorem 17:*

- *If $h_{\ell_1}(\varrho) = \varrho$ and $g_{\ell_2}(\varrho) = 1-\varrho$ for all $\ell_1 = 1,2,\ldots,n$, and $\ell_2 = 1,2,\ldots,m$, we obtain*

$$|\mathcal{T}_2(\Theta;\chi_1,\chi_2)| \leq (\chi_2-\chi_1)\left[\frac{2}{p+1}\left(\frac{2^{p+1}+1}{6^{p+1}}\right)\right]^{\frac{1}{p}}\left[\frac{|\Theta'(\chi_1)|^q+|\Theta'(\chi_2)|^q}{2}\right]^{\frac{1}{q}}.$$

- *If $h_{\ell_1}(\varrho) = \varrho^{\ell_1}$ and $g_{\ell_2}(\varrho) = (1-\varrho)^{\ell_2}$ for all $\ell_1 = 1,2,\ldots,n$, and $\ell_2 = 1,2,\ldots,m$, we obtain*

$$|\mathcal{T}_2(\Theta;\chi_1,\chi_2)| \leq (\chi_2-\chi_1)\left[\frac{2}{p+1}\left(\frac{2^{p+1}+1}{6^{p+1}}\right)\right]^{\frac{1}{p}} \times \left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\frac{1}{\ell_1+1}\right)|\Theta'(\chi_2)|^q + \left(\frac{1}{m}\sum_{\ell_2=1}^{m}\frac{1}{\ell_2+1}\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}.$$

- *If* $h_{\ell_1}(\varrho) = \varrho^s$ *and* $g_{\ell_2}(\varrho) = (1-\varrho)^s$ *for all* $\ell_1 = 1, \ldots, n$, $\ell_2 = 1, \ldots, m$, *and* $s \in (0,1]$, *we obtain*

$$|\mathcal{T}_2(\Theta;\chi_1,\chi_2)| \le (\chi_2-\chi_1)\left[\frac{2}{p+1}\left(\frac{2^{p+1}+1}{6^{p+1}}\right)\right]^{\frac{1}{p}}\left[\frac{|\Theta'(\chi_1)|^q+|\Theta'(\chi_2)|^q}{s+1}\right]^{\frac{1}{q}}.$$

Theorem 18. *Suppose that* $1 \le n \le m$, *where* $n, m \in \mathbb{N}$, *and assume that* $h_{\ell_1}, g_{\ell_2} : [0,1] \to [0,+\infty)$ *are continuous functions for all* $\ell_1 = 1,2,\ldots,n$ *and* $\ell_2 = 1,2,\ldots,m$, *and* $\Theta : [\chi_1,\chi_2] \to \mathrm{R}$ *be a differentiable function on* (χ_1,χ_2), *such that* $\Theta' \in \mathcal{L}[\chi_1,\chi_2]$. *If* $|\Theta'|^q$ *is the* (n,m)*–generalized convex function with respect to* $\mathcal{D}$ *on* $[\chi_1,\chi_2]$, *then for* $q > 1$, *we have*

$$|\mathcal{T}_2(\Theta;\chi_1,\chi_2)| \le (\chi_2-\chi_1)\left(\frac{5}{36}\right)^{1-\frac{1}{q}} \tag{29}$$

$$\times\left\{\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(\frac{A_{\ell_1}^{(1)}}{6}-B_{\ell_1}^{(1)}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(\frac{C_{\ell_2}^{(1)}}{6}-D_{\ell_2}^{(1)}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right.$$

$$+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(B_{\ell_1}^{(2)}-\frac{A_{\ell_1}^{(2)}}{6}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(D_{\ell_2}^{(2)}-\frac{C_{\ell_2}^{(2)}}{6}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(\frac{5A_{\ell_1}^{(3)}}{6}-B_{\ell_1}^{(3)}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(\frac{5C_{\ell_2}^{(3)}}{6}-D_{\ell_2}^{(3)}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}$$

$$\left.+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(B_{\ell_1}^{(4)}-\frac{5A_{\ell_1}^{(4)}}{6}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(D_{\ell_2}^{(4)}-\frac{5C_{\ell_2}^{(4)}}{6}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\right\},$$

where $A_{\ell_1}^{(k)}$, $B_{\ell_1}^{(k)}$, $C_{\ell_2}^{(k)}$, *and* $D_{\ell_2}^{(k)}$ *for all* $k = 1,2,3,4$ *are defined as in Theorem 16.*

Proof. By using Lemma 3, the well-known power mean inequality and (n,m)–generalized convexity $|\Theta'|^q$ with respect to $\mathcal{D}$, we have

$$|\mathcal{T}_2(\Theta;\chi_1,\chi_2)| \le (\chi_2-\chi_1)\left\{\int_0^{\frac{1}{2}}\left|\varrho-\frac{1}{6}\right||\Theta'(\varrho\chi_2+(1-\varrho)\chi_1)|d\varrho\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}\left|\varrho-\frac{5}{6}\right||\Theta'(\varrho\chi_2+(1-\varrho)\chi_1)|d\varrho\right\}$$

$$\le (\chi_2-\chi_1)\left[\left(\int_0^{\frac{1}{2}}\left|\varrho-\frac{1}{6}\right|d\varrho\right)^{1-\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\left|\varrho-\frac{5}{6}\right|d\varrho\right)^{1-\frac{1}{q}}\right]$$

$$\times\left\{\left(\int_0^{\frac{1}{2}}\left|\varrho-\frac{1}{6}\right||\Theta'(\varrho\chi_2+(1-\varrho)\chi_1)|^q d\varrho\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\left|\varrho-\frac{5}{6}\right||\Theta'(\varrho\chi_2+(1-\varrho)\chi_1)|^q d\varrho\right)^{\frac{1}{q}}\right\}$$

$$\le (\chi_2-\chi_1)\left[\left(\int_0^{\frac{1}{2}}\left|\varrho-\frac{1}{6}\right|d\varrho\right)^{1-\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\left|\varrho-\frac{5}{6}\right|d\varrho\right)^{1-\frac{1}{q}}\right]$$

$$\times\left\{\left[\int_0^{\frac{1}{2}}\left|\varrho-\frac{1}{6}\right|\left(\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}(\varrho)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}(\varrho)\right)|\Theta'(\chi_1)|^q\right)d\varrho\right]^{\frac{1}{q}}\right.$$

$$+\left[\int_{\frac{1}{2}}^{1}\left|\varrho-\frac{5}{6}\right|\left(\left(\frac{1}{n}\sum_{\ell_1=1}^{n}h_{\ell_1}(\varrho)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}g_{\ell_2}(\varrho)\right)|\Theta'(\chi_1)|^q\right)d\varrho\right]^{\frac{1}{q}}\Bigg\}$$

$$=(\chi_2-\chi_1)\left(\frac{5}{36}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg\{\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(\frac{A_{\ell_1}^{(1)}}{6}-B_{\ell_1}^{(1)}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(\frac{C_{\ell_2}^{(1)}}{6}-D_{\ell_2}^{(1)}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(B_{\ell_1}^{(2)}-\frac{A_{\ell_1}^{(2)}}{6}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(D_{\ell_2}^{(2)}-\frac{C_{\ell_2}^{(2)}}{6}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(\frac{5A_{\ell_1}^{(3)}}{6}-B_{\ell_1}^{(3)}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(\frac{5C_{\ell_2}^{(3)}}{6}-D_{\ell_2}^{(3)}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(B_{\ell_1}^{(4)}-\frac{5A_{\ell_1}^{(4)}}{6}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(D_{\ell_2}^{(4)}-\frac{5C_{\ell_2}^{(4)}}{6}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\Bigg\},$$

which ends our proof. □

Corollary 9. *If we take $h_{\ell_1}(\varrho)=\varrho^{\ell_1}$ and $g_{\ell_2}(\varrho)=(1-\varrho)^{\ell_2}$ in Theorem 18 for all $\ell_1=1,2,\dots,n$ and $\ell_2=1,2,\dots,m$, we obtain*

$$|\mathcal{T}_2(\Theta;\chi_1,\chi_2)|\leq(\chi_2-\chi_1)\left(\frac{5}{36}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg\{\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(\frac{U_{\ell_1}^{(1)}}{6}-V_{\ell_1}^{(1)}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(\frac{P_{\ell_2}^{(1)}}{6}-Q_{\ell_2}^{(1)}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(V_{\ell_1}^{(2)}-\frac{U_{\ell_1}^{(2)}}{6}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(Q_{\ell_2}^{(2)}-\frac{P_{\ell_2}^{(2)}}{6}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(\frac{5U_{\ell_1}^{(3)}}{6}-V_{\ell_1}^{(3)}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(\frac{5P_{\ell_2}^{(3)}}{6}-Q_{\ell_2}^{(3)}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{1}{n}\sum_{\ell_1=1}^{n}\left(V_{\ell_1}^{(4)}-\frac{5U_{\ell_1}^{(4)}}{6}\right)\right)|\Theta'(\chi_2)|^q+\left(\frac{1}{m}\sum_{\ell_2=1}^{m}\left(Q_{\ell_2}^{(4)}-\frac{5P_{\ell_2}^{(4)}}{6}\right)\right)|\Theta'(\chi_1)|^q\right]^{\frac{1}{q}}\Bigg\},$$

where $U_{\ell_1}^{(k)}$, $V_{\ell_1}^{(k)}$, $P_{\ell_2}^{(k)}$ and $Q_{\ell_2}^{(k)}$ for all $k=1,2,3,4$ are as defined in Corollary 7.

5. Conclusions

In this article, we studied algebraic properties of a new generic class of functions called the (n,m)–generalized convex function; based on this, we proposed HH inequalities. Moreover, we obtained new midpoint-type inequalities of Ostrowski and Simpson based on our new definition, using well-known integral identities. Finally, we observed that the new, defined convex function is a powerful type of function used to investigate various inequalities in the real analysis field.

Author Contributions: Conceptualization, A.K. and P.O.M.; methodology, A.K., R.P.A. and P.O.M.; software, K.N.; K.M.A. and Y.S.H.; validation, A.K., P.O.M. and Y.S.H.; formal analysis, R.P.A., P.O.M., K.N. and K.M.A.; investigation, A.K., P.O.M. and Y.S.H.; resources, P.O.M., K.N. and K.M.A.; data curation, K.N. and Y.S.H.; writing—original draft preparation, A.K., P.O.M. and R.P.A.; writing—review and editing, A.K., P.O.M., K.M.A. and Y.S.H.; visualisation, A.K., R.P.A. and P.O.M.; supervision, R.P.A., P.O.M., K.N. and Y.S.H.; project administration, P.O.M. and K.N.; funding acquisition, K.N. All authors have read and agreed to the published version of the manuscript.

Funding: Not applicable.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This Research was supported by Taif University Researchers Supporting Project Number (TURSP2020/217), Taif University, Taif, Saudi Arabia, and the National Science, Research, and Innovation Fund (NSRF), Thailand.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kashuri, A.; Meftah, B.; Mohammed, P.O.; Lupaş, A.A.; Abdalla, B.; Hamed, Y.S.; Abdeljawad, T. Fractional weighted Ostrowski-type inequalities and their applications. *Symmetry* **2021**, *13*, 968. [CrossRef]
2. Kasamsetty, K.; Ketkar, M.; Sapatnekar, S.S. A new class of convex functions for delay modeling and its application to the transistor sizing problem [CMOS gates]. *IEEE Trans.-Comput.-Aided Des. Integr. Circuits Syst.* **2020**, *19*, 779–788. [CrossRef]
3. Awan, M.U.; Noor, M.A.; Noor, K.I.; Khan, A.G. Some new classes of convex functions and inequalities. *Miskolc Math. Notes* **2018**, *19*, 77–94. [CrossRef]
4. Hudzik, H.; Maligranda, L. Some remarks on s–convex functions. *Aequationes Math.* **1994**, *48*, 100–111. [CrossRef]
5. Eftekhari, N. Some remarks on (s, m)–convexity in the second sense. *J. Math. Inequal.* **2014**, *8*, 489–495. [CrossRef]
6. Kadakal, M.; İşcan, İ. Exponential type convexity and some related inequalities. *J. Inequal. Appl.* **2020**, *1*, 82. [CrossRef]
7. Agarwal, P.; Choi, J. Fractional calculus operators and their image formulas. *J. Korean Math. Soc.* **2016**, *53*, 1183–1210. [CrossRef]
8. Rekhviashvili, S.; Pskhu, A.; Agarwal, P.; Jain, S. Application of the fractional oscillator model to describe damped vibrations. *Turk. J. Phys.* **2019**, *43*, 236–242. [CrossRef]
9. Dragomir, S.S.; Fitzpatrik, S. The Hadamard's inequality for s–convex functions in the second sense. *Demonstr. Math.* **1999**, *32*, 687–696.
10. Han, J.; Mohammed, P.O.; Zeng, H. Generalized fractional integral inequalities of Hermite–Hadamard-type for a convex function. *Open Math.* **2020**, *18*, 794–806. [CrossRef]
11. Rashid, S.; Noor, M.A.; Noor, K.I.; Akdemir, A.O. Some new generalizations for exponentially s–convex functions and inequalities via fractional operators. *Int. J. Sci. Innov. Tech.* **2014**, *1*, 1–12. [CrossRef]
12. İşcan, İ. Hermite–Hadamard–Fejér type inequalities for convex functions via fractional integrals. *Stud. Univ. Babeş-Bolyai Math.* **2015**, *60*, 355–366.
13. Abdeljawad, T.; Mohammed, P.O.; Kashuri, A. New modified conformable fractional integral inequalities of Hermite–Hadamard type with applications. *J. Funct. Spaces* **2020**, *2020*, 4352357. [CrossRef]
14. Fernandez, A.; Mohammed, P.O. Hermite–Hadamard inequalities in fractional calculus defined using Mittag–Leffler kernels. *Math. Meth. Appl. Sci.* **2020**, *44*, 8414–8431. [CrossRef]
15. Ujević, N. Sharp inequalities of Simpson type and Ostrowski type. *Comput. Math. Appl.* **2004**, *48*, 145–151. [CrossRef]
16. Liu, W.; Wen, W.; Park, J. Ostrowski type fractional integral inequalities for MT–convex functions. *Miskolc Math. Notes* **2015**, *16*, 249–256. [CrossRef]
17. Kaijser, S.; Nikolova, L.; Persson, L.E.; Wedestig, A. Hardy type inequalities via convexity. *Math. Inequal. Appl.* **2005**, *8*, 403–417. [CrossRef]
18. Rashid, S.; Jarad, F.; Noor, M.A.; Noor, K.I.; Baleanu, D.; Liu, J.B. On Grüss inequalities within generalized k–fractional integrals. *Adv. Differ. Equ.* **2020**, *2020*, 203. [CrossRef]
19. Mohammed, P.O.; Sarikaya, M.Z.; Baleanu, D. On the generalized Hermite–Hadamard Inequalities via the tempered fractional integrals. *Symmetry* **2020**, *12*, 595. [CrossRef]
20. Kalsoom, H.; Rashid, S.; Idrees, M.; Safdar, F.; Akram, S.; Baleanu, D.; Chu, Y.M. Post quantum integral inequalities of Hermite–Hadamard-type associated with co-ordinated higher-order generalized strongly pre-invex and quasi-pre-invex mappings. *Symmetry* **2020**, *12*, 443. [CrossRef]
21. Bombardelli, M.; Varošanec, S. Properties of h–convex functions related to the Hermite–Hadamard–Fejér inequalities. *Comput. Math. Appl.* **2009**, *58*, 1869–1877. [CrossRef]

22. Toply, T.; Kadakal, M.; İşcan, İ. On n–polynomial convexity and some related inequalities. *AIMS Math.* **2020**, *5*, 1304–1318. [CrossRef]
23. Rashid, S.; İşcan, İ.; Baleanu, D.; Chu, Y.M. Generation of new fractional inequalities via n–polynomials s–type convexity with applications. *Adv. Differ. Equ.* **2020**, *2020*, 264. [CrossRef]
24. Hadamard, J. Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.* **1893**, *58*, 171–215.
25. Dragomir, S.S.; Agarwal, R.P.; Cerone, P. On Simpson's inequality and applications. *J. Inequal. Appl.* **2000**, *5*, 533–579. [CrossRef]
26. Sarikaya, M.Z.; Yildirim, H. On Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals. *Miskolc Math. Notes* **2017**, *17*, 1049–1059. [CrossRef]
27. Alomari, M.; Darus, M.; Dragomir, S.S.; Cerone, P. Ostrowski type inequalities for functions whose derivatives are s–convex in the second sense. *Appl. Math. Lett.* **2010**, *23*, 1071–1076. [CrossRef]
28. Qaisar, S.; He, C.; Hussain, S. A generalizations of Simpson's type inequality for differentiable functions using (α, m)–convex functions and applications. *J. Inequal. Appl.* **2013**, *2013*, 158. [CrossRef]