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Bipolar Fuzzy Set Theory Applied to the Certain Ideals in BCI-Algebras

N. Abughazalah [1], G. Muhiuddin [2,*], Mohamed E. A. Elnair [2,3] and A. Mahboob [4]

1 Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; nhabughazala@pnu.edu.sa
2 Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia; abomunzir124@gmail.com
3 Department of Mathematics and Physics, Gezira University, P.O. Box 20, Wad Medani 2667, Sudan
4 Department of Mathematics, Madanapalle Institute of Technology & Science, Madanapalle 517325, India; khanahsan56@gmail.com
* Correspondence: chishtygm@gmail.com

Abstract: The study of symmetry is one of the most important and beautiful themes uniting various areas of contemporary arithmetic. Algebraic structures are useful structures in pure mathematics for learning a geometrical object's symmetries. In this paper, we introduce new concepts in an algebraic structure called BCI-algebra, where we present the concepts of bipolar fuzzy (closed) *BCI*-positive implicative ideals and bipolar fuzzy (closed) *BCI*-commutative ideals of BCI-algebras. The relationship between bipolar fuzzy (closed) *BCI*-positive implicative ideals and bipolar fuzzy ideals is investigated, and various conditions are provided for a bipolar fuzzy ideal to be a bipolar fuzzy *BCI*-positive implicative ideal. Furthermore, conditions are presented for a bipolar fuzzy (closed) ideal to be a bipolar fuzzy *BCI*-commutative ideal.

Keywords: *BCI*-algebra; ideal; commutative ideal; closed ideal; bipolar fuzzy *BCI*-positive implicative ideal; bipolar fuzzy *BCI*-commutative ideal; bipolar fuzzy *BCI*-closed ideal

MSC: 06D72; 06F35; 03G25

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1. Introduction

The study of symmetry is one of the most important and beautiful themes uniting various areas of contemporary arithmetic. Algebraic structures are useful structures in pure mathematics for learning a geometrical object's symmetries. For example, the most important functions in ring theory are those that preserve the ring operation, which are referred to as homomorphism. Another algebraic structure viz. the theory of groups is also used to provide a broad theory of symmetry. There are various sorts of symmetries that may be studied using the theory of groups, which is already widely utilized as an algebraic tool.

The *BCI*-algebras and *BCK*-algebas are important classes of logical algebras (see [1–4] for more details). The notion of fuzzy sets and various operations on it were initially introduced by Zadeh in [5] (see [6,7] for more information on fuzzy sets). Many studies have been done on fuzzy set structure. For example, fuzzy ideals in *BCI*-algebras were studied by Liu in [8]. In [9], Meng et al. introduced the concept of "fuzzy implicative ideals" of *BCK*-algebras while Jun [10] gave the notion of "closed fuzzy ideals" in *BCI*-algebras. Kordi et al. studied fuzzy (p-ideals, H-ideals, BCI-positive implicative ideals) [11], and Jun et al. [12] considered fuzzy commutative ideals in *BCI*-algebras.

In 1998, Zhang was the first to initiate the concept of bipolar fuzzy sets [13] as a generalization of fuzzy sets, which were introduced by Zadeh in 1965, and later, the author introduced bipolar fuzzy logic [14]. Fuzzy sets characterize each element in a given set over a unit interval while the bipolar fuzzy sets characterize the elements over the extended interval $[-1,1]$. Intuitionistic fuzzy sets characterize elements over the interval $[0,1]$ such that the sum of the membership degree and non-membership degree ranges over the interval $[0,1]$. We refer the reader to Lee's paper [15] where a nice comparison between these concepts is made. In *BCH*-algebras, Jun et al. [16] investigated the ideas of bipolar fuzzy subalgebras and bipolar fuzzy closed ideals. Muhiuddin et al. [17] established the ideas of bipolar fuzzy closed, bipolar fuzzy positive implicative, and bipolar fuzzy implicative ideals of *BCK*-algebras. The concept of bipolar fuzzy a-ideals of *BCI*-algebras was proposed by Lee and Jun [18]. The ideas of doubt bipolar fuzzy subalgebras and (closed) doubt bipolar fuzzy ideals were developed, and the associated characteristics of these notions were studied by Al-Masarwah [19]. Jana et al. [20] proposed $(\in, \in \vee q)$-bipolar fuzzy subalgebras and $(\in, \in \vee q)$-bipolar fuzzy ideals, which were described in terms of $\in$-bipolar fuzzy soft sets and q-bipolar fuzzy soft sets of *BCK/BCI*-algebras. Different aspects in bipolar fuzzy structures have been studied in different algebras by many authors (see for e.g., [21–34]). More concepts related to this study have been studied in [35–38].

Motivated by the work done in this area, and using the notion introduced by Liu et al. [39], Al-Kadi et al. introduced, in [40], the notion of bipolar fuzzy BCI-implicative ideals of a BCI-algebra. That is, a bipolar fuzzy set $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in a BCI-algebra Ω is said to be a bipolar fuzzy BCI-implicative ideal if it satisfies the following assertions: (1) $\widetilde{\mathcal{U}_n}(0) \leq \widetilde{\mathcal{U}_n}(\vartheta)$, $\widetilde{\mathcal{U}_p}(0) \geq \widetilde{\mathcal{U}_p}(\vartheta)$, (2) $\widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \leq \widetilde{\mathcal{U}_n}((((\vartheta * \kappa) * \kappa) * (0 * \kappa)) * \hbar) \vee \widetilde{\mathcal{U}_n}(\hbar)$, and (3) $\widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \geq \widetilde{\mathcal{U}_p}((((\vartheta * \kappa) * \kappa) * (0 * \kappa)) * \hbar) \wedge \widetilde{\mathcal{U}_p}(\hbar)$, $\forall\, \vartheta, \kappa, \hbar \in \Omega$.

In this paper, we continue to study bipolar fuzzy structure of different kinds of ideals in BCI-algebras. In fact, the notions of bipolar fuzzy (closed) BCI-positive implicative ideals and bipolar fuzzy (closed) BCI-commutative ideals of BCI-algebras are introduced. The associated characteristics of bipolar fuzzy (closed) BCI-positive implicative ideals and bipolar fuzzy ideals are considered, and several conditions are presented under which a bipolar fuzzy ideal becomes a bipolar fuzzy BCI-positive implicative ideal. Furthermore, certain conditions are given under which a bipolar fuzzy (closed) ideal is a bipolar fuzzy BCI-commutative ideal.

2. Preliminaries

In this section, we collect the following notions to develop our main results.

Definition 1. *A nonempty set "Ω" together with a binary operation "$*$" and a constant* 0 *is called a "BCI-algebra" if it satisfies the following conditions; for all* $\vartheta, \kappa, \hbar \in \Omega$,
(K_1) $((\vartheta * \kappa) * (\vartheta * \hbar)) * (\hbar * \kappa) = 0$,
(K_2) $(\vartheta * (\vartheta * \kappa)) * \kappa = 0$,
(K_3) $\vartheta * \vartheta = 0$,
(K_4) $\vartheta * \kappa = 0$ *and* $\kappa * \vartheta = 0 \Rightarrow \vartheta = \kappa$.

The following are true in a BCI-algebra Ω.
(P_1) $\vartheta * 0 = \vartheta$
(P_2) $(\vartheta * \kappa) * \hbar = (\vartheta * \hbar) * \kappa$
(P_3) $\vartheta \leq \kappa \Rightarrow \vartheta * \hbar \leq \kappa * \hbar$ and $\hbar * \kappa \leq \hbar * \vartheta$
(P_4) $0 * (\vartheta * \kappa) = (0 * \vartheta) * (0 * \kappa)$
(P_5) $0 * (0 * (\vartheta * \kappa)) = 0 * (\kappa * \vartheta)$
(P_6) $(\vartheta * \hbar) * (\kappa * \hbar) \leq (\vartheta * \kappa)$
(P_7) $\vartheta * (\vartheta * (\vartheta * \kappa)) = \vartheta * \kappa$
for any $\vartheta, \kappa, \hbar \in \Omega$ (see [3] for more details).

For brevity, Ω denotes a BCI-algebra. We remind the reader of the following definitions that are taken from [8,12,41,42].

A nonempty subset A of Ω is called an *ideal* of Ω if it satisfies
(I_1) $0 \in A$,

(I_2) $\forall\, \vartheta, \kappa \in \Omega, \vartheta * \kappa \in A, \kappa \in A \Rightarrow \vartheta \in A$.

A nonempty subset A of Ω is called a *BCI-positive implicative ideal* of Ω if it satisfies (I_1) and

(I_3) $\forall\, \vartheta, \kappa, \hbar \in \Omega, ((\vartheta * \hbar) * \hbar) * (\kappa * \hbar) \in A, \kappa \in A \Rightarrow \vartheta * \hbar \in A$.

A nonempty subset A of Ω is called a *BCI-commutative ideal* of Ω if it satisfies (I_1) and

(I_4) $\forall\, \vartheta, \kappa, \hbar \in \Omega, (\vartheta * \kappa) * \hbar \in A, \hbar \in A \Rightarrow \vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa)))) \in A$.

A fuzzy set $\widetilde{\mathcal{U}}$ in Ω is a map from Ω to $[0,1]$. A fuzzy set $\widetilde{\mathcal{U}}$ in Ω is called a *fuzzy ideal* of Ω if it satisfies for all $\vartheta, \kappa, \hbar \in \Omega$,

(F_1) $\widetilde{\mathcal{U}}(0) \geq \widetilde{\mathcal{U}}(\vartheta)$, and

(F_2) $\widetilde{\mathcal{U}}(\vartheta) \geq \widetilde{\mathcal{U}}(\vartheta * \kappa) \wedge \widetilde{\mathcal{U}}(\kappa)$.

A fuzzy set $\widetilde{\mathcal{U}}$ in Ω is called a *fuzzy BCI-positive implicative ideal* of Ω if it satisfies for all $\vartheta, \kappa, \hbar \in \Omega$, (F_1) and (F_3) $\widetilde{\mathcal{U}}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \wedge \widetilde{\mathcal{U}}(\kappa)$.

A fuzzy set $\widetilde{\mathcal{U}}$ in Ω is called a *fuzzy BCI-commutative ideal* of Ω if it satisfies for all $\vartheta, \kappa, \hbar \in \Omega$, (F_1) and (F_4) $\widetilde{\mathcal{U}}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa)))) \geq \widetilde{\mathcal{U}}((\vartheta * \kappa) * \hbar) \wedge \widetilde{\mathcal{U}}(\hbar)$.

A bipolar fuzzy set in Ω is denoted by $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$, where $\widetilde{\mathcal{U}_n} : \Omega \to [-1,0]$ and $\widetilde{\mathcal{U}_p} : \Omega \to [0,1]$.

Definition 2 ([28]). *A bipolar fuzzy set $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω is called a bipolar fuzzy ideal of Ω if it satisfies the following assertions:*

(BF_1) $(\forall\, \vartheta \in \Omega)$ $(\widetilde{\mathcal{U}_n}(0) \leq \widetilde{\mathcal{U}_n}(\vartheta),\ \widetilde{\mathcal{U}_p}(0) \geq \widetilde{\mathcal{U}_p}(\vartheta))$;

(BF_2) $(\forall\, \vartheta, \kappa \in \Omega)$ $\widetilde{\mathcal{U}_n}(\vartheta) \leq \widetilde{\mathcal{U}_n}(\vartheta * \kappa) \vee \widetilde{\mathcal{U}_n}(\kappa)$;

(BF_3) $(\forall\, \vartheta, \kappa \in \Omega)$ $\widetilde{\mathcal{U}_p}(\vartheta) \geq \widetilde{\mathcal{U}_p}(\vartheta * \kappa) \wedge \widetilde{\mathcal{U}_p}(\kappa)$.

3. Bipolar Fuzzy BCI-Positive Implicative Ideal

In this section, we begin with the following definition to obtain our results.

Definition 3. *A BFS $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω is said to be a bipolar fuzzy BCI-positive implicative ideal (BF-BCI-PII) of Ω if it satisfies (BF_1) and*

(BF_7) $\widetilde{\mathcal{U}_n}(\vartheta * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \vee \widetilde{\mathcal{U}_n}(\kappa)$,

(BF_8) $\widetilde{\mathcal{U}_p}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \wedge \widetilde{\mathcal{U}_p}(\kappa)$,

$\forall\, \vartheta, \kappa, \hbar \in \Omega$.

Example 1. *Consider a BCI-algebra $\Omega = \{0, j, k\}$ under the $*$ operation defined by table:*

$*$	0	j	k
0	0	k	j
j	j	0	k
k	k	j	0

Define a BFS $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω as:

Ω	0	j	k
$\widetilde{\mathcal{U}_n}$	−0.6	−0.3	−0.3
$\widetilde{\mathcal{U}_p}$	0.7	0.4	0.4

Then, $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BF-BCI-PII of Ω.

By taking $\hbar = 0$ in (BF_7) and (BF_8), we find the following.

Corollary 1. *Every BF-BCI-PII is a BFI.*

The converse of Corollary 1 is not true, as shown in the following example.

Example 2. *Consider a BCI-algebra $\Omega = \{0, j, k, l, m\}$ under the $*$ operation defined by table:*

$*$	0	j	k	l	m
0	0	0	0	0	0
j	j	0	j	0	0
k	k	k	0	0	0
l	l	l	l	0	0
m	m	l	m	j	0

Define a BFS $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω as:

Ω	0	j	k	l	m
$\widetilde{\mathcal{U}_n}$	-0.6	-0.4	-0.6	-0.4	-0.4
$\widetilde{\mathcal{U}_p}$	0.6	0.3	0.6	0.3	0.3

Then, $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BFI of Ω but is not a BF-BCI-PII of Ω as $\widetilde{\mathcal{U}_n}(m * l) = \widetilde{\mathcal{U}_n}(j) = -0.4 \nleq -0.6 = \widetilde{\mathcal{U}_n}(((m * l) * l) * (0 * l)) \vee \widetilde{\mathcal{U}_n}(0) = \widetilde{\mathcal{U}_n}(0)$.

Definition 4. *A BFS $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω is said to be a bipolar fuzzy closed BCI-positive implicative ideal (BFC-BCI-PII) of Ω if it satisfies (BF_1), (BF_7), (BF_8) and (BF_9) $\widetilde{\mathcal{U}_n}(0 * \vartheta) \leq \widetilde{\mathcal{U}_n}(\vartheta)$ and $\widetilde{\mathcal{U}_p}(0 * \vartheta) \geq \widetilde{\mathcal{U}_p}(\vartheta)$, $\forall\, \vartheta \in \Omega$.*

Example 3. *Consider Example 1, where $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BF-BCI-PII of Ω and $\widetilde{\mathcal{U}_n}(0 * \vartheta) = \widetilde{\mathcal{U}_n}(\vartheta)$, $\widetilde{\mathcal{U}_p}(0 * \vartheta) = \widetilde{\mathcal{U}_p}(\vartheta)$, $\forall\, \vartheta \in \Omega$. Thus $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BFC-BCI-PII of Ω.*

The following result gives the consequence of Corollary 1.

Corollary 2. *Every BFC-BCI-PII is a BFI.*

The converse of Corollary 2 is not true. Example 2 validates it.

Lemma 1 ([28])**.** *A BFS $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω is a BFI of $\Omega \Leftrightarrow$ for all $\vartheta, \kappa, \hbar \in \Omega$, $(\vartheta * \kappa) * \hbar = 0$ implies $\widetilde{\mathcal{U}_n}(\vartheta) \leq \widetilde{\mathcal{U}_n}(\kappa) \vee \widetilde{\mathcal{U}_n}(\hbar)$ and $\widetilde{\mathcal{U}_p}(\vartheta) \geq \widetilde{\mathcal{U}_p}(\kappa) \wedge \widetilde{\mathcal{U}_p}(\hbar)$.*

Lemma 2 ([28])**.** *A BFS $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω is a BFI of $\Omega \Leftrightarrow$ for all $\vartheta, \kappa, \hbar \in \Omega$, $\vartheta * \kappa = 0$ implies $\widetilde{\mathcal{U}_n}(\vartheta) \leq \widetilde{\mathcal{U}_n}(\kappa)$ and $\widetilde{\mathcal{U}_p}(\vartheta) \geq \widetilde{\mathcal{U}_p}(\kappa)$.*

Theorem 1. *Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a BFI of Ω. The following assertions are equivalent:*

(1) *$\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BF-BCI-PII of Ω.*
(2) *$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) = \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (0 * \hbar))$ and $\widetilde{\mathcal{U}_p}(\vartheta * \hbar) = \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (0 * \hbar))$, $\forall\, \vartheta, \hbar \in \Omega$.*
(3) *$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (0 * \hbar))$ and $\widetilde{\mathcal{U}_p}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (0 * \hbar))$, $\forall\, \vartheta, \hbar \in \Omega$.*

Proof. $(1 \Rightarrow 2)$ Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a *BF-BCI-PII* of Ω. Then, for any $\vartheta, \kappa, \hbar \in \Omega$, we have

$$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \vee \widetilde{\mathcal{U}_n}(\kappa)$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \wedge \widetilde{\mathcal{U}_p}(\kappa).$$

Take $\kappa = 0$, so

$$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)) \vee \widetilde{\mathcal{U}_n}(0)$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)) \wedge \widetilde{\mathcal{U}_p}(0).$$

That is,

$$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)) \quad (1)$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)). \quad (2)$$

On the other hand, it follows from P_6 and P_1 that $((\vartheta * \hbar) * \hbar) * (0 * \hbar) \leq (\vartheta * \hbar)$. Therefore

$$\widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)) \leq \widetilde{\mathcal{U}_n}(\vartheta * \hbar) \quad (3)$$

and

$$\widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)) \geq \widetilde{\mathcal{U}_p}(\vartheta * \hbar) \quad (4)$$

From (1) and (3), (2) and (4), we have

$$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) = \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (0 * \hbar))$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \hbar) = \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)),$$

as required.

($2 \Rightarrow 3$) Suppose that $\widetilde{\mathcal{U}_n}(\vartheta * \hbar) = \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (0 * \hbar))$ and $\widetilde{\mathcal{U}_p}(\vartheta * \hbar) = \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (0 * \hbar))$, $\vartheta, \hbar \in \Omega$. Then, we have

$$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (0 * \hbar))$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)).$$

($3 \Rightarrow 1$) Assume that

$$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)) \quad (5)$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)) \quad (6)$$

for all $\vartheta, \kappa, \hbar \in \Omega$. From P_6 and K_1, we obtain

$$(((\vartheta * \hbar) * \hbar) * (0 * \hbar)) * (((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \leq r.$$

By using Lemma 1, we have

$$\widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \vee \widetilde{\mathcal{U}_n}(\kappa) \quad (7)$$

and

$$\widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \wedge \widetilde{\mathcal{U}_p}(\kappa). \quad (8)$$

From (5) and (7), (6) and (8), we have

$$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \vee \widetilde{\mathcal{U}_n}(\kappa)$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \wedge \widetilde{\mathcal{U}_p}(\kappa).$$

Hence, $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a *BF-BCI-PII* of Ω. □

Theorem 2. *Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a BFI of Ω. Then, $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BF-BCI-PII of Ω if for all $\vartheta, \kappa \in \Omega$, $\widetilde{\mathcal{U}_n}(\vartheta * (\vartheta * (\kappa * (\kappa * \vartheta)))) \leq \widetilde{\mathcal{U}_n}((\vartheta * (\vartheta * \kappa)) * (\kappa * \vartheta))$ and $\widetilde{\mathcal{U}_p}(\vartheta * (\vartheta * (\kappa * (\kappa * \vartheta)))) \geq \widetilde{\mathcal{U}_p}((\vartheta * (\vartheta * \kappa)) * (\kappa * \vartheta))$.*

Proof. Assume that

$$\widetilde{\mathcal{U}_n}(\vartheta*(\vartheta*(\kappa*(\kappa*\vartheta)))) \leq \widetilde{\mathcal{U}_n}((\vartheta*(\vartheta*\kappa))*(\kappa*\vartheta))$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta*(\vartheta*(\kappa*(\kappa*\vartheta)))) \geq \widetilde{\mathcal{U}_p}((\vartheta*(\vartheta*\kappa))*(\kappa*\vartheta)),$$

for all $\vartheta, \kappa \in \Omega$. Therefore

$$\widetilde{\mathcal{U}_n}((\vartheta*\kappa)*((\vartheta*\kappa)*(\vartheta*(\vartheta*(\vartheta*\kappa))))) \leq \widetilde{\mathcal{U}_n}(((\vartheta*\kappa)*((\vartheta*\kappa)*\vartheta))*(\vartheta*(\vartheta*\kappa))) \quad (9)$$

and

$$\widetilde{\mathcal{U}_p}((\vartheta*\kappa)*((\vartheta*\kappa)*(\vartheta*(\vartheta*(\vartheta*\kappa))))) \geq \widetilde{\mathcal{U}_p}(((\vartheta*\kappa)*((\vartheta*\kappa)*\vartheta))*(\vartheta*(\vartheta*\kappa))). \quad (10)$$

From P_7, K_3 and P_1, we have

$$(\vartheta*\kappa)*((\vartheta*\kappa)*(\vartheta*(\vartheta*(\vartheta*\kappa)))) = (\vartheta*\kappa)*((\vartheta*\kappa)*(\vartheta*\kappa)) = \vartheta*\kappa$$

and on the other hand, from P_2, K_3 and P_7, we have

$$\begin{aligned}((\vartheta*\kappa)*((\vartheta*\kappa)*\vartheta))*(\vartheta*(\vartheta*\kappa)) &= ((\vartheta*\kappa)*((\vartheta*\vartheta)*\kappa))*(\vartheta*(\vartheta*\kappa)) \\ &= ((\vartheta*\kappa)*(0*\kappa))*(\vartheta*(\vartheta*\kappa)) \\ &= ((\vartheta*\kappa)*(\vartheta*(\vartheta*\kappa)))*(0*\kappa) \\ &= ((\vartheta*(\vartheta*(\vartheta*\kappa)))*\kappa)*(0*\kappa) \\ &= ((\vartheta*\kappa)*\kappa)*(0*\kappa).\end{aligned}$$

Therefore

$$\widetilde{\mathcal{U}_n}((\vartheta*\kappa)*((\vartheta*\kappa)*(\vartheta*(\vartheta*(\vartheta*\kappa))))) = \widetilde{\mathcal{U}_n}(\vartheta*\kappa) \quad (11)$$

and

$$\widetilde{\mathcal{U}_p}((\vartheta*\kappa)*((\vartheta*\kappa)*(\vartheta*(\vartheta*(\vartheta*\kappa))))) = \widetilde{\mathcal{U}_p}(\vartheta*\kappa). \quad (12)$$

In addition,

$$\widetilde{\mathcal{U}_n}(((\vartheta*\kappa)*((\vartheta*\kappa)*\vartheta))*(\vartheta*(\vartheta*\kappa))) = \widetilde{\mathcal{U}_n}(((\vartheta*\kappa)*\kappa)*(0*\kappa)) \quad (13)$$

and

$$\widetilde{\mathcal{U}_p}(((\vartheta*\kappa)*((\vartheta*\kappa)*\vartheta))*(\vartheta*(\vartheta*\kappa))) = \widetilde{\mathcal{U}_p}(((\vartheta*\kappa)*\kappa)*(0*\kappa)). \quad (14)$$

Substitute (11), (13) in (9) and (12), (16) in (10),

$$\widetilde{\mathcal{U}_n}(\vartheta*\kappa) \leq \widetilde{\mathcal{U}_n}(((\vartheta*\kappa)*\kappa)*(0*\kappa))$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta*\kappa) \geq \widetilde{\mathcal{U}_p}(((\vartheta*\kappa)*\kappa)*(0*\kappa)).$$

Thus, from Theorem 1, $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a *BF-BCI-PII* of Ω. □

Similarly, we can prove the following.

Corollary 3. *Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a BFI of Ω. Then $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BF-BCI-PII of Ω if for all $\vartheta, \kappa \in \Omega$, $\widetilde{\mathcal{U}_n}(\vartheta*(\vartheta*(\kappa*(\kappa*\vartheta)))) = \widetilde{\mathcal{U}_n}((\vartheta*(\vartheta*\kappa))*(\kappa*\vartheta))$ and $\widetilde{\mathcal{U}_p}(\vartheta*(\vartheta*(\kappa*(\kappa*\vartheta)))) = \widetilde{\mathcal{U}_p}((\vartheta*(\vartheta*\kappa))*(\kappa*\vartheta))$.*

Theorem 3. *Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a BFI of Ω. The following statements are equivalent:*

(1) *$\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BF-BCI-PII of Ω.*

(2) $\widetilde{\mathcal{U}_n}((\vartheta * \hbar) * \kappa) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$ *and* $\widetilde{\mathcal{U}_p}((\vartheta * \hbar) * \kappa) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)), \forall\, \vartheta, \kappa, \hbar \in \Omega$.

Proof. $(1 \Rightarrow 2)$ Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a *BF-BCI-PII* of Ω. Then, from Theorem 1, for any $\vartheta, \hbar \in \Omega$, we have

$$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)) \tag{15}$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (0 * \hbar)). \tag{16}$$

From P_2, K_1, P_6, P_1 and K_3, we have

$$\begin{aligned}
&((((\vartheta * \kappa) * \hbar) * \hbar) * (0 * \hbar)) * (((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \\
&= ((((\vartheta * \hbar) * \kappa) * \hbar) * (0 * \hbar)) * (((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \\
&= ((((\vartheta * \hbar) * \hbar) * \kappa) * (0 * \hbar)) * (((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \\
&= ((((\vartheta * \hbar) * \hbar) * (0 * \hbar)) * \kappa) * (((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \\
&= ((((\vartheta * \hbar) * \hbar) * (0 * \hbar)) * (((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))) * \kappa \\
&\leq ((\kappa * \hbar) * (0 * \hbar)) * \kappa \\
&\leq (\kappa * 0) * \kappa \\
&= \kappa * \kappa \\
&= 0.
\end{aligned}$$

By using Lemma 1, we obtain

$$\widetilde{\mathcal{U}_n}((((\vartheta * \kappa) * \hbar) * \hbar) * (0 * \hbar)) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \vee \widetilde{\mathcal{U}_n}(0)$$

and

$$\widetilde{\mathcal{U}_p}((((\vartheta * \kappa) * \hbar) * \hbar) * (0 * \hbar)) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \wedge \widetilde{\mathcal{U}_p}(0).$$

That is,

$$\widetilde{\mathcal{U}_n}((((\vartheta * \kappa) * \hbar) * \hbar) * (0 * \hbar)) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \tag{17}$$

and

$$\widetilde{\mathcal{U}_p}((((\vartheta * \kappa) * \hbar) * \hbar) * (0 * \hbar)) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)). \tag{18}$$

By using (15) and (17), (16) and (18), we obtain

$$\widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$$

and

$$\widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)).$$

Hence, by P_2, we have

$$\widetilde{\mathcal{U}_n}((\vartheta * \hbar) * \kappa) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$$

and

$$\widetilde{\mathcal{U}_p}((\vartheta * \hbar) * \kappa) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)).$$

$(2 \Rightarrow 1)$ Assume that

$$\widetilde{\mathcal{U}_n}((\vartheta * \hbar) * \kappa) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$$

and

$$\widetilde{\mathcal{U}_p}((\vartheta * \hbar) * \kappa) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)).$$

Since $\widetilde{\mathcal{U}_n}(\vartheta) \leq \widetilde{\mathcal{U}_n}(\vartheta * \kappa) \vee \widetilde{\mathcal{U}_n}(\kappa)$ and $\widetilde{\mathcal{U}_p}(\vartheta) \geq \widetilde{\mathcal{U}_p}(\vartheta * \kappa) \wedge \widetilde{\mathcal{U}_p}(\kappa)$, so we have

$$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) \leq \widetilde{\mathcal{U}_n}((\vartheta * \hbar) * \kappa) \vee \widetilde{\mathcal{U}_n}(\kappa)$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}_p}((\vartheta * \hbar) * \kappa) \wedge \widetilde{\mathcal{U}_p}(\kappa).$$

By the assumption, we have

$$\widetilde{\mathcal{U}_n}(\vartheta * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \vee \widetilde{\mathcal{U}_n}(\kappa)$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \wedge \widetilde{\mathcal{U}_p}(\kappa).$$

Hence $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ a *BF-BCI-PII* of Ω. □

Theorem 4. *Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a BFI of Ω. The following assertions are equivalent:*

(1) *$\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω is a BF-BCI-PII.*
(2) *$\widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$ and $\widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$.*
(3) *$\widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar) = \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$ and $\widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar) = \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$.*
(4) *$\widetilde{\mathcal{U}_n}(\vartheta * \kappa) = \widetilde{\mathcal{U}_n}(((\vartheta * \kappa) * \kappa) * (0 * \kappa))$ and $\widetilde{\mathcal{U}_p}(\vartheta * \kappa) = \widetilde{\mathcal{U}_p}(((\vartheta * \kappa) * \kappa) * (0 * \kappa))$.*
(5) *$\widetilde{\mathcal{U}_n}(\vartheta * \kappa) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \kappa) * \kappa) * (0 * \kappa))$ and $\widetilde{\mathcal{U}_p}(\vartheta * \kappa) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \kappa) * \kappa) * (0 * \kappa))$.*

Proof. $(1 \Rightarrow 2)$ Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω be a *BF-BCI-PII*. Then, by using Theorem 1, for any $\vartheta, \kappa, \hbar \in \Omega$, we have

$$\widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar) \leq \widetilde{\mathcal{U}_n}((((\vartheta * \kappa) * \hbar) * \hbar) * (0 * \hbar))$$

and

$$\widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar) \geq \widetilde{\mathcal{U}_p}((((\vartheta * \kappa) * \hbar) * \hbar) * (0 * \hbar)).$$

Now, using P_2 and P_6, we obtain

$$\begin{aligned}(((\vartheta * \kappa) * \hbar) * \hbar) * (0 * \hbar) &= (((\vartheta * \hbar) * \hbar) * \kappa) * ((\kappa * \hbar) * \kappa) \\ &\leq ((\vartheta * \hbar) * \hbar) * (\kappa * \hbar).\end{aligned}$$

Therefore, by Lemma 2, we have

$$\widetilde{\mathcal{U}_n}((((\vartheta * \kappa) * \hbar) * \hbar) * (0 * \hbar)) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$$

and

$$\widetilde{\mathcal{U}_p}((((\vartheta * \kappa) * \hbar) * \hbar) * (0 * \hbar)) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)).$$

Thus,

$$\widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$$

and

$$\widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)).$$

$(2 \Rightarrow 3)$ Assume that $\widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$ and $\widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$ for all $\vartheta, \kappa, \hbar \in \Omega$. Now $((\vartheta * \hbar) * \hbar) * (\kappa * \hbar) \leq (\vartheta * \hbar) * \kappa = (\vartheta * \kappa) * \hbar$.

Therefore, by Lemma 2, we have

$$\widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \leq \widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar)$$

and

$$\widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)) \geq \widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar).$$

Thus

$$\widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar) = \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$$

and

$$\widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar) = \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar)).$$

$(3 \Rightarrow 4)$ Assume that $\widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar) = \widetilde{\mathcal{U}_n}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$ and $\widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar) = \widetilde{\mathcal{U}_p}(((\vartheta * \hbar) * \hbar) * (\kappa * \hbar))$ for $\vartheta, \kappa, \hbar \in \Omega$. Take $\hbar = \kappa$ and $\kappa = 0$, so

$$\widetilde{\mathcal{U}_n}((\vartheta * 0) * \kappa) = \widetilde{\mathcal{U}_n}(((\vartheta * \kappa) * \kappa) * (0 * \kappa))$$

and

$$\widetilde{\mathcal{U}_p}((\vartheta * 0) * \kappa) = \widetilde{\mathcal{U}_p}(((\vartheta * \kappa) * \kappa) * (0 * \kappa)).$$

Therefore

$$\widetilde{\mathcal{U}_n}(\vartheta * \kappa) = \widetilde{\mathcal{U}_n}(((\vartheta * \kappa) * \kappa) * (0 * \kappa))$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \kappa) = \widetilde{\mathcal{U}_p}(((\vartheta * \kappa) * \kappa) * (0 * \kappa)).$$

$(4 \Rightarrow 5)$ trivially holds.
$(5 \Rightarrow 1)$ Assume that $\widetilde{\mathcal{U}_n}(\vartheta * \kappa) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \kappa) * \kappa) * (0 * \kappa))$ and $\widetilde{\mathcal{U}_p}(\vartheta * \kappa) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \kappa) * \kappa) * (0 * \kappa))$ for any $\vartheta, \kappa \in \Omega$. As $(((\vartheta * \kappa) * \kappa) * (0 * \kappa)) * (((\vartheta * \kappa) * \kappa) * (\hbar * \kappa)) \leq (\hbar * \kappa) * (0 * \kappa) \leq \hbar * 0 = \hbar$. So, by Lemma 1, we have

$$\widetilde{\mathcal{U}_n}(((\vartheta * \kappa) * \kappa) * (0 * \kappa)) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \kappa) * \kappa) * (\hbar * \kappa)) \vee \widetilde{\mathcal{U}_n}(\hbar)$$

and

$$\widetilde{\mathcal{U}_p}(((\vartheta * \kappa) * \kappa) * (0 * \kappa)) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \kappa) * \kappa) * (\hbar * \kappa)) \wedge \widetilde{\mathcal{U}_p}(\hbar).$$

Thus, $\widetilde{\mathcal{U}_n}(\vartheta * \kappa) \leq \widetilde{\mathcal{U}_n}(((\vartheta * \kappa) * \kappa) * (\hbar * \kappa)) \vee \widetilde{\mathcal{U}_n}(\hbar)$ and $\widetilde{\mathcal{U}_p}(\vartheta * \kappa) \geq \widetilde{\mathcal{U}_p}(((\vartheta * \kappa) * \kappa) * (\hbar * \kappa)) \wedge \widetilde{\mathcal{U}_p}(\hbar)$. Hence, $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a *BF-BCI-PII* of Ω. □

Theorem 5. *Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a BFI of Ω. If $\widetilde{\mathcal{U}_n}(\vartheta * (\vartheta * (\kappa * (\kappa * \vartheta)))) \leq \widetilde{\mathcal{U}_n}((\vartheta * (\vartheta * \kappa)) * (\kappa * \vartheta))$ and $\widetilde{\mathcal{U}_p}(\vartheta * (\vartheta * (\kappa * (\kappa * \vartheta)))) \geq \widetilde{\mathcal{U}_p}((\vartheta * (\vartheta * \kappa)) * (\kappa * \vartheta))$ for any $\vartheta, \kappa \in \Omega$, then $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω is a BF-BCI-PII of Ω.*

Proof. Consider $(a * (a * \vartheta)) * (\vartheta * a)$. Substituting a by $\vartheta * \kappa$, then using K_1 and P_7, we have

$$\begin{aligned}(a * (a * \vartheta)) * (\vartheta * a) &= ((\vartheta * \kappa) * ((\vartheta * \kappa) * \vartheta)) * (\vartheta * (\vartheta * \kappa)) \\ &= ((\vartheta * \kappa) * (0 * \kappa)) * (\vartheta * (\vartheta * \kappa)) \\ &= ((\vartheta * \kappa) * (\vartheta * (\vartheta * \kappa))) * (0 * \kappa) \\ &= ((\vartheta * (\vartheta * (\vartheta * \kappa))) * \kappa) * (0 * \kappa) \\ &= ((\vartheta * \kappa) * \kappa) * (0 * \kappa).\end{aligned}$$

Therefore,

$$\widetilde{\mathcal{U}_n}((a * (a * \vartheta)) * (\vartheta * a)) = \widetilde{\mathcal{U}_n}(((\vartheta * \kappa) * \kappa) * (0 * \kappa))$$

and

$$\widetilde{\mathcal{U}_p}((a * (a * \vartheta)) * (\vartheta * a)) = \widetilde{\mathcal{U}_p}(((\vartheta * \kappa) * \kappa) * (0 * \kappa)).$$

Similarly, from P_7 and P_1, we have

$$\begin{aligned} a*(a*(\vartheta*(\vartheta*a))) &= (\vartheta*\kappa)*((\vartheta*\kappa)*(\vartheta*(\vartheta*(\vartheta*\kappa)))) \\ &= (\vartheta*\kappa)*((\vartheta*\kappa)*(\vartheta*\kappa)) = (\vartheta*\kappa)*0 \\ &= \vartheta*\kappa. \end{aligned}$$

Therefore $\widetilde{\mathcal{U}_n}(a*(a*(\vartheta*(\vartheta*a)))) = \widetilde{\mathcal{U}_n}(\vartheta*\kappa)$ and $\widetilde{\mathcal{U}_p}(a*(a*(\vartheta*(\vartheta*a)))) = \widetilde{\mathcal{U}_p}(\vartheta*\kappa)$. By given hypothesis, we have

$$\widetilde{\mathcal{U}_n}(a*(a*(\vartheta*(\vartheta*a)))) \leq \widetilde{\mathcal{U}_n}((a*(a*\vartheta))*(\vartheta*a))$$

and

$$\widetilde{\mathcal{U}_p}(a*(a*(\vartheta*(\vartheta*a)))) \geq \widetilde{\mathcal{U}_p}((a*(a*\vartheta))*(\vartheta*a)).$$

Thus, $\widetilde{\mathcal{U}_n}(\vartheta*\kappa) \leq \widetilde{\mathcal{U}_n}(((\vartheta*\kappa)*\kappa)*(0*\kappa))$ and $\widetilde{\mathcal{U}_p}(\vartheta*\kappa) \geq \widetilde{\mathcal{U}_p}(((\vartheta*\kappa)*\kappa)*(0*\kappa))$. Hence, by Theorem 1, $\widetilde{\mathcal{U}}$ is a *BF-BCI-PICI* of Ω. □

4. Bipolar Fuzzy BCI-Commutative Ideal

In this section, the concept of bipolar fuzzy BCI-commutative ideals is introduced, and several properties are investigated.

Definition 5. *A BFS $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω is said to be a bipolar fuzzy BCI-commutative ideal (BF-BCI-CI) of Ω if it satisfies (BF_1) and*
(BF_9) $\widetilde{\mathcal{U}_n}(\vartheta*((\kappa*(\kappa*\vartheta))*(0*(0*(\vartheta*\kappa))))) \leq \widetilde{\mathcal{U}_n}((\vartheta*\kappa)*\hbar) \vee \widetilde{\mathcal{U}_n}(\hbar)$,
(BF_{10}) $\widetilde{\mathcal{U}_p}(\vartheta*((\kappa*(\kappa*\vartheta))*(0*(0*(\vartheta*\kappa))))) \geq \widetilde{\mathcal{U}_p}((\vartheta*\kappa)*\hbar) \wedge \widetilde{\mathcal{U}_p}(\hbar)$,
for all $\vartheta, \kappa, \hbar \in \Omega$.

Example 4. *Consider a BCI-algebra $(\Omega; *, 0)$ where $\Omega = \{0, j, k, l\}$ and $*$ given by the Cayley table: Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a BFS in Ω represented by:*

*	0	j	k	l
0	0	0	0	0
j	j	0	0	j
k	k	j	0	k
l	l	l	l	0

Ω	0	j	k	l
$\widetilde{\mathcal{U}_n}$	−0.4	−0.4	−0.2	−0.3
$\widetilde{\mathcal{U}_p}$	0.9	0.9	0.6	0.8

Then, by routine calculations, $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BF-BCI-CI of Ω.

Definition 6. *A BFS $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ in Ω is called a bipolar fuzzy closed BCI-commutative ideal (BFC-BCI-CI) of Ω if it satisfies (BF_1), (BF_9), (BF_{10}) and (BF_{11}) $\widetilde{\mathcal{U}_n}(0*\vartheta) \leq \widetilde{\mathcal{U}_n}(\vartheta)$ and $\widetilde{\mathcal{U}_p}(0*\vartheta) \geq \widetilde{\mathcal{U}_p}(\vartheta)$, for all $\vartheta \in \Omega$.*

Example 5. *Consider Example 4, where $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BF-BCI-CI of Ω. Also $\widetilde{\mathcal{U}_n}(0*\vartheta) = \widetilde{\mathcal{U}_n}(0) \leq \widetilde{\mathcal{U}_n}(\vartheta)$ and $\widetilde{\mathcal{U}_p}(0*\vartheta) = \widetilde{\mathcal{U}_n}(0) \geq \widetilde{\mathcal{U}_p}(\vartheta)$ for all $\vartheta \in \{0, j, k, l\}$. Thus, $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BFC-BCI-CI of Ω.*

Theorem 6. *Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a BFI of Ω. The following statements are equivalent:*

(1) *$\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BF-BCI-CI of Ω.*

(2) $\widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \leq \widetilde{\mathcal{U}_n}(\vartheta * \kappa)$ *and* $\widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \geq \widetilde{\mathcal{U}_p}(\vartheta * \kappa), \forall\, \vartheta, \kappa \in \Omega.$

(3) $\widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) = \widetilde{\mathcal{U}_n}(\vartheta * \kappa)$ *and* $\widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) = \widetilde{\mathcal{U}_p}(\vartheta * \kappa), \forall\, \vartheta, \kappa \in \Omega.$

Proof. $(1 \Rightarrow 2)$ Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a *BF-BCI-CI* of Ω. Then, for any $\vartheta, \kappa, \hbar \in \Omega$, we have

$$\widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \leq \widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar) \vee \widetilde{\mathcal{U}_n}(\hbar)$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \geq \widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar) \wedge \widetilde{\mathcal{U}_p}(\hbar).$$

Substitute 0 for $\hbar$, so we obtain

$$\widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \leq \widetilde{\mathcal{U}_n}((\vartheta * \kappa) * 0) \vee \widetilde{\mathcal{U}_n}(0)$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \geq \widetilde{\mathcal{U}_p}((\vartheta * \kappa) * 0) \wedge \widetilde{\mathcal{U}_p}(0).$$

Therefore,

$$\widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \leq \widetilde{\mathcal{U}_n}(\vartheta * \kappa)$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \geq \widetilde{\mathcal{U}_p}(\vartheta * \kappa).$$

$(2 \Rightarrow 3)$ Assume that for $\vartheta, \kappa \in \Omega$,

$$\widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \leq \widetilde{\mathcal{U}_n}(\vartheta * \kappa) \tag{19}$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \geq \widetilde{\mathcal{U}_p}(\vartheta * \kappa). \tag{20}$$

As $(\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))) = (\kappa * (\kappa * \vartheta)) * (0 * (\kappa * \vartheta)) \leq \kappa$, using P_5 and P_6, So, by P_3, we have $\vartheta * \kappa \leq \vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))$. By Lemma 2, we have

$$\widetilde{\mathcal{U}_n}(\vartheta * \kappa) \leq \widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \tag{21}$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * \kappa) \geq \widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))). \tag{22}$$

From (19) and (21), (20) and (22), we obtain

$$\widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) = \widetilde{\mathcal{U}_n}(\vartheta * \kappa)$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) = \widetilde{\mathcal{U}_p}(\vartheta * \kappa).$$

$(3 \Rightarrow 1)$ Assume that for all $\vartheta, \kappa \in \Omega$,

$$\widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) = \widetilde{\mathcal{U}_n}(\vartheta * \kappa) \tag{23}$$

and

$$\widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) = \widetilde{\mathcal{U}_p}(\vartheta * \kappa). \tag{24}$$

Since $(\vartheta * \kappa) * ((\vartheta * \kappa) * \hbar) \leq t$, using K_1. Therefore, by Lemma 1, we have

$$\widetilde{\mathcal{U}_n}(\vartheta * \kappa) \leq \widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar) \vee \widetilde{\mathcal{U}_n}(\hbar) \tag{25}$$

and
$$\widetilde{\mathcal{U}_p}(\vartheta * \kappa) \geq \widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar) \wedge \widetilde{\mathcal{U}_p}(\hbar). \tag{26}$$

Substitute (23) in (25) and (24) in (26), so
$$\widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \leq \widetilde{\mathcal{U}_n}((\vartheta * \kappa) * \hbar) \vee \widetilde{\mathcal{U}_n}(\hbar)$$
and
$$\widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \geq \widetilde{\mathcal{U}_p}((\vartheta * \kappa) * \hbar) \wedge \widetilde{\mathcal{U}_p}(\hbar).$$

Hence, $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a *BF-BCI-CI* of Ω. □

Theorem 7. *Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a bipolar fuzzy closed ideal of Ω. Then, $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a BF-BCI-CI of $\Omega \Leftrightarrow$*

(1) $\widetilde{\mathcal{U}_n}(\vartheta * (\kappa * (\kappa * \vartheta))) \leq \widetilde{\mathcal{U}_n}(\vartheta * \kappa)$,

(2) $\widetilde{\mathcal{U}_p}(\vartheta * (\kappa * (\kappa * \vartheta))) \geq \widetilde{\mathcal{U}_p}(\vartheta * \kappa)$,

for all $\vartheta, \kappa \in \Omega$.

Proof. Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a *BF-BCI-CI* of Ω. Since $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a *BFCI* of Ω, so for any $\vartheta, \kappa \in \Omega, \widetilde{\mathcal{U}_n}(0 * (\vartheta * \kappa)) \leq \widetilde{\mathcal{U}_n}(\vartheta * \kappa)$ and $\widetilde{\mathcal{U}_p}(0 * (\vartheta * \kappa)) \geq \widetilde{\mathcal{U}_p}(\vartheta * \kappa)$. From K_1 and P_5, we obtain
$$(\vartheta * (\kappa * (\kappa * \vartheta))) * (\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \leq 0 * (\vartheta * \kappa).$$

So, by Lemma 1, we have
$$\widetilde{\mathcal{U}_n}(\vartheta * (\kappa * (\kappa * \vartheta))) \leq \widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \vee \widetilde{\mathcal{U}_n}(0 * (\vartheta * \kappa))$$
and
$$\widetilde{\mathcal{U}_p}(\vartheta * (\kappa * (\kappa * \vartheta))) \geq \widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) \wedge \widetilde{\mathcal{U}_p}(0 * (\vartheta * \kappa)).$$

By Theorem 6, we have
$$\widetilde{\mathcal{U}_n}(\vartheta * (\kappa * (\kappa * \vartheta))) \leq \widetilde{\mathcal{U}_n}(\vartheta * \kappa) \vee \widetilde{\mathcal{U}_n}(0 * (\vartheta * \kappa)) = \widetilde{\mathcal{U}_n}(\vartheta * \kappa)$$
and
$$\widetilde{\mathcal{U}_p}(\vartheta * (\kappa * (\kappa * \vartheta))) \geq \widetilde{\mathcal{U}_p}(\vartheta * \kappa) \wedge \widetilde{\mathcal{U}_p}(0 * (\vartheta * \kappa)) = \widetilde{\mathcal{U}_p}(\vartheta * \kappa).$$

($\Leftarrow$) Let $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ be a *BFCI* of Ω satisfying the conditions $\widetilde{\mathcal{U}_n}(\vartheta * (\kappa * (\kappa * \vartheta))) \leq \widetilde{\mathcal{U}_n}(\vartheta * \kappa)$ and $\widetilde{\mathcal{U}_p}(\vartheta * (\kappa * (\kappa * \vartheta))) \geq \widetilde{\mathcal{U}_p}(\vartheta * \kappa)$, for all $\vartheta, \kappa \in \Omega$. From K_1, P_5 and P_6, we have $(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) * (\vartheta * (\kappa * (\kappa * \vartheta))) \leq 0 * (0 * (\vartheta * \kappa))$. By Lemma 1, we have
$$\begin{aligned}\widetilde{\mathcal{U}_n}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) &\leq \widetilde{\mathcal{U}_n}(\vartheta * (\kappa * (\kappa * \vartheta))) \vee \widetilde{\mathcal{U}_n}(0 * (0 * (\vartheta * \kappa))) \\ &\leq \widetilde{\mathcal{U}_n}(\vartheta * \kappa) \vee \widetilde{\mathcal{U}_n}(0 * (0 * (\vartheta * \kappa)))) \\ &= \widetilde{\mathcal{U}_n}(\vartheta * \kappa)\end{aligned}$$
and
$$\begin{aligned}\widetilde{\mathcal{U}_p}(\vartheta * ((\kappa * (\kappa * \vartheta)) * (0 * (0 * (\vartheta * \kappa))))) &\geq \widetilde{\mathcal{U}_p}(\vartheta * (\kappa * (\kappa * \vartheta))) \wedge \widetilde{\mathcal{U}_p}(0 * (0 * (\vartheta * \kappa))) \\ &\geq \widetilde{\mathcal{U}_p}(\vartheta * \kappa) \wedge \widetilde{\mathcal{U}_p}(0 * (0 * (\vartheta * \kappa)))) \\ &= \widetilde{\mathcal{U}_p}(\vartheta * \kappa).\end{aligned}$$

Hence, by Theorem 6, $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}_n}, \widetilde{\mathcal{U}_p})$ is a *BF-BCI-CI* of Ω. □

5. Conclusions

The "world of science" and its "related fields" have accomplished such complicated processes for which consistent and complete information is not always conceivable. For the last few decades, a number of theories and postulates have been introduced by many researchers to handle indeterminate constituents in science and technologies. These theories include "the theory of probability", "interval mathematics", "fuzzy set theory", "neutrosophic set theory", "intuitionistic fuzzy set theory", "bipolar fuzzy set theory", etc. In the present paper, we applied the bipolar fuzzy set theory to an algebraic structure called BCI-algebra where the concepts of bipolar fuzzy (closed) BCI-positive implicative ideals and bipolar fuzzy (closed) BCI-commutative ideals of BCI-algebras are introduced. Moreover, the relationship between bipolar fuzzy (closed) BCI-positive implicative ideals and bipolar fuzzy ideals is investigated, and various conditions are provided for a bipolar fuzzy ideal to be a bipolar fuzzy BCI-positive implicative ideal. Furthermore, conditions are presented for a bipolar fuzzy (closed) ideal to be a bipolar fuzzy BCI-commutative ideal. Finally, the relationships among bipolar fuzzy BCI-implicative ideals, bipolar fuzzy BCI-positive implicative ideals and bipolar fuzzy BCI-commutative ideals are investigated. In future work, one may extend these concepts to various algebras *BL*-algebras, *MTL*-algebras, *R0*-algebras, *MV*-algebras, *EQ*-algebras, lattice implication algebras, etc.

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