

Article

Stochastic Finite-Time Stability for Stochastic Nonlinear Systems with Stochastic Impulses

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Abstract: In this paper, some novel stochastic finite-time stability criteria for stochastic nonlinear systems with stochastic impulse effects are established. The results in this paper blackgeneralized the related results in from two aspects: 1. the model in is the deterministic systems, which means that the noise effect that can be described as a symmetric Markov process Brownian motion is considered in our models; 2. the stochastic finite-time stability criterion is established in this paper, not the asymptotic stability and the input-to-state stability that are studied in the form literature. Finally, an example is given to show the significance blackand usefulness of our results.

Keywords: stochastic impulse; stochastic nonlinear system; stochastic finite-time stability

1. Introduction

Impulsive stochastic nonlinear systems is an important class of hybrid stochastic systems, which can be used in multi-agent systems, dynamic networks and impulsive control. For instance, Ref. [1] considered the exponential consensus for stochastic multi-agent systems by using impulsive control. Ref. [2] studied the synchronization of stochastic dynamical networks under the impulsive control approach. Ref. [3] obtained some leader-following consensus results for impulsive stochastic delayed multi-agent systems. Stability is one of the most key properties in the research of the related questions in the literature [1–3]. Up to date now, most of the stability criteria have been established for impulsive stochastic nonlinear systems with deterministic impulse effects, i.e., the impulses occur at the deterministic instants. For example, Refs. [4–6] considered the asymptotic stability for impulsive stochastic (delayed) systems by using different approaches. Moreover, Refs. [7,8] considered the exponential stability for impulsive stochastic delay differential systems. Ref. [9] established the exponential stability for neutral impulsive stochastic delay differential systems. Ref. [10] designed an impulsive controller for stochastic recurrent neural networks. Ref. [11] studied the stability for impulsive stochastic differential equations driven by G-Brownian motion. For the other asymptotic stability criteria of stochastic nonlinear systems with impulse effects at deterministic instants, please refer to the references in [5,6]. In many cases, however, it can be found that the impulses often occur at random instants. From this point, Refs. [12,13] studied the asymptotic stability for stochastic nonlinear (delayed) systems with impulse effects at random times by using the stochastic processes theory. For the stability and its related application in some others fields, such as the teaching model of education information, children’s mental heath prevention and control, please see the [14–20] and the reference therein.

Recently, Ref. [21] considered the asymptotic stability and input-to-state stability for deterministic nonlinear systems with stochastic impulses. Here, the stochastic impulses are determined by a continuous-time Markov chain. However, there are still some limitations for the result of this model. Firstly, the noise is ignored in the model of [21], which cannot describe the stochastic phenomena in the real world. As is known to all, the noise effect can be described by a Brownian motion, which is a symmetric Markov process. Additionally, asymptotic stability cannot ensure that the system state trend to equilibrium point in a



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finite time. However, in the real world, such as industrial production and the robustness for the systems, whether the system state can trend to equilibrium point in a finite time is crucial. In order to describe whether the system state can trend to equilibrium point in a finite time, the concept finite-time stability has been proposed. Refs. [22,23] considered the finite-time stability for deterministic autonomous and non-autonomous nonlinear systems, respectively. Ref. [24] studied the finite-time stability for deterministic nonlinear delayed systems. By using the probability theory, Refs. [25,26] provide a new stability concept, which is called stochastic finite-time stability for stochastic nonlinear systems. There are some important results for stochastic finite-time stability. For example, Ref. [27] obtained some probability properties for stochastic finite-time stable stochastic nonlinear systems. Ref. [26] obtained some stochastic finite-time stability and stochastic finite-time instability criteria for autonomous stochastic systems. After that, Refs. [28–34] considered how to design the finite-time state feedback stabilizers for some stochastic nonlinear systems by using the stability criteria in [26] and the backstepping technology. Recently, Refs. [35–37] derived some more widely used stochastic finite-time stable criteria under the framework of weak solutions for stochastic differential equations.

It should be noted that for stochastic nonlinear systems with stochastic impulse effects, whether the systems states can reach the equilibrium point in a finite time is still unknown. Thus, motivated by the above analysis and discussions, in this paper, we focus on the stochastic finite-time stability for stochastic nonlinear systems with stochastic impulse effects. Compared to [21], the main differences we considered in this paper are that the model we considered contains noise effects and the type of the stability we considered in this paper is the stochastic finite-time stability. By using the Markov chain theory, stochastic analysis theory and Lyapunov method, we overcome the difficulties that arise from the noise effects and the stochastic impulse effects. Some useful criteria are established.

The rest of the paper is organized as follows: In Section 2, we introduce the model of stochastic nonlinear systems with stochastic impulse effects, and then we give some definitions. In Section 3, a novel stochastic finite-time stability criterion is established for stochastic nonlinear systems with stochastic impulse effects. Some important remarks are also displayed. An example is provided in Section 4 in order to demonstrate the significance and usefulness of our results. Finally, in Section 5, we give some more general remarks for the main results of this paper.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual condition (i.e., it is right continuous and \mathcal{F}_0 contains all \mathbb{P} -null sets). We use $\mathbb{E}[\cdot]$ to denote the expectation operator with respect to \mathbb{P} . Let $B(t) = (B_1(t), B_2(t), \dots, B_m(t))^T$ be an m -dimensional Brownian motion defined on a complete probability space. Ψ denotes all the functions that are continuously twice differentiable in $x \in \mathbb{R}^n$ and once differentiable in t . Φ denotes all the functions that are continuously twice differentiable in $x \in \mathbb{R}^n \setminus \{0\}$ and once differentiable in t . The symbol “ C ” denotes a constant whose precise value is not important.

In this section, we consider the following stochastic nonlinear system with stochastic impulse effects:

$$\begin{cases} dx(t) = f(t, x(t))dt + g(t, x(t))dB(t), & t \geq t_0 \geq 0, t \neq \tau_k, \\ x(\tau_k) = H_{r(\tau_k)}(x(\tau_k^-)), & t = \tau_k, k = 1, 2, \dots, \end{cases} \quad (1)$$

where $\{r(\tau_k), k = 1, 2, \dots\}$ is the embedded chain for the Markov chain $\{r(t), t \geq t_0\}$, which takes values in $\Gamma = \{1, 2, \dots, N\}$. There are N different impulse gains, and $r(\tau_k) = i$ implies that the i th impulse occurs at instant τ_k (Convention: $\tau_0 = t_0$.)

We need the definition of stochastic finite-time stability. See, e.g., [35,36].

Definition 1. The solution $x(t, x_0)$ of (1) is stochastic finite-time stable if the following two properties meet:

(1) *Stable in probability:* For any $r > 0$, we have

$$\lim_{x_0 \rightarrow 0} \mathbb{P} \left(\sup_{t \geq t_0} |x(t, x_0)| \geq r \right) = 0$$

(2) *Finite-time attractiveness in probability:* The random time $\varrho_0 = \inf\{t \geq t_0 : x(t) = 0\}$ is finite a.s., and $x(t + \varrho_0) = 0$, a.s. $\forall t \geq 0$.

Remark 1. Note that if $H_k(\tau_k, x(\tau_k^-)) = 0$ for some $k = 1, 2, \dots$, then according to the definition of trivial solution, we can see that the finite-time attractiveness in probability holds obviously. Thus, in order to avoid the trivial, we assume that $H_k(\tau_k, x(\tau_k^-)) \neq 0$ for all $k = 1, 2, \dots$. Moreover, we can see that the drift coefficient $f(t, x)$ and the diffusion coefficient $g(t, x)$ of System (1) cannot satisfy the local Lipschitz condition simultaneously. Otherwise, according to Lemma 3.2, Page 120 in [38], it is nonsense to consider the stochastic finite-time stability for System (1) because the system state cannot reach the equilibrium point in a finite time.

Define the operator $\mathcal{L}U$ from $\mathbb{R}^+ \times \mathbb{R}^n$ to \mathbb{R} by

$$\mathcal{L}U(t, x) = \frac{\partial U(t, x)}{\partial x} + \frac{\partial U(t, x)}{\partial x} f(t, x) + \frac{1}{2} \text{trace}[g^T(t, x) \frac{\partial^2 U(t, x)}{\partial x^2} g(t, x)].$$

3. Main Results

In order to study the stability for System (1), the existence of the solution for System (1) should be considered previously. The following proposition gives the existence of the solution for System (1).

Proposition 1. If there exist a positive definite and radially unbounded function $U \in \Psi$, constants $\alpha \geq 0$, $\beta_{r(\tau_k)} > 0$ and $\theta > 0$ satisfy:

(A) For all $t \geq t_0$ and $t \neq \tau_k, k = 1, 2, \dots$,

$$\mathcal{L}U(t, x(t)) \leq \alpha U(t, x(t)).$$

(B) For all $t = \tau_k, k = 1, 2, \dots$,

$$U(\tau_k, x(\tau_k)) \leq \beta_{r(\tau_k)} U(\tau_k^-, x(\tau_k^-)).$$

(C) $\theta \leq \inf_i \frac{q_i}{q_i + \alpha}$, where $\theta = \max_i \sum_{j \neq i} p_{ij} \beta_j$.

Then, for any initial data x_0 , the global weak solution exists for System (1).

Proof. First of all, we prove the existence of the solution for System (1). For any $t \geq t_0$, we use $N(t)$ to denote the impulse number during the time interval $[0, t)$. If there are k impulse instants during the time interval $[0, t)$, for $i = 0, 1, 2, \dots, k$, then we define

$$\nu_m^k = \begin{cases} \inf\{t > \tau_k : |x(t)| > m\}, & \text{if } |x(\tau_k)| \leq m, \\ \tau_k, & \text{if } |x(\tau_k)| > m. \end{cases}$$

According to Theorem 1 in [39], there exists a weak solution on $[t_0, \tau_1 \wedge \nu_\infty^0)$. Note that for any $t \in [t_0, \tau_1)$,

$$\begin{aligned}
 & \mathbb{E}[U(x(v_m^0 \wedge t))e^{-\alpha(v_m^0 \wedge t)}] \\
 & \geq U(m)\mathbb{E}[\mathbf{I}_{\{v_m^0 < t\}}e^{-\alpha v_m^0}] \\
 & \geq U(m)\mathbb{E}[\mathbf{I}_{\{v_m^0 < t\}}e^{-\alpha t}] \\
 & \geq U(m)\mathbb{E}[\mathbf{I}_{\{v_m^0 < t\}}e^{-\alpha \tau_1}] \\
 & = U(m)\mathbb{E}[\mathbf{I}_{\{v_m^0 < t\}}\mathbb{E}[e^{-\alpha \tau_1} | \mathcal{F}_{v_m^0}]] \\
 & \geq U(m)\mathbb{P}(v_m^0 < t) \inf_i \frac{q_i}{q_i + \alpha}.
 \end{aligned}$$

Moreover, by using Itô’s formula, we obtain

$$U(x(v_m^0 \wedge t))e^{-\alpha(v_m^0 \wedge t)} \leq U(x_0) + \int_{t_0}^{v_m^0 \wedge t} \frac{\partial U}{\partial x} g(s, x(s))e^{-\alpha s} dB(s),$$

Noting that $t_0 \leq v_m^0 \wedge t \leq t$, it follows from Theorem 1.28 in [38] that

$$\mathbb{E}\left(\int_{t_0}^{v_m^0 \wedge t} \frac{\partial U}{\partial x} g(s, x(s))e^{-\alpha s} dB(s)\right) = \mathbb{E}\left(\int_{t_0}^t \frac{\partial U}{\partial x} g(s, x(s))e^{-\alpha s} \mathbf{I}_{\{s \leq v_m^0\}} dB(s)\right) = 0,$$

and so

$$\mathbb{E}[U(x(v_m^0 \wedge t))e^{-\alpha(v_m^0 \wedge t)}] \leq U(x_0),$$

which implies that $\mathbb{P}(v_m^0 < t) \leq \frac{U(x_0)}{U(m)}$. Letting $t \rightarrow \tau_1^-$, and then letting $m \rightarrow \infty$, we can obtain that $\mathbb{P}(v_\infty^0 \geq \tau_1) = 1$, which verifies that the weak solution exists in $[t_0, \tau_1)$. The existence in $t = \tau_1$ is obvious.

Next, we assume that for any $i = 1, 2, \dots, k - 1$, the solution exists in $[\tau_i, \tau_{i+1})$, and then we show that the solution exists in $[\tau_k, \tau_{k+1})$. On one hand, we can conclude that

$$\begin{aligned}
 & \mathbb{E}[U(x(v_m^k \wedge t))e^{-\alpha(v_m^k \wedge t)}] \\
 & \geq U(m)\mathbb{E}[\mathbf{I}_{\{v_m^k < t\}}e^{-\alpha v_m^k}] \\
 & \geq U(m)\mathbb{E}[\mathbf{I}_{\{v_m^k < t\}}e^{-\alpha t}] \\
 & \geq U(m)\mathbb{E}[\mathbf{I}_{\{v_m^k < t\}}e^{-\alpha \tau_{k+1}}] \\
 & = U(m)\mathbb{E}[\mathbf{I}_{\{v_m^k < t\}}\mathbb{E}[e^{-\alpha \tau_{k+1}} | \mathcal{F}_{v_m^k}]] \\
 & = U(m)\mathbb{E}[\mathbf{I}_{\{v_m^k < t\}}\mathbb{E}[e^{-\alpha \tau_{k+1}} | r(\tau_k), r(\tau_{k-1}), \dots, r(\tau_1)]] \\
 & \geq U(m)\mathbb{P}(v_m^k < t) \left(\inf_i \frac{q_i}{q_i + \alpha}\right)^{k+1}.
 \end{aligned}$$

On the other hand, by using Itô’s formula, it follows that

$$U(x(v_m^k \wedge t))e^{-\alpha(v_m^k \wedge t)} \leq U(x(\tau_k))e^{-\alpha \tau_k} + \int_{\tau_k}^{v_m^k \wedge t} \frac{\partial U}{\partial x} g(s, x(s))e^{-\alpha s} dB(s).$$

Noting that $t_0 \leq \tau_k \leq v_m^k \wedge t \leq t$, it follows from Theorem 5.16 in [38] that

$$\mathbb{E}\left(\int_{\tau_k}^{v_m^k \wedge t} \frac{\partial U}{\partial x} g(s, x(s))e^{-\alpha s} dB(s)\right) = \mathbb{E}\left(\int_{\tau_k}^t \frac{\partial U}{\partial x} g(s, x(s))e^{-\alpha s} \mathbf{I}_{\{s \leq v_m^k\}} dB(s)\right) = 0.$$

By using the fact that the solution exists on the interval $[\tau_i, \tau_{i+1})$, $i = 1, 2, \dots, k - 1$, it can be derived that

$$\begin{aligned}
 & \mathbb{E}[U(x(v_m^k \wedge t))e^{-\alpha(v_m^k \wedge t)}] \\
 & \leq \mathbb{E}[U(x(\tau_k))e^{-\alpha\tau_k}] \\
 & \leq \mathbb{E}[\beta_{r(\tau_k)}U(x(\tau_k^-))e^{-\alpha\tau_k}] \\
 & = \mathbb{E}[U(x(\tau_k^-))e^{-\alpha\tau_k}\mathbb{E}[\beta_{r(\tau_k)}|\mathcal{F}_{\tau_k^-}]] \\
 & = \mathbb{E}[U(x(\tau_k^-))e^{-\alpha\tau_k}\mathbb{E}[\beta_{r(\tau_k)}|r(\tau_{k-1})]] \\
 & \leq \theta \mathbb{E}[U(x(\tau_k^-))e^{-\alpha\tau_k}] \\
 & \leq \dots \\
 & \leq \theta^k U(x_0).
 \end{aligned}$$

Thus, we have

$$\mathbb{P}(v_m^k < t) \leq \frac{1}{\theta} \frac{U(x_0)}{U(m)} \left(\frac{\theta}{\inf_i \frac{q_i}{q_i + \alpha}} \right)^{k+1} \leq C \frac{U(x_0)}{U(m)}. \tag{2}$$

By the arbitrariness of t , it follows that

$$\mathbb{P}(v_m^k < \tau_{k+1}) \leq \frac{U(x_0)}{U(m)}.$$

Letting $m \rightarrow \infty$ and using the total probability formula, we obtain $\mathbb{P}(v_\infty^{N(t)} < \tau_{N(t)+1}) = 0$. Thus, we have $\mathbb{P}(v_\infty^{N(t)} > \tau_{N(t)+1}) = 1$, which implies that for any $t \geq t_0$, the solution of System (1) exists on the intervals $[\tau_i, \tau_{i+1})$, $i = 0, 1, 2, \dots, N(t)$. For the arbitrariness of t , and noting that $N(t) \rightarrow \infty$ as $t \rightarrow \infty$, we can see that the solution of System (1) exists on $[t_0, \infty)$. \square

Remark 2. It should be noted that in Remark 2.1 in [33], it is only necessary to consider the existence of a solution to System (1) because the uniqueness of the solution is a very restrictive condition in the research of stochastic finite-time stability, which may be an obstacle for further applications. Moreover, in [39], the existence of the global weak solution was established, and then the asymptotic stability and the asymptotic stability of the weak solution were also obtained for the stochastic nonlinear systems with non-local Lipschitz coefficients.

It should be noted that when $g(t, x(t)) \equiv 0$ in System (1), the asymptotic stability and the input-to-state stability were considered in [21]. In the conclusion part of [21], the authors mentioned that it is valuable to extend the results to stochastic systems. The following result shows that under some mild conditions, not only the stability for System (1) can be obtained, but also the stability can be achieved in a finite time.

Theorem 1. Under the conditions (A–C), if there exist another positive definite function $V \in \Phi$, constants $c \geq 0$, $\hat{\beta}_{r(\tau_k)}$ and $\hat{\theta} > 0$ satisfying:

(D) For all $t \geq t_0$ and $t \neq \tau_k, k = 1, 2, \dots$,

$$\mathcal{L}V(t, x(t)) \leq -c.$$

(E) For all $t = \tau_k, k = 1, 2, \dots$,

$$V(\tau_k, x(\tau_k)) \leq \hat{\beta}_{r(\tau_k)} V(\tau_k^-, x(\tau_k^-)).$$

(F) $\hat{\theta} := \max_i \sum_{j \neq i} p_{ij} \hat{\beta}_j \leq 1$.

Then, the solution of System (1) is stochastic finite-time stable.

Proof. Firstly, we prove the stability in probability for the solution of System (1). Define $\sigma_r = \inf\{t : |x(t)| > r\}$. By using the total probability formula, we can see that for any $t \geq t_0$,

$$\mathbb{P}(\sigma_r < t) = \sum_{k=0}^{\infty} \mathbb{P}(\sigma_r < t | N(t) = k) \mathbb{P}(N(t) = k).$$

However,

$$\mathbb{P}(\sigma_r < t | N(t) = k) = \sum_{i=0}^k \mathbb{P}(\tau_i \leq \sigma_r < \tau_{i+1} \wedge t | N(t) = k).$$

According to Equation (2), it follows that for any $i = 0, 1, 2 \dots, k$,

$$\mathbb{P}(\tau_i \leq \sigma_r < \tau_{i+1} \wedge t | N(t) = k) \leq C \frac{U(x_0)}{U(r)} \left(\frac{\theta}{\inf_i \frac{q_i}{q_i + \alpha}} \right)^i,$$

which implies

$$\begin{aligned} \mathbb{P}(\sigma_r < t) &\leq C \sum_{k=0}^{\infty} \sum_{i=0}^k \frac{U(x_0)}{U(r)} \left(\frac{\theta}{\inf_i \frac{q_i}{q_i + \alpha}} \right)^i \mathbb{P}(N(t) = k) \\ &\leq C \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{U(x_0)}{U(r)} \left(\frac{\theta}{\inf_i \frac{q_i}{q_i + \alpha}} \right)^i \mathbb{P}(N(t) = k) \\ &\leq C \sum_{i=0}^{\infty} \frac{U(x_0)}{U(r)} \left(\frac{\theta}{\inf_i \frac{q_i}{q_i + \alpha}} \right)^i \\ &= C \frac{U(x_0)}{U(r)}. \end{aligned}$$

Now letting $t \rightarrow \infty$ and then letting $x_0 \rightarrow 0$, we can see that $\lim_{x_0 \rightarrow 0} \mathbb{P}(\sigma_r < \infty) = 0$, i.e., the solution is stable in probability.

Additionally, we prove the solution is finite-time attractive in probability. Assume that $x_0 \neq 0$. Note that for any increasing sequence $\{k_n\}_{n=1}^{\infty}$, ξ_{k_n} increases as k_n . It is not hard to see that $\{\xi_{\infty} = \infty\} \subset \cap_{n=1}^{\infty} \cup_{k_n=1}^{\infty} \{\xi_{k_n} \wedge k_n > \tau_n\}$. Define $A_n = \cap_{i=1}^n \cup_{k_i=1}^{\infty} \{\xi_{k_i} \wedge k_i > \tau_i\}$, $A = \lim_{n \rightarrow \infty} A_n$.

For any $t, u \in [\tau_{n-1} \wedge \xi_{k_{n-1}} \wedge k_{n-1}, \tau_n \wedge \xi_{k_n} \wedge k_n)$, by using Itô's formula and condition (D), we obtain

$$V(x(u)) \mathbf{I}_{A_{n-1}} \leq V(x(t)) \mathbf{I}_{A_{n-1}} - c(u - t) \mathbf{I}_{A_{n-1}} + \left(\int_t^u \frac{\partial V}{\partial x} g(s, x(s)) dB(s) \right) \mathbf{I}_{A_{n-1}}.$$

Note that $A_{n-1} \in \mathcal{F}_{\tau_{n-1} \wedge \xi_{k_{n-1}} \wedge k_{n-1}} \subset \mathcal{F}_t$, and so we have

$$\begin{aligned} \mathbb{E} \left(\int_t^u \frac{\partial V}{\partial x} g(s, x(s)) dB(s) \mathbf{I}_{A_{n-1}} \right) &= \mathbb{E} \left(\mathbb{E} \left(\int_t^u \frac{\partial V}{\partial x} g(s, x(s)) dB(s) \mathbf{I}_{A_{n-1}} \middle| \mathcal{F}_t \right) \right) \\ &= \mathbb{E} \left(\mathbf{I}_{A_{n-1}} \mathbb{E} \left(\int_t^u \frac{\partial V}{\partial x} g(s, x(s)) dB(s) \middle| \mathcal{F}_t \right) \right) \\ &= 0, \end{aligned}$$

which implies

$$\mathbb{E}[V(x(u)) \mathbf{I}_{A_{n-1}}] \leq \mathbb{E}[V(x(t)) \mathbf{I}_{A_{n-1}}] - c \mathbb{E}[(u - t) \mathbf{I}_{A_{n-1}}].$$

Taking $u \rightarrow \tau_n^- \wedge \zeta_{k_n} \wedge k_n$ and $t \rightarrow \tau_{n-1} \wedge \zeta_{k_{n-1}} \wedge k_{n-1}$, according to the dominate convergence theorem, we obtain

$$\begin{aligned} & \mathbb{E}[V(x(\tau_n^- \wedge \zeta_{k_n} \wedge k_n))\mathbf{I}_{A_{n-1}}] \\ & \leq \mathbb{E}[V(x(\tau_{n-1} \wedge \zeta_{k_{n-1}} \wedge k_{n-1}))\mathbf{I}_{A_{n-1}}] - c\mathbb{E}[(\tau_n^- \wedge \zeta_{k_n} \wedge k_n - \tau_{n-1} \wedge \zeta_{k_{n-1}} \wedge k_{n-1})\mathbf{I}_{A_{n-1}}]. \end{aligned}$$

Since

$$\begin{aligned} & \mathbb{E}[V(x(\tau_{n-1} \wedge \zeta_{k_{n-1}} \wedge k_{n-1}))\mathbf{I}_{A_{n-1}}] \\ & = \mathbb{E}[\hat{\beta}_{r(\tau_{n-1})}V(x(\tau_{n-1}^- \wedge \zeta_{k_{n-1}} \wedge k_{n-1}))\mathbf{I}_{A_{n-1}}] \\ & \leq \mathbb{E}[\hat{\beta}_{r(\tau_{n-1})}V(x(\tau_{n-1}^- \wedge \zeta_{k_{n-1}} \wedge k_{n-1}))\mathbf{I}_{A_{n-2}}] \\ & \leq \mathbb{E}[\mathbb{E}[\hat{\beta}_{r(\tau_{n-1})}V(x(\tau_{n-1}^- \wedge \zeta_{k_{n-1}} \wedge k_{n-1}))\mathbf{I}_{A_{n-2}}|\mathcal{F}_{\tau_{n-1}^-}]] \\ & \leq \mathbb{E}[V(x(\tau_{n-1}^- \wedge \zeta_{k_{n-1}} \wedge k_{n-1}))\mathbf{I}_{A_{n-2}}E[\hat{\beta}_{r(\tau_{n-1})}|\mathcal{F}_{\tau_{n-1}^-}]] \\ & \leq \hat{\theta}\mathbb{E}[V(x(\tau_{n-1}^- \wedge \zeta_{k_{n-1}} \wedge k_{n-1}))\mathbf{I}_{A_{n-2}}], \end{aligned}$$

we have

$$\begin{aligned} & \mathbb{E}[V(x(\tau_n^- \wedge \zeta_{k_n} \wedge k_n))\mathbf{I}_{A_{n-1}}] \\ & \leq \hat{\theta}\mathbb{E}V(x(\tau_{n-1}^- \wedge \zeta_{k_{n-1}} \wedge k_{n-1})\mathbf{I}_{A_{n-2}}) - c\mathbb{E}((\tau_n^- \wedge \zeta_{k_n} \wedge k_n - \tau_{n-1} \wedge \zeta_{k_{n-1}} \wedge k_{n-1})\mathbf{I}_{A_{n-1}}) \\ & \leq \dots \\ & \leq \hat{\theta}^{n-1}V(x_0) - c \sum_{i=0}^{n-1} \hat{\theta}^i \mathbb{E}[(\tau_{n-i}^- \wedge \zeta_{k_{n-i}} \wedge k_{n-i} - \tau_{n-i-1} \wedge \zeta_{k_{n-i-1}} \wedge k_{n-i-1})\mathbf{I}_{A_{n-i-1}}]. \end{aligned}$$

It follows that

$$\begin{aligned} V(x_0) & \geq c \sum_{i=1}^n \hat{\theta}^{1-i} \mathbb{E}[(\tau_i^- \wedge \zeta_{k_i} \wedge k_i - \tau_{i-1} \wedge \zeta_{k_{i-1}} \wedge k_{i-1})\mathbf{I}_{A_{i-1}}] \\ & = c \sum_{i=0}^{n-1} \hat{\theta}^{-i} \mathbb{E}[(\tau_{i+1}^- \wedge \zeta_{k_{i+1}} \wedge k_{i+1} - \tau_i \wedge \zeta_{k_i} \wedge k_i)\mathbf{I}_{A_i}] \\ & \geq c \sum_{i=0}^{n-1} \hat{\theta}^{-i} \mathbb{E}[(\tau_{i+1} - \tau_i)\mathbf{I}_A]. \end{aligned}$$

Assuming that $\mathbb{P}(A) > 0$. Letting $n \rightarrow \infty$, from the monotone convergence theorem and the fact that $\inf_i \mathbb{E}[(\tau_{i+1} - \tau_i)\mathbf{I}_A] > 0$, it follows that

$$\begin{aligned} V(x_0) & \geq c \sum_{i=1}^{\infty} \hat{\theta}^{-i} \mathbb{E}[(\tau_{i+1} - \tau_i)\mathbf{I}_A] \\ & \geq c \sum_{i=1}^{\infty} \hat{\theta}^{-i} \inf_i \mathbb{E}[(\tau_{i+1} - \tau_i)\mathbf{I}_A] \\ & = \infty, \end{aligned}$$

which yields a contradiction. Here, we use the fact that $\tau_{i+1} - \tau_i$ is an exponential distribution random variable. Its parameter takes value in the set $\{q_1, q_2, \dots, q_N\}$. Thus, we can obtain $\mathbb{P}(A) = 0$, which implies that the solution is finite-time attractive in probability. Hence, according to Definition 1, we see that the solution is stochastic finite-time stable. \square

Remark 3. From our result, we can see that the transition probability of the embedded chain and impulse gains play some important roles in keeping the stochastic finite-time stability. We compare our result with Corollary 3.3 in [21]. Firstly, in [21], the noise effect is ignored because there are some essential difficulties in handling the stochastic integral and stopping time sequence

$\{\tau_k, k = 1, 2, \dots\}$. We use the stochastic analysis theory to overcome these difficulties. Additionally, our result shows that not only the stochastic systems can keep stable but also stochastic finite-time stable, which is more widely used in the real world.

Remark 4. In [12,13], the author also obtained some useful stability criteria for impulsive stochastic nonlinear systems with impulse effects at random times. As is known to all, the impulse effects can be divided into two types: the stabilizing impulse (i.e., $\hat{\beta} < 1$) and the destabilizing impulse (i.e., $\hat{\beta} > 1$). It should be noted that the impulse effect in [12,13] is either the stabilizing impulse or the destabilizing impulse in all the impulse instants, which means that the the stabilizing impulse or the destabilizing impulse cannot exist in the impulse instants $\{\tau_k\}_{k=1}^\infty$ simultaneously. However, in this paper, the impulse gain takes value at the set Γ subjected to the Markov chain $r(t)$, which means that the impulse type may be different in every impulse instant (i.e., multiply impulse effects). Moreover, only the asymptotic stability is considered in [12,13]. However, in this paper, the stochastic finite-time stability criterion is established.

Remark 5. There are also some researchers that considered the stability for the deterministic systems with multiple impulse effects; see [40] and the references therein. In [40], only the asymptotic stability and input-to-state stability are considered. Moreover, the noise effect is also ignored, and the impulse gains are all deterministic. However, in this paper, the model we considered is a stochastic system, and the impulse effects are all stochastic. Meanwhile, we find that under some mild conditions, not only the stability for System (1) can be obtained, but also the stability can be achieved in a finite time.

The following corollary can be checked more easily than Theorem 1. Since the proof is similar to [35] Theorem 2, we omit it.

Corollary 1. Theorem 1 still holds if conditions (D–F) are replaced by
 (D') For all $t \geq t_0$ and $t \neq \tau_k, k = 1, 2, \dots$,

$$K(V(t, x))[\mathcal{L}V(t, x) + cK(V(t, x))] \leq \frac{1}{2}K'(V(t, x))|V_x(t, x)g(t, x)|^2.$$

(E') For all $t = \tau_k, k = 1, 2, \dots$,

$$\int_0^{V(\tau_k, x(\tau_k))} \frac{ds}{K(s)} \leq \tilde{\beta}_{r(\tau_k)} \int_0^{V(\tau_k, x(\tau_k^-))} \frac{ds}{K(s)}.$$

(F') $\tilde{\theta} := \max_i \sum_{j \neq i} p_{ij} \tilde{\beta}_j \leq 1$.

Here, $K(\cdot)$ is a piecewise continuous function satisfying $K(s) > 0$ and $K'(s) \geq 0$.

4. Apply to a Stochastic Nonlinear Systems with Stochastic Impulses

In this section, we consider an example to illustrate the validity of our result.

Example 1. Consider the stochastic nonlinear system with stochastic impulse effects as follows:

$$\begin{cases} dx(t) = [-0.4x(t) - 0.5x(t)^{\frac{1}{3}}]dt + x(t)dB(t), & t \geq t_0 \geq 0, \quad t \neq \tau_k \\ x_{\tau_k} = I_{r(\tau_k)}x(\tau_k^-), & t = \tau_k \end{cases} \tag{3}$$

where $x_0 = 0.5$. The Markov chain $r(t)$ take values in $\Gamma = \{1, 2, 3, 4\}$, and the generator Q of $r(t)$ is given by

$$Q = \begin{bmatrix} -2 & 0.5 & 0.8 & 0.7 \\ 0.2 & -1 & 0.3 & 0.5 \\ 0.4 & 0.6 & -2 & 1 \\ 0.2 & 0.1 & 0.7 & -1 \end{bmatrix}.$$

The random impulse $I_{r(\tau_k)}$ takes values in $\Sigma = \{1.1, 1.2, 0.4, 0.5\}$.

Take the Lyapunov function $U(t, x(t)) = x(t)^2$. By using Itô's formula, we can obtain $\mathcal{L}U(t, x(t)) = 0.2x(t)^2 - x(t)^{\frac{4}{3}} \leq 0.2x(t)^2 = 0.2U(t, x(t))$; thus, $\alpha = 0.2$. By a direct computation, it follows that $\theta = 0.8 < \inf_i \frac{q_i}{q_i + \alpha} = 0.83$. Choose another Lyapunov function $V(t, x(t)) = |x(t)|$ and $K(s) = s^{\frac{1}{3}}$, then $\mathcal{L}U(t, x(t)) \leq -0.5V(t, x(t))^{\frac{1}{3}}$. It is not difficult to check that $\tilde{\theta} = 0.8405 < 1$ by using the definition of $\tilde{\theta}$. Thus, from Corollary 1, System (3) is stochastic finite-time stable. See Figure 1.

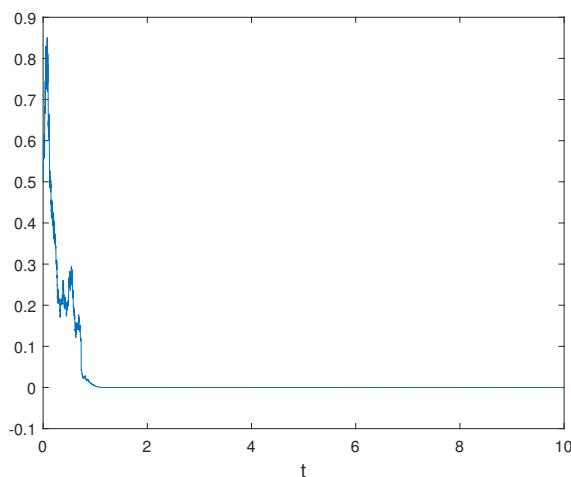


Figure 1. States for System (3).

Remark 6. Figure 1 displays the states for System (3). We can see that the states for System (3) can reach the equilibrium point in a finite time, which is different from the asymptotic convergence of the impulsive stochastic systems considered in the literature [5–13].

Remark 7. A numerical simulation for the stochastic differential equation and the stability analysis for the numerical scheme is a very hot topic in the research of stability for stochastic differential equation, see the monograph [41].

5. Conclusions

In this paper, the stochastic finite-time stability for stochastic nonlinear systems with stochastic impulse effects is considered. By using the probability theory and Lyapunov approach, some useful and easily checked stochastic finite-time stability criteria are obtained. Compare with the recent achievements in the field, the impulse gain in our model is more complex. It can be seen that the transition probability of the Markov chain and the impulse gain play some important roles in keeping the stochastic finite-time stability for the related systems. Meanwhile, the criteria also provide the approach to designing the impulse gain and the transition probability to achieve the stochastic finite-time stability. Moreover, an example is provided to show the efficiency of our result. There are some questions that still need to be considered. Firstly, how to research the same question in the frame of stochastic delayed systems. Additionally, how to use the theoretical results of the stochastic finite-time synchronization for the impulsive stochastic neural networks.

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