symmetry

MDPI

Article

Simpson's and Newton's Type Inequalities for (α, m)-Convex Functions via Quantum Calculus

Jarunee Soontharanon [1], Muhammad Aamir Ali [2], Hüseyin Budak [3], Kamsing Nonlaopon [4,*] and Zoya Abdullah [5]

1 Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; jarunee.s@sci.kmutnb.ac.th
2 Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China; mahr.muhammad.aamir@gmail.com
3 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey; hsyn.budak@gmail.com
4 Department of Mathematics, Faculty of Science and Arts, Khon Kaen University, Khon Kaen 40002, Thailand
5 Department of Mathematics, COMSATS University Islamabad Sahiwal Campus, Sahiwal 57000, Pakistan; zoya@cuisahiwal.edu.pk
* Correspondence: nkamsi@kku.ac.th

Abstract: In this paper, we give the generalized version of the quantum Simpson's and quantum Newton's formula type inequalities via quantum differentiable (α, m)-convex functions. The main advantage of these new inequalities is that they can be converted into quantum Simpson and quantum Newton for convex functions, Simpson's type inequalities (α, m)-convex function, and Simpson's type inequalities without proving each separately. These inequalities can be helpful in finding the error bounds of Simpson's and Newton's formulas in numerical integration. Analytic inequalities of this type as well as particularly related strategies have applications for various fields where symmetry plays an important role.

Keywords: Simpson's inequalities; Newton's inequalities; quantum calculus; (α, m)-convex functions

check for updates

Citation: Soontharanon, J.; Ali, M.A.; Budak, H.; Nonlaopon, K.; Abdullah, Z. Simpson's and Newton's Type Inequalities for (α, m)-Convex Functions via Quantum Calculus. *Symmetry* **2022**, *14*, 736. https://doi.org/10.3390/sym14040736

Academic Editor: Alexander Zaslavski

Received: 17 March 2022
Accepted: 31 March 2022
Published: 3 April 2022

1. Introduction

Hudzik and Maligranda [1] introduced the s-convexity idea of generalized convexity, which is defined as: A mapping $\Upsilon : [0, \infty) \to \mathbb{R}$ is called $\Upsilon \in K_s^i$ or s-convex if the following inequality:

$$\Upsilon(\delta x + uy) \leq \delta^s \Upsilon(x) + u^s \Upsilon(y)$$

holds for all $x, y \in [0, \infty)$, $s \in (0, 1)$, and $\delta, u \in [0, 1]$.

Note that the above class of functions is called s-convex in the first sense if $u^s + \delta^s = 1$ and is represented by K_s^1, while this class of functions is called s-convex in the second sense if $u + \delta = 1$ and is represented by K_s^2.

Convexity in the context of integral inequalities is a fascinating research topic, given how much attention has been dedicated to the concept of convexity and its various manifestations in recent years. Three of the most important inequalities associated with the integral mean of a convex function are Hermite's inequality, Hadamard's and Jensen's inequalities, and Hilbert's and Hardy's inequalities; see [2–6]. The Hermite–Hadamard inequality is a necessary and sufficient condition for a function to be convex. The well-known Hermite–Hadamard result is as follows:

$$\Upsilon\left(\frac{\varphi_1 + \varphi_2}{2}\right) \leq \frac{1}{\varphi_2 - \varphi_1} \int_{\varphi_1}^{\varphi_2} \Upsilon(x)dx \leq \frac{\Upsilon(\varphi_1) + \Upsilon(\varphi_2)}{2}.$$

This double inequality is a development of the concept of convexity, and it readily follows from Jensen's inequality.

In [7], Dragomir and Fitzpatrick proved the following Hermite–Hadamard inequality for s-convex mapping in the second sense:

$$2^{s-1}\Upsilon\left(\frac{\varphi_1+\varphi_2}{2}\right) \leq \frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\Upsilon(x)dx \leq \frac{\Upsilon(\varphi_1)+\Upsilon(\varphi_2)}{1+s}.$$

Definition 1 ([8])**.** *A function* $\Upsilon : [0,\varphi_2) \to \mathbb{R}$ *is called* (α, m)*-convex if the inequality*

$$\Upsilon(\delta x + m(1-\delta)y) \leq \delta^{\alpha}\Upsilon(x) + m(1-\delta^{\alpha})\Upsilon(y)$$

holds for all $x, y \in [0,\varphi_2)$, $\delta \in [0,1]$, $(\alpha, m) \in [0,1]^2$, *and* $m \in (0,1]$.

On the other hand, several studies in the q-analysis field are underway, starting with Euler, in order to achieve excellence in quantum computing. q-calculus is the link between physics and mathematics and contains a wide range of applications in many fields such as mathematics—including numerical theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other fields—as well as mechanics, theory of relativity, and quantum theory [9,10]. q-calculus also has many applications in quantum information theory, which is a mixture of computer science, information theory, philosophy, and cryptography, among other things [11,12]. Euler is the founder of this important branch of mathematics. Newton used the q parameter in his work on the endless series. The q -calculus, which is known to have no calculation limits, was introduced by Jackson [13] in a systematic way. In 1966, a q-analogue of the q-fractional integrals and q-Riemann–Liouville fractional were introduced by Al-Salam [14]. Since then, related research has been growing steadily. In particular, the left quantum difference operator and the left quantum integral were introduced by Tariboon and Ntouyas in 2013 [15]. In 2020, Bermudo et al. presented the concept of right quantum derivative and right quantum integral in [16].

Quantum and post-quantum integrals for various sorts of functions have also been used to investigate many integral inequalities. Using quantum derivatives and integrals, the authors demonstrated the Hermite–Hadamard integral inequality and their left–right estimates for convex and co-ordinated convex functions in [16–24]. Noor et al. presented a generalized version of q-integral inequalities in [25]. Ali et al. [26] established some trapezoid type inequalities for twice q-differentiable convex functions. Using the Green function, Khan et al. [27] discovered the quantum Hermite–Hadamard inequality. Budak et al. [28], Ali et al. [29,30], and Vivas-Cortez et al. [31] discovered new quantum Simpson's and quantum Newton's type inequalities for convex and co-ordinated convex functions. Sial et al. [32] used the right q-integral and derivative to show several of Simpson's and Newton's type inequalities for (α, m)-convex functions.

We give several new Simpson's and Newton's formula type inequalities for (α, m)-convex functions utilizing the left q-integral and derivative, inspired by existing research. These inequalities have the advantage that they could converted into quantum Simpson's and quantum Newton's inequalities for convex functions [28], classical Simpson's type inequalities for (α, m)-convex functions [33], and classical Simpson's type inequalities for convex functions [34] without having to prove each one separately. These inequalities can be very helpful in numerical integration formulas such as Simpson's and Newton's formulas.

2. Preliminaries of q-Calculus and Some Inequalities

We provide some basics in quantum calculus and related integral inequalities in this section. Let $0 < q < 1$ be a constant throughout the paper.

Quantum numbers are expressed as follows:

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{n=0}^{n-1} q^n. \tag{1}$$

The quantum integral, namely the q-Jackson integral, given by Jackson for a mapping Y over $[0, \varphi_2]$ and defined as follows:

$$\int_0^{\varphi_2} \Upsilon(x) d_q x = (1-q)\varphi_2 \sum_{n=0}^{\infty} q^n \Upsilon(\varphi_2 q^n) \tag{2}$$

and in [13], he defined another q-integral over $[\varphi_1, \varphi_2]$ as:

$$\int_{\varphi_1}^{\varphi_2} \Upsilon(x) d_q x = \int_0^{\varphi_2} \Upsilon(x) d_q x - \int_0^{\varphi_1} \Upsilon(x) d_q x. \tag{3}$$

Definition 2. *In* [15] *The left q-derivative of $\Upsilon : [\varphi_1, \varphi_2] \to \mathbb{R}$ at $x \in [\varphi_1, \varphi_2]$ is described as:*

$${}_{\varphi_1}D_q \Upsilon(x) = \begin{cases} \dfrac{\Upsilon(x) - \Upsilon(qx + (1-q)\varphi_1)}{(1-q)(x-\varphi_1)}, & \text{if } x \neq \varphi_1; \\ \lim\limits_{x \to \varphi_1} {}_{\varphi_1}D_q \Upsilon(x), & \text{if } x = \varphi_1. \end{cases} \tag{4}$$

For more details about q-derivatives, one can consult [10,13,15].

Definition 3. *In* [15] *The left q-integral for $\Upsilon : [\varphi_1, \varphi_2] \to \mathbb{R}$ at $z \in [\varphi_1, \varphi_2]$ is described as:*

$$\int_{\varphi_1}^{z} \Upsilon(x)_{\varphi_1} d_q x = (1-q)(z-\varphi_1) \sum_{n=0}^{\infty} q^n \Upsilon(q^n z + (1-q^n)\varphi_1). \tag{5}$$

For more details about q-integrals, one can consult [10,13,15].

The q-Hermite–Hadamard inequality established by Alp et al. is presented as follows:

Theorem 1 ([17]). *Let $\Upsilon : [\varphi_1, \varphi_2] \to \mathbb{R}$ be a convex mapping, then we have the following equation:*

$$\Upsilon\left(\frac{q\varphi_1 + \varphi_2}{[2]_q}\right) \leq \frac{1}{\varphi_2 - \varphi_1} \int_{\varphi_1}^{\varphi_2} \Upsilon(x)_{\varphi_1} d_q x \leq \frac{q\Upsilon(\varphi_1) + \Upsilon(\varphi_2)}{[2]_q}. \tag{6}$$

Definition 4 ([16]). *The right q-derivative for $\Upsilon : [\varphi_1, \varphi_2] \to \mathbb{R}$ at $x \in [\varphi_1, \varphi_2]$ is described as follows:*

$${}^{\varphi_2}D_q \Upsilon(x) = \frac{\Upsilon(qx + (1-q)\varphi_2) - \Upsilon(x)}{(1-q)(\varphi_2 - x)}, \quad x \neq \varphi_2.$$

Definition 5 ([16]). *The right q-integral for $\Upsilon : [\varphi_1, \varphi_2] \to \mathbb{R}$ at $z \in [\varphi_1, \varphi_2]$ is described as follows:*

$$\int_z^{\varphi_2} \Upsilon(x)^{\varphi_2} d_q x = (1-q)(\varphi_2 - z) \sum_{k=0}^{\infty} q^k \Upsilon\left(q^k z + \left(1-q^k\right)\varphi_2\right).$$

Another version of the q-Hermite–Hadamard inequality established by Bermudo et al. is presented as:

Theorem 2 ([16]). *Let $\Upsilon : [\varphi_1, \varphi_2] \to \mathbb{R}$ be a convex mapping, then we have the following equation:*

$$\Upsilon\left(\frac{\varphi_1 + q\varphi_2}{[2]_q}\right) \leq \frac{1}{\varphi_2 - \varphi_1} \int_{\varphi_1}^{\varphi_2} \Upsilon(x)^{\varphi_2} d_q x \leq \frac{\Upsilon(\varphi_1) + q\Upsilon(\varphi_2)}{[2]_q}. \tag{7}$$

Now, we give a new Lemma which can help us to prove the identities in the next section.

Lemma 1. *For continuous functions $\Upsilon, g : [\varphi_1, \varphi_2] \to \mathbb{R}$, the following equality holds true:*

$$\int_0^c g(t)_a D_q f(tb+(1-t)a)d_q t = \frac{g(t)f(tb+(1-t)a)}{b-a}\bigg|_0^c - \frac{1}{b-a}\int_0^c D_q g(t) f(qtb+(1-qt)a)d_q t.$$

Proof. The Lemma can be shown by straightforward calculations, therefore it is omitted. □

3. Identities

In this section, we establish two integral identities that have an important role in proving the primary outcomes of this paper.

Lemma 2. *Let $\Upsilon : [\varphi_1, \varphi_2] \to \mathbb{R}$ be a q-differentiable function. If ${}_{\varphi_1}D_q\Upsilon$ is continuous and integrable on $[\varphi_1, \varphi_2]$, then one has the following identity:*

$$\begin{aligned}
&\frac{1}{6}\left[\Upsilon(m\varphi_1) + 4\Upsilon\left(\frac{m\varphi_1+\varphi_2}{2}\right) + \Upsilon(\varphi_2)\right] - \frac{1}{(\varphi_2 - m\varphi_1)}\int_{m\varphi_1}^{\varphi_2} \Upsilon(x)\ {}_{m\varphi_1}d_q x \\
&= (\varphi_2 - m\varphi_1)\left[\int_0^{\frac{1}{2}}\left(q\delta - \frac{1}{6}\right)\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)\ d_q\delta \right. \\
&\quad \left. + \int_{\frac{1}{2}}^{1}\left(q\delta - \frac{5}{6}\right){}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)d_q\delta\right].
\end{aligned} \tag{8}$$

Proof. From (3), we have:

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left(q\delta - \frac{1}{6}\right)\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)d_q\delta + \int_{\frac{1}{2}}^{1}\left(q\delta - \frac{5}{6}\right)\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)\ d_q\delta \\
&= \int_0^{\frac{1}{2}}\left(q\delta - \frac{1}{6}\right)\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)d_q\delta + \int_0^{1}\left(q\delta - \frac{5}{6}\right)\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)d_q\delta \\
&\quad - \int_0^{\frac{1}{2}}\left(q\delta - \frac{5}{6}\right)\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)d_q\delta \\
&= I_1 + I_2 - I_3.
\end{aligned} \tag{9}$$

Using the Lemma 1, we have the following:

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}}\left(q\delta - \frac{1}{6}\right)\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)d_q\delta \\
&= \left(q\delta - \frac{1}{6}\right)\frac{\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)}{\varphi_2 - m\varphi_1}\bigg|_0^{\frac{1}{2}} - \frac{q}{\varphi_2 - m\varphi_1}\int_0^{\frac{1}{2}}\Upsilon(q\delta\varphi_2 + m(1-q\delta)\varphi_1)d_q\delta \\
&= \frac{3q-1}{6(\varphi_2 - m\varphi_1)}\Upsilon\left(\frac{m\varphi_1+\varphi_2}{2}\right) + \frac{1}{6(\varphi_2 - m\varphi_1)}\Upsilon(m\varphi_1) - \frac{q}{\varphi_2 - m\varphi_1}\int_0^{\frac{1}{2}}\Upsilon(q\delta\varphi_2 + m(1-q\delta)\varphi_1)d_q\delta.
\end{aligned} \tag{10}$$

Similarly, we have:

$$\begin{aligned}
I_2 &= \int_0^{1}\left(q\delta - \frac{5}{6}\right)\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)d_q\delta \\
&= \frac{6q-5}{6(\varphi_2 - m\varphi_1)}\Upsilon(\varphi_2) + \frac{5}{6(\varphi_2 - m\varphi_1)}\Upsilon(m\varphi_1) - \frac{q}{\varphi_2 - m\varphi_1}\int_0^{1}\Upsilon(q\delta\varphi_2 + m(1-q\delta)\varphi_1)d_q\delta \\
&= \frac{6q-5}{6(\varphi_2 - m\varphi_1)}\Upsilon(\varphi_2) + \frac{5}{6(\varphi_2 - m\varphi_1)}\Upsilon(m\varphi_1) - \frac{1}{(\varphi_2 - m\varphi_1)^2}\int_{m\varphi_1}^{\varphi_2}\Upsilon(x)\ {}_{m\varphi_1}d_q x + \frac{1-q}{\varphi_2 - m\varphi_1}\Upsilon(\varphi_2)
\end{aligned} \tag{11}$$

and

$$I_3 = \int_0^{\frac{1}{2}} \left(q\delta - \frac{5}{6}\right) {}_{\varphi_1}D_q \Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)\, d_q\delta$$
$$= \frac{3q-5}{6(\varphi_2 - m\varphi_1)} \Upsilon\left(\frac{m\varphi_1 + \varphi_2}{2}\right) + \frac{5}{6}\Upsilon(m\varphi_1) - \frac{q}{\varphi_2 - m\varphi_1} \int_0^{\frac{1}{2}} \Upsilon(q\delta\varphi_2 + m(1-q\delta)\varphi_1)\, d_q\delta. \quad (12)$$

Thus, from (10)–(12), we have the following:

$$I_1 + I_2 - I_3 = \frac{1}{6(\varphi_2 - m\varphi_1)}\left[\Upsilon(m\varphi_1) + 4\Upsilon\left(\frac{m\varphi_1 + \varphi_2}{2}\right) + \Upsilon(\varphi_2)\right] - \frac{1}{(\varphi_2 - m\varphi_1)^2}\int_{m\varphi_1}^{\varphi_2} \Upsilon(x)\ {}_{m\varphi_1}d_q x \quad (13)$$

and we obtain the required equality (8) by multiplying $(\varphi_2 - m\varphi_1)$ on both sides of (13). The proof is completed. □

Remark 1. *In Lemma 2, we have the following:*

(i) With $\alpha = m = 1$, we regain (Lemma 2 in [28]).
(ii) With the limit as $q \to 1^-$, we regain (Lemma 2.1 in [33]).
(iii) With $\alpha = m = 1$, along with the limit as $q \to 1^-$, we regain (Lemma 1 in [34]).

Lemma 3. *Let $\Upsilon : [\varphi_1, \varphi_2] \to \mathbb{R}$ be a q-differentiable function on (φ_1, φ_2). If ${}_{\varphi_1}D_q\Upsilon$ is continuous and integrable on $[\varphi_1, \varphi_2]$, then one has the following identity:*

$$\frac{3}{8}\left[\frac{\Upsilon(m\varphi_1)}{3} + \Upsilon\left(\frac{2m\varphi_1 + \varphi_2}{3}\right) + \Upsilon\left(\frac{m\varphi_1 + 2\varphi_2}{3}\right) + \frac{\Upsilon(\varphi_2)}{3}\right] - \frac{1}{(\varphi_2 - m\varphi_1)}\int_{m\varphi_1}^{\varphi_2} \Upsilon(x)_{m\varphi_1} d_q x$$
$$= (\varphi_2 - m\varphi_1)\left[\int_0^{\frac{1}{3}} \left(q\delta - \frac{1}{8}\right) {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1) d_q\delta\right.$$
$$+ \int_{\frac{1}{3}}^{\frac{2}{3}} \left(q\delta - \frac{1}{2}\right) {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)\, d_q\delta$$
$$\left. + \int_{\frac{2}{3}}^{1} \left(q\delta - \frac{7}{8}\right) {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)\, d_q\delta\right]. \quad (14)$$

Proof. The desired result can be attained if the same steps used in the proof of Lemma 2 are used in this proof. □

Remark 2. *With $\alpha = m = 1$ in Lemma 3, we regain (Lemma 3 in [28]).*

4. Simpson's 1/3 Formula Type Inequalities

In this section, we establish some inequalities associated with Simpson's 1/3 formula for differentiable functions.

Theorem 3. *Under the assumption of Lemma 2, if $\left|{}_{\varphi_1}D_q\Upsilon\right|$ is an (α, m)-convex mapping over $[\varphi_1, \varphi_2]$, then we have the following Simpson type inequality:*

$$\left|\frac{1}{6}\left[\Upsilon(m\varphi_1) + 4\Upsilon\left(\frac{m\varphi_1 + \varphi_2}{2}\right) + \Upsilon(\varphi_2)\right] - \frac{1}{(\varphi_2 - m\varphi_1)}\int_{m\varphi_1}^{\varphi_2} \Upsilon(x)\ {}_{m\varphi_1}d_q x\right|$$
$$\le (\varphi_2 - m\varphi_1)\left[(\Omega_1(\alpha;q) + \Omega_3(\alpha;q))\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right| + (\Omega_2(\alpha;q) + \Omega_4(\alpha;q))m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|\right],$$

where

$$\Omega_1(\alpha;q) = \int_0^{\frac{1}{2}} \left|qt - \frac{1}{6}\right| t^\alpha\, d_q t$$

$$
= \begin{cases} \dfrac{1}{6\cdot 2^{\alpha+1}[\alpha+1]_q} - \dfrac{q}{2^{\alpha+2}[\alpha+2]_q}, & 0<q<\frac{1}{3}; \\ \dfrac{2-(3q)^{\alpha+1}}{6\cdot(6q)^{\alpha+1}[\alpha+1]_q} - \dfrac{q\left((3q)^{\alpha+2}-2\right)}{(6q)^{\alpha+2}[\alpha+2]_q}, & \frac{1}{3}\le q<1, \end{cases}
$$

$$
\Omega_2(\alpha;q) = \int_0^{\frac{1}{2}} \left|qt-\frac{1}{6}\right|(1-t^\alpha)d_qt
$$

$$
= \begin{cases} \dfrac{1-2q}{12[2]_q} - \dfrac{1}{6\cdot 2^{\alpha+1}[\alpha+1]_q} + \dfrac{q}{2^{\alpha+2}[\alpha+2]_q}, & 0<q<\frac{1}{3}; \\ \dfrac{6q-1}{36[2]_q} - \dfrac{2-(3q)^{\alpha+1}}{6\cdot(6q)^{\alpha+1}[\alpha+1]_q} + \dfrac{q\left((3q)^{\alpha+2}-2\right)}{(6q)^{\alpha+2}[\alpha+2]_q}, & \frac{1}{3}\le q<1, \end{cases}
$$

$$
\Omega_3(\alpha;q) = \int_{\frac{1}{2}}^{1} \left|qt-\frac{5}{6}\right|t^\alpha\, d_qt
$$

$$
= \begin{cases} \dfrac{5(2^{\alpha+1}-1)}{6\cdot 2^{\alpha+1}[\alpha+1]_q} + \dfrac{q(1-2^{\alpha+2})}{2^{\alpha+2}[\alpha+2]_q}, & 0<q<\frac{5}{6}; \\ \dfrac{5}{6[\alpha+1]_q}\left(\dfrac{2\cdot 5^{\alpha+1}-3^{\alpha+1}}{(6q)^{\alpha+1}}-1\right) + \dfrac{q}{[\alpha+2]_q}\left(\dfrac{3^{\alpha+2}-2\cdot 5^{\alpha+2}}{(6q)^{\alpha+2}}+1\right), & \frac{5}{6}\le q<1, \end{cases}
$$

$$
\Omega_4(\alpha;q) = \int_{\frac{1}{2}}^{1} \left|qt-\frac{5}{6}\right|(1-t^\alpha)\, d_qt
$$

$$
= \begin{cases} \dfrac{5-4q}{12[2]_q} - \dfrac{5(2^{\alpha+1}-1)}{6\cdot 2^{\alpha+1}[\alpha+1]_q} - \dfrac{q(1-2^{\alpha+2})}{2^{\alpha+2}[\alpha+2]_q}, & 0<q<\frac{5}{6}; \\ \dfrac{5}{36[2]_q} - \dfrac{5}{6[\alpha+1]_q}\left(\dfrac{2\cdot 5^{\alpha+1}-3^{\alpha+1}}{(6q)^{\alpha+1}}-1\right) - \dfrac{q}{[\alpha+2]_q}\left(\dfrac{3^{\alpha+2}-2\cdot 5^{\alpha+2}}{(6q)^{\alpha+2}}+1\right), & \frac{5}{6}\le q<1. \end{cases}
$$

Proof. By taking the modulus in (8) and using the (α, m)-convexity of $\left|{}_{\varphi_1}D_q\Upsilon\right|$, we have the following:

$$
\left|\frac{1}{6}\left[\Upsilon(m\varphi_1)+4\Upsilon\left(\frac{m\varphi_1+\varphi_2}{2}\right)+\Upsilon(\varphi_2)\right] - \frac{1}{(\varphi_2-m\varphi_1)}\int_{m\varphi_1}^{\varphi_2}\Upsilon(x)\ {}_{m\varphi_1}d_qx\right|
$$

$$
\le (\varphi_2-m\varphi_1)\left[\int_0^{\frac{1}{2}}\left|\left(q\delta-\frac{1}{6}\right)\right|\left|\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2+m(1-\delta)\varphi_1)\ \right|d_q\delta\right.
$$

$$
\left.+\int_{\frac{1}{2}}^{1}\left|\left(q\delta-\frac{5}{6}\right)\right|\left|\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2+m(1-\delta)\varphi_1)\right|\, d_q\delta\right]
$$

$$
\le (\varphi_2-m\varphi_1)\left[\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|\int_0^{\frac{1}{2}}\left|\left(q\delta-\frac{1}{6}\right)\right|\delta^\alpha\, d_q\delta + m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|\int_0^{\frac{1}{2}}\left|\left(q\delta-\frac{1}{6}\right)\right|(1-\delta^\alpha)\, d_q\delta\right.
$$

$$
\left.+\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|\int_{\frac{1}{2}}^{1}\left|\left(q\delta-\frac{5}{6}\right)\right|\delta^\alpha\, d_q\delta + m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|\int_{\frac{1}{2}}^{1}\left|\left(q\delta-\frac{1}{6}\right)\right|(1-\delta^\alpha)\, d_q\delta\right]
$$

$$
= (\varphi_2-m\varphi_1)\left[(\Omega_1(\alpha;q)+\Omega_3(\alpha;q))\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right| + (\Omega_2(\alpha;q)+\Omega_4(\alpha;q))m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|\right].
$$

Thus, the proof is completed. □

Remark 3. *In Theorem 3, we have:*

(i) With $\alpha = m = 1$, we regain (Theorem 4 in [28]).
(ii) With the limit as $q \to 1^-$, we regain (Theorem 2.2 in [33]).
(iii) With $\alpha = m = 1$, along with the limit as $q \to 1^-$, we regain (Corollary 1 in [34]).

Theorem 4. *Under the assumption of Lemma 2, if* $\left|{}_{\varphi_1}D_q\Upsilon\right|^s$*,* $s \geq 1$ *is an* (α, m)*-convex mapping over* $[\varphi_1, \varphi_2]$*, then we have the following Simpson type inequality:*

$$\left|\frac{1}{6}\left[\Upsilon(m\varphi_1) + 4\Upsilon\left(\frac{m\varphi_1+\varphi_2}{2}\right) + \Upsilon(\varphi_2)\right] - \frac{1}{(\varphi_2 - m\varphi_1)}\int_{m\varphi_1}^{\varphi_2}\Upsilon(x)\ {}_{m\varphi_1}d_qx\right|$$
$$\leq (\varphi_2 - m\varphi_1)\left[\Omega_5^{1-\frac{1}{s}}(q)\left(\Omega_1(\alpha;q)\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s + \Omega_2(\alpha;q)m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\Omega_6^{1-\frac{1}{s}}(q)\left(\Omega_3(\alpha;q)\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s + \Omega_4(\alpha;q)m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s\right)^{\frac{1}{s}}\right], \tag{15}$$

where $\Omega_i(\alpha;q), i = 1,2,3,4$ *are defined in Theorem 3 as:*

$$\Omega_5(q) = \int_0^{\frac{1}{2}}\left|q\delta - \frac{1}{6}\right|d_q\delta = \begin{cases}\dfrac{1-2q}{12[2]_q}, & 0 < q < \frac{1}{3};\\ \dfrac{6q-1}{36[2]_q}, & \frac{1}{3} \leq q < 1,\end{cases}$$

and

$$\Omega_6(q) = \int_{\frac{1}{2}}^1\left|q\delta - \frac{5}{6}\right|d_q\delta = \begin{cases}\dfrac{5-4q}{12[2]_q}, & 0 < q < \frac{5}{6};\\ \dfrac{5}{36[2]_q}, & \frac{5}{6} \leq q < 1.\end{cases}$$

Proof. By taking the modulus in (8) and using the power mean inequality, we have the following:

$$\left|\frac{1}{6}\left[\Upsilon(m\varphi_1) + 4\Upsilon\left(\frac{m\varphi_1+\varphi_2}{2}\right) + \Upsilon(\varphi_2)\right] - \frac{1}{(\varphi_2 - m\varphi_1)}\int_{m\varphi_1}^{\varphi_2}\Upsilon(x)_{m\varphi_1}d_qx\right|$$
$$\leq (\varphi_2 - m\varphi_1)\left[\int_0^{\frac{1}{2}}\left|\left(q\delta - \frac{1}{6}\right)\right|\left|\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)\ \right|d_q\delta\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left|\left(q\delta - \frac{5}{6}\right)\right|\left|\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)\right|\ d_q\delta\right]$$
$$\leq (\varphi_2 - m\varphi_1)\left[\left(\int_0^{\frac{1}{2}}\left|\left(q\delta - \frac{1}{6}\right)\right|d_q\delta\right)^{1-\frac{1}{s}}\left(\int_0^{\frac{1}{2}}\left|\left(q\delta - \frac{1}{6}\right)\right|\left|\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)\ \right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1\left|\left(q\delta - \frac{5}{6}\right)\right|\ d_q\delta\right)^{1-\frac{1}{s}}\left(\int_{\frac{1}{2}}^1\left|\left(q\delta - \frac{5}{6}\right)\right|\left|\ {}_{\varphi_1}D_q\Upsilon(\delta\varphi_2 + m(1-\delta)\varphi_1)\right|^s\ d_q\delta\right)^{\frac{1}{s}}\right].$$

Now, by applying the (α, m)-convexity, we have:

$$\left|\frac{1}{6}\left[\Upsilon(m\varphi_1) + 4\Upsilon\left(\frac{m\varphi_1+\varphi_2}{2}\right) + \Upsilon(\varphi_2)\right] - \frac{1}{(\varphi_2 - m\varphi_1)}\int_{m\varphi_1}^{\varphi_2}\Upsilon(x)\ {}_{m\varphi_1}d_qx\right|$$
$$\leq (\varphi_2 - m\varphi_1)\left[\left(\int_0^{\frac{1}{2}}\left|\left(q\delta - \frac{1}{6}\right)\right|d_q\delta\right)^{1-\frac{1}{s}}\right.$$
$$\times\left(\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s\int_0^{\frac{1}{2}}\left|\left(q\delta - \frac{1}{6}\right)\right|\delta^\alpha\ d_q\delta + m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s\int_0^{\frac{1}{2}}\left|\left(q\delta - \frac{1}{6}\right)\right|(1-\delta^\alpha)\ d_q\delta\right)^{\frac{1}{s}}$$
$$+\left(\int_{\frac{1}{2}}^1\left|\left(q\delta - \frac{5}{6}\right)\right|d_q\delta\right)^{1-\frac{1}{s}}$$

$$\times\left(\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s\int_{\frac{1}{2}}^{1}\left|\left(q\delta-\frac{5}{6}\right)\right|\delta^\alpha\,d_q\delta+m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s\int_{\frac{1}{2}}^{1}\left|\left(q\delta-\frac{1}{6}\right)\right|(1-\delta^\alpha)\,d_q\delta\right)^{\frac{1}{s}}\Bigg]$$

$$=(\varphi_2-m\varphi_1)\Bigg[\Omega_5^{1-\frac{1}{s}}(q)\left(\Omega_1(\alpha;q)\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s+\Omega_2(\alpha;q)m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s\right)^{\frac{1}{s}}$$

$$+\Omega_6^{1-\frac{1}{s}}(q)\left(\Omega_3(\alpha;q)\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s+\Omega_4(\alpha;q)m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s\right)^{\frac{1}{s}}\Bigg].$$

Thus, the proof is completed. □

Remark 4. *In Theorem 4, we have the following:*

(i) With $\alpha = m = 1$, we regain (Theorem 6 in [28]).
(ii) With the limit as $q \to 1^-$, we regain (Theorem 2.10 in [33]).
(iii) With $\alpha = m = 1$, along with the limit as $q \to 1^-$, we regain (Theorem 7 for $s = 1$ in [34]).

Theorem 5. *Under the assumption of Lemma 2, if $s > 1$ is a real number and $\left|{}_{\varphi_1}D_q\Upsilon\right|^s$ is an (α, m)-convex mapping over $[\varphi_1, \varphi_2]$, then we have the following Simpson type inequality:*

$$\left|\frac{1}{6}\left[\Upsilon(m\varphi_1)+4\Upsilon\left(\frac{m\varphi_1+\varphi_2}{2}\right)+\Upsilon(\varphi_2)\right]-\frac{1}{(\varphi_2-m\varphi_1)}\int_{m\varphi_1}^{\varphi_2}\Upsilon(x)\,{}_{m\varphi_1}d_qx\right|$$
$$\leq\frac{(\varphi_2-m\varphi_1)}{6}\Bigg[2^{1-\frac{1}{r}}\left(\frac{1}{2^{\alpha+1}[\alpha+1]_q}\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s+\frac{2^\alpha[\alpha+1]_q-1}{2^{\alpha+1}[\alpha+1]_q}m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s\right)^{\frac{1}{s}}$$
$$+\left(5^r-2^{r-1}\right)^{\frac{1}{r}}\left(\frac{2^{\alpha+1}-1}{2^{\alpha+1}[\alpha+1]_q}\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s+\frac{2^\alpha\left([\alpha+1]_q-2\right)+1}{2^{\alpha+1}[\alpha+1]_q}m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s\right)^{\frac{1}{s}}\Bigg],\tag{16}$$

where $s^{-1} + r^{-1} = 1$.

Proof. Taking the modulus in Lemma 2 and applying Hölder's inequality, we have the following:

$$\left|\frac{1}{6}\left[\Upsilon(m\varphi_1)+4\Upsilon\left(\frac{m\varphi_1+\varphi_2}{2}\right)+\Upsilon(\varphi_2)\right]-\frac{1}{(\varphi_2-m\varphi_1)}\int_{m\varphi_1}^{\varphi_2}\Upsilon(x)\,{}_{m\varphi_1}d_qx\right|$$
$$\leq(\varphi_2-m\varphi_1)\left[\left(\int_0^{\frac{1}{2}}\left|\left(q\delta-\frac{1}{6}\right)\right|^r d_q\delta\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left|\,{}_{\varphi_1}D_q\Upsilon(\delta\varphi_2+m(1-\delta)\varphi_1)\,\right|^s d_q\delta\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\left|\left(q\delta-\frac{5}{6}\right)\right|^r d_q\delta\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^{1}\left|\,{}_{\varphi_1}D_q\Upsilon(\delta\varphi_2+m(1-\delta)\varphi_1)\,\right|^s d_q\delta\right)^{\frac{1}{s}}\right].$$

Now, by applying the (α, m)-convexity of $\left|{}_{\varphi_1}D_q\Upsilon\right|^s$, we have:

$$\left|\frac{1}{6}\left[f(ma)+4f\left(\frac{ma+b}{2}\right)+f(b)\right]-\frac{1}{(b-ma)}\int_{ma}^{b}f(x)_{ma}d_qx\right|$$
$$\leq(b-ma)\left[\left(\int_0^{\frac{1}{2}}\left|\left(qt-\frac{1}{6}\right)\right|^r d_qt\right)^{\frac{1}{r}}\left(\left|{}_aD_qf(b)\right|^s\int_0^{\frac{1}{2}}t^\alpha\,d_qt+m\left|{}_aD_qf(a)\right|^s\int_0^{\frac{1}{2}}(1-t^\alpha)\,d_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\left|\left(qt-\frac{5}{6}\right)\right|^r d_qt\right)^{\frac{1}{r}}\left(\left|{}_aD_qf(b)\right|^s\int_{\frac{1}{2}}^{1}t^\alpha\,d_qt+m\left|{}_aD_qf(a)\right|^s\int_{\frac{1}{2}}^{1}(1-t^\alpha)\,d_qt\right)^{\frac{1}{s}}\right]$$

$$\leq \frac{(b-ma)}{6}\left[2^{1-\frac{1}{r}}\left(\frac{1}{2^{\alpha+1}[\alpha+1]_q}\left|{}_aD_qf(b)\right|^s+\frac{2^\alpha[\alpha+1]_q-1}{2^{\alpha+1}[\alpha+1]_q}m\left|{}_aD_qf(a)\right|^s\right)^{\frac{1}{s}}\right.$$

$$\left.+\left(5^r-2^{r-1}\right)^{\frac{1}{r}}\left(\frac{2^{\alpha+1}-1}{2^{\alpha+1}[\alpha+1]_q}\left|{}_aD_qf(b)\right|^s+\frac{2^\alpha\left([\alpha+1]_q-2\right)+1}{2^{\alpha+1}[\alpha+1]_q}m\left|{}_aD_qf(a)\right|^s\right)^{\frac{1}{s}}\right],$$

where one can easily observe that:

$$\begin{aligned}\int_0^{\frac{1}{2}}\left|\left(q\delta-\frac{1}{6}\right)\right|^r d_q\delta &= \frac{1-q}{2}\sum_{n=0}^{\infty}q^n\left|\frac{q^{n+1}}{2}-\frac{1}{6}\right|^r \\ &\leq \frac{1-q}{2}\sum_{n=0}^{\infty}q^n\left|\frac{1}{2}-\frac{1}{6}\right|^r \\ &= \frac{1}{2.3^r}\end{aligned}$$

and similarly, we have:

$$\int_{\frac{1}{2}}^{1}\left|\left(q\delta-\frac{5}{6}\right)\right|^r d_q\delta \leq \frac{5^r-2^{r-1}}{6}.$$

Thus, the proof is completed. □

Remark 5. *With $\alpha = m = 1$ in Theorem 5, we regain (Theorem 5 in [28]).*

5. Simpson's 3/8 Formula Type Inequalities

In this section, we establish some inequalities associated with Simpson's 3/8 formula for differentiable functions.

Theorem 6. *Under the assumption of Lemma 3, if $\left|{}_{\varphi_1}D_q\Upsilon\right|$ is an (α, m)-convex mapping over $[\varphi_1, \varphi_2]$, then we have the following Newton's type inequality:*

$$\begin{aligned}&\left|\frac{3}{8}\left[\frac{\Upsilon(m\varphi_1)}{3}+\Upsilon\left(\frac{2m\varphi_1+\varphi_2}{3}\right)+\Upsilon\left(\frac{m\varphi_1+2\varphi_2}{3}\right)+\frac{\Upsilon(\varphi_2)}{3}\right]-\frac{1}{(\varphi_2-m\varphi_1)}\int_{m\varphi_1}^{\varphi_2}\Upsilon(x)\ {}_{m\varphi_1}d_qx\right| \\ &\leq (\varphi_2-m\varphi_1)\left[\begin{array}{c}(\Omega_7(\alpha;q)+\Omega_9(\alpha;q)+\Omega_{11}(\alpha;q))\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right| \\ +(\Omega_8(\alpha;q)+\Omega_{10}(\alpha;q)+\Omega_{12}(\alpha;q))m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|\end{array}\right],\end{aligned} \quad (17)$$

where

$$\begin{aligned}\Omega_7(\alpha;q) &= \int_0^{\frac{1}{3}}\left|qt-\frac{1}{8}\right|t^\alpha d_qt \\ &= \begin{cases}\dfrac{1}{8\times 3^{\alpha+1}[\alpha+1]_q}-\dfrac{q}{3^{\alpha+2}[\alpha+2]_q}, & 0<q<\frac{3}{8}; \\ \dfrac{3^{\alpha+1}-4(8q)^{\alpha+1}}{4\times(24q)^{\alpha+1}[\alpha+1]_q}-\dfrac{q\left((8q)^{\alpha+2}-2\cdot 3^{\alpha+2}\right)}{(24q)^{\alpha+2}[\alpha+2]_q}, & \frac{3}{8}\leq q<1,\end{cases}\end{aligned}$$

$$\begin{aligned}\Omega_8(\alpha;q) &= \int_0^{\frac{1}{3}}\left|qt-\frac{1}{8}\right|(1-t^\alpha)d_qt \\ &= \begin{cases}\dfrac{3-5q}{72[2]_q}-\dfrac{1}{8\times 3^{\alpha+1}[\alpha+1]_q}+\dfrac{q}{3^{\alpha+2}[\alpha+2]_q}, & 0<q<\frac{3}{8}; \\ \dfrac{20q-3}{288[2]_q}-\dfrac{3^{\alpha+1}-4(8q)^{\alpha+1}}{4\times(24q)^{\alpha+1}[\alpha+1]_q}+\dfrac{q\left((8q)^{\alpha+2}-2\times 3^{\alpha+2}\right)}{(24q)^{\alpha+2}[\alpha+2]_q}, & \frac{3}{8}\leq q<1,\end{cases}\end{aligned}$$

$$\Omega_9(\alpha;q) = \int_{\frac{1}{3}}^{\frac{2}{3}} \left|qt - \frac{1}{2}\right| t^\alpha \, d_q t$$
$$= \begin{cases} \frac{2^{\alpha+1}-1}{2\times 3^{\alpha+1}[\alpha+1]_q} + \frac{q\left(1-2^{\alpha+1}\right)}{3^{\alpha+2}[\alpha+2]_q}, & 0 < q < \frac{3}{4}; \\ \frac{2\times 3^{\alpha+1} - (2q)^{\alpha+1}\left(1+2^{\alpha+1}\right)}{2\times (6q)^{\alpha+1}[\alpha+1]_q} + \frac{q\left(-2\times 3^{\alpha+2} - (2q)^{\alpha+2}\left(1+2^{\alpha+2}\right)\right)}{(6q)^{\alpha+2}[\alpha+2]_q}, & \frac{3}{4} \leq q < 1, \end{cases}$$

$$\Omega_{10}(\alpha;q) = \int_{\frac{1}{3}}^{\frac{2}{3}} \left|qt - \frac{1}{2}\right| (1-t^\alpha) \, d_q t$$
$$= \begin{cases} \frac{3-3q}{18[2]_q} - \frac{2^{\alpha+1}-1}{2\times 3^{\alpha+1}[\alpha+1]_q} - \frac{q\left(1-2^{\alpha+1}\right)}{3^{\alpha+2}[\alpha+2]_q}, & 0 < q < \frac{3}{4}; \\ \frac{q}{18[2]_q} - \frac{2\times 3^{\alpha+1} - (2q)^{\alpha+1}\left(1+2^{\alpha+1}\right)}{2\times (6q)^{\alpha+1}[\alpha+1]_q} - \frac{q\left(-2\times 3^{\alpha+2} - (2q)^{\alpha+2}\left(1+2^{\alpha+2}\right)\right)}{(6q)^{\alpha+2}[\alpha+2]_q}, & \frac{3}{4} \leq q < 1, \end{cases}$$

$$\Omega_{11}(\alpha;q) = \int_{\frac{2}{3}}^{1} \left|qt - \frac{7}{8}\right| t^\alpha \, d_q t$$
$$= \begin{cases} \frac{7\left(3^{\alpha+1}-2^{\alpha+1}\right)}{8\times 3^{\alpha+1}[\alpha+1]_q} - \frac{q\left(2^{\alpha+2}-32\right)}{3^{\alpha+2}[\alpha+2]_q}, & 0 < q < \frac{7}{8}; \\ \frac{7}{8[\alpha+1]_q}\left(2\left(\frac{7}{8q}\right)^{\alpha+1} - \left(\frac{2}{3}\right)^{\alpha+1} - 1\right) + \frac{q}{[\alpha+2]_q}\left(\left(\frac{2}{3}\right)^{\alpha+2} - 2\left(\frac{7}{8q}\right)^{\alpha+2} + 1\right), & \frac{7}{8} \leq q < 1, \end{cases}$$

and

$$\Omega_{12}(\alpha;q) = \int_{\frac{2}{3}}^{1} \left|qt - \frac{7}{8}\right| (1-t^\alpha) \, d_q t$$
$$= \begin{cases} \frac{21-19q}{72[2]_q} - \frac{7\left(3^{\alpha+1}-2^{\alpha+1}\right)}{8\times 3^{\alpha+1}[\alpha+1]_q} + \frac{q\left(2^{\alpha+2}-32\right)}{3^{\alpha+2}[\alpha+2]_q}, & 0 < q < \frac{7}{8}; \\ \frac{21-4q}{288[2]_q} - \frac{7}{8[\alpha+1]_q}\left(2\left(\frac{7}{8q}\right)^{\alpha+1} - \left(\frac{2}{3}\right)^{\alpha+1} - 1\right) - \frac{q}{[\alpha+2]_q}\left(\left(\frac{2}{3}\right)^{\alpha+2} - 2\left(\frac{7}{8q}\right)^{\alpha+2} + 1\right), & \frac{7}{8} \leq q < 1. \end{cases}$$

Proof. By addressing equality (14), the proof of this theorem follows the same lines as that of the proof of Theorem 3. □

Remark 6. *With $\alpha = m = 1$ in Theorem 6, we regain (Theorem 7 in [28]).*

Theorem 7. *Under the assumption of Lemma 3, if $s \geq 1$ is a real number and $\left|{}_{\varphi_1}D_q\Upsilon\right|^s$ is an (α, m)-convex mapping over $[\varphi_1, \varphi_2]$, then we have the following Newton's type inequality:*

$$\left|\frac{3}{8}\left[\frac{\Upsilon(m\varphi_1)}{3} + \Upsilon\left(\frac{2m\varphi_1+\varphi_2}{3}\right) + \Upsilon\left(\frac{m\varphi_1+2\varphi_2}{3}\right) + \frac{\Upsilon(\varphi_2)}{3}\right] - \frac{1}{(\varphi_2 - m\varphi_1)}\int_{m\varphi_1}^{\varphi_2} \Upsilon(x)_{m\varphi_1} d_q x\right|$$
$$\leq (\varphi_2 - m\varphi_1)\left[\Omega_{13}^{1-\frac{1}{s}}(q)\left(\Omega_7(\alpha;q)\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s + \Omega_8(\alpha;q)m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s\right)^{\frac{1}{s}}\right.$$
$$+ \Omega_{14}^{1-\frac{1}{s}}(q)\left(\Omega_9(\alpha;q)\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s + \Omega_{10}(\alpha;q)m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s\right)^{\frac{1}{s}}$$
$$\left.+\Omega_{15}^{1-\frac{1}{s}}(q)\left(\Omega_{11}(\alpha;q)\left|{}_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s + \Omega_{12}(\alpha;q)m\left|{}_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s\right)^{\frac{1}{s}}\right], \tag{18}$$

where $\Omega_j(\alpha;q), j = 7,8,9,\ldots,12$ *are defined in Theorem 6 as:*

$$\Omega_{13}(q) = \int_0^{\frac{1}{3}} \left| q\delta - \frac{1}{8} \right| \delta^\alpha d_q\delta = \begin{cases} \frac{3-5q}{72[2]_q}, & 0 < q < \frac{3}{8}; \\ \frac{20q-3}{288[2]_q}, & \frac{3}{8} \leq q < 1, \end{cases}$$

$$\Omega_{14}(q) = \int_{\frac{1}{3}}^{\frac{2}{3}} \left| q\delta - \frac{1}{2} \right| d_q\delta = \begin{cases} \frac{3-3q}{18[2]_q}, & 0 < q < \frac{3}{4}; \\ \frac{q}{18[2]_q}, & \frac{3}{4} \leq q < 1, \end{cases}$$

and

$$\Omega_{15}(q) = \int_{\frac{2}{3}}^{1} \left| q\delta - \frac{7}{8} \right| d_q\delta = \begin{cases} \frac{21-19q}{72[2]_q}, & 0 < q < \frac{7}{8}; \\ \frac{21-4q}{288[2]_q}, & \frac{7}{8} \leq q < 1. \end{cases}$$

Proof. By addressing equality (14), the proof of this theorem follows the same lines as that of the proof of Theorem 4. □

Remark 7. *With* $\alpha = m = 1$ *in Theorem 7, we regain (Theorem 9 in [28]).*

Theorem 8. *Under the assumption of Lemma 3, if* $s > 1$ *is a real number and* $\left|_{\varphi_1}D_q\Upsilon\right|^s$ *is an* (α, m)*-convex mapping over* $[\varphi_1, \varphi_2]$*, then we have the following Newton's type inequality:*

$$\begin{aligned} &\left| \frac{3}{8}\left[\frac{\Upsilon(m\varphi_1)}{3} + \Upsilon\left(\frac{2m\varphi_1+\varphi_2}{3}\right) + \Upsilon\left(\frac{m\varphi_1+2\varphi_2}{3}\right) + \frac{\Upsilon(\varphi_2)}{3} \right] - \frac{1}{(\varphi_2 - m\varphi_1)} \int_{m\varphi_1}^{\varphi_2} \Upsilon(x)_{m\varphi_1} d_q x \right| \\ &\leq (\varphi_2 - m\varphi_1)\left[\left(\frac{5^r}{3.8^r}\right)^{\frac{1}{r}} \left(\frac{1}{3^{\alpha+1}[\alpha+1]_q} \left|_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s + \frac{3^\alpha[\alpha+1]_q - 1}{3^{\alpha+1}[\alpha+1]_q} m \left|_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s \right)^{\frac{1}{s}} \right. \\ &+ \left(\frac{2.3^r - 1}{3.6^r}\right)^{\frac{1}{r}} \left(\frac{2^{\alpha+1}-1}{3^{\alpha+1}[\alpha+1]_q} \left|_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s + \frac{3^\alpha[\alpha+1]_q - 2^{\alpha+1} + 1}{3^{\alpha+1}[\alpha+1]_q} m \left|_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s \right)^{\frac{1}{s}} \\ &\left. + \left(\frac{3.7^r - 2}{3.8^r}\right)^{\frac{1}{r}} \left(\frac{2^{\alpha+1}-2^{\alpha+1}}{3^{\alpha+1}[\alpha+1]_q} \left|_{\varphi_1}D_q\Upsilon(\varphi_2)\right|^s + \frac{3^\alpha\left([\alpha+1]_q - 3\right) + 2^{\alpha+1}}{3^{\alpha+1}[\alpha+1]_q} m \left|_{\varphi_1}D_q\Upsilon(\varphi_1)\right|^s \right)^{\frac{1}{s}} \right], \end{aligned} \tag{19}$$

where $s^{-1} + r^{-1} = 1$.

Proof. By addressing equality (14), the proof of this theorem follows the same lines as that of the proof of Theorem 5. □

Remark 8. *With* $\alpha = m = 1$ *in Theorem 8, we regain (Theorem 8 in [28]).*

6. Conclusions

The major goal of this paper was to prove two quantum integral identities in order to establish some new quantum Simpson's and quantum Newton's formula type inequalities for differentiable (α, m)-convex functions. We also demonstrated that the newly established inequalities for convex functions might be transformed into classical Simpson's inequalities and quantum Simpson's and quantum Newton's inequalities for convex functions without having to prove each one independently. The results for symmetric functions may be obtained by using the concepts of symmetric convex functions, which will be further investigated in future work. It is a new and interesting problem that the researcher can obtain similar inequalities for other kinds of convexity and co-ordinated (α, m)-convexity in their future work.

Author Contributions: Conceptualization, J.S., M.A.A., H.B., K.N. and Z.A.; methodology, J.S., M.A.A., H.B., K.N. and Z.A.; software, J.S., M.A.A., H.B., K.N. and Z.A.; validation, J.S., M.A.A., H.B., K.N. and Z.A.; formal analysis, J.S., M.A.A., H.B., K.N. and Z.A.; investigation, J.S., M.A.A., H.B., K.N. and Z.A.; resources, J.S., M.A.A., H.B., K.N. and Z.A.; data curation, J.S., M.A.A., H.B., K.N. and Z.A.; writing—original draft preparation, J.S., M.A.A., H.B., K.N. and Z.A.; writing—review and editing, J.S., M.A.A., H.B., K.N. and Z.A.; visualization, J.S., M.A.A., H.B., K.N. and Z.A.; supervision, J.S., M.A.A., H.B., K.N. and Z.A.; project administration, J.S., M.A.A., H.B., K.N. and Z.A.; funding acquisition, J.S., M.A.A., H.B., K.N. and Z.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Mongkut's University of Technology North Bangkok. Contract No. KMUTNB-64-KNOW-36.

Data Availability Statement: Not applicable.

Acknowledgments: We thanks to worthy referees and editor for their valuable comments. The second author is also very thankful to Chinese Government for providing Scholarship in his higher studies.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hudzik, H.; Maligrada, L. Some remarks on s-convex functions. *Aequ. Math.* **1994**, *48*, 100–111. [CrossRef]
2. AlNemer, G.; Ahmed, I.S.; Zakarya, M.; El-Hamid, H.A.A.; Hoda, A.; Bazighifan, O.; Rezk, H.M. Some New Reverse Hilbert's Inequalities on Time Scales. *Symmetry* **2021**, *13*, 2431. [CrossRef]
3. AlNemer, G.; Kenawy, M.; Zakarya, M.; Cesarano, C.; Rezk, H.M. Generalizations of Hardy's type inequalities via conformable calculus. *Symmetry* **2021**, *13*, 242. [CrossRef]
4. Dragomir, S.S.; Pearce, C.E.M. *Selected Topics on Hermite—Hadamard Inequalities and Applications*; RGMIA Monographs, Victoria University: Footscray, Australia, 2000.
5. Pećarixcx, J.E.; Proschan, F.; Tong, Y.L. *Convex Functions, Partial Orderings and Statistical Applications*; Academic Press: Boston, MA, USA, 1992.
6. El-Hamid, H.A.A.; Rezk, H.M.; Ahmed, A.M.; AlNemer, G.; Zakarya, M.; Saify, H.A.E. Some dynamic Hilbert-type inequalities for two variables on time scales. *J. Inequal. Appl.* **2021**, *2021*, 1–21. [CrossRef]
7. Dragomir, S.S.; Fitzpatrick, S. The Hadamard's inequality for s-convex functions in the second sense. *Demonstr. Math.* **1999**, *32*, 687–696.
8. Mihesan, V.G. A generalization of the convexity. In *Seminar on Functional Equations*; Approx. and Convex.: Cluj-Napoca, Romania, 1993.
9. Ernst, T. A *Comprehensive Treatment of q -Calculus*; Springer: Berlin/Heidelberg, Germany, 2012.
10. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: Berlin/Heidelberg, Germany, 2001.
11. Benatti, F.; Fannes, M.; Floreanini, R.; Petritis, D. *Quantum Information, Computation and Cryptography: An Introductory Survey of Theory, Technology and Experiments*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2010.
12. Bokulich, A.; Jaeger, G. *Philosophy of Quantum Information Theory and Entaglement*; Cambridge Uniersity Press: Cambridge, UK, 2010.
13. Jackson, F.H. On a q-definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
14. Al-Salam, W. Some fractional q-integrals and q -derivatives. *Proc. Edinb. Math. Soc.* **1966**, *15*, 135–140. [CrossRef]
15. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* **2013**, *2013*, 1–19. [CrossRef]
16. Bermudo, S.; Kórus, P.; Valdés, J.N. On q-Hermite—Hadamard inequalities for general convex functions. *Acta Math. Hung.* **2020**, *162*, 364–374. [CrossRef]
17. Alp, N.; Sarikaya, M.Z.; Kunt, M.; İşcan, I. q -Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud Univ. Sci.* **2018**, *30*, 193–203. [CrossRef]
18. Alp, N.; Sarikaya, M.Z. Hermite Hadamard's type inequalities for co-ordinated convex functions on quantum integral. *Appl. Math. E-Notes* **2020**, *20*, 341–356.
19. Budak, H.; Ali, M.A.; Tarhanaci, M. Some new quantum Hermite—Hadamard-like inequalities for coordinated convex functions. *J. Optim. Theory Appl.* **2020**, *186*, 899–910. [CrossRef]
20. Jhanthanam, S.; Tariboon, J.; Ntouyas, S.K.; Nonlaopon, K. On q-Hermite—Hadamard inequalities for differentiable convex functions. *Mathematics* **2019**, *7*, 632. [CrossRef]
21. Kalsoom, H.; Amer, M.; Junjua, M.U.; Hussain, S.; Shahzadi, G. Some (p, q)-estimates of Hermite—Hadamard—Type inequalities for coordinated convex and quasi-convex functions. *Mathematics* **2019**, *7*, 683. [CrossRef]

22. Kalsoom, H.; Rashid, S.; Idress, M.; Chu, Y.-M.; Baleanu, D. Two—Variable quantum integral inequalities of Simpson—Type based on higher—Order generalized strongly preinvex and quasi—Preinvex functions. *Symmetry* **2020**, *12*, 51. [CrossRef]
23. Liu, W.; Hefeng, Z. Some quantum estimates of Hermite—Hadamard inequalities for convex functions. *J. Appl. Anal. Comput.* **2016**, *7*, 501–522.
24. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum estimates for Hermite—Hadamard inequalities. *Appl. Math. Comput.* **2015**, *251*, 675–679. [CrossRef]
25. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum integral inequalities via preinvex functions. *Appl. Math. Comput.* **2015**, *269*, 242–251. [CrossRef]
26. Ali, M.A.; Budak, H.; Abbas, M.; Chu, Y.-M. Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second q^b-derivatives. *Adv. Differ. Equ.* **2021**, *2021*, 7. [CrossRef]
27. Khan, M.A.; Noor, M.; Nwaeze, E.R.; Chu, Y.-W. Quantum Hermite—Hadamard inequality by means of a Green function. *Adv. Differ. Equ.* **2020**, *2020*, 99. [CrossRef]
28. Budak, H.; Erden, S.; Ali, M.A. Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. *Math. Meth. Appl. Sci.* **2020**, *44*, 378–390. [CrossRef]
29. Ali, M.A.; Budak, H.; Zhang, Z.; Yildrim, H. Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus. *Math. Meth. Appl. Sci.* **2021**, *44*, 4515–4540. [CrossRef]
30. Ali, M.A.; Abbas, M.; Budak, H.; Agarwal, P.; Murtaza, G.; Chu, Y.-W. New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions. *Adv. Differ. Equ.* **2021**, *2021*, 64. [CrossRef]
31. Vivas-Cortez, M.; Ali, M.A.; Kashuri, A.; Sial, I.B.; Zhang, Z. Some New Newton's Type Integral Inequalities for Co-Ordinated Convex Functions in Quantum Calculus. *Symmetry* **2020**, *12*, 1476. [CrossRef]
32. Sial, I.B.; Mei, S.; Ali, M.A.; Nonlaopon, K. On Some Generalized Simpson's and Newton's Inequalities for (α, m)-Convex Functions in q-Calculus. *Mathematics* **2021**, *9*, 3266. [CrossRef]
33. Qaisar, S.; He, C.; Hussain, S. A generalizations of Simpson's type inequality for differentiable functions using (α, m)-convex functions and applications. *J. Inequal. Appl.* **2013**, *2013*, 158. [CrossRef]
34. Alomari, M.; Darus, M.; Dragomir, S.S. New inequalities of Simpson's type for s-convex functions with applications. *RGMIA Res. Rep. Coll.* **2009**, *12*, 9.