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Subordination Involving Regular Coulomb Wave Functions

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Abstract: The functions $\sqrt{1+z}$, e^z, $1+Az$, $A \in (0,1]$ map the unit disc $\mathbb{D}$ to a domain which is symmetric about the x-axis. The Regular Coulomb wave function (*RCWF*) $\mathtt{F}_{L,\eta}$ is a function involving two parameters L and η, and $\mathtt{F}_{L,\eta}$ is symmetric about these. In this article, we derive conditions on the parameter L and η for which the normalized form $\mathtt{f}_L$ of $\mathtt{F}_{L,\eta}$ are subordinated by $\sqrt{1+z}$. We also consider the subordination by e^z and $1+Az$, $A \in (0,1]$. A few more subordination properties involving RCWF are discussed, which leads to the star-likeness of normalized Regular Coulomb wave functions.

Keywords: star-like function; Janowski star-like; differential subordination; Regular Coulombwave functions

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Citation: Mondal, S.R. Subordination Involving Regular Coulomb Wave Functions. *Symmetry* **2022**, *14*, 1007. https://doi.org/10.3390/sym14051007

Academic Editor: Boris Malomed

Received: 8 April 2022
Accepted: 10 May 2022
Published: 16 May 2022

1. Introduction

The Regular Coulomb wave function (*RCWF*) defined in the complex plane is an entire function and closely associated with the well-known classical Bessel function. The Coulomb wave functions have a rich literature (See [1–10] and references therein) in terms of mathematical and numerical research articles and its applications in various branches of physics, especially in nuclear physics. The symmetrical property of *RCWF* is established in [11]. Entire functions have good geometric characterizations in the unit disc. In this sense, the exploration of the geometric nature of Coulomb wave functions is limited [4,12]. The aim of this article is to contribute some results on the geometric properties of *RCWF*.

The Coulomb differential equation [13] is a second-order differential equation of the form

$$\frac{d^2w}{dz^2} + \left(1 - \frac{2\eta}{z} - \frac{L(L+1)}{z^2}\right)w = 0, \quad \eta, \quad z \in \mathbb{C}, \tag{1}$$

that asserts two independent solutions, namely regular and irregular Coulomb wave functions. In terms of Kummer confluent hypergeometric functions ${}_1F_1$, the *RCWF* is defined as

$$F_{L,\eta}(z) := z^{L+1}e^{-iz}C_L(\eta)\,{}_1F_1(L+1+i\eta, 2L+2; 2iz) = C_L(\eta)\sum_{n=0}^{\infty} a_{L,n}z^{n+L+1}. \tag{2}$$

In this case, $z, \eta, L \in \mathbb{C}$ and

$$C_L(\eta) = \frac{2^L e^{\frac{\pi\eta}{2}}\left|\Gamma(L+1+i\eta)\right|}{\Gamma(2L+2)},$$

$$a_{L,0} = 1, \quad a_{L,1} = \frac{\eta}{L+1}, \quad a_{L,n} = \frac{2\eta a_{L,n-1} - a_{L,n-2}}{n(n+2L+1)}, n \in \{2, 3, \dots\}.$$

For the study of the geometric characterization of *RCWF*, we need a normalized form such as as $h(0) = 1$ or $h(0) = 0 = h'(0) - 1$. Clearly, $F_{L,\eta}$ defined in (2) does not process such a normalization. Hence, we consider the following two normalizations:

$$\mathtt{f}_L(z) = C_L^{-1}(\eta) z^{-L-1} F_{L,\eta}(z) = 1 + \tfrac{\eta}{L+1} z + \ldots\ldots \tag{3}$$
$$\mathtt{g}_L(z) = C_L^{-1}(\eta) z^{-L} F_{L,\eta}(z) = z\mathtt{f}_L(z) = z + \tfrac{\eta}{L+1} z^2 + \ldots\ldots \tag{4}$$

By a calculation, it can be shown from (1) that f_L satisfies the differential equation

$$z^2 \mathrm{y}''(z) + 2(L+1) z \mathrm{y}'(z) + (z^2 - 2\eta z)\mathrm{y}(z) = 0, \tag{5}$$

while the function g_L is the solution of the differential equation

$$z^2 \mathrm{y}''(z) + 2Lz\mathrm{y}'(z) + (z^2 - 2\eta z - 2L)\mathrm{y}(z) = 0. \tag{6}$$

Let $\mathcal{A}$ denote the class of normalized analytic functions f in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ satisfying $f(0) = 0 = f'(0) - 1$. Denote by $\mathcal{S}^*$ and $\mathcal{C}$, respectively, the widely studied subclasses of $\mathcal{A}$ consisting of univalent (one-to-one) star-like and convex functions. Geometrically, $f \in \mathcal{S}^*$ if the linear segment tw, $0 \le t \le 1$, lies completely in $f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$, while $f \in \mathcal{C}$ if $f(\mathbb{D})$ is a convex domain. Related to these subclasses is the Cáratheòdory class $\mathcal{P}$ consisting of analytic functions p satisfying $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$. Analytically, $f \in \mathcal{S}^*$ if $zf'(z)/f(z) \in \mathcal{P}$, while $f \in \mathcal{C}$ if $1 + zf''(z)/f'(z) \in \mathcal{P}$.

For two analytic functions f and g in $\mathbb{D}$, the function f is *subordinate* to g, written as $f \prec g$, or $f(z) \prec g(z)$, $z \in \mathbb{D}$, if there is an analytic self-map ω of $\mathbb{D}$ satisfying $\omega(0) = 0$ and $f(z) = g(\omega(z))$, $z \in \mathbb{D}$.

Consider now the class $\mathcal{P}[\varphi]$ of analytic functions $p(z) = 1 + c_1 z + \cdots$ in $\mathbb{D}$ satisfying $p(z) \prec \varphi(z)$, where φ is an analytic function with a positive real impact on $\mathbb{D}$, $\varphi(0) = 1$ and $\varphi'(0) > 0$. This article considers three different φ, namely $\varphi(z) = (1 + Az)/(1 + Bz)$, $\varphi(z) = \sqrt{1+z}$ and $\varphi(z) = e^z$.

For $-1 \le B < A \le 1$ and $\varphi(z) = (1 + Az)/(1 + Bz)$, denote the class as $\mathcal{P}[A, B]$. This family $\mathcal{P}[A, B]$ has been widely studied by several authors and most notably by Janowski in [14], and the class is also referred to as *Janowski class of functions*. The class $\mathcal{P}[A, B]$ contains several known classes of functions for judicious choices of A and B. For instance, if $0 \le \beta < 1$, then $\mathcal{P}[1 - 2\beta, -1]$ is the class of functions $p(z) = 1 + c_1 z + \cdots$ satisfying $\operatorname{Re} p(z) > \beta$ in $\mathbb{D}$. In the limiting case $\beta = 0$, the class reduces to the classical Cáratheòdory class $\mathcal{P}$.

The class of Janowski star-like functions $\mathcal{S}^*[A, B]$ consists of $f \in \mathcal{A}$ satisfying

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}[A, B],$$

while the Janowski convex functions $\mathcal{C}[A, B]$ are functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}[A, B].$$

For $0 \le \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] := \mathcal{S}^*(\beta)$ is the classical class of *star-like functions of order* β; $\mathcal{S}^*[1 - \beta, 0] := \mathcal{S}^*_\beta = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < 1 - \beta\}$, and $\mathcal{S}^*[\beta, -\beta] := \mathcal{S}^*[\beta] = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \beta |zf'(z)/f(z) + 1|\}$. These are all classes that have been widely studied; see, for example, the works of [14–16].

The next important class is related to the right half of lemniscate of Bernoulli given by $\{w : |w^2 - 1| = 1\}$. The functions $p(z) = 1 + c_1 z + \cdots$ in $\mathbb{D}$ satisfying $p(z) \prec \sqrt{1+z}$ are known as lemniscate Cáratheòdory function, and the corresponding class is denoted by $\mathcal{P}_{\mathcal{L}}$. A lemniscate Cáratheòdory function is also Cáratheòdory function and hence univalent. The lemniscate star-like class $\mathcal{S}_{\mathcal{L}}$ consists of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec$

$\sqrt{1+z}$. The class $\mathcal{K}_{\mathcal{L}} = \{f \in \mathcal{A} : 1 + (zf''(z))/f'(z) \prec \sqrt{1+z}\}$ is known as a class of lemniscate convex functions.

The third important class that is considered in sequence relates with the exponential functions e^z. The functions $p(z) = 1 + c_1 z + \cdots$ in $\mathbb{D}$ satisfying $p(z) \prec e^z$ is known as the exponential Cáratheòdory function, and the corresponding class is denoted by $\mathcal{P}_{\mathcal{E}}$. The exponential star-like class $\mathcal{S}_{\mathcal{E}}$ consists of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec e^z$. The class $\mathcal{K}_{\mathcal{E}} = \{f \in \mathcal{A} : 1 + (zf''(z))/f'(z) \prec e^z\}$ is known as a class of exponential convex functions.

The inclusion properties of special functions in the geometric classes are well known [4,12,17–22]. However, there are limited articles regarding the inclusion of RCWF in the classes of geometric functions theory. It is proved in [4] that for $L, \eta \in \mathbb{C}$, the function $z \to \mathrm{g}_L$ is leminiscate star like provided that

$$(\sqrt{2} - 1)|2L - 1| + 2|\eta| < \frac{\sqrt{2}}{4}.$$

It is also shown that $z \to \mathrm{g}_L$ is exponentially star like provided that

$$(e - 1)|2L - 1| + 2|\eta| < \frac{e - 1}{e^2}.$$

The radius of star-likeness, univalency, is discussed in [12] by using the Weierstrassian canonical product expansion of RCWF. It is proved that for $L > -1$ and $\eta \leq 0$, the radius of star-likeness of the order $\beta \in (0, 1]$ for the functions $z \to \mathrm{g}_L$ is the smallest positive root of

$$(L + \beta)\mathtt{F}_{L,\eta}(r) - r\mathtt{F}'_{L,\eta}(r) = 0.$$

The star-likeness of g_L is discussed in ([12], Theorem 4). The conditions for which $\mathrm{Re}(\mathrm{g}_L) > 0$ were also found in same results ([12], Theorem 4). However, it seems that the obtained condition for which $\mathrm{Re}(\mathrm{g}_L) > 0$ is not correct with that fact that $\mathrm{g}_L(0) = 0$, while as per the requirement of ([12], Lemma 2), it should be $\mathrm{g}_L(0) = 1$. The aim of this study is to contribute more results related to the inclusion of normalized RCWF in the classes of univalent functions theory. In Section 2, we state a proof of the results in which the function $\mathtt{f}_L$ is subordinated by three functions, $\sqrt{1+z}$, $1 + Az$ and e^z. We explain our results through a graphical representation in some special cases. The star-likeness for g_L in the shifted disc $1 + Az$ is also considered.

Throughout this study, we used the principle of differential subordination [23,24], which is an important tool in the investigation of various classes of analytic functions to proof main result.

Lemma 1 ([23,24])**.** *Let $\Omega \subset \mathbb{C}$ and $\Psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfy*

$$\Psi(i\rho, \sigma; z) \notin \Omega$$

for $z \in \mathbb{D}$, and real ρ, σ such that $\sigma \leq -(1 + \rho^2)/2$. If p is analytic in $\mathbb{D}$ with $p(0) = 1$, and $\Psi(p(z), zp'(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $\mathrm{Re}\, p(z) > 0$ *in $\mathbb{D}$.*

Lemma 2 ([25])**.** *Let $\Omega \subset \mathbb{C}$, and $\Psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfy*

$$\Psi(r, s, t; z) \notin \Omega$$

whenever $z \in \mathbb{D}$, and for $m \geq n \geq 1$, $-\pi/4 \leq \theta \leq \pi/4$,

$$r = \sqrt{2\cos(2\theta)}e^{i\theta}, \quad s = \frac{m\, e^{3i\theta}}{2\sqrt{2\cos(2\theta)}} \quad \text{and} \quad \mathrm{Re}\Big((t + s)e^{-3i\theta}\Big) \geq \frac{3m^2}{8\sqrt{2\cos(2\theta)}}. \tag{7}$$

If $\Psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $p(z) \prec \sqrt{1+z}$ in $\mathbb{D}$.

Lemma 3 ([21])**.** *Let $\Omega \subset \mathbb{C}$, and $\Psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfy $\Psi(r,s,t;z) \notin \Omega$ whenever $z \in \mathbb{D}$, and for $m \geq 1, \theta \in (0, 2\pi)$,*

$$r = e^{e^{i\theta}}, \quad s = me^{i\theta}e^{e^{i\theta}} \quad \text{and} \quad \operatorname{Re}\left(1 + \frac{t}{s}\right) \geq m(1 + \cos(\theta). \tag{8}$$

If $\Psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $p(z) \prec e^z$ in $\mathbb{D}$.

The following Lemma holds for r and s as stated in Lemma 2.

Lemma 4. *Consider s and r as in (7) with $m \geq 1$. For any $\alpha \in \mathbb{R}$, the following inequalities are true:*

1. $|s + r^2 - \alpha|^2 \geq \left(\frac{1}{2\sqrt{2}} + 2 - \alpha\right)^2$
2. $|r - \alpha|^2 \leq (\alpha - \sqrt{2})^2$

Proof. For the r and s along with $m \geq 1$, we have

$$\begin{aligned}
&|s + r^2 - \alpha|^2 \\
&\quad = \left| \frac{me^{3i\theta}}{2\sqrt{2\cos(2\theta)}} + 2\cos(2\theta)e^{2i\theta} - \alpha \right|^2 \\
&\quad = \left| \frac{me^{3i\theta}}{2\sqrt{2\cos(2\theta)}} + 2\cos^2(2\theta) + i\sin(4\theta) - \alpha \right|^2 \\
&\quad = \left| \frac{me^{3i\theta}}{2\sqrt{2\cos(2\theta)}} + e^{4i\theta} - (\alpha - 1) \right|^2 \\
&\quad = \left| \frac{m}{2\sqrt{2\cos(2\theta)}} + e^{i\theta} - (\alpha - 1)e^{-3i\theta} \right|^2 \\
&\quad = \left(\frac{m}{2\sqrt{2\cos(2\theta)}} + \cos(\theta) - (\alpha - 1)\cos(3\theta) \right)^2 + \left(\sin(\theta) + (\alpha - 1)\sin(3\theta) \right)^2 \\
&\quad \geq \left(\frac{1}{2\sqrt{2\cos(2\theta)}} + \cos(\theta) - (\alpha - 1)\cos(3\theta) \right)^2 + \left(\sin(\theta) + (\alpha - 1)\sin(3\theta) \right)^2 := g(\theta)
\end{aligned}$$

A calculation shows

$$\begin{aligned}
g'(\theta) = \tfrac{1}{4}\Big(&2\sqrt{2}\sec^{\frac{3}{2}}(2\theta)((2\alpha - 1)\sin(\theta) + (\alpha - 1)\sin(5\theta)) \\
&+ \tan(2\theta)\sec(2\theta)(8\alpha + 8(\alpha - 1)\cos(4\theta) - 7)\Big).
\end{aligned}$$

For a fixed $\alpha > 0$, $g'(\theta)$ has a zero only at $\theta = 0$ in $(-\pi/4, \pi/4)$, and further for $\alpha > 3\left(5 + 2\sqrt{2}\right)/(16 + 7\sqrt{2}) \approx 0.906785$,

$$g''(\theta = 0) = \frac{1}{4}\left(8(\alpha - 1) + 24\alpha + 2\sqrt{2}(7\alpha - 6) - 22\right) > 0.$$

This implies that g has a local minimum at $\theta = 0$ and hence

$$|s + r^2 - \alpha|^2 \geq g(0) = \left(\frac{1}{2\sqrt{2}} + 2 - \alpha\right)^2.$$

$$\begin{aligned}|r-\alpha|^2 =&\left|\sqrt{2\cos(2\theta)}-\alpha e^{-i\theta}\right|^2\\ =&\left(\sqrt{2\cos(2\theta)}-\alpha\cos(\theta)\right)^2+\alpha^2\sin^2(\theta)\\ =&2\cos(2\theta)-2\alpha\cos(\theta)\sqrt{2\cos(2\theta)}+\alpha^2\\ \leq&\ \alpha^2-2\sqrt{2}\alpha+2.\end{aligned}$$

This complete the proof. □

2. Geometric Properties of Coulomb Wave Functions (CWF)

2.1. Subordination by $\sqrt{1+z}$

In [4], sufficient conditions based on L and η is derived for which $z\mathrm{g}_L'(z)/\mathrm{g}_L(z) \prec \sqrt{1+z}$, that is g_L, is Lemniscate star like. This section derives conditions on L and η for which $\mathtt{f}_L(z) \prec \sqrt{1+z}$, which we termed as $\mathtt{f}_L$ is the Lemniscate Catathéodory function.

Theorem 1. *For $\eta, L \in \mathbb{C}$, suppose that*

$$\mathrm{Re}(2L+1) > \max\left\{0, 8|\eta|+\frac{13}{4}\right\}, \tag{9}$$

then $\mathtt{f}_L(z) \prec \sqrt{1+z}$.

Proof. Suppose that $\mathrm{p}(z) = \mathtt{f}_L(z)$. Since $\mathtt{f}_L$ is the solution of the differential equation (5), p is the solution of

$$z^2\mathrm{p}''(z)+2(L+1)z\mathrm{p}'(z)+(z^2-2\eta z)\mathrm{p}(z)=0. \tag{10}$$

Let $\Omega = \{0\} \subset \mathbb{C}$ and define $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ as

$$\psi(r,s,t;z) := t+2(L+1)s+(z^2-2\eta z)r. \tag{11}$$

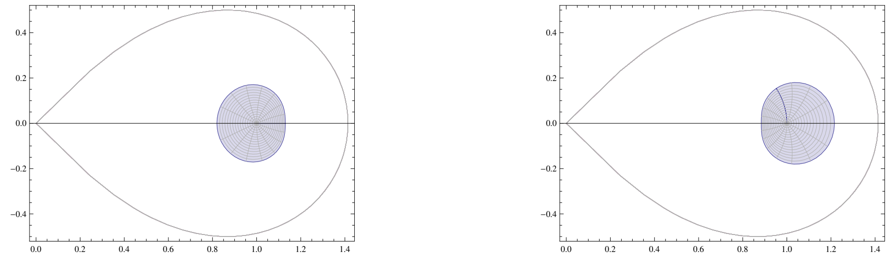
It is clear from (10) that $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$. We shall apply Lemma 2 to show $\psi(r,s,t;z) \notin \Omega$, which further implies $p(z) \prec \sqrt{1+z}$.

For r, s, t as given (7), it follows from (11) that

$$\begin{aligned}|\psi(r,s,t;z)| =& \left|(t+s)+(2L+1)s+(z^2-2\eta z)r\right|\\ >& \left|(t+s)e^{-3i\theta}+\frac{(2L+1)m}{2\sqrt{2cos(2\theta)}}\right|-|z-2\eta|\sqrt{2cos(2\theta)}\\ >& \frac{3m^2}{8\sqrt{2cos(2\theta)}}+\frac{\mathrm{Re}\,(2L+1)m}{2\sqrt{2cos(2\theta)}}-|z-2\eta|\sqrt{2cos(2\theta)}\\ =& \frac{1}{8\sqrt{2cos(2\theta)}}\left[3m^2+4m\,\mathrm{Re}\,(2L+1)-16-32|\eta|\right]>0,\end{aligned}$$

provided $\mathrm{Re}(2L+1) > (13+32|\eta|)/4$. □

It is well known that for univalent function g, if $f \prec g$, then $f(D) \subset g(D)$. Using this fact, we chose some real and complex η, and validated Theorem 1. For the first case, consider $\eta = 1, i$, and L is a real number. Using Theorem 1, in both cases, $L > 5.25$. This fact is represented in Figure 1.



(**a**) For $L = 5.25$ and $\eta = 1$ (**b**) For $L = 5.25$ and $\eta = i$

Figure 1. Image of $\mathtt{f}_L(\mathbb{D})$ for $L = 5.25$.

We consider another case by taking $\eta = (1 + i)$. By Theorem 1, in this case for real $L > 6.78185$, and Figure 2 validate the result.

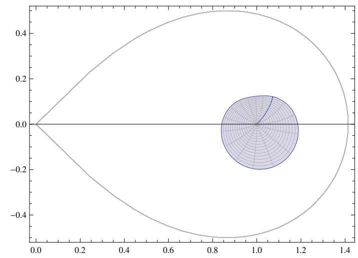


Figure 2. Image of $\mathtt{f}_L(\mathbb{D})$ for $L = 6.78185$.

2.2. Subordination by $1 + Az$

In this part we proved that result related to the subordination $\mathtt{f}_L \prec 1 + Az$ and $z\mathtt{g}'_L(z)/\mathtt{g}_L(z) \prec 1 + Az$ which describe the nature of $\mathtt{f}_L$ and $\mathtt{g}_L$ in the disc center at $(1, 0)$ and radius A. The results in this section are proved by using Lemma 2.

Theorem 2. *For $\eta, L \in \mathbb{C}$ and $A \in (0, 1]$, suppose that*

$$4A\,\mathrm{Re}(L) > 2|\eta|(A + 1) - 3A + \tfrac{5}{8} \tag{12}$$

Then, $\mathtt{f}_L(z) \prec 1 + Az$.

Proof. Consider

$$q(z) = \sqrt{\tfrac{1}{A}(\mathtt{f}_L(z) + A - 1)}. \tag{13}$$

A simplification gives

$$\mathtt{f}_L(z) = Aq^2(z) - A + 1, \quad \mathtt{f}'_L(z) = 2Aq'(z)q(z) \quad \mathtt{f}''_L(z) = 2Aq''(z)q(z) + 2A(q'(z))^2.$$

From (5), it follows that

$$2Az^2q''(z)q(z) + 2A(zq'(z))^2 + 4A(L + 1)zq'(z)q(z) + (z^2 - 2\eta z)(Aq^2(z) - A + 1) = 0.$$

Let $\Omega = \{0\} \subset \mathbb{C}$ and define $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ as

$$\psi(r, s; z) := 2Atr + 2As^2 + 4A(L + 1)sr + (z^2 - 2\eta z)(Ar^2 - A + 1). \tag{14}$$

It is clear from (14) that $\psi(q(z), zq'(z); z) \in \Omega$. We shall apply Lemma 2 to show that $\psi(r, s, t; z) \notin \Omega$, which further implies $q(z) \prec \sqrt{1 + z}$.

Now, for $-\pi/4 \leq \theta \leq \pi/4$, let

$$r = \sqrt{2\cos(2\theta)}e^{i\theta}, \quad s = \frac{me^{3i\theta}}{2\sqrt{2\cos(2\theta)}}.$$

It follows by elementary trigonometric identities that

$$r^2 - 1 = 2\cos(2\theta)e^{2i\theta} - 1 = (2\cos^2(2\theta) - 1) + i2\cos(2\theta)\sin(2\theta) = e^{4i\theta}.$$

Substitute r and s in (18) and a simplification leads to

$$\begin{aligned}
|\psi(r,s,t;z)| &= |2Atr + 2As^2 + 4A(L+1)sr + (z^2 - 2\eta z)(Ar^2 - A + 1)| \\
&= |2Ar(t+s) + 2As^2 + A(4L+2)sr + (z^2 - 2\eta z)(Ar^2 - A + 1)| \\
&> |e^{4i\theta}|\left(\sqrt{2\cos(2\theta)}\operatorname{Re}(t+s)e^{-3i\theta} + 2A\frac{m^2\operatorname{Re}(e^{2i\theta})}{2\cos(2\theta)} + A\operatorname{Re}(4L+2)m\right) \\
&\qquad - A(1+2|\eta|)|r^2 - 1| - (1+2|\eta|) \\
&> \frac{3m^2}{8} + 2Am^2 + A\operatorname{Re}(4L+2)m - A(1+2|\eta|)|e^{4i\theta}| - (1+2|\eta|) \\
&> 4A\operatorname{Re}(L) + 3A - 2|\eta|(A+1) - \tfrac{5}{8} > 0.
\end{aligned}$$

By Lemma 2, it is proved that $q(z) \prec \sqrt{1+z}$ which is equivalent to

$$\sqrt{\tfrac{1}{A}(\mathtt{f}_L + A - 1)} = \sqrt{1 + w(z)}, \tag{15}$$

for some analytic function $w(z)$ such that $|w(z)| < 1$. A simplification of (15) gives

$$\tfrac{1}{A}(\mathtt{f}_L + A - 1) = 1 + w(z) \implies \mathtt{f}_L = 1 + Aw(z) \implies \mathtt{f}_L \prec 1 + Az.$$

This complete the proof. □

Again to validate the Theorem 2, we fix $A = 1/2$ and $\eta = 1$. Lets L be real and then as per Theorem 2, $L > 17/16 \approx 1.1$. The Figure 3 indicates that the lower bound for L is possible sharp.

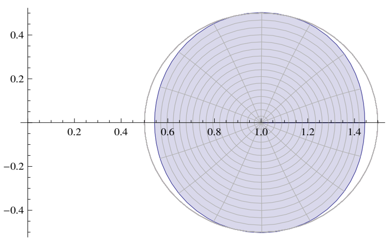


Figure 3. Image of $\mathtt{f}_L(\mathbb{D})$ for $L = 1.1$ and $\eta = 1$.

Our next result is about the starlikenes of g_L in the disc $1 + Az$.

Theorem 3. *For $\eta, L \in \mathbb{C}$ and $A \in (0,1]$, suppose that*

$$\operatorname{Re}(2L - 1) > A - 2 + \frac{2|\eta|}{A}. \tag{16}$$

Then, $z\mathrm{g}_L'(z)/\mathrm{g}_L(z) \prec 1 + Az$.

Proof. To prove the result consider

$$q(z) = \sqrt{\frac{1}{A}\left(\frac{z\mathrm{g}_L'(z)}{\mathrm{g}_L(z)} + A - 1\right)}. \tag{17}$$

A calculation yield

$$\begin{aligned}\frac{z\mathrm{g}_L'(z)}{\mathrm{g}_L(z)} &= Aq^2(z) - A + 1\\ \frac{z^2\mathrm{g}_L''(z)}{\mathrm{g}_L(z)} &= 2Azq'(z)q(z) - (Aq^2(z) - A + 1) + (Aq^2(z) - A + 1)^2\end{aligned}$$

From (6) it follows that

$$\begin{aligned}&\frac{z^2\mathrm{g}''(z)}{\mathrm{g}(z)} + 2L\frac{z\mathrm{g}'(z)}{\mathrm{g}(z)} + (z^2 - 2\eta z - 2L) = 0\\ \Longrightarrow\ &2Azq'(z)q(z) + (Aq^2(z) - A + 1)^2 + A(2L-1)(q^2(z) - 1) + z^2 - 2\eta z - 1 = 0\\ \Longrightarrow\ &2Azq'(z)q(z) + A^2(q^2(z) - 1)^2 + 2A(q^2(z) - 1) + A(2L-1)(q^2(z) - 1) + z^2 - 2\eta z = 0.\end{aligned}$$

Let $\Omega = \{0\} \subset \mathbb{C}$ and define $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ as

$$\psi(r,s;z) := 2Ars + A^2(r^2-1)^2 + 2A(r^2-1) + A(2L-1)(r^2-1) + z^2 - 2\eta z. \tag{18}$$

It is clear from (18) that $\psi(q(z), zq'(z); z) \in \Omega$. We shall apply Lemma 2 and proceed similar to the proof of Theorem 2 to show that $\psi(r,s,t;z) \notin \Omega$. Substitute r and s into (18), and a simplification leads to

$$\begin{aligned}|\psi(r,s;z)| &= |2Ars + A^2(r^2-1)^2 + 2A(r^2-1) + A(2L-1)(r^2-1) + z^2 - 2\eta z|\\ &= |me^{4i\theta} + A^2e^{8i\theta} + 2Ae^{4i\theta} + A(2L-1)e^{4i\theta} + z^2 - 2\eta z|\\ &> \operatorname{Re}(m + A^2e^{4i\theta} + 2A + A\operatorname{Re}(2L-1)) - 1 - 2|\eta|\\ &= m + A^2\cos(4\theta) + 2A + A\operatorname{Re}(2L-1) - 1 - 2|\eta|\\ &> -A^2 + 2A + A\operatorname{Re}(2L-1) - 2|\eta| > 0,\end{aligned}$$

provided $\operatorname{Re}(2L-1) > A - 2 + 2|\eta|/A$.

In view of Lemma 2, it concludes that $q(z) \prec \sqrt{1+z}$, which is equivalent to

$$\sqrt{\frac{1}{A}\left(\frac{z\mathrm{g}_L'(z)}{\mathrm{g}_L(z)} + A - 1\right)} = \sqrt{1 + w(z)}$$

for some analytic functions $w(z)$ such that $|w(z)| < 1$. A simplification gives

$$\frac{z\mathrm{g}_L'(z)}{\mathrm{g}_L(z)} = 1 + Aw(z) \implies \frac{z\mathrm{g}_L'(z)}{\mathrm{g}_L(z)} \prec 1 + Az.$$

This concludes the result. □

2.3. Subordination by e^z

In this part, we derive sufficient conditions on L and η for which $\mathtt{f}_L(z) \prec e^z$. The exponential starlikeness of $z\mathrm{g}_L'(z)/\mathrm{g}_L(z)$ is discussed in [4]. It is worthy to note here that exponential starlikeness is equivalent to $z\mathrm{g}_L'(z)/\mathrm{g}_L(z) \prec e^z$.

Theorem 4. *For $\eta, L \in \mathbb{C}$, suppose that*

$$\operatorname{Re}(L) \geq |\eta| \tag{19}$$

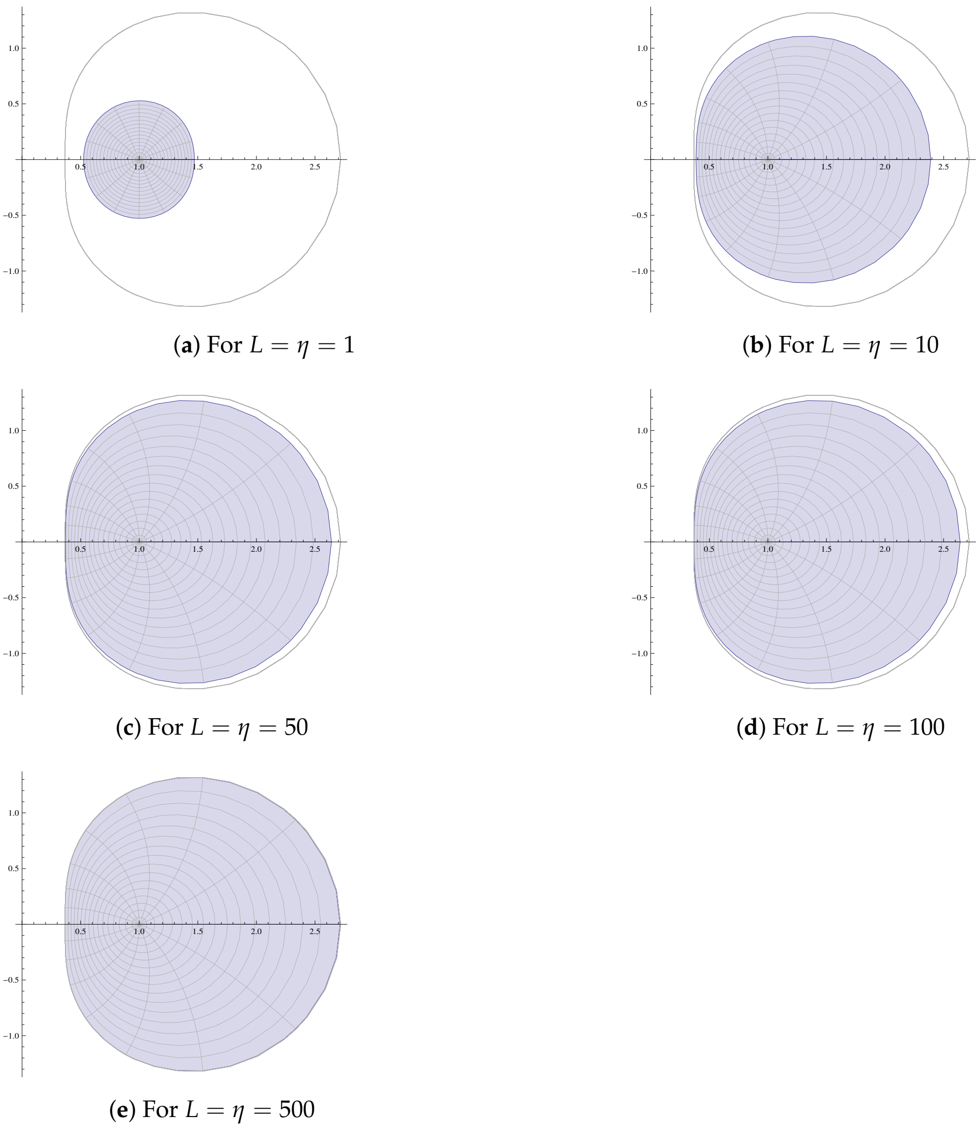
Then $\mathtt{f}_L(z) \prec e^z$.

Proof. To prove the theorem, it is enough to consider the function $\Psi(r,s,t;z)$ as defined in (11) and then apply Lemma 3 to show that $\Psi(r,s,t;z) \notin \Omega$ for r, s and t as given in (8). For $m \geq 1$ and $\mathrm{Re}(2L+1) > 0$, it follows that

$$\begin{aligned}|\psi(r,s,t;z)| &= \left|(t+s) + (2L+1)s + (z^2 - 2\eta z)r\right| \\ &> e^{\cos(\theta)}\left(\left|(t+s)e^{-i\theta}e^{-e^{i\theta}} + (2L+1)m\right| - 1 - 2|\eta|\right) \\ &> e^{\cos(\theta)}\left(\mathrm{Re}(t+s)e^{-i\theta}e^{-e^{i\theta}} + \mathrm{Re}(2L+1)m - 1 - 2|\eta|\right) \\ &> e^{\cos(\theta)}\left(m(1+\cos(\theta)) + 2\,\mathrm{Re}(L)m + m - 1 - 2|\eta|\right) > 0,\end{aligned}$$

provided $\mathrm{Re}(L) \geq |\eta|$. Lemma 3 implies $\Psi(r,s,t;z) \notin \Omega$ and hence $\mathtt{f}_L(z) \prec e^z$. This completes the proof. □

We validate this result graphically by taking real L and η and $L = \eta = 1, 10, 50, 100, 500$, and all of the case are presented in Figure 4. It is evident from Figure 4 that $\mathtt{f}_L(\mathbb{D}) \subset e^{\mathbb{D}}$, and for larger $L = \eta$.



(**a**) For $L = \eta = 1$ (**b**) For $L = \eta = 10$ (**c**) For $L = \eta = 50$ (**d**) For $L = \eta = 100$ (**e**) For $L = \eta = 500$

Figure 4. Cases for $\mathtt{f}_L(z) \prec e^z$ for $L = \eta$.

2.4. Subordination by $(1 + Az)/(1 + Bz)$

Theorem 5. *Let* $L, \eta \in \mathbb{C}$ *and* $-1 \le B < A \le 1$. *Suppose that*

$$(A-B)^2\big((1+B)\,\mathrm{Re}(L) + (1-B)\,\mathrm{Im}(L)\big)^2 < \Bigg[\bigg((1+2|\eta|)(1+B)^2 - 2A - 1 + B\bigg)$$

$$\times \bigg((1+2|\eta|)(1-B)^2 - 2(A-B)\big((1-B)\,\mathrm{Re}(L) - (1+B)\,\mathrm{Im}(L)\big) \tag{20}$$

$$+ 2(1+AB) - (A-B) + (1-A)^2\bigg)\Bigg] \tag{21}$$

Then,

$$\frac{z g_L'(z)}{g_L(z)} \prec \frac{1+Az}{1+Bz},$$

provided $(1+B)z g_L'(z) \neq (1+A) g_L(z)$.

Proof. Define the function

$$p(z) := -\frac{(1-B)z g_L'(z) - (1-A) g_L(z)}{(1+B) z g_L'(z) - (1+A) g_L(z)}.$$

A series of calculation and simplification leads to

$$\frac{z g_L'(z)}{g_L(z)} = \frac{(1+A)p(z) + 1 - A}{(1+B)p(z) + 1 - B}$$

$$\frac{z^2 g_L''(z)}{g_L(z)} = \frac{2(A-B) z p'(z)}{((1+B)p(z) + 1 - B)^2} - \frac{(1+A)p(z) + 1 - A}{(1+B)p(z) + 1 - B} + \left(\frac{(1+A)p(z) + 1 - A}{(1+B)p(z) + 1 - B}\right)^2$$

From (6), it follows that

$$\frac{z^2 g_L''(z)}{g_L(z)} + 2L \frac{z g_L'(z)}{g_L(z)} + (z^2 - 2\eta z - 2L) = 0$$

$$\Longrightarrow \frac{2(A-B) z p'(z)}{((1+B)p(z) + 1 - B)^2} + (2L-1)\frac{(A-B)(p(z)-1)}{(1+B)p(z) + 1 - B}$$

$$+ \left(\frac{(1+A)p(z) + 1 - A}{(1+B)p(z) + 1 - B}\right)^2 + (z^2 - 2\eta z) = 0$$

$$\Longrightarrow 2(A-B) z p'(z) + ((1+A)p(z) + 1 - A)^2 + (z^2 - 2\eta z)((1+B)p(z) + 1 - B)^2$$

$$+ \bigg((2L(A-B) - (1+A))p(z) - 2L(A-B) - A + 1\bigg)\bigg((1+B)p(z) + 1 - B\bigg) = 0$$

Let $\Omega = \{0\} \subset \mathbb{C}$ and define $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$

$$\psi(r, s; z) := 2(A-B)s + +((1+A)r + 1 - A)^2 + (z^2 - 2\eta z)((1+B)r + 1 - B)^2$$

$$+ \bigg((2L(A-B) - (1+A))r - 2L(A-B) - A + 1\bigg)\bigg((1+B)r + 1 - B\bigg)$$

Denoting $L_1 = \mathrm{Re}(L)$ and $L_2 = \mathrm{Im}(L)$, we have

$$
\begin{aligned}
&\operatorname{Re}\psi(i\rho,\sigma;z)\\
&< (A-B)(1+\rho^2)+\operatorname{Re}((1+A)i\rho+1-A)^2+\operatorname{Re}((z^2-2\eta z)((1+B)i\rho+1-B)^2)\\
&\qquad+\operatorname{Re}\Big((2L(A-B)-(1+A))i\rho-2L(A-B)-A+1\Big)\Big((1+B)i\rho+1-B\Big)\\
&= -(A-B)(1+\rho^2)-(1+A)\rho^2+(1-A)^2+(1+2|\eta|)((1+B)^2\rho^2+(1-B)^2)\\
&\ -2(A-B)((1+B)L_1+(1-B)L_2)\rho-2(A-B)((1-B)L_1+(1+B)L_2)+2(1+AB)\\
&= \Big(-(A-B)-(1+A)+(1+2|\eta|)(1+B)^2\Big)\rho^2-\Big(2(A-B)((1+B)L_1+(1-B)L_2)\Big)\rho\\
&\quad+(1+2|\eta|)(1-B)^2-2(A-B)\big((1-B)L_1-(1+B)L_2\big)+2(1+AB)-(A-B)+(1-A)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Delta_1}{4} &= (A-B)^2\big((1+B)L_1+(1-B)L_2\big)^2-\Bigg[\Big((1+2|\eta|)(1+B)^2-2A-1+B\Big)\\
&\quad\times\Big((1+2|\eta|)(1-B)^2-2(A-B)\big((1-B)L_1-(1+B)L_2\big)+2(1+AB)-(A-B)+(1-A)^2\Big)\Bigg]
\end{aligned}
$$

Clearly, by the given hypothesis, $\Delta_1 < 0$, and hence $\operatorname{Re}\psi(i\rho,\sigma;z) < 0$. This implies that $\Psi(r,s,t;z) \notin \Omega$. From Lemma 1 it follows

$$
\begin{aligned}
p(z) \prec \frac{1+z}{1-z} &\implies -\frac{(1-B)zg_L'(z)-(1-A)g_L(z)}{(1+B)zg_L'(z)-(1+A)g_L(z)} \prec \frac{1+z}{1-z}\\
&\implies \frac{zg_L'(z)}{g_L(z)} \prec \frac{1+Az}{1+Bz}.
\end{aligned}
$$

This completes the proof. □

By choosing $A = -B = 1$ in Theorem 5, we have the following result on the star-likeness of g_L.

Corollary 1. *For $L, \eta \in \mathbb{C}$, suppose that $|\eta| < \operatorname{Re}(L) - \frac{1}{3}(\operatorname{Im}(L))^2 - \frac{1}{4}$. Then, g_L is star like provided that $g_L \neq 0$ for $z \in \mathbb{D}$.*

Remark 1. *The condition for the star-likeness of g_L is provided in ([12], Theorem 4) which is the same as stated in Corollary 1.*

3. Conclusions

This study finds the conditions for the parameters L and η for which the normalized function $\mathtt{f}_L(z) = C_L^{-1}(\eta)z^{-L-1}F_{L,\eta}(z)$ is subordinated by three different functions $\sqrt{1+z}$, $1+Az$, and e^z.

We already interpreted Theorems 1, 2 and 4 graphically. Figure 3 describes the sharpness of Theorem 2. On the other hand, Figure 4 indicates the situation related to Theorem 4. However, we were unable to obtain examples with a similar sharpness using Theorem 1. Thus, it is expected show some improvement in the obtained results. For example, let $\eta = 2i$, and then by Theorem 4, $\mathtt{f}_L(z) \prec e^z$ for $\operatorname{Re}(L) > |\eta| = 2$. However, Figure 5 indicates that if L is real, then it can be lower than 1 for $\eta = 2i$.

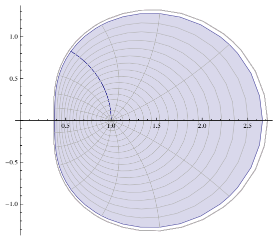


Figure 5. Image of $\mathtt{f}_L(\mathbb{D})$ for $\eta = 2i$ and $L = 1.05$.

Similarly, for $\eta = 2i$, Theorem 1 implies that $\mathtt{f}_L(z) \prec \sqrt{1+z}$ for

$$\mathrm{Re}(L) > 8|\eta| + \frac{13}{4} = 9.125.$$

However, for real L, it follows from Figure 6 that L can be down to 5.5.

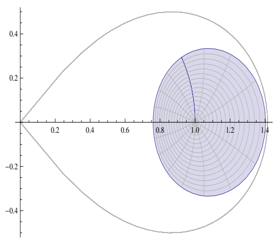


Figure 6. Image of $\mathtt{f}_L(\mathbb{D})$ for $\eta = 2i$ and $L = 1.05$.

From the above discussion, we can finally conclude that Theorems 1, 2 and 4 are completely valid with respect to the stated hypothesis. However, for some special choice of parameter η, there is a possibility for improvement.

This article also derives the conditions for the star-likeness of $\mathtt{g}_L$ in the disc $(1+Az)/(1+Bz)$ and $1+Az$. With the special case for $A = 1$ and $B = -1$, the results lead to a known result ([12], Theorem 4).

Funding: This work was supported through the Annual Funding track by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Project No. AN00097].

Institutional Review Board Statement: Not Applicable.

Informed Consent Statement: Not Applicable.

Data Availability Statement: Not Applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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