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Qualitative Study for a Delay Quadratic Functional Integro-Differential Equation of Arbitrary (Fractional) Orders

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Abstract: Symmetry analysis has been applied to solve many differential equations, although determining the symmetries can be computationally intensive compared to other solution methods. In this work, we study some operators which keep the set of solutions invariant. We discuss the existence of solutions for two initial value problems of a delay quadratic functional integro-differential equation of arbitrary (fractional) orders and its corresponding integer orders equation. The existence of the maximal and the minimal solutions is proved. The sufficient condition for the uniqueness of the solutions is given. The continuous dependence of the unique solution on some data is studied. The continuation of the arbitrary (fractional) orders problem to the integer order problem is investigated.

Keywords: quadratic functional integral equation; existence of solutions; maximal and minimal solutions; continuous dependence; continuation properties

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1. Introduction

Differential and integral equations of fractional order have been investigated in many literature studies and monographs [1–7].

Quadratic integral equations have achieved high attention because of their useful application and problems concerning the real world. These types of equations have been studied by many authors and in different classes, see [8–20]. Each of these monographs contains existence results, but their main objectives were to present special methods or techniques and results concerning various existences for certain quadratic integral equations.

In [21], an infinite system of singular integral equations was discussed. In [22], some integro-differential equations of fractional orders involving Carathéodory nonlinearities were studied. In [18], the existence of at least a positive nondecreasing solution for an initial value problem of a quadratic integro-differential equation by applying the technique of measure of noncompactness was established.

Recently, the existence results for fractional order quadratic functional integro-differential equation were studied and some attractivity results were obtained [23].

Consider the two initial value problems of the delay quadratic functional integro-differential equation of arbitrary (fractional) orders

$$\frac{dx}{dt} = f\left(t, D^{\alpha}x(t). \int_0^{\phi(t)} g(s, x(s))ds\right), \qquad a.e.\ \ t \in (0,1] \tag{1}$$

and its corresponding integer orders equation

$$\frac{dx}{dt} = f\left(t, \frac{dx}{dt}. \int_0^{\phi(t)} g(s, x(s))ds\right), \qquad t \in (0,1] \tag{2}$$

with the initial data

$$x(0) = x_0, \tag{3}$$

where D^α is the Caputo fractional derivative of order $\alpha \in (0, 1)$.

Here we are concerned with the initial value problem of the delay quadratic functional integro-differential equation of arbitrary (fractional) orders (1) and (3) and its corresponding integer orders Equations (2) and (3). The existence of solutions is proved. The maximal and the minimal solutions are studied. Next, the sufficient condition for the uniqueness of the solution is given. The continuous dependence of the unique solution on the initial data x_0, the function g and on the delay function ϕ are studied.

Finally, the necessary condition for the continuation as $\alpha \to 1$ of the problem (1) with (3) to the initial value problem of the integer-orders Equations (2) and (3) is studied.

2. Existence of Solution

Let $I = [0, 1]$ and suppose the following conditions:

(i) $\phi : I \to I$, $\phi(t) \leq t$ is continuous and increasing.

(ii) $f : I \times R \to R$ is measurable in $t \in I$ for any $x \in R$ and continuous in $x \in R$ for all $t \in I$. Moreover, there exist a bounded measurable function $v : I \to R$ and a positive constant b_1 such that

$$|f(t, x)| \leq |v(t)| + b_1|x| \leq f^* + b_1|x|, \quad f^* = \sup_{t \in I} |v(t)|.$$

(iii) $g : I \times R \to R$ is measurable in $t \in I$ for any $x \in R$ and continuous in $x \in R$ for all $t \in I$. Moreover, there exists a bounded measurable function $m : I \to R$ and a positive constant b_2 such that

$$|g(t, x)| \leq |m(t)| + b_2|x| \leq a + b_2|x|, \quad a = \sup_{t \in I} |m(t)|.$$

(iv) There exists a positive root r_α of the algebraic equation

$$\frac{b_1 b_2 r_\alpha^2}{\Gamma(2-\alpha)\Gamma(1+\alpha)} + \left(\frac{b_1 a + b_1 b_2 |x_0|}{\Gamma(2-\alpha)} - 1\right) r_\alpha + \frac{f^*}{\Gamma(2-\alpha)} = 0. \tag{4}$$

Lemma 1. *Problem (1) with (3) is equivalent to the integral equation*

$$x(t) = x_0 + I^\alpha y(t) \tag{5}$$

where y is the solution of the integral equation

$$y(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau) d\theta\right) ds. \tag{6}$$

Proof. Let x be a solution of (1) with (3). Operating by $I^{1-\alpha}$ on both sides of the Equation (1); we can obtain

$$D^\alpha x(t) = I^{1-\alpha} \frac{dx}{dt} = I^{1-\alpha} f(t, D^\alpha x(t). \int_0^{\phi(t)} g(s, x(s)) ds).$$

Let $D^\alpha x(t) = y(t)$; we obtain

$$x(t) = x_0 + I^\alpha y(t)$$

and

$$y(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)d\tau)d\theta\right) ds.$$

Let $y \in C(I)$ be a solution of (6); then

$$x(t) = x_0 + I^\alpha y(t) = x_0 + I^\alpha I^{1-\alpha} f(t, y(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta)ds)$$
$$= x_0 + \int_0^t f(s, D^\alpha x(s). \int_0^{\phi(s)} g(\theta, x(\theta))d\theta)ds$$

and

$$\frac{dx}{dt} = f(t, D^\alpha x(t). \int_0^{\phi(t)} g(s, x(s))ds), \quad a.e.\ t \in (0, 1]$$

with $x(0) = x_0$ □

Now, we have the following existences theorem.

Theorem 1. *Let the assumptions (i)–(iv) be satisfied; then problem (1) with (3) has at least one solution $x \in C(I)$.*

Proof. Let Q_{r_α} be the closed ball

$$Q_{r_\alpha} = \{y \in C(I) : \|y\| \le r_\alpha\},\ \ r_\alpha = \frac{1}{\Gamma(2-\alpha)}(f^* + b_1 a r_\alpha + b_1 b_2 |x_0| r_\alpha + \frac{b_1 b_2 r_\alpha^2}{\Gamma(1+\alpha)})$$

and the operator F

$$Fy(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)d\tau)d\theta\right) ds.$$

Now, let $y \in Q_{r_\alpha}$; then

$$\begin{aligned}
|Fy(t)| &= \left| \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)d\tau)d\theta\right) ds \right| \\
&\le \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \left(f^* + b_1 |y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)d\tau)d\theta|\right) ds \\
&\le \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \left(f^* + b_1 \|y\|.(a + b_2|x_0| + \frac{b_2\|y\|}{\Gamma(1+\alpha)})\right) ds \\
&\le \frac{1}{\Gamma(2-\alpha)}(f^* + b_1 a r_\alpha + b_1 b_2 |x_0| r_\alpha + \frac{b_1 b_2 r_\alpha^2}{\Gamma(1+\alpha)}) = r_\alpha
\end{aligned}$$

and

$$\|Fy\| \le \frac{1}{\Gamma(2-\alpha)}(f^* + b_1 a r_\alpha + b_1 b_2 |x_0| r_\alpha + \frac{b_1 b_2 r_\alpha^2}{\Gamma(1+\alpha)}) = r_\alpha.$$

This proves that $F : Q_{r_\alpha} \to Q_{r_\alpha}$ and the class of functions $\{Fy\}$ is uniformly bounded on Q_{r_α}.

Now, let $y \in Q_{r_\alpha}$ and $t_1, t_2 \in I$, such that $t_2 > t_1$ and $| t_1 - t_2 | \leq \delta$; then

$$\begin{aligned}
|Fy(t_2) - Fy(t_1)| &= \Big| \int_0^{t_2} \frac{(t_2 - s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau) d\theta\Big) ds \\
&- \int_0^{t_1} \frac{(t_1 - s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau) d\theta\Big) ds \Big| \\
&\leq \Big| \int_0^{t_1} \frac{(t_2 - s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau) d\theta\Big) ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau) d\theta\Big) ds \\
&- \int_0^{t_1} \frac{(t_1 - s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau) d\theta\Big) ds \Big| \\
&\leq \Big| \int_0^{t_1} \frac{(t_2 - s)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{(t_1 - s)^{-\alpha}}{\Gamma(1-\alpha)} (f^* + b_1 a r_\alpha + b_1 b_2 |x_0| r_\alpha + \frac{b_1 b_2 r_\alpha^2}{\Gamma(1+\alpha)}) ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{-\alpha}}{\Gamma(1-\alpha)} (f^* + b_1 a r_\alpha + b_1 b_2 |x_0| r_\alpha + \frac{b_1 b_2 r_\alpha^2}{\Gamma(1+\alpha)}) ds \Big| \\
&\leq \int_0^{t_1} \Big| \frac{(t_2 - s)^{\alpha} - (t_1 - s)^{\alpha}}{\Gamma(1-\alpha)(t_1 - s)^{\alpha}(t_2 - s)^{\alpha}} \Big| (f^* + b_1 a r_\alpha + b_1 b_2 |x_0| r_\alpha + \frac{b_1 b_2 r_\alpha^2}{\Gamma(1+\alpha)}) ds \\
&+ \int_{t_1}^{t_2} \Big| \frac{(t_2 - s)^{\alpha}}{\Gamma(1-\alpha)(t_2 - s)^{\alpha}} \Big| (f^* + b_1 a r_\alpha + b_1 b_2 |x_0| r_\alpha + \frac{b_1 b_2 r_\alpha^2}{\Gamma(1+\alpha)}) ds.
\end{aligned}$$

This means that the class of functions $\{Fy\}$ is equicontinuous on Q_{r_α} and by the Arzela–Ascoli Theorem [13], the operator F is relatively compact.

Now, let $\{y_n\} \subset Q_{r_\alpha}$, and $y_n \to y$; then

$$Fy_n(t) = \int_0^t \frac{(t - s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y_n(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_n(\tau) d\tau) d\theta\Big) ds$$

and

$$\lim_{n\to\infty} Fy_n(t) = \lim_{n\to\infty} \int_0^t \frac{(t - s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y_n(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_n(\tau) d\tau) d\theta\Big) ds.$$

Applying the Lebesgue dominated convergence theorem [13], then from our assumptions we get

$$\begin{aligned}
\lim_{n\to\infty} Fy_n(t) &= \int_0^t \frac{(t - s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, \lim_{n\to\infty} y_n(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} \lim_{n\to\infty} y_n(\tau) d\tau) d\theta)\Big) ds \\
&= \int_0^t \frac{(t - s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau) d\theta\Big) ds = Fy(t).
\end{aligned}$$

This means that $Fy_n(t) \to Fy(t)$. Hence the operator F is continuous. Now, by the Schauder fixed point theorem [13] there exists at least one fixed point $y \in Q_{r_\alpha} \subset C(I)$ of the integral Equation (6). Consequently there exists at least one solution $x \in C(I)$ of the problem (1) with (3). □

2.1. Maximal and Minimal Solutions

Lemma 2. *Let the assumptions of Theorem 1 be satisfied. Assume that $x,\ y$ are two continuous functions on I satisfying*

$$\begin{aligned} x(t) &\leq \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, x(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} x(\tau) d\tau) d\theta\right) ds, \\ y(t) &\geq \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau) d\theta\right) ds \end{aligned}$$

where one of them is strict. Let the functions f and g be monotonically nondecreasing; then

$$x(t) < y(t), \quad t > 0. \tag{7}$$

Proof. Let the conclusion (7) be not true; then there exists t_1 such that
$x(t_1) = y(t_1), \quad t_1 > 0$ and $x(t) < y(t) \quad 0 < t < t_1$.
From the monotonicity of f and g, we get

$$\begin{aligned} x(t_1) &\leq \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, x(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} x(\tau) d\tau) d\theta\right) ds \\ &< \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau) d\theta\right) ds \\ &= y(t_1). \end{aligned}$$

Hence $x(t_1) < y(t_1)$. This contradicts the fact that $x(t_1) = y(t_1)$; then $x(t) < y(t)$, $t \in I$. □

Theorem 2. *Let the assumptions of Theorem 1 be satisfied. If f and g are monotonic nondecreasing functions, then the problem (1) with (3) has maximal and minimal solutions.*

Proof. Firstly, we prove the existence of the maximal solution of Equation (6).
Let $\epsilon > 0$; then

$$y_\epsilon(t) = \epsilon + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y_\epsilon(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_\epsilon(\tau) d\tau) d\theta\right) ds. \tag{8}$$

It is easy to show that Equation (8) has a solution $y_\epsilon \in C(I)$.
Now, let $\epsilon_1, \epsilon_2 > 0$ such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$; then

$$\begin{aligned} y_{\epsilon_1}(t) &= \epsilon_1 + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y_{\epsilon_1}(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_1}(\tau) d\tau) d\theta\right) ds \\ &> \epsilon_2 + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y_{\epsilon_1}(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_1}(\tau) d\tau) d\theta\right) ds \end{aligned}$$

and from Lemma 2, we obtain

$$y_{\epsilon_2}(t) < y_{\epsilon_1}(t),\ t \in I.$$

Now, the family $\{y_\epsilon(t)\}$ is uniformly bounded as follows:

$$\begin{aligned} |y_\epsilon(t)| &\leq \epsilon + \left| \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y_\epsilon(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_\epsilon(\tau) d\tau) d\theta\right) ds \right| \\ &\leq \epsilon + r_\alpha = r_\alpha^*. \end{aligned}$$

Also, the family $\{y_\epsilon(t)\}$ is equicontinuous as follows:

$$\begin{aligned}
|y_\epsilon(t_2)-y_\epsilon(t_1)| &= \Big|\epsilon+\int_0^{t_2}\frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}f\Big(s,y_\epsilon(s).\int_0^{\phi(s)}g(\theta,x_0+\int_0^{\theta}\frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}y_\epsilon(\tau)d\tau)d\theta\Big)ds\\
&- \epsilon-\int_0^{t_1}\frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)}f\Big(s,y_\epsilon(s).\int_0^{\phi(s)}g(\theta,x_0+\int_0^{\theta}\frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}y_\epsilon(\tau)d\tau)d\theta\Big)ds\Big|\\
&\le \int_0^{t_1}|\frac{(t_2-s)^{\alpha}-(t_1-s)^{\alpha}}{\Gamma(1-\alpha)(t_1-s)^{\alpha}(t_2-s)^{\alpha}}|(f^*+b_1ar_\alpha+b_1b_2|x_0|r_\alpha+\frac{b_1b_2r_\alpha^2}{\Gamma(1+\alpha)})ds\\
&+ \int_{t_1}^{t_2}|\frac{(t_2-s)^{\alpha}}{\Gamma(1-\alpha)(t_2-s)^{\alpha}}|(f^*+b_1ar_\alpha+b_1b_2|x_0|r_\alpha+\frac{b_1b_2r_\alpha^2}{\Gamma(1+\alpha)})ds.
\end{aligned}$$

Then $\{y_\epsilon(t)\}$ is equicontinuous and uniformly bounded on I; then $\{y_\epsilon\}$ is relatively compact by the Arzela–Ascoli theorem [13]; then there exists a decreasing sequence ϵ_n such that $\epsilon_n\to 0$, $n\to\infty$ and $\lim_{n\to\infty} y_{\epsilon_n}(t)$ exists uniformly on I; let $\lim_{n\to\infty} y_{\epsilon_n}(t)=q(t)$.

Now, form the continuity of f, g and the Lebesgue dominated convergence theorem [13]; we have

$$\int_0^{t}\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}f\Big(s,y_{\epsilon_n}(s).\int_0^{\phi(s)}g(\theta,x_0+\int_0^{\theta}\frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}y_{\epsilon_n}(\tau)d\tau)d\theta\Big)ds\to$$
$$\int_0^{t}\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}f\Big(s,q(s).\int_0^{\phi(s)}g(\theta,x_0+\int_0^{\theta}\frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}q(\tau)d\tau)d\theta\Big)ds$$

Then

$$\begin{aligned}
q(t) &= \lim_{n\to\infty} y_{\epsilon_n}(t)\\
&= \int_0^{t}\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}f\Big(s,q(s).\int_0^{\phi(s)}g(\theta,x_0+\int_0^{\theta}\frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}q(\tau)d\tau)d\theta\Big)ds
\end{aligned}$$

which implies that $q(t)$ is a solution of Equation (6).

Finally, let us prove that $q(t)$ is the maximal solution of Equation (6). To do this, let $y(t)$ be any solution of Equation (6); then

$$\begin{aligned}
y_\epsilon(t) &= \epsilon+\int_0^{t}\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}f\Big(s,y_\epsilon(s).\int_0^{\phi(s)}g(\theta,x_0+\int_0^{\theta}\frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}y_\epsilon(\tau)d\tau)d\theta\Big)ds\\
&> \int_0^{t}\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}f\Big(s,y(s).\int_0^{\phi(s)}g(\theta,x_0+\int_0^{\theta}\frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)d\tau)d\theta\Big)ds
\end{aligned}$$

Applying Lemma 2, we get

$$y(t)<y_\epsilon(t),\ t\in I.$$

From the uniqueness of the maximal solution, it is clear that $y_\epsilon(t)\to q(t)$ uniformly on I as $\epsilon\to 0$; thus q is the maximal solution of Equation (6).

By a similar way we can prove the existence of the minimal solution. Consequently there exist maximal and minimal solutions of problem (1) with (3). □

2.2. Uniqueness of the Solution

Now, consider the following assumptions:

$(ii)^*$ $f,\ g: I\times R\to R$ are measurable in $t\in I$ $\forall x\in R$ and satisfy

$$|f(t,x)-f(t,y)|\le b_1|x-y|,\ t\in I,\ x,y\in R. \tag{9}$$

$$|g(t,x)-g(t,y)|\le b_2|x-y|,\ t\in I,\ x,y\in R.$$

From the assumption $(ii)^*$ we have

$$|f(t,x)| \leq |f(t,0)| + b_1|x|$$

and

$$|f(t,x)| \leq f^* + b_1|x|, \quad where \;\; f^* = \sup_{t\in I}|f(t,0)|.$$

Moreover, we get

$$|g(t,x)| \leq |g(t,0)| + b_2|x|$$

and

$$|g(t,x)| \leq a + b_2|x|, \quad where \;\; a = \sup_{t\in I}|f(t,0)|.$$

So, we can prove the following Lemma.

Lemma 3. *The assumption $(ii)^*$ implies assumptions (ii) and (iii).*

Theorem 3. *Let assumptions (i), $(ii)^*$ and (iv) be satisfied. If*

$$\frac{2b_1b_2r_\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)} + \frac{b_1a + b_1b_2|x_0|}{\Gamma(2-\alpha)} < 1, \tag{10}$$

then the solution of problems (1) and (3) is unique.

Proof. From Lemma 3 the assumptions of Theorem 1 are satisfied and the solution of integral Equation (6) exists. Let y_1, y_2 be two solutions of integral Equation (6); then

$$\begin{aligned}
|y_2(t) - y_1(t)| &= \Big|\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y_2(s).\int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_2(\tau)d\tau)d\theta\Big)ds \\
&- \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y_1(s).\int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_1(\tau)d\tau)d\theta\Big)ds\Big| \\
&\leq b_1\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\Big(|y_2(s)|.\int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y_2(\tau)|d\tau)d\theta \\
&- |y_1(s)|.\int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y_1(\tau)|d\tau)d\theta\Big)ds \\
&\leq b_1\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\Big(|y_2(s)|.\int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y_2(\tau)|d\tau)d\theta \\
&- |y_2(s)|.\int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y_1(\tau)|d\tau)d\theta \\
&+ |y_2(s)|.\int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y_1(\tau)|d\tau)d\theta \\
&- |y_1(s)|.\int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y_1(\tau)|d\tau)d\theta\Big)ds \\
&\leq b_1\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\Big(\|y_2\|.b_2\int_0^{\phi(s)}\int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y_2(\tau) - y_1(\tau)|d\tau d\theta \\
&+ \|y_2 - y_1\|.(a + b_2|x_0| + \frac{b_2\|y_1\|}{\Gamma(1+\alpha)})\Big)ds \\
&\leq \frac{b_1}{\Gamma(2-\alpha)}\Big(\|y_2 - y_1\|\frac{b_2r_\alpha}{\Gamma(1+\alpha)} + \|y_2 - y_1\|(a + b_2|x_0| + \frac{b_2r_\alpha}{\Gamma(1+\alpha)})\Big).
\end{aligned}$$

Hence,

$$\|y_2 - y_1\| \left(1 - (\frac{2b_1 b_2 r_\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)} + \frac{b_1 a + b_1 b_2 |x_0|}{\Gamma(2-\alpha)})\right) \leq 0.$$

Then the solution of Equation (6) is unique. Consequently, the solution of problem (1) with (3) is unique. □

2.3. Continuous Dependence

2.3.1. Continuous Dependence on the Initial Data x_0

Theorem 4. *Let the assumptions of Theorem 3 be satisfied; then the unique solution of problem (1) with (3) depends continuously on the parameter x_0.*

Proof. Let $\delta > 0$ be given such that $|x_0 - x_0^*| \leq \delta$ and let x^* be the solution of (1) with (3), corresponding to initial value x_0^*; then

$$\begin{aligned} |x(t) - x^*(t)| &= |x_0 + I^\alpha y(t) - x_0^* - I^\alpha y^*(t)| \\ &\leq |x_0 - x_0^*| + I^\alpha |y(t) - y^*(t)| \ \leq \ \delta + \frac{\|y - y^*\|}{\Gamma(1+\alpha)}. \end{aligned}$$

But

$$\begin{aligned} |y(t) - y^*(t)| &= \Big| \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau) d\theta\right) ds \\ &- \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y^*(s). \int_0^{\phi(s)} g(\theta, x_0^* + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y^*(\tau) d\tau) d\theta\right) ds \Big| \\ &\leq b_1 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \left(|y(s)|. \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y(\tau)| d\tau) d\theta \right. \\ &- \left. |y^*(s)|. \int_0^{\phi(s)} g(\theta, x_0^* + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y^*(\tau)| d\tau) d\theta \right) ds \\ &\leq b_1 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \left(\|y\| b_2 (\int_0^{\phi(s)} |x_0 - x_0^*| d\theta \right. \\ &+ \left. \int_0^{\phi(s)} \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y(\tau) - y^*(\tau)| d\tau d\theta + \|y - y^*\|.(a + b_2|x_0^*| + \frac{b_2 \|y^*\|}{\Gamma(1+\alpha)}) \right) ds \\ &\leq \frac{b_1}{\Gamma(2-\alpha)} \left(b_2 r_\alpha (\delta + \frac{\|y - y^*\|}{\Gamma(1+\alpha)}) + \|y - y^*\| (a + b_2|x_0^*| + \frac{b_2 r_\alpha}{\Gamma(1+\alpha)}) \right). \end{aligned}$$

Hence,

$$\|y - y^*\| \left(1 - (\frac{2b_1 b_2 r_\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)} + \frac{b_1 a + b_1 b_2 |x_0^*|}{\Gamma(2-\alpha)})\right) \leq \frac{b_1 b_2 r_\alpha \delta}{\Gamma(2-\alpha)},$$

Then

$$\|y - y^*\| \ \leq \ \frac{\frac{b_1 b_2 r_\alpha \delta}{\Gamma(2-\alpha)}}{1 - (\frac{2b_1 b_2 r_\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)} + \frac{b_1 a + b_1 b_2 |x_0^*|}{\Gamma(2-\alpha)})} = \epsilon_1$$

and

$$\|x - x^*\| \ \leq \ \delta + \frac{\epsilon_1}{\Gamma(1+\alpha)} = \epsilon.$$

□

Theorem 5. *Let the assumptions of Theorem 3 be satisfied; then the unique solution of problem (1) with (3) depends continuously on the function g.*

Proof. Let $\delta > 0$ be given such that $|g(t,x(t)) - g^*(t,x(t))| \leq \delta$ and let x^* be the solution of (1) with (3), corresponding to $g^*(t,x(t))$; then

$$\begin{aligned} |x(t) - x^*(t)| &= |x_0 + I^\alpha y(t) - x_0 - I^\alpha y^*(t)| \\ &\leq I^\alpha |y(t) - y^*(t)| \leq \frac{\|y - y^*\|}{\Gamma(1+\alpha)}. \end{aligned}$$

But

$$\begin{aligned} |y(t) - y^*(t)| &= \Bigg| \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)d\tau)d\theta\right)ds \\ &- \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y^*(s). \int_0^{\phi(s)} g^*(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y^*(\tau)d\tau)d\theta\right)ds \Bigg| \\ &\leq b_1 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \Bigg(|y(s)|. \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y(\tau)|d\tau)d\theta \\ &- |y^*(s)|. \int_0^{\phi(s)} g^*(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y^*(\tau)|d\tau)d\theta \Bigg| ds \\ &\leq b_1 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \Bigg(\|y\|. \int_0^{\phi(s)} (g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y(\tau)|d\tau) \\ &- g^*(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y^*(\tau)|d\tau))d\theta \\ &+ |y(s) - y^*(s)|.(\int_0^{\phi(s)} g^*(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y^*(\tau)|d\tau)d\theta) \Bigg) ds \\ &\leq b_1 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} (r_\alpha \delta + \|y - y^*\|.(a + b_2|x_0| + \frac{b_2\|y^*\|}{\Gamma(1+\alpha)}))ds \\ &\leq \frac{b_1 r_\alpha \delta}{\Gamma(2-\alpha)} + \frac{b_1\|y - y^*\|}{\Gamma(2-\alpha)}(a + b_2|x_0| + \frac{b_2 r_\alpha}{\Gamma(1+\alpha)}) \\ &\leq \frac{b_1 r_\alpha \delta}{\Gamma(2-\alpha)} + \frac{b_1\|y - y^*\|}{\Gamma(2-\alpha)}(a + b_2|x_0| + \frac{b_2 r_\alpha}{\Gamma(1+\alpha)}) + \frac{b_1 b_2 \|y - y^*\| r_\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)}. \end{aligned}$$

Hence,

$$\|y - y^*\| \left(1 - (\frac{2b_1 b_2 r_\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)} + \frac{b_1 a + b_1 b_2 |x_0|}{\Gamma(2-\alpha)})\right) \leq \frac{b_1 r_\alpha \delta}{\Gamma(2-\alpha)},$$

Then

$$\|y - y^*\| \leq \frac{\frac{b_1 r_\alpha \delta}{\Gamma(2-\alpha)}}{1 - (\frac{2b_1 b_2 r_\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)} + \frac{b_1 a + b_1 b_2 |x_0|}{\Gamma(2-\alpha)})} = \epsilon_1$$

and

$$\|x - x^*\| \leq \frac{\epsilon_1}{\Gamma(1+\alpha)} = \epsilon.$$

□

2.3.2. Continuous Dependence on the Delay Function ϕ

Theorem 6. *Let the assumptions of Theorem 3 be satisfied; then the unique solution of problem (1) with (3) depends continuously on the delay function ϕ.*

Proof. Let $\delta > 0$ be given such that $|\phi(t) - \phi^*(t)| \leq \delta$ and let x^* be the solution of (1) with (3), corresponding to $\phi^*(t)$; then

$$\begin{aligned} |x(t) - x^*(t)| &= |x_0 + I^\alpha y(t) - x_0 - I^\alpha y^*(t)| \\ &\leq I^\alpha |y(t) - y^*(t)| \leq \frac{\|y - y^*\|}{\Gamma(1+\alpha)}. \end{aligned}$$

But

$$\begin{aligned} |y(t) - y^*(t)| &= \Bigg| \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y(s). \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau) d\theta\Big) ds \\ &- \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(s, y^*(s). \int_0^{\phi^*(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y^*(\tau) d\tau) d\theta\Big) ds \Bigg| \\ &\leq b_1 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \Big(|y(s)|. \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y(\tau)| d\tau) d\theta \\ &- |y^*(s)|. \int_0^{\phi^*(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y^*(\tau)| d\tau) d\theta \Big) ds \\ &\leq b_1 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \Big(\|y\|.(\int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y(\tau)| d\tau) d\theta \\ &- \int_0^{\phi^*(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y^*(\tau)| d\tau) d\theta) \\ &+ |y(s) - y^*(s)|. \int_0^{\phi^*(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y^*(\tau)| d\tau) d\theta \Big| ds \\ &\leq b_1 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \Big(r_\alpha (\int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y(\tau)| d\tau) d\theta \\ &- \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y^*(\tau)| d\tau) d\theta + \int_0^{\phi(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y^*(\tau)| d\tau) d\theta \\ &- \int_0^{\phi^*(s)} g(\theta, x_0 + \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y^*(\tau)| d\tau) d\theta) + |y(s) - y^*(s)|.(a + b_2|x_0| + \frac{b_2\|y^*\|}{\Gamma(1+\alpha)}) \Big) ds \\ &\leq b_1 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \Big(r_\alpha b_2 \int_0^{\phi(s)} \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y(\tau) - y^*(\tau)| d\tau d\theta \\ &+ r_\alpha \int_{\phi^*(s)}^{\phi(s)} (|m(\theta)| + b_2|x_0| + \frac{b_2|y^*(\theta)|}{\Gamma(1+\alpha)}) d\theta + \|y - y^*\|(a + b_2|x_0| + \frac{b_2\|y^*\|}{\Gamma(1+\alpha)}) \Big) \\ &\leq \frac{b_1}{\Gamma(2-\alpha)} \Big(\frac{b_2\|y - y^*\| r_\alpha}{\Gamma(1+\alpha)} + (a + b_2|x_0| + \frac{b_2\|y^*\|}{\Gamma(1+\alpha)}) |\phi - \phi^*| r_\alpha \Big) \\ &+ \|y - y^*\|(a + b_2|x_0| + \frac{b_2\|y^*\|}{\Gamma(1+\alpha)}) \\ &\leq \frac{\|y - y^*\| b_1 b_2 r_\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)} + (a + b_2|x_0| + \frac{b_2 r_\alpha}{\Gamma(1+\alpha)}) \frac{b_1 r_\alpha \delta}{\Gamma(2-\alpha)} \\ &+ \|y - y^*\|(\frac{b_1 a + b_1 b_2 |x_0|}{\Gamma(2-\alpha)} + \frac{b_1 b_2 r_\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)}). \end{aligned}$$

Hence,

$$\|y - y^*\| \Big(1 - (\frac{2b_1 b_2 r_\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)} + \frac{b_1 a + b_1 b_2|x_0|}{\Gamma(2-\alpha)}) \Big) \leq (a + b_2|x_0| + \frac{b_2 r_\alpha}{\Gamma(1+\alpha)}) \frac{b_1 r_\alpha \delta}{\Gamma(2-\alpha)},$$

Then

$$\|y - y^*\| \leq \frac{(a + b_2|x_0| + \frac{b_2 r_\alpha}{\Gamma(1+\alpha)})\frac{b_1 r_\alpha \delta}{\Gamma(2-\alpha)}}{1 - (\frac{2b_1 b_2 r_\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)} + \frac{b_1 a + b_1 b_2 |x_0|}{\Gamma(2-\alpha)})} = \epsilon_1$$

and

$$\|x - x^*\| \leq \frac{\epsilon_1}{\Gamma(1+\alpha)} = \epsilon.$$

□

Example 1. *Consider the following initial value problem*

$$\frac{dx}{dt} = \frac{t^3}{96} + \frac{1}{2} D^{\frac{1}{2}} x(t). \int_0^{t^\beta} (\frac{s}{4} + \frac{1}{2} x(s)) ds. \quad t \in (0, 1] \tag{11}$$

with initial data

$$x(0) = 1. \tag{12}$$

Then

$$f(t, D^\alpha x(t). \int_o^{\phi(t)} g(s, x(s)) ds) = \frac{t^3}{96} + \frac{1}{2} D^{\frac{1}{2}} x(t). \int_o^{t^\beta} (\frac{s}{4} + \frac{1}{2} x(s)) ds \quad t \in I, \ \beta \geq 1,$$

$$g(t, x(t)) = \frac{t}{4} + \frac{1}{2} x(t) \text{and } \phi(t) = t^\beta \quad t \in I, \ \beta \geq 1.$$

It is clear that all assumptions of Theorem 1 are verified, for $t = 1$ then

$$f^* = \frac{1}{96}, \ a = \frac{1}{4}, \ b_1 = b_2 = \frac{1}{2} \text{ and } \alpha = \frac{1}{2}.$$

From (4) we can deduce that r_α satisfies the quadratic equation

$$(2-\alpha)(1+\alpha) b_1 b_2 r_\alpha^2 + ((2-\alpha) b_1 a + (2-\alpha) b_1 b_2 |x_0| - 1) r_\alpha + (2-\alpha) f^* = 0$$

and

$$\frac{9}{16} r_\alpha^2 - \frac{7}{16} r_\alpha + \frac{1}{64} = 0;$$

then $r_\alpha = 0.04$ and $r_\alpha = 0.74$. Then the initial value problems (11) and (12) have at least one solution.

3. Integer-Orders Problem

Consider now the initial value problems (2) and (3) under the assumptions $(i), (iii)$ and the following assumption:

$(ii)^{**}$ $f : I \times R \to R$ is continuous and there exists an integrable function $v : I \to R$ and a positive constant b_1 such that

$$|f(t, x)| \leq |v(t)| + b_1 |x| \leq f^* + b_1 |x|, \quad f^* = \sup_{t \in I} |v(t)|.$$

$(iv)^*$ There exists a positive root r_1 of the algebraic equation

$$b_1 b_2 r_1^2 + (b_1 a + b_1 b_2 |x_0| - 1) r_1 + f^* = 0. \tag{13}$$

Lemma 4. *Let the assumptions (i), $(ii)^{**}$ and (iii) be satisfied; then the continuation of Equation (1) as $\alpha \to 1$ is Equation (2).*

Proof. From Theorem 1 the solution y of integral Equation (6) exists and is continuous and from Lemma 1 $\frac{d}{dt}x(t)$ exists and is continuous. Then from the properties of the fractional derivative [7] we have $D^{\alpha}x(t) \to \frac{d}{dt}x(t)$ as $\alpha \to 1$. Then Equation (1)→(2) as $\alpha \to 1$. □

Now, the following lemma can be proved.

Lemma 5. *Problems (2) and (3) are equivalent to the integral equation*

$$x(t) = x_0 + \int_0^t y(s)ds \tag{14}$$

where

$$y(t) = f\Big(t, y(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)ds\Big). \tag{15}$$

Now, we have the following existences theorem.

Theorem 7. *Let assumptions (i), $(ii)^*$, $(ii)^{**}$, (iii) and $(iv)^*$ be satisfied; then problems (2) and (3) have at least one solution $x \in Q_{r_1} \in C(I)$.*

Proof. Let Q_{r_1} be the closed ball

$$Q_{r_1} = \{y \in C(I) : \|y\| \le r_1\}, \;\; r_1 \; = \; f^* + b_1 a r_1 + b_1 b_2 |x_0| r_1 + b_1 b_2 r_1^2$$

and define the operator F by

$$Fy(t) = f\Big(t, y(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)ds\Big).$$

Now, let $y \in Q_{r_1}$; then

$$\begin{aligned} |Fy(t)| &= \Big|f(t, y(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)ds)\Big| \\ &\le \Big|f^* + b_1|y(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)ds|\Big| \\ &\le f^* + b_1\|y\|(a + b_2|x_0| + b_2\|y\|) \\ &\le f^* + b_1 a r_1 + b_1 b_2|x_0|r_1 + b_1 b_2 r_1^2 = r_1 \end{aligned}$$

and

$$\|Fy\| \;\; \le \;\; f^* + b_1 a r_1 + b_1 b_2 |x_0| r_1 + b_1 b_2 r_1^2 \; = \; r_1.$$

Now, let $y \in Q_{r_1}$ and define $\theta_1(\delta) = \sup_{y \in Q_{r_1}} \{|f(t_2, y(t)) - f(t_1, y(t))| : \; t_1, t_2 \in I, \; t_1 < t_2, \; |t_2 - t_1| < \delta, \; ||y|| \le r_1\}, \theta_2(\delta) = \sup_{u,v \in Q_{r_1}} \{|f(t, u) - f(t, v)| : \; t \in I, \; |u - v| < \epsilon, |u|, |v| \in [0, r_1]\}$; then from the uniform continuity of the function $f : I \times Q_{r_1} \to R$, and our assumptions, we deduce that $\theta_1(\delta), \; \theta_2(\delta) \to 0$ as $\delta \to 0$ independently of $y \in Q_{r_1}$. Then we have

$$
\begin{aligned}
|Fy(t_2)-Fy(t_1)| &= \Big|f(t_2,y(t_2)).\int_0^{\phi(t_2)} g(s,x_0+\int_0^s y(\theta)d\theta)ds) \\
&- f(t_1,y(t_1)).\int_0^{\phi(t_1)} g(s,x_0+\int_0^s y(\theta)d\theta)ds)\Big| \\
&\leq \Big|f(t_2,y(t_2)).\int_0^{\phi(t_2)} g(s,x_0+\int_0^s y(\theta)d\theta)ds) \\
&- f(t_1,y(t_2)).\int_0^{\phi(t_2)} g(s,x_0+\int_0^s y(\theta)d\theta)ds) \\
&+ f(t_1,y(t_2)).\int_0^{\phi(t_2)} g(s,x_0+\int_0^s y(\theta)d\theta)ds) \\
&- f(t_1,y(t_2)).\int_0^{\phi(t_1)} g(s,x_0+\int_0^s y(\theta)d\theta)ds) \\
&+ f(t_1,y(t_2)).\int_0^{\phi(t_1)} g(s,x_0+\int_0^s y(\theta)d\theta)ds) \\
&- f(t_1,y(t_1)).\int_0^{\phi(t_1)} g(s,x_0+\int_0^s y(\theta)d\theta)ds)\Big| \\
&\leq \theta_1(\delta)+b_1r_1(\int_{\phi(t_1)}^{\phi(t_2)}|m(s)|ds+b_2|x_0|(t_2-t_1)+b_2r_1(t_2-t_1))+\theta_2(\delta).
\end{aligned}
$$

This means that the class of functions $\{Fy\}$ is equicontinuous on Q_{r_1} and by the Arzela–Ascoli theorem [13], the operator F is relatively compact. Now, let $\{y_n\} \subset Q_{r_1}$, and $y_n \to y$; then

$$
\lim_{n\to\infty} Fy_n(t) = \lim_{n\to\infty} f\left(t, y_n(t).\int_0^{\phi(t)} g(s,x_0+\int_0^s y_n(\theta)d\theta)ds\right).
$$

Applying the Lebesgue dominated convergence theorem [13], from our assumptions we get

$$
\begin{aligned}
\lim_{n\to\infty} Fy_n(t) &= \left(t, \lim_{n\to\infty} y_n(t).\int_0^{\phi(t)} g(s,x_0+\int_0^s \lim_{n\to\infty} y_n(\theta)d\theta)ds\right) \\
&= f\left(t, y(t).\int_0^{\phi(t)} g(s,x_0+\int_0^s y(\theta)d\theta)ds\right) = Fy(t).
\end{aligned}
$$

This means that $Fy_n(t) \to Fy(t)$. Hence, the operator F is continuous.

Then by the Schauder fixed point theorem [13] there exists at least one fixed point $y \in C(I)$ of Equation (15). Consequently, there exists at least one solution $x \in C(I)$ of problems (2) and (3). □

3.1. Maximal and Minimal Solutions

By the same way as Lemma 2 and Theorem 2, we can prove Lemma 6 and Theorem 8.

Lemma 6. *Let the assumptions of Theorem 7 be satisfied. Assume that x, y are two continuous functions on I satisfying*

$$
\begin{aligned}
x(t) &\leq f\left(t, x(t).\int_0^{\phi(t)} g(s,x_0+\int_0^s x(\theta)d\theta)ds\right), \\
y(t) &\geq f\left(t, y(t).\int_0^{\phi(t)} g(s,x_0+\int_0^s y(\theta)d\theta)ds\right)
\end{aligned}
$$

where one of them is strict. Let the functions f and g be monotonically nondecreasing; then

$$
x(t) < y(t), \quad t > 0.
$$

Theorem 8. *Let the assumptions of Theorem 7 be satisfied. If f and g are monotonic nondecreasing functions, then problems (2) and (3) have maximal and minimal solutions.*

3.2. Uniqueness of the Solution

Theorem 9. *Let assumptions $(i),(ii)^*$ and $(iv)^*$ be satisfied. If*

$$2b_1b_2r_1 + b_1a + b_1b_2|x_0| < 1, \tag{16}$$

then the solution of problems (2) and (3) is unique.

Proof. Let y_1, y_2 be two solutions of functional integral Equation (15); then

$$\begin{aligned}
|y_2(t) - y_1(t)| &= \Big| f(t, y_2(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_2(\theta)d\theta)ds) \\
&- f(t, y_1(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_1(\theta)d\theta)ds) \Big| \\
&\le b_1 \Big| y_2(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_2(\theta)d\theta)ds - y_1(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_1(\theta)d\theta)ds \Big| \\
&\le b_1 \Big| y_2(t). \int_0^{\phi(t)} \Big(g(s, x_0 + \int_0^s y_2(\theta)d\theta) - g(s, x_0 + \int_0^s y_1(\theta)d\theta) \Big) ds \\
&+ (y_2(t) - y_1(t)). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_1(\theta)d\theta) \Big| \\
&\le b_1b_2\|y_2\|\|y_2 - y_1\| + b_1\|y_2 - y_1\|(a + b_2|x_0| + b_2\|y_1\|).
\end{aligned}$$

Hence,

$$\|y_2 - y_1\|(1 - (2b_1b_2r_1 + b_1a + b_1b_2|x_0|)) \le 0.$$

Then the solution of = functional integral Equation (15) is unique. Consequently, the solution of problems (2) and (3) is unique. □

3.3. Continuous Dependence

Let $\alpha \to 1$. By the same way as Theorems 4–6, we can prove that the unique solution of problems (2) and (3) depends continuously on the parameter x_0 and on the functions g, ϕ.

Remark 1. *We notice that under the assumption $(ii)^*$ integral Equations (14) and (15) are the continuation of the two integral Equations (5) and (6) as $\alpha \to 1$.*

Remark 2. *We notice that, under the assumption $(ii)^*$, we can deduce the continuation of algebraic Equations (4)–(13) as $\alpha \to 1$.*

Remark 3. *Under assumption $(ii)^*$, we can deduce the continuation of assumption (16) is the continuation of assumption (10) as $\alpha \to 1$.*

Example 2. *Consider the following initial value problem of the delay quadratic integro-differential equation*

$$\frac{dx}{dt} = \frac{t^3}{96} + \frac{1}{2}\frac{dx}{dt}. \int_0^{t^\beta} (\frac{s}{4} + \frac{1}{2}x(s))ds. \quad t \in (0, 1]$$

with initial data

$$x(0) = 1.$$

Here,

$$f(t, \frac{dx}{dt}. \int_0^{\phi(t)} g(s, x(s))ds) = \frac{t^3}{96} + \frac{1}{2}\frac{dx}{dt}. \int_0^{t^\beta} (\frac{s}{4} + \frac{1}{2}x(s))ds \qquad t \in I, \ \beta \geq 1,$$
$$g(t, x(t)) = \frac{t}{4} + \frac{1}{2}x(s) and \phi(t) = t^\beta \qquad t \in I, \ \beta \geq 1.$$

It is clear that our assumptions of Theorem (7) are satisfied for $t = 1$*; then* $f^* = \frac{1}{96}$, $a = \frac{1}{4}$ *and* $b_1 = b_2 = \frac{1}{2}$ *and* r_1 *satisfies*

$$b_1 b_2 r_1^2 + (b_1 a + b_1 b_2 |x_0| - 1)r_1 + f^* = 0$$
$$\frac{1}{4}r_1^2 - \frac{5}{8}r_1 + \frac{1}{96} = 0;$$

then $r_1 = 0.02$. *Therefore, by applying this to Theorem 7, the given initial value problem has a unique solution.*

4. Continuation Theorem

Now, for $\alpha \in (0, 1]$ we can combine Theorems 1 and 7 in the following theorem.

Theorem 10. *Let* $\alpha \in (0, 1]$. *Let the assumptions* (i), $(ii)^*$, $(iii)^*$, (iv) *and* $(iv)^*$ *be satisfied; then initial value problems (1) and (3) have a unique solution* $x \in C(I)$.

Conclusions

Quadratic integro-differential equations have been discussed in many literature studies, for instance [18,21,22,24–26]. Many real problems have been modelled by Integro-differential equations and have been studied in different classes. Various techniques have been applied such as measure of noncompactness, Schauder's fixed point theorem and Banach contraction mapping.

In this paper, we have investigated the existences of the solutions of the initial value problem of the delay quadratic functional integro-differential equation of fractional of arbitrary (fractional) orders (1) with (3) and we have proved the existence of the maximal and minimal solutions. Moreover, we have discussed the uniqueness and the continuous dependence of the solution on x_0, the function g and on the delay function ϕ.

For the continuation of problem (1) with (3) to problems (2) and (3) as $\alpha \to 1$, we have shown that the function f should satisfy the Lipschitz condition (9).

Finally, problem (1) with (3) can be studied for all values of $\alpha \in (0, 1]$ when the function f satisfies the Lipschitz condition (9). Moreover, some examples have been demonstrated to verify the results.

We can also extend the results presented in this paper to more generalized fractional differential equations.

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