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Some New Generalizations of Reverse Hilbert-Type Inequalities on Time Scales

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Abstract: This manuscript develops the study of reverse Hilbert-type inequalities by applying reverse Hölder inequalities on $\mathbb{T}$. We generalize the reverse inequality of Hilbert-type with power two by replacing the power with a new power β, $\beta > 1$. The main results are proved by using Specht's ratio, chain rule and Jensen's inequality. Our results (when $\mathbb{T} = \mathbb{N}$) are essentially new. Symmetrical properties play an essential role in determining the correct methods to solve inequalities.

Keywords: reverse Hilbert-type inequalities; Specht's ratio; time scales; reverse Hölder inequalities

MSC: 26D10; 26D15; 34N05; 47B38; 39A12

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1. Introduction

In [1], Hardy established that

$$\sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Xi_i F_m}{i+m} \le \frac{\pi}{\sin\frac{\pi}{l}}\left(\sum_{i=1}^{\infty}\Xi_i^l\right)^{\frac{1}{l}}\left(\sum_{m=1}^{\infty}F_m^q\right)^{\frac{1}{q}}, \tag{1}$$

where $\Xi_i, F_m \ge 0$ with $0 < \sum_{i=1}^{\infty}\Xi_i^l < \infty$, $0 < \sum_{m=1}^{\infty}F_m^q < \infty$ and $l > 1$, $1/l + 1/q = 1$. The continuous form (see [2]) of (1) is

$$\int_0^{\infty}\int_0^{\infty}\frac{\varphi(\vartheta)\psi(y)}{\vartheta+y}d\vartheta dy \le \frac{\pi}{\sin\frac{\pi}{l}}\left(\int_0^{\infty}\varphi^l(\vartheta)d\vartheta\right)^{\frac{1}{l}}\left(\int_0^{\infty}\psi^q(y)dy\right)^{\frac{1}{q}}, \tag{2}$$

where $\varphi, \psi \ge 0$ are measurable functions such that $0 < \int_0^{\infty}\varphi^l(\vartheta)d\vartheta < \infty$ and $0 < \int_0^{\infty}\psi^q(y)dy < \infty$. The constant $\pi/\sin(\pi/l)$ in both (1) and (2) is sharp. In [2], Hardy showed that if $d > 1$, $q > 1$, $1/d + 1/q \ge 1$ and $0 < \lambda = 2 - (1/d + 1/q) \le 1$, then

$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Xi_i F_n}{(i+n)^{\lambda}} \le K(d,q)\left(\sum_{i=1}^{\infty}\Xi_i^d\right)^{\frac{1}{d}}\left(\sum_{n=1}^{\infty}F_n^q\right)^{\frac{1}{q}}.$$

In [3], Hölder proved that

$$\sum_{k=1}^{n} \zeta_k y_k \leq \left(\sum_{k=1}^{n} \zeta_k^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{k=1}^{n} y_k^{\beta}\right)^{\frac{1}{\beta}}, \tag{3}$$

where (ζ_k) and (y_k) are positive sequences and $\alpha, \beta > 1$ such that $1/\alpha + 1/\beta = 1$. The continuous form of (3) is

$$\int_{\varrho}^{b} \psi(\tau)\varpi(\tau)d\tau \leq \left(\int_{\varrho}^{b} \psi^{\alpha}(\tau)d\tau\right)^{\frac{1}{\alpha}} \left(\int_{\varrho}^{b} \varpi^{\beta}(\tau)d\tau\right)^{\frac{1}{\beta}},$$

where $\alpha, \beta > 1$ such that $1/\alpha + 1/\beta = 1$ and $\psi, \varpi \in C((\varrho, b), \mathbb{R}^+)$.

In [4], Zhao and Cheung proved that if $\psi(\zeta), \varpi(\zeta) \geq 0$ are continuous functions and $\psi^{1/\alpha}(\zeta)\varpi^{1/\beta}(\zeta)$ is integrable on $[\varrho, c]$, then

$$\left(\int_{\varrho}^{c} \psi^{\alpha}(\zeta)d\zeta\right)^{\frac{1}{\alpha}} \left(\int_{\varrho}^{c} \varpi^{\beta}(\zeta)d\zeta\right)^{\frac{1}{\beta}} \leq \int_{\varrho}^{c} S\left(\frac{Y\psi^{\alpha}(\zeta)}{X\varpi^{\beta}(\zeta)}\right)\psi(\zeta)\varpi(\zeta)d\zeta,$$

with

$$X = \int_{\varrho}^{c} \psi^{\alpha}(\zeta)d\zeta,\ Y = \int_{\varrho}^{c} \varpi^{\beta}(\zeta)d\zeta,\ \alpha > 1 \text{ and } \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

where $S(.)$ is Specht's ratio function (see [5]) and defined as

$$S(u) = \frac{u^{1/(u-1)}}{e \log u^{1/(u-1)}}, u \neq 1 \quad \text{and} \quad S(1) = 1.$$

In [4], the authors proved that if $\psi, \varpi \in C((\varrho, c), \mathbb{R}^+)$ and $m > 0$, then

$$\int_{\varrho}^{c} \frac{\psi^{m+1}(\zeta)}{\varpi^{m}(\zeta)}d\zeta \leq \frac{\left(\int_{\varrho}^{c} S\left(\frac{G\psi^{m+1}(\zeta)}{F\varpi^{m+1}(\zeta)}\right)\psi(\zeta)d\zeta\right)^{m+1}}{\left(\int_{\varrho}^{c} \varpi(\zeta)d\zeta\right)^{m}}, \tag{4}$$

where

$$G = \int_{\varrho}^{c} \varpi(\zeta)d\zeta \text{ and } F = \int_{\varrho}^{c} \frac{\psi^{m+1}(\zeta)}{\varpi^{m}(\zeta)}d\zeta.$$

In addition, they proved the discrete case of (4) and established that

$$\sum_{i=1}^{\infty} \frac{\varrho_i^{m+1}}{b_i^{m}} \leq \frac{\sum_{i=1}^{\infty} S\left(\frac{B\varrho_i^{m+1}}{Ab_i^{m+1}}\right)\varrho_i}{\left(\sum_{i=1}^{\infty} b_i\right)^{m}},$$

where $B = \sum_{i=1}^{\infty} b_i$ and $A = \sum_{i=1}^{\infty} \varrho_i^{m+1}/b_i^{m}$.

In 2019, Zhao and Cheung [6] studied the reverse Hilbert inequalities and proved that if $0 \leq d, q \leq 1$ and $\{\lambda_i\}_1^k$, $\{\psi_n\}_1^r$ are nonnegative and decreasing sequences of real numbers with $k, r \in \mathbb{N}$, then

$$
\begin{aligned}
&\sum_{i=1}^{k}\sum_{n=1}^{r}\frac{S_{d,q,k,r,i,n}\left(\sum\limits_{s=1}^{i}\lambda_s\right)^{d}\left(\sum\limits_{t=1}^{n}\psi_t\right)^{q}}{(in)^{\frac{1}{2}}}\\
&\geq 2C(d,q,k,r)\left(\sum_{i=1}^{k}\left[\lambda_i\left(\sum_{s=1}^{i}\lambda_s\right)^{d-1}\right]^2(k-i+1)\right)^{\frac{1}{2}} \\
&\times\left(\sum_{n=1}^{r}\left[\psi_n\left(\sum_{t=1}^{n}\psi_t\right)^{q-1}\right]^2(r-n+1)\right)^{\frac{1}{2}},
\end{aligned}
\tag{5}
$$

where

$$
C(d,q,r,s)=\frac{1}{2}dq(kr)^{\frac{1}{2}},
$$

and

$$
\begin{aligned}
S_{d,q,k,r,i,n} &= S\left(\frac{k\sum\limits_{s=1}^{i}\left[\lambda_s\left(\sum\limits_{\tau=1}^{s}\lambda_\tau\right)^{d-1}\right]^2}{\sum\limits_{s=1}^{k}(k-s+1)\left[\lambda_s\left(\sum\limits_{\tau=1}^{s}\lambda_\tau\right)^{d-1}\right]^2}\right)\\
&\times S\left(\frac{i\left[\lambda_u\left(\sum\limits_{\tau=1}^{u}\lambda_\tau\right)^{d-1}\right]^2}{\sum\limits_{s=1}^{i}\left[\lambda_s\left(\sum\limits_{\tau=1}^{s}\lambda_\tau\right)^{d-1}\right]^2}\right)\\
&\times S\left(\frac{r\sum\limits_{t=1}^{n}\left[\psi_t\left(\sum\limits_{\tau=1}^{t}\psi_\tau\right)^{q-1}\right]^2}{\sum\limits_{t=1}^{r}(r-t+1)\left[\psi_t\left(\sum\limits_{\tau=1}^{t}\psi_\tau\right)^{q-1}\right]^2}\right)\\
&\times S\left(\frac{n\left[\psi_v\left(\sum\limits_{\tau=1}^{v}\psi_\tau\right)^{q-1}\right]^2}{\sum\limits_{t=1}^{n}\left[\psi_t\left(\sum\limits_{\tau=1}^{t}\psi_\tau\right)^{q-1}\right]^2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&S\left(\frac{i\left[\lambda_u\left(\sum\limits_{\tau=1}^{u}\lambda_\tau\right)^{d-1}\right]^2}{\sum\limits_{s=1}^{i}\left[\lambda_s\left(\sum\limits_{\tau=1}^{s}\lambda_\tau\right)^{d-1}\right]^2}\right)\\
&=\max\left\{S\left(\frac{i\left[\lambda_1\left(\sum\limits_{\tau=1}^{1}\lambda_\tau\right)^{d-1}\right]^2}{\sum\limits_{s=1}^{i}\left[\lambda_s\left(\sum\limits_{\tau=1}^{s}\lambda_\tau\right)^{d-1}\right]^2}\right)\right.\\
&\left.;S\left(\frac{i\left[\lambda_i\left(\sum\limits_{\tau=1}^{i}\lambda_\tau\right)^{d-1}\right]^2}{\sum\limits_{s=1}^{i}\left[\lambda_s\left(\sum\limits_{\tau=1}^{s}\lambda_\tau\right)^{d-1}\right]^2}\right)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&S\left(\frac{n\left[\psi_v\left(\sum_{\tau=1}^{v}\psi_\tau\right)^{q-1}\right]^2}{\sum_{t=1}^{n}\left[\psi_t\left(\sum_{\tau=1}^{t}\psi_\tau\right)^{q-1}\right]^2}\right)\\
&=\max\left\{S\left(\frac{n\left[\psi_1\left(\sum_{\tau=1}^{1}\psi_\tau\right)^{q-1}\right]^2}{\sum_{t=1}^{n}\left[\psi_t\left(\sum_{\tau=1}^{t}\psi_\tau\right)^{q-1}\right]^2}\right)\right.\\
&\left.;S\left(\frac{n\left[\psi_n\left(\sum_{\tau=1}^{n}\psi_\tau\right)^{q-1}\right]^2}{\sum_{t=1}^{n}\left[\psi_t\left(\sum_{\tau=1}^{t}\psi_\tau\right)^{q-1}\right]^2}\right)\right\}.
\end{aligned}
$$

In addition, they proved that if $\{\lambda_i\}_1^k$, $\{\varpi_n\}_1^r$ are nonnegative sequences and $\{d_i\}_1^k$, $\{q_n\}_1^r$ are positive sequences with $k, r \in \mathbb{N}$, then

$$
\begin{aligned}
&\sum_{i=1}^{k}\sum_{n=1}^{r}\frac{S_{k,r,i,n}\phi(\Lambda_i)\psi(\Omega_n)}{(in)^{\frac{1}{2}}}\\
&\geq 2N(k,r)\left(\sum_{s=1}^{k}\left[d_s\phi\left(\frac{\lambda_s}{d_s}\right)\right]^2(k-s+1)\right)^{\frac{1}{2}}\\
&\times\left(\sum_{t=1}^{r}\left[q_t\psi\left(\frac{\varpi_t}{q_t}\right)\right]^2(r-t+1)\right)^{\frac{1}{2}},
\end{aligned}
\tag{6}
$$

with

$$
N(k,r)=\frac{1}{2}\left(\sum_{i=1}^{k}\left(\frac{\phi(D_i)}{D_i}\right)^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)^{\frac{1}{2}},
$$

$$
\begin{aligned}
S_{k,r,i,n}=&S\left(\frac{\left(\sum_{s=1}^{k}\left[d_s\phi\left(\frac{\lambda_s}{d_s}\right)\right]^2(k-s+1)\right)\left(\frac{\phi(D_i)}{D_i}\right)^2}{\left(\sum_{i=1}^{k}\left(\frac{\phi(D_i)}{D_i}\right)^2\right)\left(\sum_{s=1}^{i}\left[d_s\phi\left(\frac{\lambda_s}{d_s}\right)\right]^2\right)}\right)\\
&\times S\left(\frac{\left(\sum_{t=1}^{r}\left[q_t\psi\left(\frac{\varpi_t}{q_t}\right)\right]^2(r-t+1)\right)\left(\frac{\psi(Q_n)}{Q_n}\right)^2}{\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)\left(\sum_{t=1}^{n}\left[q_t\psi\left(\frac{\varpi_t}{q_t}\right)\right]^2\right)}\right),
\end{aligned}
$$

$$
\Lambda_i=\sum_{s=1}^{i}S\left(\frac{i\left[d_s\phi\left(\frac{\lambda_s}{d_s}\right)\right]^2}{\sum_{s=1}^{i}\left[d_s\phi\left(\frac{\lambda_s}{d_s}\right)\right]^2}\right)\lambda_s,
$$

$$
\Omega_n=\sum_{t=1}^{n}S\left(\frac{n\left[q_t\psi\left(\frac{\varpi_t}{q_t}\right)\right]^2}{\sum_{t=1}^{n}\left[q_t\psi\left(\frac{\varpi_t}{q_t}\right)\right]^2}\right)\varpi_t;
$$

$$D_i = \sum_{s=1}^{i} S\left(\frac{i\left[d_s\phi\left(\frac{\lambda_s}{d_s}\right)\right]^2}{\sum_{s=1}^{i}\left[d_s\phi\left(\frac{\lambda_s}{d_s}\right)\right]^2}\right)d_s;$$

and

$$Q_n = \sum_{t=1}^{n} S\left(\frac{n\left[q_t\psi\left(\frac{\varpi_t}{q_t}\right)\right]^2}{\sum_{t=1}^{n}\left[q_t\psi\left(\frac{\varpi_t}{q_t}\right)\right]^2}\right)q_t,$$

where ϕ, ψ are nonnegative, concave and supermultiplicative functions.

In [6], the authors proved that if $\{\lambda_i\}_1^k$, $\{\varpi_n\}_1^r$ are nonnegative sequences with $k, r \in \mathbb{N}$, then

$$\begin{aligned} &\sum_{i=1}^{k}\sum_{n=1}^{r}\frac{S_{k,r,i,n}\Lambda_i\Omega_n}{(in)^{\frac{1}{2}}} \\ &\geq (kr)^{\frac{1}{2}}\left(\sum_{i=1}^{k}\lambda_i^2(k-i+1)\right)^{\frac{1}{2}}\left(\sum_{n=1}^{r}\varpi_n^2(r-n+1)\right)^{\frac{1}{2}}, \end{aligned} \tag{7}$$

with

$$S_{k,r,i,n} = S\left(\frac{\sum_{s=1}^{k}\lambda_s^2(k-s+1)}{k\left(\sum_{s=1}^{i}\lambda_s^2\right)}\right)S\left(\frac{\sum_{t=1}^{r}\varpi_t^2(r-t+1)}{r\left(\sum_{t=1}^{n}\varpi_t^2\right)}\right),$$

$$\Lambda_i = \sum_{s=1}^{i} S\left(\frac{i\lambda_s^2}{\sum_{s=1}^{i}\lambda_s^2}\right)\lambda_s \text{ and } \Omega_n = \sum_{t=1}^{n} S\left(\frac{n\varpi_t^2}{\sum_{t=1}^{n}\varpi_t^2}\right)\varpi_t.$$

Furthermore, many authors studied the inequalities of Hilbert-type, see [7–15].

In the last decades, the time scale theory was discovered which is a unification of the continuous calculus and discrete calculus. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Many authors established some dynamic inequalities of Hilbert-type on time scales. For example, in 2021, AlNemer et al. [16] studied some reversed dynamic inequalities of Hilbert-type and proved that if $a \in \mathbb{T}$, $0 \leq \alpha, \beta \leq 1$ and λ, ψ are nonnegative and decreasing functions, then the inequality

$$\begin{aligned} &\int_a^{\sigma(s)}\int_a^{\sigma(r)}\frac{S_{\alpha,\beta,t,\xi,r,s}\left(\int_a^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{\alpha}\left(\int_a^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{\beta}}{(\sigma(t)-a)^{\frac{1}{2}}(\sigma(\xi)-a)^{\frac{1}{2}}}\Delta t\Delta\xi \\ \geq\ & 2C(\alpha,\beta,r,s)\left(\int_a^{\sigma(r)}\left[\lambda(t)\left(\int_a^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{\alpha-1}\right]^2(\sigma(r)-t)\Delta t\right)^{\frac{1}{2}} \\ &\times\left(\int_a^{\sigma(s)}\left[\psi(\xi)\left(\int_a^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{\beta-1}\right]^2(\sigma(s)-\xi)\Delta\xi\right)^{\frac{1}{2}}, \end{aligned} \tag{8}$$

holds for all $r, s \in [a, \infty]_{\mathbb{T}}$, with

$$C(\alpha,\beta,r,s) = \frac{1}{2}\alpha\beta(\sigma(r)-a)^{\frac{1}{2}}(\sigma(s)-a)^{\frac{1}{2}},$$

and

$$
\begin{aligned}
S_{\alpha,\beta,t,\xi,r,s} \quad = \quad & S\left(\frac{(\sigma(t)-a)\left[\lambda(\zeta)\left(\int_a^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{\alpha-1}\right]^2}{\int_a^{\sigma(t)}\left[\lambda(\varkappa)\left(\int_a^{\sigma(\varkappa)}\lambda(\tau)\Delta\tau\right)^{\alpha-1}\right]^2\Delta\varkappa}\right)\\
&\times S\left(\frac{(\sigma(\xi)-a)\left[\psi(\eta)\left(\int_a^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{\beta-1}\right]^2}{\int_a^{\sigma(\xi)}\left[\psi(z)\left(\int_a^{\sigma(z)}\psi(\tau)\Delta\tau\right)^{\beta-1}\right]^2\Delta z}\right)\\
&\times S\left(\frac{(\sigma(r)-a)\int_a^{\sigma(t)}\left[\lambda(\zeta)\left(\int_a^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{\alpha-1}\right]^2}{\int_a^{\sigma(r)}\left[\lambda(\varkappa)\left(\int_a^{\sigma(\varkappa)}\lambda(\tau)\Delta\tau\right)^{\alpha-1}\right]^2(\sigma(r)-\varkappa)\Delta\varkappa}\right)\\
&\times S\left(\frac{(\sigma(s)-a)\int_a^{\sigma(\xi)}\left[\psi(\eta)\left(\int_a^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{\beta-1}\right]^2}{\int_a^{\sigma(s)}\left[\psi(z)\left(\int_a^{\sigma(z)}\psi(\tau)\Delta\tau\right)^{\beta-1}\right]^2(\sigma(s)-z)\Delta z}\right).
\end{aligned}
$$

Such that

$$
\begin{aligned}
& S\left(\frac{(\sigma(t)-a)\left[\lambda(\zeta)\left(\int_a^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{\alpha-1}\right]^2}{\int_a^{\sigma(t)}\left[\lambda(\varkappa)\left(\int_a^{\sigma(\varkappa)}\lambda(\tau)\Delta\tau\right)^{\alpha-1}\right]^2\Delta\varkappa}\right)\\
= \quad & \max\left\{S\left(\frac{(\sigma(t)-a)\left[\lambda(a)\left(\int_a^{\sigma(a)}\lambda(\tau)\Delta\tau\right)^{\alpha-1}\right]^2}{\int_a^{\sigma(t)}\left[\lambda(\varkappa)\left(\int_a^{\sigma(\varkappa)}\lambda(\tau)\Delta\tau\right)^{\alpha-1}\right]^2\Delta\varkappa}\right)\right.\\
& \left.;S\left(\frac{(\sigma(t)-a)\left[\lambda(t)\left(\int_a^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{\alpha-1}\right]^2}{\int_a^{\sigma(t)}\left[\lambda(\varkappa)\left(\int_a^{\sigma(\varkappa)}\lambda(\tau)\Delta\tau\right)^{\alpha-1}\right]^2\Delta\varkappa}\right)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
& S\left(\frac{(\sigma(\xi)-a)\left[\psi(\eta)\left(\int_a^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{\beta-1}\right]^2}{\int_a^{\sigma(\xi)}\left[\psi(z)\left(\int_a^{\sigma(z)}\psi(\tau)\Delta\tau\right)^{\beta-1}\right]^2\Delta z}\right)\\
= \quad & \max\left\{S\left(\frac{(\sigma(\xi)-a)\left[\psi(a)\left(\int_a^{\sigma(a)}\psi(\tau)\Delta\tau\right)^{\beta-1}\right]^2}{\int_a^{\sigma(\xi)}\left[\psi(z)\left(\int_a^{\sigma(z)}\psi(\tau)\Delta\tau\right)^{\beta-1}\right]^2\Delta z}\right)\right.\\
& \left.;S\left(\frac{(\sigma(\xi)-a)\left[\psi(\xi)\left(\int_a^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{\beta-1}\right]^2}{\int_a^{\sigma(\xi)}\left[\psi(z)\left(\int_a^{\sigma(z)}\psi(\tau)\Delta\tau\right)^{\beta-1}\right]^2\Delta z}\right)\right\},
\end{aligned}
$$

where the function $S(.)$ is the Specht ratio (see [5]) which is defined as follows:

$$S(h)=\frac{h^{1/(h-1)}}{e\log h^{1/(h-1)}},h\neq 1,\qquad S(1)=1.$$

The aim of this manuscript is to use reverse Hölder inequalities with Specht's ratio on time scales $\mathbb{T}$ to establish some new generalizations of reverse Hilbert-type inequalities. In particular, we generalize the inequality (8) by replacing the power 2 with a new power β, $\beta>1$.

The following is a breakdown of the paper's structure. In Section 2, we cover some fundamentals of time scale theory as well as several time scale lemmas that will be useful in Section 3, where we prove our findings. As specific examples (when $\mathbb{T}=\mathbb{N}$), our major results yield (5)–(7) proven by Zhao and Cheung [6]. In addition, we obtain the inequality (8) proved by AlNemer et al. [16].

2. Definitions and Basic Lemmas

A time scale $\mathbb{T}$ is defined as an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ and the forward jump operator is defined by: $\sigma(\tau):=\inf\{r\in\mathbb{T}:r>\tau\}$. The set of all such rd-continuous functions is ushered by $C_{rd}(\mathbb{T},\mathbb{R})$ and for any function $U:\mathbb{T}\to\mathbb{R}$, the notation $U^{\sigma}(\tau)$ denotes $U(\sigma(\tau))$.

The derivatives of $U\varpi$ and U/ϖ (where $\varpi\varpi^{\sigma}\neq 0$) are given by

$$(U\varpi)^{\Delta}=U^{\Delta}\varpi+U^{\sigma}\varpi^{\Delta}=U\varpi^{\Delta}+U^{\Delta}\varpi^{\sigma},\ \left(\frac{U}{\varpi}\right)^{\Delta}=\frac{U^{\Delta}\varpi-U\varpi^{\Delta}}{\varpi\varpi^{\sigma}}.$$

The integration by parts formula on $\mathbb{T}$ is

$$\int_{v_0}^{v}\lambda(\tau)\varphi^{\Delta}(\tau)\Delta\tau=[\lambda(\tau)\varphi(\tau)]_{v_0}^{v}-\int_{v_0}^{v}\lambda^{\Delta}(\tau)\varphi^{\sigma}(\tau)\Delta\tau. \tag{9}$$

The time scales chain rule is

$$(\varpi\circ\varphi)^{\Delta}(\tau)=\varpi^{'}(\varphi(\varkappa))\varphi^{\Delta}(\tau),\ \text{where}\ \varkappa\in[\tau,\sigma(\tau)],$$

where it is supposed that $\varpi:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $\varphi:\mathbb{T}\to\mathbb{R}$ is Δ-differentiable. For further information on the time scale calculus, see [17,18].

Definition 1 ([19])**.** *A function $G:J\to\mathbb{R}^{+}$ is supermultiplicative if*

$$G(\varkappa s)\geq G(\varkappa)G(s),\qquad\forall\varkappa,s\in J\subset\mathbb{R}. \tag{10}$$

Inequality (10) holds with equality if G is the identity map (i.e., $G(\varkappa)=\varkappa$). G is said to be a submultiplicative function if the last inequality has the opposite sign.

Lemma 1. *If $\varrho\in\mathbb{T}$, λ is a nonnegative rd-continuous function and $0<\gamma\leq 1$, then*

$$\left(\int_{\varrho}^{\sigma(s)}\lambda(\tau)\Delta\tau\right)^{\gamma}\geq\gamma\int_{\varrho}^{\sigma(s)}\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{\gamma-1}\lambda(\vartheta)\Delta\vartheta. \tag{11}$$

Proof. Using the time scales chain rule on the term $\int_{\varrho}^{\vartheta}\lambda(\tau)\Delta\tau$, we obtain

$$\left[\left(\int_{\varrho}^{\vartheta}\lambda(\tau)\Delta\tau\right)^{\gamma}\right]^{\Delta}=\gamma\left(\int_{\varrho}^{\zeta}\lambda(\tau)\Delta\tau\right)^{\gamma-1}\lambda(\vartheta),\quad\zeta\in[\vartheta,\sigma(\vartheta)]. \tag{12}$$

Since $\zeta\leq\sigma(\vartheta)$, then we have (note $0<\gamma\leq 1$) that

$$\left(\int_{\varrho}^{\zeta}\lambda(\tau)\Delta\tau\right)^{\gamma-1}\geq\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{\gamma-1}, \tag{13}$$

Substituting (13) into (12), we see

$$\left[\left(\int_{\varrho}^{\vartheta}\lambda(\tau)\Delta\tau\right)^{\gamma}\right]^{\Delta}\geq\gamma\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{\gamma-1}\lambda(\vartheta). \tag{14}$$

Integrating (14) over ϑ from ϱ to $\sigma(s)$, we have

$$\int_{\varrho}^{\sigma(s)}\left[\left(\int_{\varrho}^{\vartheta}\lambda(\tau)\Delta\tau\right)^{\gamma}\right]^{\Delta}\Delta\vartheta\geq\gamma\int_{\varrho}^{\sigma(s)}\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{\gamma-1}\lambda(\vartheta)\Delta\vartheta.$$

This means that

$$\left(\int_{\varrho}^{\sigma(s)}\lambda(\tau)\Delta\tau\right)^{\gamma}\geq\gamma\int_{\varrho}^{\sigma(s)}\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{\gamma-1}\lambda(\vartheta)\Delta\vartheta,$$

which is (11). □

Lemma 2 (Specht's ratio [5])**.** *Let α,β be positive numbers, $d>1$ and $1/d+1/q=1$. Then,*

$$S\left(\frac{\alpha}{\beta}\right)\alpha^{1/d}\beta^{1/q}\geq\frac{\alpha}{d}+\frac{\beta}{q}, \tag{15}$$

where

$$S(u)=\frac{u^{1/(u-1)}}{e\log u^{1/(u-1)}},u\neq 1.$$

Lemma 3 ([5])**.** *Let $S(.)$ be as defined in Lemma 2. Then, $S(l)$ is strictly decreasing for $0<l<1$ and strictly increasing for $l>1$. In addition, the following equations are true*

$$S(1)=1 \text{ and } S(l)=S(\frac{1}{l})\ \forall l>0.$$

Lemma 4 ([20], when $\alpha=1$)**.** *If $f,g\in C([\varrho,c]_{\mathbb{T}},\mathbb{R}^{+})$ such that f^{γ}, g^{ν} are Δ-integrable on $[\varrho,c]_{\mathbb{T}}$ and let $\beta>1$ and $1/\beta+1/\nu=1$, then*

$$\int_{\varrho}^{c}S\left(\frac{Yf^{\beta}(\zeta)}{Xg^{\nu}(\zeta)}\right)f(\zeta)g(\zeta)\Delta\zeta \geq\left(\int_{\varrho}^{c}f^{\beta}(\zeta)\Delta\zeta\right)^{\frac{1}{\beta}}\left(\int_{\varrho}^{c}g^{\nu}(\zeta)\Delta\zeta\right)^{\frac{1}{\nu}}, \tag{16}$$

where $X=\int_{\varrho}^{c}f^{\beta}(\zeta)\Delta\zeta$ and $Y=\int_{\varrho}^{c}g^{\nu}(\zeta)\Delta\zeta$.

Lemma 5 (Jensen's inequality)**.** *Let $\zeta_0,\zeta\in\mathbb{T}$ and $r_0,d\in\mathbb{R}$. If $\lambda\in C_{rd}([\zeta_0,\zeta]_{\mathbb{T}},\mathbb{R})$, $\varphi:[\zeta_0,\zeta]_{\mathbb{T}}\to(r_0,d)$ is rd-continuous and $\Psi:(r_0,d)\to\mathbb{R}$ is continuous and convex, then*

$$\Psi\left(\frac{1}{\int_{\zeta_0}^{\zeta}\lambda(\tau)\Delta\tau}\int_{\zeta_0}^{\zeta}\lambda(\tau)\varphi(\tau)\Delta\tau\right)\leq\frac{1}{\int_{\zeta_0}^{\zeta}\lambda(\tau)\Delta\tau}\int_{\zeta_0}^{\zeta}\lambda(\tau)\Psi(\varphi(\tau))\Delta\tau. \tag{17}$$

Lemma 6. *Let $\varrho\in\mathbb{T}$, $\lambda,\psi\geq 0$ be decreasing functions and $0<d,q\leq 1$, $\beta>1$. Then,*

$$\begin{aligned}
&S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\\
&=\max\left\{S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\varrho)\left(\int_{\varrho}^{\sigma(\varrho)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\right.\\
&\left.;S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\right\}, \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
&S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)\\
&=\max\left\{S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\varrho)\left(\int_{\varrho}^{\sigma(\varrho)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)\right.\\
&\left.;S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\xi)\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)\right\}. \qquad (19)
\end{aligned}$$

Proof. We have for $\vartheta \leq y$ that

$$\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau \leq \int_{\varrho}^{\sigma(y)}\lambda(\tau)\Delta\tau,$$

and then (where $0 < d \leq 1$),

$$\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1} \geq \left(\int_{\varrho}^{\sigma(y)}\lambda(\tau)\Delta\tau\right)^{d-1}.$$

Since λ is decreasing, we have

$$\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta} \geq \left[\lambda(y)\left(\int_{\varrho}^{\sigma(y)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta},$$

thus the function $\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}$ is decreasing. Therefore, we have for $\varrho \leq \vartheta$ that

$$\left[\lambda(\varrho)\left(\int_{\varrho}^{\sigma(\varrho)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta} \geq \left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}. \qquad (20)$$

Integrating (20) over ϑ from ϱ to $\sigma(t)$, we obtain

$$
\begin{aligned}
&(\sigma(t)-\varrho)\left[\lambda(\varrho)\left(\int_{\varrho}^{\sigma(\varrho)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta} \\
\geq\ & \int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta,
\end{aligned}
$$

and then,

$$
\frac{(\sigma(t)-\varrho)\left[\lambda(\varrho)\left(\int_{\varrho}^{\sigma(\varrho)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\geq 1. \tag{21}
$$

Since the function $\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}$ is decreasing, we obtain that

$$
\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\geq\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}.
$$

Integrating the last inequality over ϑ from ϱ to $\sigma(t)$, we have

$$
\begin{aligned}
&\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta \\
&\geq\int_{\varrho}^{\sigma(t)}\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta \\
&=(\sigma(t)-\varrho)\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta},
\end{aligned}
$$

and then,

$$
\frac{(\sigma(t)-\varrho)\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\leq 1. \tag{22}
$$

From (21) and (22), we observe

$$
\begin{aligned}
&\frac{(\sigma(t)-\varrho)\left[\lambda(\varrho)\left(\int_{\varrho}^{\sigma(\varrho)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\geq\ldots\geq 1 \\
&\geq\ldots\geq\frac{(\sigma(t)-\varrho)\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}.
\end{aligned}
$$

Since $S(.)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$, we find that one of

$$
S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\varrho)\left(\int_{\varrho}^{\sigma(\varrho)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right),
$$

and

$$S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right),$$

is maximum (where $S(1) = 1$), and it is in the form

$$\begin{aligned}
&S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\\
&=\max\left\{S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\varrho)\left(\int_{\varrho}^{\sigma(\varrho)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\right.\\
&\left.;S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\right\},
\end{aligned}$$

which is (18). Similarly, with respect to the decreasing function ψ when $0 < q \leq 1$, we have

$$\begin{aligned}
&S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)\\
&=\max\left\{S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\varrho)\left(\int_{\varrho}^{\sigma(\varrho)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)\right.\\
&\left.;S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\xi)\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)\right\},
\end{aligned}$$

which is (19). □

3. Main Results

Theorem 1. *Let* $\varrho \in \mathbb{T}$, $0 \leq d, q \leq 1$ *and* λ, ψ *be nonnegative and decreasing functions. If* $\beta > 1$, $\nu > 1$ *with* $1/\beta + 1/\nu = 1$, *then*

$$\int_{\varrho}^{\sigma(s)}\int_{\varrho}^{\sigma(r)} \frac{S_{d,q,t,\xi,r,s,\beta}\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d}\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q}}{(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\xi)-\varrho)^{\frac{1}{\nu}}}\Delta t\Delta\xi$$
$$\geq \nu C(d,q,r,s)\left(\int_{\varrho}^{\sigma(r)}\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}(\sigma(r)-t)\Delta t\right)^{\frac{1}{\beta}} \quad (23)$$
$$\times\left(\int_{\varrho}^{\sigma(s)}\left[\psi(\xi)\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}(\sigma(s)-\xi)\Delta\xi\right)^{\frac{1}{\beta}},$$

where

$$C(d,q,r,s,\nu)=\frac{1}{\nu}dq(\sigma(r)-\varrho)^{\frac{1}{\nu}}(\sigma(s)-\varrho)^{\frac{1}{\nu}},$$

and

$$S_{d,q,t,\xi,r,s,\beta}=S\left(\frac{(\sigma(r)-\varrho)\int_{\varrho}^{\sigma(t)}\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(r)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta}\right)$$
$$\times S\left(\frac{(\sigma(s)-\varrho)\int_{\varrho}^{\sigma(\xi)}\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(s)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}(\sigma(s)-y)\Delta y}\right)$$
$$\times S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)$$
$$\times S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right),$$

such that

$$S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)$$
$$=\max\left\{S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\varrho)\left(\int_{\varrho}^{\sigma(\varrho)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\right.$$
$$\left.;S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\right\},$$

and

$$S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)$$
$$=\max\left\{S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\varrho)\left(\int_{\varrho}^{\sigma(\varrho)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)\right.$$
$$\left.;S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\xi)\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)\right\}.$$

Proof. Applying (11) with $\gamma = d$, we obtain

$$\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d}\geq d\int_{\varrho}^{\sigma(t)}\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\Delta\vartheta. \tag{24}$$

Multiplying the last inequality by

$$S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right),$$

we obtain

$$S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d}$$
$$\geq d\int_{\varrho}^{\sigma(t)}S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)$$
$$\times\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\Delta\vartheta.$$

From Lemma 6, the last inequality becomes

$$S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d}$$
$$\geq d\int_{\varrho}^{\sigma(t)}S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\tag{25}$$
$$\times\,\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\Delta\vartheta.$$

Similarly, we have for ψ and $0<q\leq 1$ that

$$S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q}$$
$$\geq q\int_{\varrho}^{\sigma(\xi)}S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)\tag{26}$$
$$\times\,\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\Delta y.$$

From (25) and (26), we see that

$$S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)$$
$$\times\,S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)$$
$$\times\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d}\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q}$$
$$\geq dq\int_{\varrho}^{\sigma(t)}S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\tag{27}$$
$$\times\,\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\times 1\Delta\vartheta$$
$$\times\int_{\varrho}^{\sigma(\xi)}S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)$$
$$\times\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\times 1\Delta y.$$

Applying (16) on the right hand side of (27), we have

$$\begin{aligned}
&S\left(\frac{(\sigma(t)-\varrho)\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta}\right)\\
&\times S\left(\frac{(\sigma(\xi)-\varrho)\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y}\right)\\
&\times\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d}\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q}\\
&\geq dq(\sigma(t)-\varrho)^{\frac{1}{\nu}}\left(\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta\right)^{\frac{1}{\beta}}\\
&\times(\sigma(\xi)-\varrho)^{\frac{1}{\nu}}\left(\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y\right)^{\frac{1}{\beta}}.
\end{aligned} \tag{28}$$

Multiplying (28) by

$$\begin{aligned}
&S\left(\frac{(\sigma(r)-\varrho)\int_{\varrho}^{\sigma(t)}\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(r)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta}\right)\\
&\times S\left(\frac{(\sigma(s)-\varrho)\int_{\varrho}^{\sigma(\xi)}\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(s)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}(\sigma(s)-y)\Delta y}\right),
\end{aligned}$$

we obtain

$$\begin{aligned}
&S_{d,q,t,\xi,r,s,\beta}\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d}\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q}\\
&\geq dq(\sigma(t)-\varrho)^{\frac{1}{\nu}}S\left(\frac{(\sigma(r)-\varrho)\int_{\varrho}^{\sigma(t)}\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(r)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta}\right)\\
&\times\left(\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta\right)^{\frac{1}{\beta}}\\
&\times S\left(\frac{(\sigma(s)-\varrho)\int_{\varrho}^{\sigma(\xi)}\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(s)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}(\sigma(s)-y)\Delta y}\right)\\
&\times(\sigma(\xi)-\varrho)^{\frac{1}{\nu}}\left(\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y\right)^{\frac{1}{\beta}}.
\end{aligned} \tag{29}$$

Dividing the two sides of (29) by $(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\xi)-\varrho)^{\frac{1}{\nu}}$ and then taking the integration over t from ϱ to $\sigma(r)$ and the integration over ξ from ϱ to $\sigma(s)$, we have

$$\begin{aligned}
&\int_{\varrho}^{\sigma(s)}\int_{\varrho}^{\sigma(r)} \frac{S_{d,q,t,\xi,r,s,\beta}\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d}\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q}}{(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\xi)-\varrho)^{\frac{1}{\nu}}}\Delta t\Delta\xi \\
&\geq dq\int_{\varrho}^{\sigma(r)} S\left(\frac{(\sigma(r)-\varrho)\int_{\varrho}^{\sigma(t)}\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(r)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta}\right) \\
&\times\left(\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta\right)^{\frac{1}{\beta}}\Delta t \\
&\times\int_{\varrho}^{\sigma(s)} S\left(\frac{(\sigma(s)-\varrho)\int_{\varrho}^{\sigma(\xi)}\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}}{\int_{\varrho}^{\sigma(s)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}(\sigma(s)-y)\Delta y}\right) \\
&\times\left(\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y\right)^{\frac{1}{\beta}}\Delta\xi.
\end{aligned} \tag{30}$$

Applying (9) on the term

$$\int_{\varrho}^{\sigma(r)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta,$$

with $u(\vartheta)=(\sigma(r)-\vartheta)$ and $v^{\Delta}(\vartheta)=\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}$, we obtain

$$\begin{aligned}
&\int_{\varrho}^{\sigma(r)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta \\
&=(\sigma(r)-\vartheta)v(\vartheta)|_{\varrho}^{\sigma(r)}+\int_{\varrho}^{\sigma(r)}v^{\sigma}(\vartheta)\Delta\vartheta,
\end{aligned}$$

where $v(\vartheta)=\int_{\varrho}^{\vartheta}\left[\lambda(\theta)\left(\int_{\varrho}^{\sigma(\theta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\theta$, and then (where $v(\varrho)=0$),

$$\begin{aligned}
&\int_{\varrho}^{\sigma(r)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta \\
&=\int_{\varrho}^{\sigma(r)}\int_{\varrho}^{\sigma(\vartheta)}\left[\lambda(\theta)\left(\int_{\varrho}^{\sigma(\theta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\theta\Delta\vartheta.
\end{aligned} \tag{31}$$

Similarly, we see that

$$\begin{aligned}
&\int_{\varrho}^{\sigma(s)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}(\sigma(s)-y)\Delta y \\
&=\int_{\varrho}^{\sigma(s)}\int_{\varrho}^{\sigma(y)}\left[\psi(\theta)\left(\int_{\varrho}^{\sigma(\theta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta\theta\Delta y.
\end{aligned} \tag{32}$$

Substituting (31) and (32) into (30) and, then, by applying (16), we observe that

$$\begin{aligned}
&\int_{\varrho}^{\sigma(s)}\int_{\varrho}^{\sigma(r)} \frac{S_{d,q,t,\xi,r,s,\beta}\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d}\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q}}{(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\xi)-\varrho)^{\frac{1}{\nu}}}\Delta t\Delta\xi \\
&\geq dq\int_{\varrho}^{\sigma(r)} S\left(\frac{(\sigma(r)-\varrho)\int_{\varrho}^{\sigma(t)}\left[\lambda(\zeta)\left(\int_{\varrho}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\zeta}{\int_{\varrho}^{\sigma(r)}\int_{\varrho}^{\sigma(\vartheta)}\left[\lambda(\theta)\left(\int_{\varrho}^{\sigma(\theta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\theta\Delta\vartheta}\right) \\
&\times\left(\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta\right)^{\frac{1}{\beta}}\times 1\Delta t \\
&\times\int_{\varrho}^{\sigma(s)} S\left(\frac{(\sigma(s)-\varrho)\int_{\varrho}^{\sigma(\xi)}\left[\psi(\eta)\left(\int_{\varrho}^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta\eta}{\int_{\varrho}^{\sigma(s)}\int_{\varrho}^{\sigma(y)}\left[\psi(\theta)\left(\int_{\varrho}^{\sigma(\theta)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta\theta\Delta y}\right) \\
&\times\left(\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y\right)^{\frac{1}{\beta}}\times 1\Delta\xi \\
&\geq dq(\sigma(r)-\varrho)^{\frac{1}{\nu}}\left(\int_{\varrho}^{\sigma(r)}\int_{\varrho}^{\sigma(t)}\left[\lambda(\vartheta)\left(\int_{\varrho}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}\Delta\vartheta\Delta t\right)^{\frac{1}{\beta}} \\
&\times(\sigma(s)-\varrho)^{\frac{1}{\nu}}\left(\int_{\varrho}^{\sigma(s)}\int_{\varrho}^{\sigma(\xi)}\left[\psi(y)\left(\int_{\varrho}^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}\Delta y\Delta\xi\right)^{\frac{1}{\beta}}.
\end{aligned} \tag{33}$$

From (31)–(33), the last inequality becomes

$$\begin{aligned}
&\int_{\varrho}^{\sigma(s)}\int_{\varrho}^{\sigma(r)} \frac{S_{d,q,t,\xi,r,s,\beta}\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d}\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q}}{(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\xi)-\varrho)^{\frac{1}{\nu}}}\Delta t\Delta\xi \\
&\geq dq(\sigma(r)-\varrho)^{\frac{1}{\nu}}\left(\int_{\varrho}^{\sigma(r)}\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}(\sigma(r)-t)\Delta t\right)^{\frac{1}{\beta}} \\
&\times(\sigma(s)-\varrho)^{\frac{1}{\nu}}\left(\int_{\varrho}^{\sigma(s)}\left[\psi(\xi)\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}(\sigma(s)-\xi)\Delta\xi\right)^{\frac{1}{\beta}} \\
&=\nu C(d,q,r,s)\left(\int_{\varrho}^{\sigma(r)}\left[\lambda(t)\left(\int_{\varrho}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{d-1}\right]^{\beta}(\sigma(r)-t)\Delta t\right)^{\frac{1}{\beta}} \\
&\times\left(\int_{\varrho}^{\sigma(s)}\left[\psi(\xi)\left(\int_{\varrho}^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{q-1}\right]^{\beta}(\sigma(s)-\xi)\Delta\xi\right)^{\frac{1}{\beta}},
\end{aligned}$$

which is (23). □

Remark 1. *If $\nu = \beta = 2$, we obtain (8) proved by AlNemer et al. [16].*

Remark 2. *When $\mathbb{T} = \mathbb{N}$, $\varrho = 1$ and $\nu = \beta = 2$, in Theorem 1, we obtain (5) as demonstrated in [6].*

Remark 3. *As a special case of Theorem 1 (when $\mathbb{T} = \mathbb{R}$), we have that if $0 \leq d, q \leq 1$ and λ, ψ are nonnegative and decreasing functions and assume that $\beta > 1$, $\nu > 1$ with $1/\beta + 1/\nu = 1$, then*

$$\int_0^s \int_0^r \frac{S_{d,q,t,\xi,r,s,\beta}\left(\int_0^t \lambda(\tau)d\tau\right)^d \left(\int_0^\xi \psi(\tau)d\tau\right)^q}{t^{\frac{1}{\nu}}\xi^{\frac{1}{\nu}}} dtd\xi$$

$$\geq \nu C(d,q,r,s)\left(\int_0^r \left[\lambda(t)\left(\int_0^t \lambda(\tau)d\tau\right)^{d-1}\right]^\beta (r-t)dt\right)^{\frac{1}{\beta}}$$

$$\times \left(\int_0^s \left[\psi(\xi)\left(\int_0^\xi \psi(\tau)d\tau\right)^{q-1}\right]^\beta (s-\xi)d\xi\right)^{\frac{1}{\beta}},$$

where

$$C(d,q,r,s,\nu) = \frac{1}{\nu} dq r^{\frac{1}{\nu}} s^{\frac{1}{\nu}},$$

and

$$S_{d,q,t,\xi,r,s,\beta} = S\left(\frac{r\int_0^t \left[\lambda(\zeta)\left(\int_0^\zeta \lambda(\tau)d\tau\right)^{d-1}\right]^\beta d\zeta}{\int_0^r \left[\lambda(\vartheta)\left(\int_0^\vartheta \lambda(\tau)d\tau\right)^{d-1}\right]^\beta (r-\vartheta)d\vartheta}\right)$$

$$\times S\left(\frac{s\int_0^\xi \left[\psi(\eta)\left(\int_0^\eta \psi(\tau)d\tau\right)^{q-1}\right]^\beta d\eta}{\int_0^s \left[\psi(y)\left(\int_0^y \psi(\tau)d\tau\right)^{q-1}\right]^\beta (s-y)dy}\right)$$

$$\times S\left(\frac{t\left[\lambda(\zeta)\left(\int_0^\zeta \lambda(\tau)d\tau\right)^{d-1}\right]^\beta}{\int_0^t \left[\lambda(\vartheta)\left(\int_0^\vartheta \lambda(\tau)d\tau\right)^{d-1}\right]^\beta d\vartheta}\right)$$

$$\times S\left(\frac{\xi\left[\psi(\eta)\left(\int_0^\eta \psi(\tau)d\tau\right)^{q-1}\right]^\beta}{\int_0^\xi \left[\psi(y)\left(\int_0^y \psi(\tau)d\tau\right)^{q-1}\right]^\beta dy}\right),$$

Theorem 2. *Let $\varrho \in \mathbb{T}$, λ, ϖ be nonnegative and d, q be positive functions. If $\phi, \psi \geq 0$ are concave and supermultiplicative functions and $\beta > 1$, $\nu > 1$ with $1/\beta + 1/\nu = 1$, then*

$$\int_\varrho^{\sigma(s)} \int_\varrho^{\sigma(r)} \frac{S_{t,r,s,\zeta,\nu,\beta}\phi(\Lambda(t))\psi(\Omega(\zeta))}{(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\zeta)-\varrho)^{\frac{1}{\nu}}} \Delta t \Delta\zeta$$

$$\geq \nu M(r,s,\nu)\left(\int_\varrho^{\sigma(r)} \left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^\beta (\sigma(r)-\vartheta)\Delta\vartheta\right)^{\frac{1}{\beta}}$$

$$\times \left(\int_\varrho^{\sigma(s)} \left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^\beta (\sigma(s)-y)\Delta y\right)^{\frac{1}{\beta}}, \tag{34}$$

holds for all $r, s \in [\varrho, \infty]_{\mathbb{T}}$, with

$$M(r,s,\nu) = \frac{1}{\nu}\left(\int_\varrho^{\sigma(r)} \left(\frac{\phi(D(t))}{D(t)}\right)^\nu \Delta t\right)^{\frac{1}{\nu}} \left(\int_\varrho^{\sigma(s)} \left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^\nu \Delta\zeta\right)^{\frac{1}{\nu}},$$

$$S_{t,r,s,\zeta,\nu,\beta} = S\left(\frac{\left(\int_{\varrho}^{\sigma(r)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta\right)\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(r)}\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}\Delta t\right)\left(\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta\right)}\right)$$
$$\times S\left(\frac{\left(\int_{\varrho}^{\sigma(s)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}(\sigma(s)-y)\Delta y\right)\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(s)}\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}\Delta\zeta\right)\left(\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y\right)}\right),$$

$$\Lambda(t) = \int_{\varrho}^{\sigma(t)} S\left(\frac{(\sigma(t)-\varrho)\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta}\right)\lambda(\vartheta)\Delta\vartheta,$$

$$\Omega(\zeta) = \int_{\varrho}^{\sigma(\zeta)} S\left(\frac{(\sigma(\zeta)-\varrho)\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}}{\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y}\right)\varpi(y)\Delta y,$$

$$D(t) = \int_{\varrho}^{\sigma(t)} S\left(\frac{(\sigma(t)-\varrho)\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta}\right)d(\vartheta)\Delta\vartheta,$$

and

$$Q(\zeta) = \int_{\varrho}^{\sigma(\zeta)} S\left(\frac{(\sigma(\zeta)-\varrho)\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}}{\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y}\right)q(y)\Delta y.$$

Proof. Using the fact that ϕ is a supermultiplicative function, applying Jensen's inequality and then applying (16), we find

$$\phi(\Lambda(t)) = \phi\left(\frac{D(t)\int_{\varrho}^{\sigma(t)} S\left(\frac{(\sigma(t)-\varrho)\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta}\right)d(\vartheta)\lambda(\vartheta)/d(\vartheta)\Delta\vartheta}{\int_{\varrho}^{\sigma(t)} S\left(\frac{(\sigma(t)-\varrho)\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta}\right)d(\vartheta)\Delta\vartheta}\right)$$
$$\geq \phi(D(t))\phi\left(\frac{\int_{\varrho}^{\sigma(t)} S\left(\frac{(\sigma(t)-\varrho)\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta}\right)d(\vartheta)\left[\frac{\lambda(\vartheta)}{d(\vartheta)}\right]\Delta\vartheta}{\int_{\varrho}^{\sigma(t)} S\left(\frac{(\sigma(t)-\varrho)\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta}\right)d(\vartheta)\Delta\vartheta}\right)$$
$$\geq \frac{\phi(D(t))}{D(t)}\int_{\varrho}^{\sigma(t)} S\left(\frac{(\sigma(t)-\varrho)\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}}{\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta}\right)d(\vartheta)\phi\left[\frac{\lambda(\vartheta)}{d(\vartheta)}\right]\times 1\Delta\vartheta$$
$$\geq \frac{\phi(D(t))}{D(t)}(\sigma(t)-\varrho)^{\frac{1}{\nu}}\left(\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta\right)^{\frac{1}{\beta}}. \tag{35}$$

Similarly, we can obtain

$$\psi(\Omega(\zeta)) \geq \frac{\psi(Q(\zeta))}{Q(\zeta)}(\sigma(\zeta)-\varrho)^{\frac{1}{\nu}}\left(\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y\right)^{\frac{1}{\beta}}. \tag{36}$$

Multiplying both sides of (35) and (36), respectively, by

$$S\left(\frac{\left(\int_{\varrho}^{\sigma(r)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta\right)\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(r)}\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}\Delta t\right)\left(\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta\right)}\right),$$

and

$$S\left(\frac{\left(\int_{\varrho}^{\sigma(s)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}(\sigma(s)-y)\Delta y\right)\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(s)}\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}\Delta\zeta\right)\left(\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y\right)}\right),$$

and then multiplying these inequalities, we obtain

$$\begin{aligned}
&S\left(\frac{\left(\int_{\varrho}^{\sigma(r)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta\right)\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(r)}\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}\Delta t\right)\left(\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta\right)}\right)\phi(\Lambda(t))\\
&\times S\left(\frac{\left(\int_{\varrho}^{\sigma(s)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}(\sigma(s)-y)\Delta y\right)\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(s)}\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}\Delta\zeta\right)\left(\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y\right)}\right)\psi(\Omega(\zeta))\\
&\geq\frac{\phi(D(t))}{D(t)}(\sigma(t)-\varrho)^{\frac{1}{\nu}}\left(\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta\right)^{\frac{1}{\beta}}\\
&\times S\left(\frac{\left(\int_{\varrho}^{\sigma(r)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta\right)\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(r)}\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}\Delta t\right)\left(\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta\right)}\right)\\
&\times\frac{\psi(Q(\zeta))}{Q(\zeta)}(\sigma(\zeta)-\varrho)^{\frac{1}{\nu}}\left(\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y\right)^{\frac{1}{\beta}}\\
&\times S\left(\frac{\left(\int_{\varrho}^{\sigma(s)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}(\sigma(s)-y)\Delta y\right)\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(s)}\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}\Delta\zeta\right)\left(\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y\right)}\right).
\end{aligned}\tag{37}$$

By dividing the two sides of (37) on $(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\zeta)-\varrho)^{\frac{1}{\nu}}$ and then taking the integration over ζ from ϱ to $\sigma(s)$ and, then, the integration over t from ϱ to $\sigma(r)$, we obtain

$$\begin{aligned}
&\int_{\varrho}^{\sigma(s)}\int_{\varrho}^{\sigma(r)}\frac{S_{t,r,s,\zeta,\nu,\beta}\phi(\Lambda(t))\psi(\Omega(\zeta))}{(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\zeta)-\varrho)^{\frac{1}{\nu}}}\Delta t\Delta\zeta \\
&\geq \int_{\varrho}^{\sigma(r)} S\left(\frac{\left(\int_{\varrho}^{\sigma(r)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta\right)\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(r)}\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}\Delta t\right)\left(\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta\right)}\right) \\
&\times \frac{\phi(D(t))}{D(t)}\left(\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta\right)^{\frac{1}{\beta}}\Delta t \\
&\times \int_{\varrho}^{\sigma(s)} S\left(\frac{\left(\int_{\varrho}^{\sigma(s)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}(\sigma(s)-y)\Delta y\right)\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(s)}\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}\Delta\zeta\right)\left(\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y\right)}\right) \\
&\times \frac{\psi(Q(\zeta))}{Q(\zeta)}\left(\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y\right)^{\frac{1}{\beta}}\Delta\zeta.
\end{aligned} \tag{38}$$

By using the integration by parts, we can see that

$$\begin{aligned}
&\int_{\varrho}^{\sigma(r)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta \\
&= \int_{\varrho}^{\sigma(r)}\int_{\varrho}^{\sigma(\vartheta)}\left[d(\theta)\phi\left(\frac{\lambda(\theta)}{d(\theta)}\right)\right]^{\beta}\Delta\theta\Delta\vartheta.
\end{aligned} \tag{39}$$

In addition, we can obtain that

$$\begin{aligned}
&\int_{\varrho}^{\sigma(s)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}(\sigma(s)-y)\Delta y \\
&= \int_{\varrho}^{\sigma(s)}\int_{\varrho}^{\sigma(y)}\left[q(\theta)\psi\left(\frac{\varpi(\theta)}{q(\theta)}\right)\right]^{\beta}\Delta\theta\Delta y.
\end{aligned} \tag{40}$$

Substituting (39) and (40) into (38), we have

$$\begin{aligned}
&\int_{\varrho}^{\sigma(s)}\int_{\varrho}^{\sigma(r)}\frac{S_{t,r,s,\zeta,\nu,\beta}\phi(\Lambda(t))\psi(\Omega(\zeta))}{(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\zeta)-\varrho)^{\frac{1}{\nu}}}\Delta t\Delta\zeta \\
&\geq \int_{\varrho}^{\sigma(r)} S\left(\frac{\left(\int_{\varrho}^{\sigma(r)}\int_{\varrho}^{\sigma(\vartheta)}\left[d(\theta)\phi\left(\frac{\lambda(\theta)}{d(\theta)}\right)\right]^{\beta}\Delta\theta\Delta\vartheta\right)\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(r)}\left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}\Delta t\right)\left(\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta\right)}\right) \\
&\times \frac{\phi(D(t))}{D(t)}\left(\int_{\varrho}^{\sigma(t)}\left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}\Delta\vartheta\right)^{\frac{1}{\beta}}\Delta t \\
&\times \int_{\varrho}^{\sigma(s)} S\left(\frac{\left(\int_{\varrho}^{\sigma(s)}\int_{\varrho}^{\sigma(y)}\left[q(\theta)\psi\left(\frac{\varpi(\theta)}{q(\theta)}\right)\right]^{\beta}\Delta\theta\Delta y\right)\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}}{\left(\int_{\varrho}^{\sigma(s)}\left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}\Delta\zeta\right)\left(\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y\right)}\right) \\
&\times \frac{\psi(Q(\zeta))}{Q(\zeta)}\left(\int_{\varrho}^{\sigma(\zeta)}\left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}\Delta y\right)^{\frac{1}{\beta}}\Delta\zeta.
\end{aligned} \tag{41}$$

Applying (16) with $\gamma = \nu = \nu$ on the R.H.S. of (41), we have

$$\begin{aligned}
&\int_{\varrho}^{\sigma(s)} \int_{\varrho}^{\sigma(r)} \frac{S_{t,r,s,\zeta,\nu,\beta}\phi(\Lambda(t))\psi(\Omega(\zeta))}{(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\zeta)-\varrho)^{\frac{1}{\nu}}}\Delta t\Delta\zeta \\
&\geq \left(\int_{\varrho}^{\sigma(r)} \left(\frac{\phi(D(t))}{D(t)}\right)^{\nu}\Delta t\right)^{\frac{1}{\nu}} \left(\int_{\varrho}^{\sigma(r)} \int_{\varrho}^{\sigma(\vartheta)} \left[d(\theta)\phi\left(\frac{\lambda(\theta)}{d(\theta)}\right)\right]^{\beta}\Delta\theta\Delta\vartheta\right)^{\frac{1}{\beta}} \\
&\times \left(\int_{\varrho}^{\sigma(s)} \left(\frac{\psi(Q(\zeta))}{Q(\zeta)}\right)^{\nu}\Delta\zeta\right)^{\frac{1}{\nu}} \left(\int_{\varrho}^{\sigma(s)} \int_{\varrho}^{\sigma(y)} \left[q(\theta)\psi\left(\frac{\varpi(\theta)}{q(\theta)}\right)\right]^{\beta}\Delta\theta\Delta y\right)^{\frac{1}{\beta}} \qquad (42) \\
&= \nu M(r,s,\nu)\left(\int_{\varrho}^{\sigma(r)} \int_{\varrho}^{\sigma(\vartheta)} \left[d(\theta)\phi\left(\frac{\lambda(\theta)}{d(\theta)}\right)\right]^{\beta}\Delta\theta\Delta\vartheta\right)^{\frac{1}{\beta}} \\
&\times \left(\int_{\varrho}^{\sigma(s)} \int_{\varrho}^{\sigma(y)} \left[q(\theta)\psi\left(\frac{\varpi(\theta)}{q(\theta)}\right)\right]^{\beta}\Delta\theta\Delta y\right)^{\frac{1}{\beta}}.
\end{aligned}$$

From (39) and (40), the Inequality (42) becomes

$$\begin{aligned}
&\int_{\varrho}^{\sigma(s)} \int_{\varrho}^{\sigma(r)} \frac{S_{t,r,s,\zeta,\nu,\beta}\phi(\Lambda(t))\psi(\Omega(\zeta))}{(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\zeta)-\varrho)^{\frac{1}{\nu}}}\Delta t\Delta\zeta \\
&\geq \nu M(r,s,\nu)\left(\int_{\varrho}^{\sigma(r)} \left[d(\vartheta)\phi\left(\frac{\lambda(\vartheta)}{d(\vartheta)}\right)\right]^{\beta}(\sigma(r)-\vartheta)\Delta\vartheta\right)^{\frac{1}{\beta}} \\
&\times \left(\int_{\varrho}^{\sigma(s)} \left[q(y)\psi\left(\frac{\varpi(y)}{q(y)}\right)\right]^{\beta}(\sigma(s)-y)\Delta y\right)^{\frac{1}{\beta}},
\end{aligned}$$

which is (34). □

Remark 4. *If $\mathbb{T} = \mathbb{N}$, $\varrho = 1$ and $\nu = \beta = 2$, in Theorem 2, then we obtain (6) as demonstrated in [6].*

By putting $\phi(\vartheta) = \vartheta$ and $\psi(y) = y$ in Theorem 2, we have the following theorem.

Theorem 3. *Assume that $\varrho \in \mathbb{T}$ and λ, ϖ are nonnegative functions and $\beta > 1$, $\nu > 1$ with $1/\beta + 1/\nu = 1$. Then, for all $r, s \in [\varrho, \infty]_{\mathbb{T}}$, we have*

$$\begin{aligned}
&\int_{\varrho}^{\sigma(s)} \int_{\varrho}^{\sigma(r)} \frac{S_{t,r,s,\zeta,\nu,\beta}\Lambda(t)\Omega(\zeta)}{(\sigma(t)-\varrho)^{\frac{1}{\nu}}(\sigma(\zeta)-\varrho)^{\frac{1}{\nu}}}\Delta t\Delta\zeta \\
&\geq \nu M(r,s,\nu)\left(\int_{\varrho}^{\sigma(r)} \lambda^{\beta}(\vartheta)(\sigma(r)-\vartheta)\Delta\vartheta\right)^{\frac{1}{\beta}} \\
&\times \left(\int_{\varrho}^{\sigma(s)} \varpi^{\beta}(y)(\sigma(s)-y)\Delta y\right)^{\frac{1}{\beta}},
\end{aligned}$$

where

$$M(r,s,\nu) = \frac{1}{\nu}(\sigma(r)-\varrho)^{\frac{1}{\nu}}(\sigma(s)-\varrho)^{\frac{1}{\nu}},$$

$$S_{t,r,s,\zeta,\nu,\beta} = S\left(\frac{\int_{\varrho}^{\sigma(r)} \lambda^{\beta}(\vartheta)(\sigma(r)-\vartheta)\Delta\vartheta}{(\sigma(r)-\varrho)\left(\int_{\varrho}^{\sigma(t)} \lambda^{\beta}(\vartheta)\Delta\vartheta\right)}\right) S\left(\frac{\left(\int_{\varrho}^{\sigma(s)} \varpi^{\beta}(y)(\sigma(s)-y)\Delta y\right)}{(\sigma(s)-\varrho)\left(\int_{\varrho}^{\sigma(\zeta)} \varpi^{\beta}(y)\Delta y\right)}\right),$$

$$\Lambda(t) = \int_{\varrho}^{\sigma(t)} S\left(\frac{(\sigma(t)-\varrho)\lambda^{\beta}(\vartheta)}{\int_{\varrho}^{\sigma(t)} \lambda^{\beta}(\vartheta)\Delta\vartheta}\right)\lambda(\vartheta)\Delta\vartheta,$$

$$\Omega(\zeta) = \int_{\varrho}^{\sigma(\zeta)} S\left(\frac{(\sigma(\zeta)-\varrho)\varpi^{\beta}(y)}{\int_{\varrho}^{\sigma(\zeta)} \varpi^{\beta}(y)\Delta y}\right)\varpi(y)\Delta y,$$

$$D(t) = \int_{\varrho}^{\sigma(t)} S\left(\frac{(\sigma(t)-\varrho)\lambda^{\beta}(\vartheta)}{\int_{\varrho}^{\sigma(t)} \lambda^{\beta}(\vartheta)\Delta\vartheta}\right)d(\vartheta)\Delta\vartheta,$$

and

$$Q(\zeta) = \int_{\varrho}^{\sigma(\zeta)} S\left(\frac{(\sigma(\zeta)-\varrho)\varpi^{\beta}(y)}{\int_{\varrho}^{\sigma(\zeta)} \varpi^{\beta}(y)\Delta y}\right)q(y)\Delta y.$$

Remark 5. *As a special case of Theorem 3, when $\mathbb{T} = \mathbb{N}$, $\varrho = 1$ and $\nu = \beta = 2$, we obtain (7) as was proved by Zhao and Cheung [6].*

4. Conclusions

In this paper, we establish some new generalizations of reverse Hilbert-type inequalities by applying reverse Hölder inequalities with the Specht ratio function on time scales. We generalize a number of those inequalities to a general time-scale measure space. In addition to this, in order to obtain some new inequalities as special cases, we also extend our inequalities to a discrete and continuous calculus. In future work, we will continue to generalize more fractional dynamic inequalities by using Specht's ratio, Kantorovich's ratio and n-tuple fractional integral. In particular, such inequalities can be introduced by using fractional integrals and fractional derivatives of the Riemann–Liouville-type on time scales. It will also be very interesting to introduce such inequalities in quantum calculations.

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