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Fixed Point of (α, β)-Admissible Generalized Geraghty F-Contraction with Application

Min Wang [1], Naeem Saleem [2], Xiaolan Liu [3,4,5,*], Arslan Hojat Ansari [6,7] and Mi Zhou [8,9,10,11,*]

1 School of Science and Physics, Mianyang Teachers' College, Mianyang 621000, China; muminzi@mtc.edu.cn
2 Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan; naeem.saleem2@gmail.com
3 College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong 643000, China
4 Key Laboratory of Higher Education of Sichuan Province for Enterprise, Informationlization and Internet of Things, Zigong 643000, China
5 South Sichuan Center for Applied Mathematics, Zigong 643000, China
6 Department of Mathematics, Karaj Brancem Islamic Azad University, Karaj 61349-37333, Iran; analsisamirmath2@gmail.com
7 Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Pretoria 0204, South Africa
8 School of Science and Technology, Sanya University, Sanya 572000, China
9 Center for Mathematical Research, University of Sanya, Sanya 572022, China
10 Academician Guoliang Chen Team Innovation Center, University of Sanya, Sanya 572022, China
11 Academician Chunming Rong Workstation, University of Sanya, Sanya 572022, China
* Correspondence: xiaolanliu@suse.edu.cn (X.-L.L.); mizhou@sanyau.edu.cn (M.Z.)

Abstract: In this paper, we introduce some new types of extended Geraghty contractions, called (α, β)-admissible generalized Geraghty F-contractions, and prove some fixed point results for such contractions in the setting of partial b-metric spaces. Moreover, based on the obtained fixed point results and the property of symmetry, we inaugurate a fixed point result for graphic generalized Geraghty F-contractions defined on partial metric spaces endowed with a directed graph. As an application, we examine the existence of a unique solution to the first-order periodic boundary value by the obtained fixed point result. Moreover, some examples are presented to illustrate the validity of the new results.

Keywords: fixed point; partial metric spaces; b-metric spaces; F-contraction; (α, β)-admissible

MSC: Primary 47H10; Secondary 54H25

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1. Introduction and Mathematical Preliminaries

Metric fixed point theory started with the prominent Banach Contraction Principle presented by Banach [1] in 1922. Due to its simplicity, usefulness and applications, it is widely used in many branches of mathematics and applied sciences. The "Banach Contraction Principle" states that, under certain conditions, a self-map T on a nonempty set X admits one or more fixed points. After that, numerous efforts have been done to generalize, improve or extend the Banach Contraction Principle. In those studies, two concerns have become the main focus of many scholars: an appropriate contraction condition and reasonable abstract metric spaces. A suitable contraction condition usually deals with many distances between various points, mainly involving the images through the operator T and its original images, such as $d(x, y)$, $d(Tx, Ty)$, $d(x, Tx)$, $d(y, Ty)$, $d(x, Ty)$, $d(y, Tx)$ and so on. Meanwhile, an excellent contraction condition can also guarantee that the Picard iterative sequence $\{T^n u_0\}$ converges to the fixed point of T for any initial point u_0. One of the celebrated generalizations of the Banach Contraction Principle was the Geraghty-contraction given by Geraghty in [2] wherein the existence of a unique fixed point of such contractions controlled by a kind of auxiliary function was investigated in the setting of complete metric spaces. After Geraghty's work, some authors have studied this theorem in several ways (see [3–5]). At the same time, the work of promoting the concept of standard metrics to various types of generalized metrics has not stopped yet. For instance, the b-metric was introduced by Bakhtin [6], which generalized the standard metric by modifying the triangular inequality condition with a real number $s \geq 1$. For further works and results in b-metric spaces, we refer to [7–14]. Moreover, another interesting generalization of the standard metric is the partial metric spaces introduced by Matthews [15], wherein self-distance of an arbitrary point need not be equal to zero. Combining the definitions of partial metric and b-metric, in 2014, Shukla [16] introduced the concept of partial b-metrics. Subsequently, Mustafa et al. [17] provided a modified version of partial b-metrics.

In 2012, Samet et al. [18] introduced the notion of α-admissible mappings and obtained some fixed point results for such mappings. One year later, in 2013, Abdeljawad [19] defined a pair of α-admissible mappings which are different from the ones in [18], and provided fixed point and common fixed point theorems. In 2013, Cho et al. [20] defined the concept of α-Geraghty contraction type mappings and proved the existence of a unique fixed point for this kind of mappings in complete metric spaces. Afterward, an extension of α-admissible mappings were presented by Chandok [21] by introducing (α, β)-admissible mappings.

Most recently, Wardowski [22] introduced the concept of F-contraction and obtained a fixed point result as a generalization of Banach Contraction Principle. After that, several authors investigated the necessity of the conditions $(F1)$–$(F3)$ and presented some weak conditions by replacing or removing some of them. For more details in this direction, we refer to [23–27].

Motivated by the above results, in this paper, a concept of (α, β)-admissible generalized Geraghty F-contractions is introduced and some fixed point results concerning such contractions are established. In addition, some examples and applications are presented to illustrate our results. Our proposed definitions and related applications are different from those introduced in [28]. In the sequel, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Z}^+$ represent the set of all real numbers, positive real numbers, natural numbers, and positive integers, respectively. Some useful definitions and auxiliary results are listed in the following.

Definition 1 ([29]). *Let Ω be a nonempty set, $s \geq 1$ be a given real number, and let ρ be a function from $\Omega \times \Omega$ into $[0, \infty)$. The pair (Ω, ρ) is said to be a b-metric space if, for all x, y, $z \in \Omega$, the following assumptions hold:*

(ρ_1) $\rho(x, y) = 0$ *if and only if* $x = y$;
(ρ_2) $\rho(x, y) = \rho(y, x)$;
(ρ_3) $\rho(x, y) \leq s[\rho(x, z) + \varrho(z, y)]$.
The number $s \geq 1$ is called the coefficient of (Ω, ρ).

Definition 2 ([15]). *A function $\varrho : X \times X \to [0, \infty)$ is called a partial metric on a nonempty set Ω if, for all $x, y, z \in \Omega$, the following assumptions hold:*

(ϱ_1) $x = y$ *if and only if* $\varrho(x, x) = \varrho(x, y) = \varrho(y, y)$;
(ϱ_2) $\varrho(x, x) \leq \varrho(x, y)$;
(ϱ_3) $\varrho(x, y) = \varrho(y, x)$;
(ϱ_4) $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y) - \varrho(z, z)$.
The pair (Ω, ϱ) is called a partial metric space.

Definition 3 ([16]). *A function $\varrho_b : X \times X \to [0, \infty)$ is called a partial b-metric on a nonempty set Ω if, for all $x, y, z \in \Omega$ and a given real number $s \geq 1$, the following assumptions hold:*

(ϱ_{b_1}) $x = y$ *if and only if* $\varrho_b(x, x) = \varrho_b(x, y) = \varrho_b(y, y)$;
(ϱ_{b_2}) $\varrho_b(x, x) \leq \varrho_b(x, y)$;
(ϱ_{b_3}) $\varrho_b(x, y) = \varrho_b(y, x)$;
(ϱ_{b_4}) $\varrho_b(x, y) \leq s[\varrho_b(x, z) + \varrho_b(z, y)] - \varrho_b(z, z)$.

The pair (Ω, ϱ_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (Ω, ϱ_b).

According to these definitions of metric spaces mentioned above, we can obtain the relations stated as follows:

$$\text{partial metric spaces} \Rightarrow b\text{-metric spaces} \Rightarrow \text{partial } b\text{-metric spaces}$$

Later, Mustafa et al. [17] modified the definition of partial b-metric by replacing (ϱ_{b_4}) by $(\varrho_{b_{4'}})$ for which each partial b-metric ϱ_b can generate a b-metric ρ_{ϱ_b}.

Definition 4 ([17])**.** *A function $\varrho_b : \Omega \times \Omega \to [0, \infty)$ is called a partial b-metric if, for all x, y, $z \in \Omega$ and a given real number $s \geq 1$, the following conditions are satisfied:*

(ϱ_{b_1}) *$x = y$ if and only if $\varrho_b(x, x) = \sigma_b(x, y) = \sigma_b(y, y)$;*
(ϱ_{b_2}) *$\varrho_b(x, x) \leq \sigma_b(x, y)$;*
(ϱ_{b_3}) *$\varrho_b(x, y) = \varrho_b(y, x)$;*
$(\varrho_{b_{4'}})$ *$\varrho_b(x, y) \leq s(\varrho_b(x, z) + \varrho_b(z, y) - \varrho_b(z, z)) + (\frac{1-s}{2})(\varrho_b(x, x) + \varrho_b(y, y))$.*

The pair (Ω, ϱ_b) is called a modified partial b-metric space. The number $s \geq 1$ is called the coefficient of (Ω, ϱ_b).

Example 1 ([16])**.** *Let $\Omega = \mathbb{R}^+$, and a function ϱ_b from $\Omega \times \Omega$ into $\mathbb{R}^+$ be defined by*

$$\varrho_b(x, y) = [\max\{x, y\}]^q + |x - y|^q,$$

for all $x, y \in \Omega$, and a constant $q > 1$. Then, (Ω, ϱ_b) is a partial b-metric space with the coefficient $s = 2^{q-1} > 1$, but it is neither a b-metric nor a partial metric space.

Example 2 ([16])**.** *Let ϱ and ρ be a partial metric and a b-metric with the coefficient $s \geq 1$ on a nonempty set Ω. Then, the function $\varrho_b : \Omega \times \Omega \to [0, \infty)$ defined by $\varrho_b(x, y) = \varrho(x, y) + \rho(x, y)$ for all $x, y \in \Omega$ is a partial b-metric on Ω with the coefficient s.*

Example 3. *Let (Ω, ϱ) be a partial metric space and p be a real number with $p \geq 1$. Then, (Ω, ϱ_b) is a partial b-metric space with the coefficient $s = 2^{p-1}$, where ϱ_b is defined by $\varrho_b(x, y) = [\varrho(x, y)]^p$.*

Proposition 1 ([17])**.** *Every partial b-metric ϱ_b on a nonempty set Ω defines a b-metric ρ_{ϱ_b}, where*

$$\rho_{\varrho_b}(x, y) = 2\varrho_b(x, y) - \varrho_b(x, x) - \varrho_b(y, y),$$

for all $x, y \in \Omega$.

Definition 5 ([17])**.** *Let (Ω, ϱ_b) be a partial b-metric space. The sequence $\{u_n\}$ in Ω and u in Ω; then,*

(1) *the sequence $\{u_n\}$ is said to be ϱ_b convergent in Ω to u if $\varrho_b(u, u) = \lim_{n\to\infty} \varrho_b(u, u_n)$;*
(2) *the sequence $\{u_n\}$ is said to be ϱ_b-Cauchy if $\lim_{n,m\to\infty} \varrho_b(u_n, u_m)$ exists (and is finite);*
(3) *(Ω, ϱ_b) is said to be ϱ_b-complete if every ϱ_b-Cauchy sequence $\{u_n\}$ in Ω ϱ_b-converges to a point u in Ω, that is,*

$$\varrho_b(u, u) = \lim_{n,m\to\infty} \varrho_b(u_n, u_m) = \lim_{n\to\infty} \varrho_b(u, u_n).$$

Lemma 1 ([17])**.** *Let (Ω, ϱ_b) be a partial b-metric space and a sequence $\{u_n\}$ in Ω. Then,*

(i) $\{u_n\}$ is a ϱ_b-Cauchy if and only if $\{u_n\}$ is a b-Cauchy with the b-metric ρ_{ϱ_b};
(ii) (Ω, ϱ_b) is ϱ_b-complete if and only if $(\Omega, \rho_{\varrho_b})$ is b-complete. Moreover,

$$\lim_{n\to\infty} \rho_{\varrho_b}(u_n, u) = 0$$

if and only if

$$\varrho_b(u,u) = \lim_{n\to\infty} \varrho_b(u_n,u) = \lim_{n,m\to\infty} \varrho_b(u_n,u_m).$$

Definition 6 ([18])**.** *Let T be a self-mapping on X and $\alpha : X \times X \to [0,\infty)$ be a function. We say that T is an α-admissible mapping if*

$$x,y \in X, \alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1.$$

Definition 7 ([21])**.** *Let Ω be a nonempty set, $T : \Omega \to \Omega$ and α, $\beta : \Omega \times \Omega \to [0,\infty)$. T is said to be an (α,β)-admissible if $\alpha(x,y) \geq 1$ and $\beta(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$ and $\beta(Tx,Ty) \geq 1$, for all $x,y \in \Omega$.*

Definition 8. *Let (Ω,ϱ_b) be a partial b-metric space, $T : \Omega \to \Omega$ and $\alpha : \Omega \times \Omega \to \Omega$. We say T satisfies α-admissible property, if a sequence $\{u_n\}$ in Ω with $u_n \to u \in \Omega$ and $\alpha(u_n,u_{n+1}) \geq 1$, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha(u_{n_k},u) \geq 1$ for all $k \geq 0$.*

Definition 9. *Let (Ω,ϱ_b) be a partial b-metric space, $T : \Omega \to \Omega$ and $\alpha : \Omega \times \Omega \to \Omega$. We say T satisfies an (α,β)-admissible property, if a sequence $\{u_n\}$ in Ω with $u_n \to u \in \Omega$, $\alpha(u_n,u_{n+1}) \geq 1$ and $\beta(u_n,u_{n+1}) \geq 1$, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha(u_{n_k},u) \geq 1$ and $\beta(u_{n_k},u) \geq 1$ for all $k \geq 0$.*

On the other hand, Wardowski [22] introduced the auxiliary functions as follows:

Let $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

$(F1)$ $\mathcal{F}$ is strictly increasing, that is, for α, $\beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies $\mathcal{F}(\alpha) < \mathcal{F}(\beta)$;

$(F2)$ for each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} \mathcal{F}(\alpha_n) = -\infty$;

$(F3)$ there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k \mathcal{F}(\alpha) = 0$.

We denote the set of all functions satisfying $(F1)$–$(F3)$ by $\digamma$.

In [23], Secelean et al. replaced the condition $(F2)$ by an equivalent but a more simpler condition $(F2')$.

$(F2')$ $\inf \mathcal{F} = -\infty$,

or, also by

$(F2'')$, there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty} \mathcal{F}(\alpha_n) = -\infty$. Most recently, Piri et al. [25] used the following condition $(F3')$ instead of $(F3)$.

$(F3')$ $\mathcal{F}$ is continuous on $(0,\infty)$.

Denote the set of all functions satisfying $(F1)$, $(F2')$ and $(F3')$ by $\triangle_{\mathcal{F}}$.

Next, we introduce some three families of functions stated as follows.

Let the function $\phi : [0,\infty) \to [0,\infty)$ satisfy (i) ϕ is non-decreasing; (ii) ϕ is continuous; (iii) $\phi(t) = 0 \Leftrightarrow t = 0$. Denote the set of functions ϕ by Φ.

Let the function $\theta : [0,+\infty) \to [0,1)$ satisfy that $\theta(t_n) \to 1 \Rightarrow t_n \to 0$. Denote the set of the functions θ by Θ.

Let the continuous function $\mathcal{D}(r_1,r_2,r_3,r_4) : {\mathbb{R}^+}^4 \to \mathbb{R}^+$ satisfy that, for all r_1, r_2, r_3, $r_4 \in \mathbb{R}^+$ if $r_i = r_j$ for $i,j = 1,2,3,4$, where $i \neq j$, then there exists $\tau > 0$ such that $\mathcal{D}(r_1,r_2,r_3,r_4) = \tau$. Denote the set of the functions $\mathcal{D}$ by $\triangle_{\mathcal{D}}$.

2. Main Results

2.1. Fixed Point Results for (α,β)-Admissible Generalized Geraghty $\mathcal{F}$-Contractions

Let us start this section by introducing the following definition.

Definition 10. *Let (Ω,ϱ_b) be a partial b-metric space and $T : \Omega \to \Omega$, $\alpha,\beta : \Omega \times \Omega \to [0,\infty)$. We say that T is an (α,β)-admissible generalized Geraghty $\mathcal{F}$-contraction of type (A) on a partial* *b-metric space Ω, if T is (α,β)-admissible, and there exist $\mathcal{F} \in \triangle_{\mathcal{F}}$, $D \in \triangle_{\mathcal{D}}$, $\theta \in \Theta$ and $\phi \in \Phi$ such that, for all $x, y \in \Omega$ and $s > 1$ with $\varrho_b(Tx, Ty) > 0$,*

$$\begin{aligned} \alpha(x,y)\beta(x,y)\mathcal{F}(s^{\epsilon}\varrho_b(Tx,Ty)) \quad \leq \quad & \theta(\phi(M_s(x,y)))\mathcal{F}(N_s(x,y)) \\ & -\mathcal{D}(\varrho_b(x,y),\varrho_b(x,Tx),\varrho_b(y,Ty),\varrho_b(Tx,Ty)), \end{aligned} \tag{1}$$

where

$$\begin{aligned} M_s(x,y) &= \max\{\varrho_b(x,y),\varrho_b(x,Tx),\varrho_b(y,Ty),\frac{\varrho_b(x,Ty)+\varrho_b(y,Tx)}{2s}\}, \\ N_s(x,y) &= \max\{\varrho_b(x,y),\varrho_b(x,Tx),\varrho_b(y,Ty)\}, \end{aligned}$$

and $\epsilon > 1$ is a constant.

Definition 11. *Let (Ω, ϱ_b) be a partial b-metric space and $T : \Omega \to \Omega$, $\alpha, \beta : \Omega \times \Omega \to [0, \infty)$. We say that T is an (α,β)-admissible generalized Geraghty $\mathcal{F}$-contraction of type (B) on a partial b-metric space Ω, if T is (α,β)-admissible, and there exist $\mathcal{F} \in \triangle_{\mathcal{F}}$, $\theta \in \Theta$ and $\phi \in \Phi$ such that, for all x, $y \in X$ and $s \geq 1$ with $\varrho_b(Tx, Ty) > 0$,*

$$\alpha(x,y)\beta(x,y)\mathcal{F}(s\varrho_b(Tx,Ty)) \leq \theta(\phi(M_s(x,y)))\mathcal{F}(N_s(x,y)), \tag{2}$$

where $M_s(x,y) = \max\{\varrho_b(x,y),\varrho_b(x,Tx),\varrho_b(y,Ty),\frac{\varrho_b(x,Ty)+\varrho_b(y,Tx)}{2s}\}$,
$N_s(x,y) = \max\{\varrho_b(x,y),\varrho_b(x,Tx),\varrho_b(y,Ty)\}$.

Remark 1. *Obviously, for each $s > 1$, we have the following relation:*

$$\begin{aligned} \alpha(x,y)\beta(x,y)\mathcal{F}(s\varrho_b(Tx,Ty)) \quad &\leq \quad \alpha(x,y)\beta(x,y)\mathcal{F}(s^{\epsilon}\varrho_b(Tx,Ty)) \\ &\leq \quad \theta(\phi(M_s(x,y)))\mathcal{F}(N_s(x,y)) \\ &\qquad -\mathcal{D}(\varrho_b(x,y),\varrho_b(x,Tx),\varrho_b(y,Ty),\varrho_b(Tx,Ty)), \\ &\leq \quad \theta(\phi(M_s(x,y)))\mathcal{F}(N_s(x,y)), \end{aligned}$$

that is, each (α,β)-admissible generalized Geraghty $\mathcal{F}$-contraction of type (A) is an (α,β)-admissible generalized Geraghty $\mathcal{F}$-contraction of type (B).

One of our main result of this paper is stated as follows.

Theorem 1. *Let (Ω, ϱ_b) be a complete partial b-metric space and T be a self mapping on X satisfying the following conditions:*
(1) *T is (α,β)-admissible;*
(2) *there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\beta(u_0, Tu_0) \geq 1$;*
(3) *T is an (α,β)-admissible generalized Geraghty $\mathcal{F}$-contraction of type (B) on (Ω, ϱ_b);*
(4) *T is continuous or T satisfies (α,β)-admissible property.*
Then, T has a fixed point $u \in X$ with $\varrho_b(u, u) = 0$ and $\{T^n u_0\}$ converges to u. Furthermore, if for all $u, v \in F(T)$, with $u \neq v$ such that $\alpha(u, Tv) \geq 1$, $\alpha(v, Tu) \geq 1$ and $\beta(u, Tv) \geq 1$, $\beta(v, Tu) \geq 1$, then T has a unique fixed point in Ω.

Proof. Let $u_0 \in \Omega$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\beta(u_0, Tu_0) \geq 1$. Define a sequence $\{u_n\}$ in X by $u_{n+1} = Tu_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $u_{n_0+1} = u_{n_0}$ for any $n_0 \in \mathbb{N} \cup \{0\}$, then u_{n_0} is a fixed point of T. Consequently, assume that $u_{n+1} \neq u_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is an (α,β)-admissible mapping, it follows from condition (2) that $\alpha(u_0, Tu_0) = \alpha(u_0, u_1) \geq 1$, $\alpha(Tu_0, Tu_1) = \alpha(u_1, u_2) \geq 1$.

By induction, we obtain $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \geq 0$.

Similarly, $\beta(u_n, u_{n+1}) \geq 1$ for all $n \geq 0$.

By taking $x = u_n$ and $y = u_{n+1}$ in (2) and due to $(F1)$, properties of θ and ϕ, we arrive at

$$\begin{array}{rcl} \mathcal{F}(\varrho_b(u_n, u_{n+1})) & \leq & \mathcal{F}(s\varrho_b(Tu_{n-1}, Tu_n)) \\ & \leq & \alpha(u_{n-1}, u_n)\beta(u_{n-1}, u_n)\mathcal{F}(s^{\epsilon}\varrho_b(Tu_{n-1}, Tu_n)) \\ & \leq & \theta(\phi(M_s(u_{n-1}, u_n))\mathcal{F}(N_s(u_{n-1}, u_n)), \end{array} \tag{3}$$

where

$$\begin{array}{rcl} M_s(u_{n-1}, u_n) & = & \max\{\varrho_b(u_{n-1}, u_n), \varrho_b(u_{n-1}, u_n), \varrho_b(u_n, u_{n+1}), \dfrac{\varrho_b(u_{n-1}, u_{n+1}) + \varrho_b(u_n, u_n)}{2s}\} \\ & = & \max\{\varrho_b(u_{n-1}, u_n), \varrho_b(u_n, u_{n+1})\}, \end{array}$$

and

$$\begin{array}{rcl} N_s(u_{n-1}, u_n) & = & \max\{\varrho_b(u_{n-1}, u_n), \varrho_b(u_{n-1}, u_n), \varrho_b(u_n, u_{n+1})\} \\ & = & \max\{\varrho_b(u_{n-1}, u_n), \varrho_b(u_n, u_{n+1})\}. \end{array}$$

If $\max\{\varrho_b(u_{n-1}, u_n), \varrho_b(u_n, u_{n+1})\} = \varrho_b(u_n, u_{n+1})$, for all $n \in \mathbb{N} \cup \{0\}$. From (3) and $\theta(\phi(\varrho_b(u_n, u_{n+1}))) < 1$, we deduce that

$$\begin{array}{rcl} \mathcal{F}(\varrho_b(u_n, u_{n+1})) & \leq & \theta(\phi(\varrho_b(u_n, u_{n+1})))\mathcal{F}(\varrho_b(u_n, u_{n+1})) \\ & < & \mathcal{F}(\varrho_b(u_n, u_{n+1})), \end{array}$$

which is a contradiction. Thus, it follows that

$$\max\{\varrho_b(u_{n-1}, u_n), \varrho_b(u_n, u_{n+1})\} = \varrho_b(u_{n-1}, u_n).$$

Again, from (3) and the definition of θ, we have

$$\begin{array}{rcl} \mathcal{F}(\varrho_b(u_n, u_{n+1})) & \leq & \theta(\phi(\varrho_b(u_{n-1}, u_n)))\mathcal{F}(\varrho_b(u_{n-1}, u_n)), \\ & \leq & \mathcal{F}(\varrho_b(u_{n-1}, u_n)), \end{array} \tag{4}$$

which gives

$$\varrho_b(u_n, u_{n+1}) \leq \varrho_b(u_{n-1}, u_n). \tag{5}$$

Hence, $\varrho_b(u_n, u_{n+1})$ is a decreasing sequence of positive real numbers. Repeating use of (5), we have

$$\varrho_b(u_n, u_{n+1}) \to r \geq 0.$$

Since $\mathcal{F} \in \triangle_{\mathcal{F}}$, by taking the limit in (4) as $n \to \infty$, we obtain

$$\mathcal{F}(r) = \lim_{n\to\infty} \theta(\phi(\varrho_b(u_{n-1}, u_n)))\mathcal{F}(r) \Leftrightarrow \lim_{n\to\infty} \theta(\phi(\varrho_b(u_{n-1}, u_n))) = 1. \tag{6}$$

Since $\theta \in \Theta$, then $\lim\limits_{n\to\infty} \phi(\varrho_b(u_{n-1}, u_n)) = 0$. From $\phi \in \Phi$ and condition (ϱ_{b_2}), we have the following

$$\lim_{n\to\infty} \varrho_b(u_n, u_{n+1}) = 0 \quad \text{and} \quad \lim_{n\to\infty} \varrho_b(u_n, u_n) = 0\,. \tag{7}$$

Now, we will prove that $\{u_n\}$ is a ϱ_b-Cauchy sequence in X. From Lemma 1, we need to prove that $\{u_n\}$ is a b-Cauchy sequence in the b-metric space (X, ρ_{ϱ_b}). Suppose that $\{u_n\}$ is not b-Cauchy. Then, there exists $\delta > 0$ and sequences of integers $\{n(k)\}$, $\{m(k)\}$ with $n(k) > m(k) \geq k$, such that, for $k = 1, 2, \ldots$, we have

$$\rho_{\varrho_b}(u_{m(k)}, u_{n(k)}) \geq \delta. \tag{8}$$

By choosing $n(k)$ to be the smallest positive integer exceeding $m(k)$ for which (8) holds, we may assume that

$$\rho_{\varrho_b}(u_{m(k)}, u_{n(k)-1}) < \delta. \tag{9}$$

Due to triangle inequality and from (8), we obtain

$$\delta \leq \rho_{\varrho_b}(u_{m(k)}, u_{n(k)}) \leq s\rho_{\varrho_b}(u_{m(k)}, u_{n(k)-1}) + s\rho_{\varrho_b}(u_{n(k)-1}, u_{n(k)}).$$

Letting $k \to \infty$, it follows from (9) that

$$\frac{\delta}{s} \leq \liminf_{k\to\infty} \rho_{\varrho_b}(u_{n(k)-1}, u_{m(k)}) \leq \limsup_{k\to\infty} \rho_{\varrho_b}(u_{n(k)-1}, u_{m(k)}) \leq \delta. \tag{10}$$

In addition, from (9) and (10), we have

$$\delta \leq \limsup_{k\to\infty} \rho_{\varrho_b}(u_{m(k)}, u_{n(k)}) \leq s\delta. \tag{11}$$

and

$$\begin{aligned}
\rho_{\varrho_b}(u_{m(k)+1}, u_{n(k)}) &\leq s\rho_{\varrho_b}(u_{m(k)+1}, u_{m(k)}) + s\rho_{\varrho_b}(u_{m(k)}, u_{n(k)}) \\
&\leq s\rho_{\varrho_b}(u_{m(k)+1}, u_{m(k)}) + s^2\rho_{\varrho_b}(u_{m(k)}, u_{n(k)-1}) + s^2\rho_{\varrho_b}(u_{n(k)-1}, u_{n(k)}) \\
&\leq s\rho_{\varrho_b}(u_{m(k)+1}, u_{m(k)}) + s^2\delta + s^2\rho_{\varrho_b}(u_{n(k)-1}, u_{n(k)}),
\end{aligned}$$

which gives

$$\limsup_{k\to\infty} \rho_{\varrho_b}(u_{m(k)+1}, u_{n(k)}) \leq s^2\delta.$$

Furthermore,

$$\limsup_{k\to\infty} \rho_{\varrho_b}(u_{m(k)+1}, u_{n(k)-1}) \leq s\rho_{\varrho_b}(u_{m(k)+1}, u_{m(k)}) + s\rho_{\varrho_b}(u_{m(k)}, u_{n(k)-1}),$$

which yields

$$\limsup_{k\to\infty} \rho_{\varrho_b}(u_{m(k)+1}, u_{n(k)-1}) \leq s\delta.$$

Utilizing Proposition 1, we have

$$\limsup_{k\to\infty} \rho_{\varrho_b}(u_{m(k)}, u_{n(k)-1}) = 2\limsup_{k\to\infty} \varrho_b(u_{m(k)}, u_{n(k)-1}).$$

Hence, by (10), we have

$$\frac{\delta}{2s} \leq \liminf_{k\to\infty} \varrho_b(u_{2m(k)}, u_{2n(k)-1}) \leq \limsup_{k\to\infty} \varrho_b(u_{m(k)}, u_{n(k)-1}) \leq \frac{\delta}{2}. \tag{12}$$

Analogously, we deduce that

$$\limsup_{k\to\infty} \varrho_b(u_{m(k)}, u_{n(k)}) \leq \frac{s\delta}{2}. \tag{13}$$

$$\frac{\delta}{2s} \leq \limsup_{k\to\infty} \varrho_b(u_{m(k)+1}, u_{n(k)}), \tag{14}$$

$$\limsup_{k\to\infty} \varrho_b(u_{m(k)+1}, u_{n(k)-1}) \leq \frac{s\delta}{2}. \tag{15}$$

Since $\mathcal{F}(\varrho_b(u_{m(k)+1}, u_{n(k)})) = \mathcal{F}(\varrho_b(Tu_{m(k)}, Tu_{n(k)-1})) > 0$, due to inequality (1), we have

$$\begin{aligned}
\mathcal{F}(\varrho_b(u_{m(k)+1}, u_{n(k)})) &\leq \mathcal{F}(s\varrho_b(Tu_{m(k)}, Tu_{n(k)-1})) \\
&\leq \alpha(u_{m(k)}, u_{n(k)-1})\beta(u_{m(k)}, u_{n(k)-1})\mathcal{F}(s\varrho_b(Tu_{m(k)}, Tu_{n(k)-1})) \\
&\leq \theta(\phi(M_s(u_{m(k)}, u_{n(k)-1})))\mathcal{F}(N_s(u_{m(k)}, u_{n(k)-1})).
\end{aligned} \tag{16}$$

Utilizing the definition of $M_s(x,y)$ and $N_s(x,y)$ along with inequalities (12)–(15), we have

$$\limsup_{k\to\infty} M_s(u_{m(k)}, u_{n(k)-1}) \leq \frac{\delta}{2}, \tag{17}$$

and

$$\limsup_{k\to\infty} N_s(u_{m(k)}, u_{n(k)-1}) \leq \frac{\delta}{2}. \tag{18}$$

Indeed,

$$\begin{aligned} M_s(u_{m(k)}, u_{n(k)-1}) &= \max\{\varrho_b(u_{m(k)}, u_{n(k)-1}), \varrho_b(u_{m(k)}, u_{m(k)+1}), \varrho_b(u_{n(k)-1}, u_{n(k)}) \\ &\quad \frac{\varrho_b(u_{m(k)}, u_{n(k)}) + \varrho_b(u_{m(k)-1}, u_{m(k)+1})}{2s}\} \\ &\leq \max\{\frac{\delta}{2}, 0, 0, \frac{1}{2s}[\frac{s\delta}{2} + \frac{s\delta}{2}]\} \leq \frac{\delta}{2}. \end{aligned}$$

By repeating the above technique, one can easily arrive at

$$\limsup_{k\to\infty} N_s(u_{m(k)}, u_{n(k)-1}) \leq \max\{\frac{\delta}{2}, 0, 0\} \leq \frac{\delta}{2}.$$

From (16) together with (17) and (18), we have

$$\begin{aligned} \mathcal{F}(s\frac{\delta}{2s}) &\leq \lim_{k\to\infty} \alpha(u_{m(k)}, u_{n(k)-1})\beta(u_{m(k)}, u_{n(k)-1})\mathcal{F}(s(\sup \varrho_b(Tu_{m(k)}, Tu_{n(k)-1}))) \\ &\leq \lim_{k\to\infty} \theta(\phi(M_s(u_{m(k)}, u_{n(k)-1})))\mathcal{F}(N_s(u_{m(k)}, u_{n(k)-1})) \\ &\leq \lim_{k\to\infty} \theta(\phi(M_s(u_{m(k)}, u_{n(k)-1})))\mathcal{F}(\frac{\delta}{2}). \end{aligned}$$

This implies that

$$1 \leq \lim_{k\to\infty} \theta(\phi(M_s(u_{m(k)}, u_{n(k)-1}))),$$

which also yields

$$\lim_{k\to\infty} \theta(\phi(M_s(u_{m(k)}, u_{n(k)-1}))) = 1.$$

Utilizing the definition of θ and ϕ, we obtain

$$\lim_{k\to\infty} \phi(M_s(u_{m(k)}, u_{n(k)-1})) = 0 \Rightarrow \lim_{k\to\infty} M_s(u_{m(k)}, u_{n(k)-1}) = 0 \Rightarrow \lim_{k\to\infty} d_p(u_{m(k)}, u_{n(k)-1}) = 0,$$

a contradiction. Thus, $\{u_n\}$ is a b-Cauchy sequence in the b-metric space $(\Omega, \rho_{\varrho_b})$, so it is a ϱ_b-Cauchy sequence in the partial $b-$metric space (Ω, ϱ_b). Since $(\Omega, \rho_{\varrho_b})$ is b-complete, then the sequence $\{u_n\}$ converges to some point $u \in X$, that is, $\lim_{k\to\infty} \rho_{\varrho_b}(u_n, u) = 0$. Again, from Lemma 1,

$$\lim_{n\to\infty} \varrho_b(u_n, u) = \lim_{n,m\to\infty} \varrho_b(u_n, u_m) = \varrho_b(u, u). \tag{19}$$

On the other hand, from (7) and condition (ϱ_{b_2}), $\lim_{n\to\infty} \varrho_b(u_n, u_n) = 0$, which yields that

$$\lim_{n\to\infty} \varrho_b(u_n, u) = \lim_{n,m\to\infty} \varrho_b(u_n, u_m) = \varrho_b(u, u) = 0.$$

Next, we will show that u is a fixed point of T.

Case 1. Suppose that T is continuous. Due to the continuity of T, we have

$$\lim_{n\to\infty} \varrho_b(Tu_n, u) = \varrho_b(Tu, u) = \varrho_b(u, u) = 0,$$

which shows that u is a fixed point of T.

Case 2. Suppose that T satisfies (α,β)-admissible property.
Since T satisfies (α,β)-admissible property, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha(u_{n_k},u)\geq 1$ and $\beta(u_{n_k},u)\geq 1$ for all $k\geq 0$.
It follows from the inequality (1) by putting $x=u_{n_k}$ and $y=u$ that

$$\begin{aligned} F(\varrho_b(u_{n_k+1},Tu)) &\leq \mathcal{F}(s\varrho_b(Tu_{n_k},Tu)) \\ &\leq \alpha(u_{n_k},u)\beta(u_{n_k},u)\mathcal{F}(s\varrho_b(Tu_{n_k},Tu)) \\ &\leq \theta(\phi(M_s(u_{n_k},u)))\mathcal{F}(N_s(u_{n_k},u)), \end{aligned} \tag{20}$$

where

$$\begin{aligned} M_s(u_{n_k},u) &= \max\{\varrho_b(u_{n_k},u),\varrho_b(u_{n_k},u_{n_k+1}),\varrho_b(u,Tu),\frac{\varrho_b(u_{n_k},Tu)+\varrho_b(u,u_{n_k+1})}{2s}\}, \\ N_s(u_n,u) &= \{\varrho_b(u_{n_k},u),\varrho_b(u_{n_k},u_{n_k+1}),\varrho_b(u,Tu)\}. \end{aligned}$$

Taking the limit as $k\to\infty$ in $M_s(u_{n_k},u)$, we have

$$\begin{aligned} \lim_{k\to\infty} M_s(u_{n_k},u) &= \lim_{k\to\infty}\max\{\varrho_b(u,u),\varrho_b(u,Tu),\varrho_b(u,u),\frac{\varrho_b(u,u)+\varrho_b(u,Tu)}{2s}\} \\ &= \varrho_b(u,Tu). \end{aligned} \tag{21}$$

By following the same arguments as mentioned above, we have

$$\lim_{k\to\infty} N_s(u_{n_k},u)=\varrho_b(u,Tu). \tag{22}$$

By taking limit $k\to\infty$ in (20) and due to equalities (21), (22) and property of $\mathcal{F}$ function, we have

$$\begin{aligned} \mathcal{F}(\varrho_b(Tu,u)) &\leq \lim_{k\to\infty}\theta(\phi(M_s(u,u_{n_k})))\mathcal{F}(\varrho_b(u,Tu)) \\ 1 &\leq \lim_{k\to\infty}\theta(\phi(M_s(u,u_{n_k}))). \end{aligned}$$

From the definitions of θ and ϕ, the above inequalities imply that

$$\lim_{k\to\infty}\theta(\phi(M_s(u,u_{n_k})))=1,$$

$$\lim_{k\to\infty}\phi(M_s(u,u_{n_k}))=0.$$

Then, we have

$$\lim_{k\to\infty} M_s(u,u_{n_k})=0.$$

This implies that $\varrho_b(Tu,u)=0$, that is, $Tu=u$. Therefore, u is a fixed point of T.

Suppose that v is another fixed point of T with $\varrho_b(Tu,Tv)>0$. From (1), together with the additional assumption, we obtain that

$$\begin{aligned} \mathcal{F}(\varrho_b(u,v))=\mathcal{F}(\varrho_b(Tu,Tv)) &\leq \alpha(u,v)\beta(u,v)F(s^{\epsilon}\varrho_b(Tu,Tv)) \\ &\leq \theta(\phi(M_s(u,v)))\mathcal{F}(N_s(u,v)), \end{aligned}$$

where

$$\begin{aligned} &M_s(u,v)=\max\{\varrho_b(u,v),\varrho_b(u,Tu),\varrho_b(v,Tv),\frac{\varrho_b(u,Tv)+\varrho_b(v,Tu)}{2}\}=\varrho_b(u,v), \\ &N_s(u,v)=\max\{\varrho_b(u,v),\varrho_b(u,Tu),\varrho_b(v,Tv)\}=\varrho_b(u,v). \end{aligned}$$

Therefore, it follows from the definition of θ and the values of $M_s(u,v)$ and $N_s(u,v)$ that

$$\begin{aligned}\mathcal{F}(\varrho_b(u,v)) &\leq \theta(\phi(\varrho_b(u,v)))\mathcal{F}(\varrho_b(u,v))\\ 1 &\leq \theta(\mathcal{F}(\varrho_b(u,v))),\end{aligned}$$

which leads to a contradiction. Hence, $\varrho_b(Tu,Tv) = \varrho_b(u,v) = 0$, that is, $u = v$. Thus, we conclude that T admits a unique fixed point. Next, we will prove that $\varrho_b(u,u) = 0$. If $\varrho_b(Tu,Tu) = \varrho_b(u,u) > 0$, then, from (1), we have

$$\begin{aligned}\mathcal{F}(\varrho_b(u,u)) = \mathcal{F}(\varrho_b(Tu,Tu)) &\leq \alpha(u,u)\beta(u,u)\mathcal{F}(s\varrho_b(Tu,Tu))\\ &\leq \theta(\phi(\varrho_b(u,u)))\mathcal{F}(\varrho_b(u,u))\\ &\leq \mathcal{F}(\varrho_b(u,u)),\end{aligned}$$

which is a contradiction. Thus, $\varrho_b(u,u) = 0$. This completes the proof of the theorem. □

Theorem 2. *Let (Ω,ϱ_b) be a complete partial b-metric space and a self mapping T defined on Ω satisfy the following conditions:*
(1) *T is α-admissible;*
(2) *there exists $u_0 \in \Omega$ such that $\alpha(u_0,Tu_0) \geq 1$;*
(3) *T satisfies the following contractive condition:*
there exist $F \in \triangle_F$, $\theta \in \Theta$ and $\phi \in \Phi$ such that, for all $x,y \in X$ and $s \geq 1$ with $\varrho_b(Tx,Ty) > 0$,

$$\alpha(x,y)F(s\varrho_b(Tx,Ty)) \leq \theta(\phi(M_s(x,y)))F(N_s(x,y)),$$

where $M_s(x,y) = \max\{\varrho_b(x,y), \varrho_b(x,Tx), \varrho_b(y,Ty), \frac{\varrho_b(x,Ty)+\varrho_b(y,Tx)}{2s}\}$,
$N_s(x,y) = \max\{\varrho_b(x,y), \varrho_b(x,Tx), \varrho_b(y,Ty)\}$;
(4) *T is continuous or T satisfies (α)-admissible property.*
Then, T has a fixed point $u \in X$ with $\varrho_b(u,u) = 0$ and $\{T^n u_0\}$ converges to u. Furthermore, if for all $u,v \in F(T)$, with $u \neq v$ such that $\alpha(u,Tv) \geq 1$, $\alpha(v,Tu) \geq 1$, then T has a unique fixed point in Ω.

Proof. Define a mapping $\beta : \Omega \times \Omega \to [0,\infty)$ as $\beta(x,y) = \begin{cases} 1, & \text{if } x,y \in \Omega,\\ 0, & \text{otherwise.}\end{cases}$
Then, the conclusion follows from Theorem 1. □

Next, the following example is presented to verify the validity of our result.

Example 4. *Let $\Omega = [0,20]$ be equipped with the partial order relation $\preceq$ defined by*

$$u \preceq v \Leftrightarrow u > v$$

and the function $\varrho_b : \Omega \times \Omega \to [0,\infty)$ is defined by

$$\varrho_b(u,v) = (\max\{u,v\})^2,$$

for all $u, v \in X$, where $s = 2$. It is obvious that (Ω,ϱ_b) is a complete partial b-metric space. Let the mapping $T : \Omega \to \Omega$ is defined by

$$Tu = \frac{1}{16}u^3 e^{-u^3}.$$

In addition, we define the mapping α, $\beta : \Omega \times \Omega \to [0,\infty)$ by

$$\alpha(u,v) = \begin{cases} 1, & u,v \in [0,1];\\ 0, & \textit{otherwise}.\end{cases}$$

and

$$\beta(u,v)=\begin{cases}1, & u,v\in[0,1];\\ 0, & otherwise.\end{cases}$$

By the definition of T, it is clear that $\alpha(Tu,Tv)\leq 1$ and $\beta(Tu,Tv)\leq 1$. In addition, there exists $u_0=0$ in X such that $\alpha(0,T0)=\alpha(0,0)\geq 1$ and $\beta(0,T0)=\beta(0,0)\geq 1$. Define $\theta:[0,\infty)\to[0,1)$ by $\theta(t)=\frac{1}{t+1}$. In addition, let $\varphi:[0,\infty)\to[0,\infty)$ be given by $\varphi(t)=\frac{t}{100}$. Let $F(t)=\lg t+t$ for all $t>0$. Without loss of generality, we may take $u,v\in X$ such that $u>v$. In order to check the contractive condition (3) of Theorem 1, we have to consider the following cases (briefly, the left-hand side is denoted by L.H.S, and the right-hand side is denoted by R.H.S):

Case I. If $u,v\in[0,1]$, then

$$\begin{aligned} L.H.S. &= \alpha(u,v)\beta(u,v)\mathcal{F}(s\varrho_b(Tu,Tv))\\ &= \mathcal{F}(s\max\{\frac{1}{16}u^3e^{-u^3},\frac{1}{16}v^3e^{-v^3}\}^2)\\ &= \mathcal{F}(s\frac{1}{16}u^6e^{-2u^3})\\ &= \lg(\frac{1}{8}u^6e^{-2u^3})+\frac{1}{8}u^6e^{-2u^3}. \end{aligned} \tag{23}$$

For R.H.S., utilizing the definitions of $M_s(u,v)$, and $N_s(u,v)$, we have that

$$\begin{aligned} M_s(u,v) &= \max\{\varrho_b(u,v),\varrho_b(u,Tu),\varrho_b(v,Tv),\frac{\varrho_b(u,Tv)+\varrho_b(v,Tu)}{2s}\},\\ &= \max\{(\max\{u,v\})^2,(\max\{u,Tu\})^2,(\max\{v,Tv\})^2,\frac{(\max\{u,Tv\})^2+(\max\{v,Tu\})^2}{2s}\}\\ &= \max\{u^2,u^2,v^2,\frac{u^2+(\max\{v,Tu\})^2}{2s}\}\\ &= u^2,\\ N_s(u,v) &= \max\{\varrho_b(u,v),\varrho_b(u,Tu),\varrho_b(v,Tv)\}\\ &= \max\{(\max\{u,v\})^2,(\max\{u,Tu\})^2,(\max\{v,Tv\})^2\}\\ &= \max\{u^2,u^2,v^2\}\\ &= u^2. \end{aligned}$$

We verify that $M_s(u,v)=u^2$ and $N_s(u,v)=u^2$, thus

$$\begin{aligned} R.H.S. &= \theta(\varphi(M_s(u,v))\mathcal{F}(N_s(u,v))\\ &= \theta(\varphi(u^2))\mathcal{F}(u^2)\\ &= \theta(\tfrac{u^2}{100})\mathcal{F}(u^2)\\ &= \tfrac{100(\lg(u^2)+u^2)}{u^2+100}, \end{aligned} \tag{24}$$

for all $u,v\in[0,1]$ and with $u>0$. The following figures (see Figures 1–4) demonstrate that R.H.S. expression (with black curve) and L.H.S. expression (with blue curve) for $u,v\in[0,1]$, which validates our inequality.

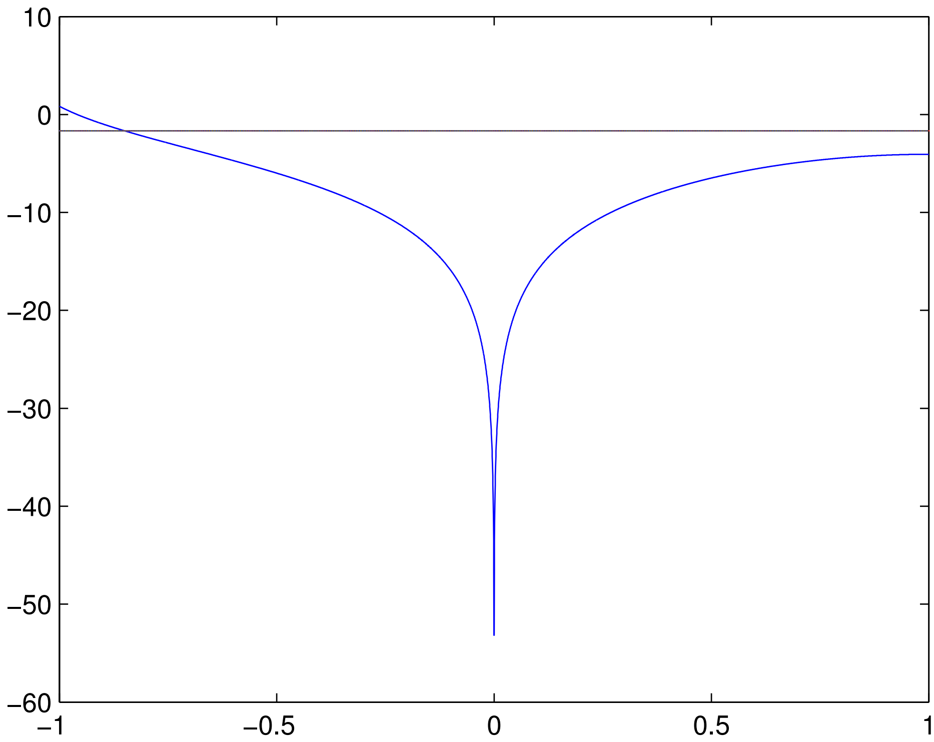

Figure 1. Plot of inequality for Case I with $s = 2$ in Example 4.

Comparing with Definition 11, setting $s = 2$, $\tau = 5$ *in condition* (1)*, and Figure 2 shows that the condition* (1) *of Definition 10 is not satisfied.*

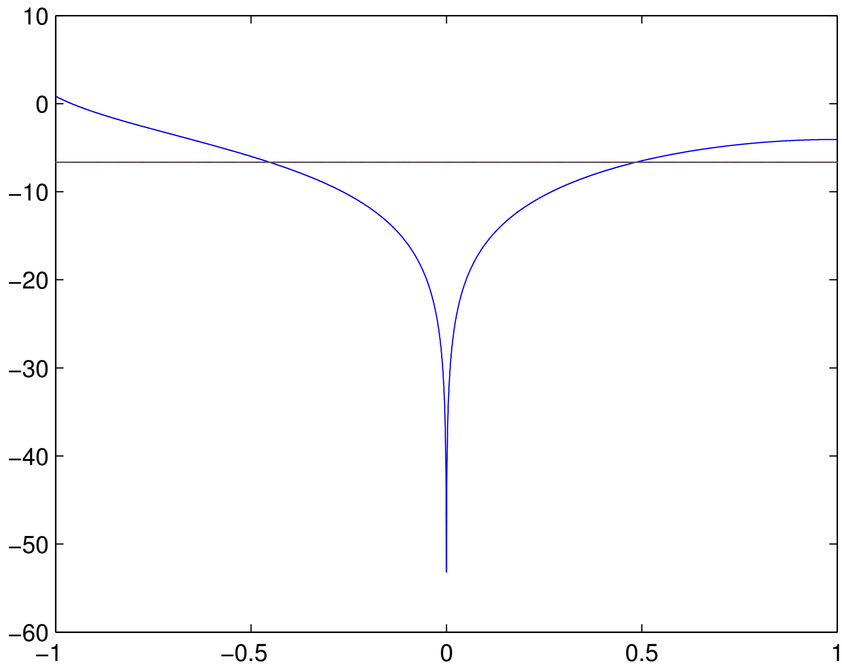

Figure 2. Plot of inequality for Case I with $s = 2$ and $\tau = 5$ for condition (1) of Definition 10 in Example 4.

If we take $s = 3$*, we can plot the figure below, and the R.H.S. expression (with black curve) dominates the L.H.S. expression (with blue curve) for* $u, v \in [0, 1]$*:*

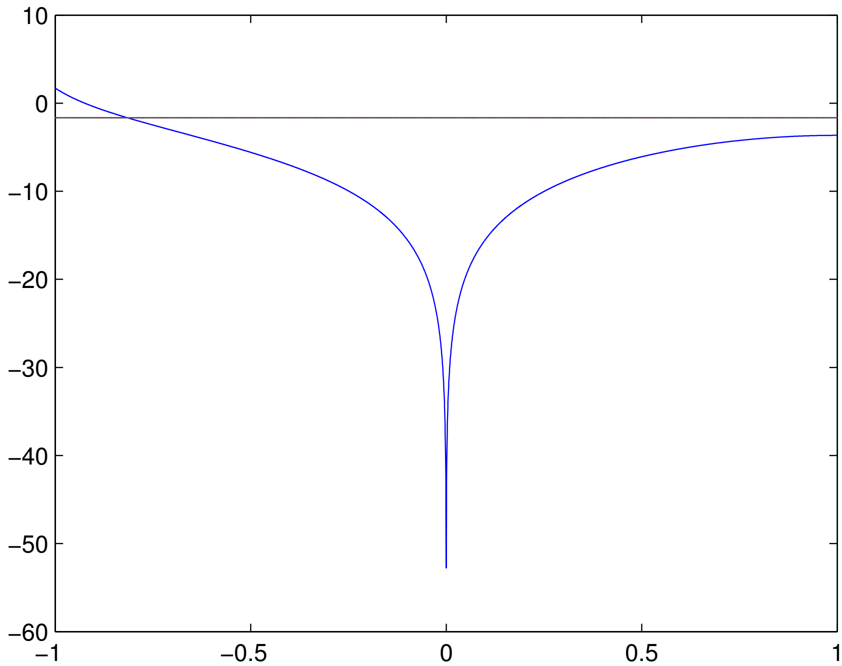

Figure 3. Plot of inequality for Case I with $s = 3$ in Example 4.

Comparing with Definition 10, setting $s = 3$, $\tau = 5$ in condition (1), Figure 4 shows that the condition (1) of Definition 10 is not satisfied.

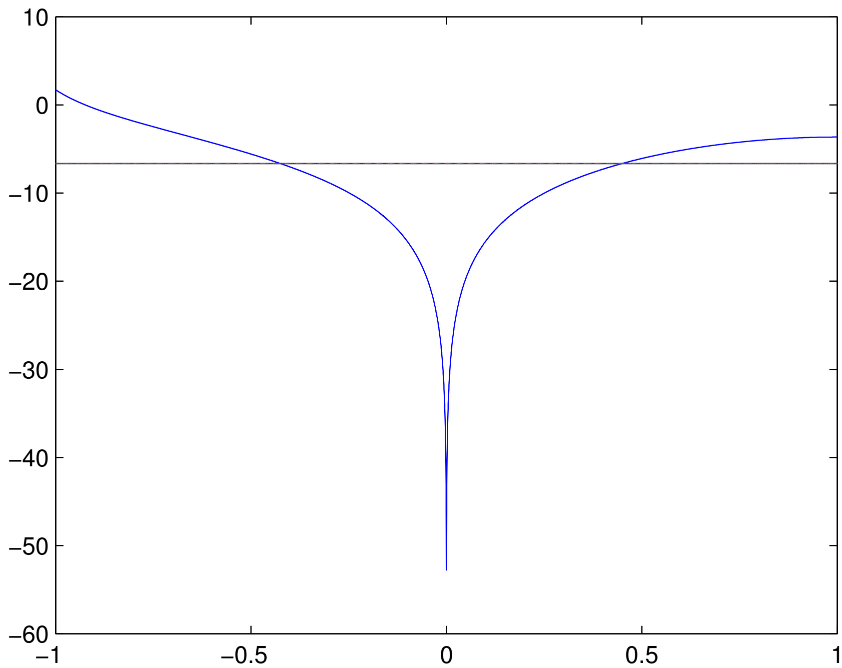

Figure 4. Plot of inequality for Case I with $s = 2$ or $s = 3$ and $\tau = 5$ for condition (1) of Definition 10 in Example 4.

Case II. If $u, v \in (1, 20]$, then $\alpha(u,v)\beta(u,v) = 0$. From (1), we have

$$\begin{aligned} L.H.S. &= \alpha(u,v)\beta(u,v)F(s\varrho_b(Tu,Tv)) = 0 \\ &\leq \frac{100(\lg(u^2)+u^2)}{u^2+100}, \end{aligned}$$

The figures below (see Figures 5 and 6) show that R.H.S. expression (with black curve) overshadows the L.H.S. expression (not appearing in the figure, since it is $v = 0$), which authenticates our inequality. Obviously, the whole figure is above the line $v = 0$. In this case, the figure of $s = 2$ is the same as that of $s = 3$, since L.H.S. are both 0, R.H.S are both $\frac{100(\lg(u^2)+u^2)}{u^2+100}$.

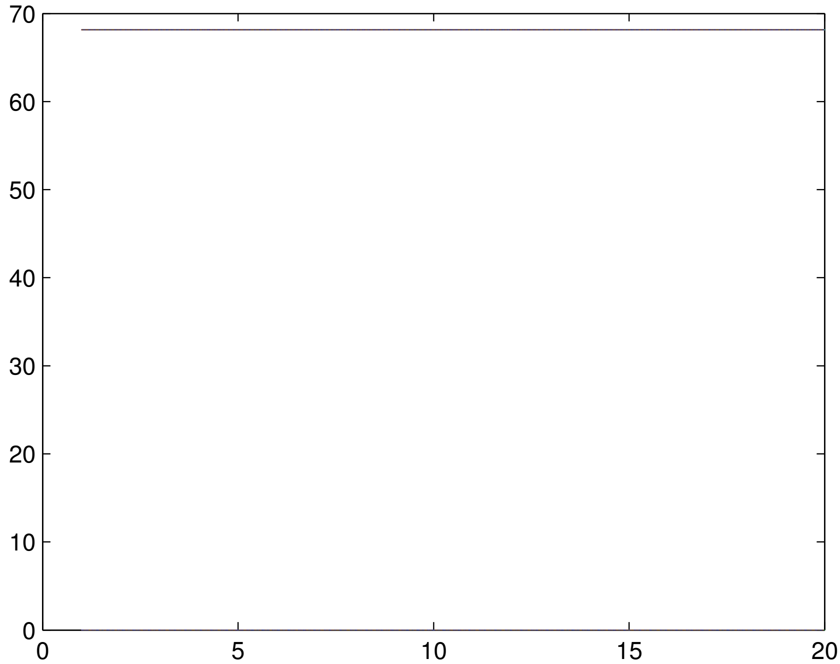

Figure 5. Plot of inequality for Case II with $s = 2$ and $s = 3$ in Example 4.

Comparing with Definition 10, setting $s = 2$ or $s = 3$, $\tau = 96$ in condition (1), L.H.S= 0, R.H.S< 0, so condition (1) in Definition 10 is not satisfied.

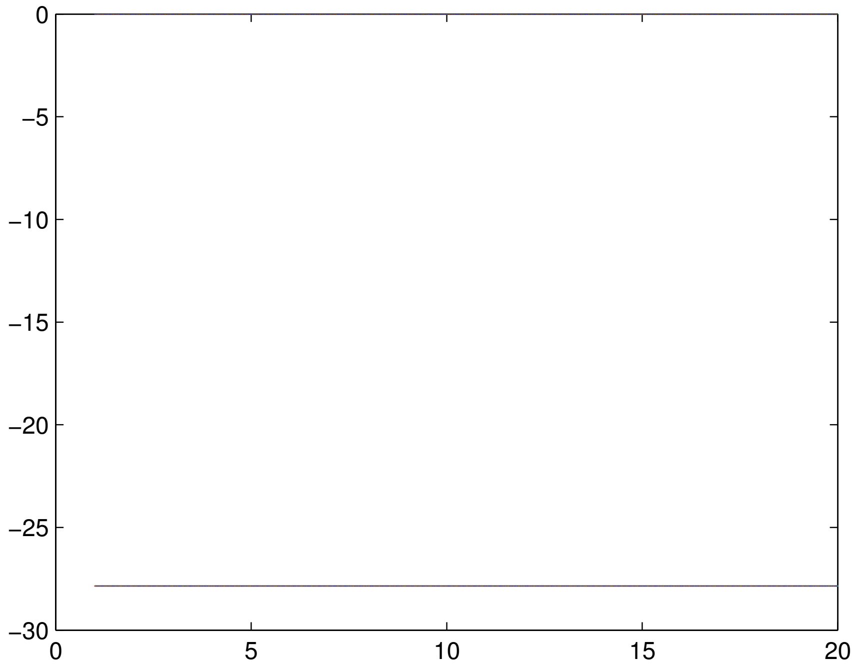

Figure 6. Plot of inequality for Case I with $s = 2$ or $s = 3$ and $\tau = 2$ for condition (1) of Definition 10 in Example 4.

Case III. If $v \in [0,1]$ and $u \in (1,20]$, then Case III is similar to Case II,

$$\begin{aligned} L.H.S. &= \alpha(u,v)\beta(u,v)\mathcal{F}(s\varrho_b(Tu,Tv)) = 0 \\ &\leq \frac{100(\lg(u^2) + u^2}{u^2+100}. \end{aligned}$$

The figures below (see Figures 7 and 8) show that R.H.S. expression (with black curve) overshadows the L.H.S. expression (not appearing in the figure, since it is $v = 0$), which authenticates our inequality. Obviously, the whole figure is above the line $v = 0$. In this case, the figure of $s = 2$ is the same as that of $s = 3$, since L.H.S. are both 0, and R.H.S are both $\frac{100(\lg(u^2)+u^2}{u^2+100}$.

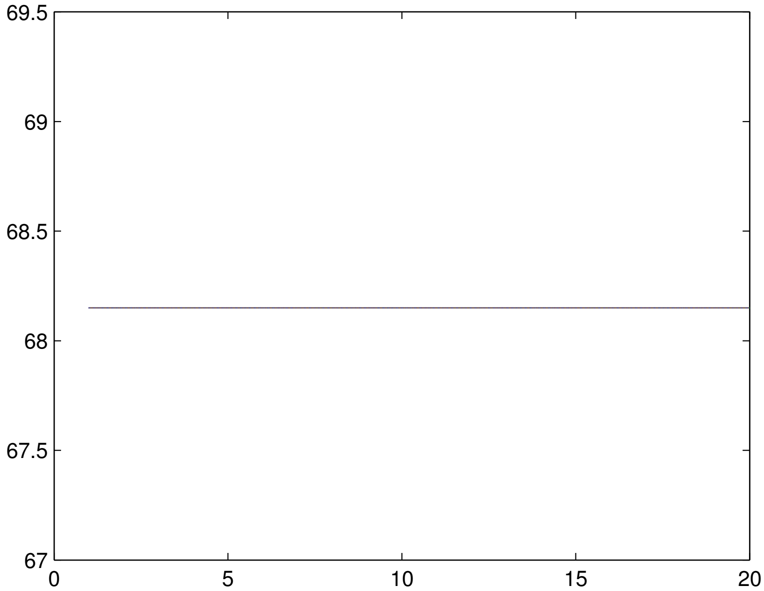

Figure 7. Plot of inequality for Case III with $s = 2$ and $s = 3$ in Example 4.

Comparing with Definition 10, setting $s = 2$ or $s = 3$, $\tau = 69$ in condition (1), L.H.S$= 0$, R.H.S < 0, so condition (1) in Definition 10 is not satisfied.

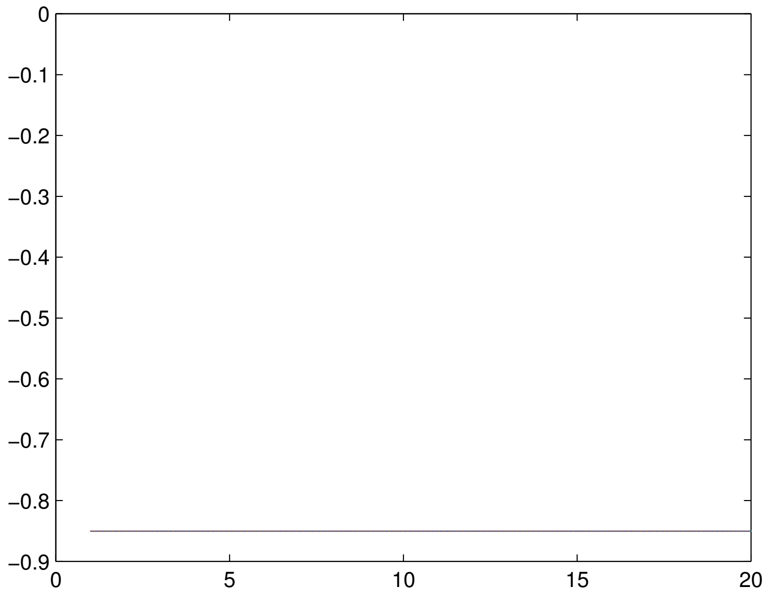

Figure 8. Plot of inequality for Case I with $s = 2$ or $s = 3$ and $\tau = 0.1$ for condition (1) of Definition 10 in Example 4.

Thus, all the conditions of Theorem 1 are fulfilled and $0 \in \Omega$ is a unique fixed point of the involved mapping T (see Figure 9).

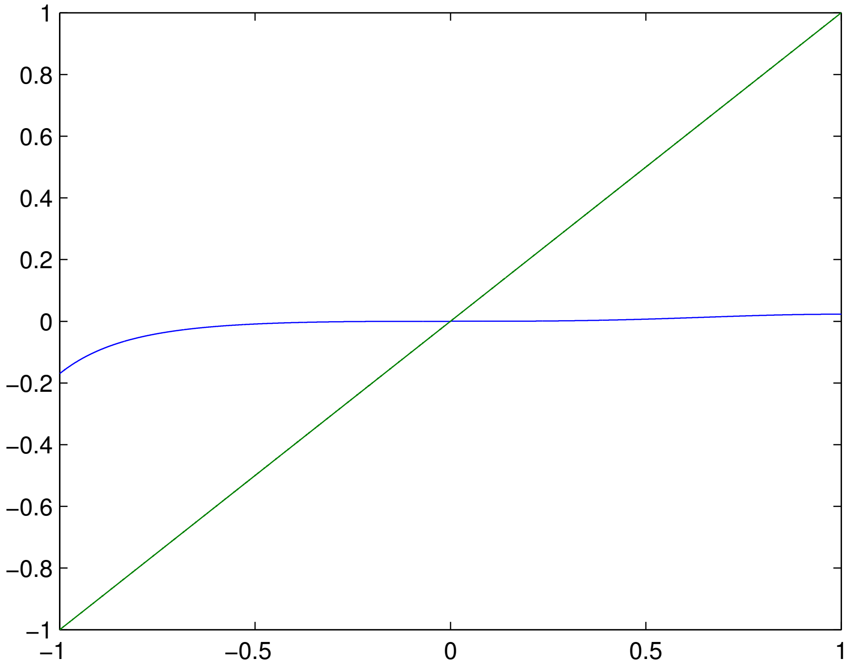

Figure 9. The fixed point of the mapping T in Example 4.

Next, we present a fixed point result for cyclic mappings in partial b-metric spaces in the following theorem.

Theorem 3. *Let (Ω, ϱ_b) be a complete partial b-metric spaces with $s \geq 1$, and A, B be two nonempty closed subsets of Ω. Let $\alpha : \Omega \times \Omega \to [0, \infty)$ and $T : A \cup B \to A \cup B$ are two mappings with $TA \subseteq B$, $TB \subseteq A$. Suppose that $\alpha(Ty, Tx) \geq 1$ if $\alpha(x, y) \geq 1$, when $x \in A, y \in B$. Furthermore, assume that T satisfies the following assumption for all $x \in A, y \in B$*

$$\alpha(x,y)\mathcal{F}(s\varrho_b(Tx,Ty)) \leq \theta(\phi(M_s(x,y)))\mathcal{F}(N_s(x,y)),$$

where $M_s(x,y) = \max\{\varrho_b(x,y), \varrho_b(x,Tx), \varrho_b(y,Ty), \frac{\varrho_b(x,Ty)+\varrho_b(y,Tx)}{2s}\}$,
$N_s(x,y) = \max\{\varrho_b(x,y), \varrho_b(x,Tx), \varrho_b(y,Ty)\}$.
If there exists $u_0 \in A$ such that $\alpha(u_0, Tu_0) \geq 1$ and either T is continuous or T satisfies α-admissible property, then T has a fixed point u in $A \cap B$ and $\{T^n u_0\}$ converges to u.
Furthermore, if for all $u, v \in F(T)$, with $u \neq v$ such that $\alpha(u, Tv) \geq 1$, $\alpha(v, Tu) \geq 1$ and $\beta(u, Tv) \geq 1$, $\beta(v, Tu) \geq 1$, then T has a unique fixed point in Ω.

Proof. Let $Y = A \cup B$ and $\beta : Y \times Y \to [0, \infty)$ be defined as $\beta(u,v) = \begin{cases} 1, & \text{if } u \in A, v \in B, \\ 0, & \text{otherwise} \end{cases}.$

It is obvious that (Y, ϱ_b) is complete. Suppose that there exists $u_0 \in A$ with $\alpha(u_0, Tu_0) \geq 1$, from the definition of β, we also have $\beta(u_0, Tu_0) \geq 1$. Hence, the hypotheses (1)–(3) of Theorem 1 hold with $X = Y$. Afterward, suppose that $\{u_n\}$ is a sequence in X satisfying $\alpha(u_{2n}, u_{2n+1}) \geq 1$ and $\beta(u_{2n}, u_{2n+1}) \geq 1$ for $n \in \mathbb{N}$ and $u_n \to u$. Hence, $u_{2n} \in A$ and $u_{2n+1} \in B$. Since B is closed, then $u \in B$ and $\alpha(u_{2n}, u) \geq 1$ and $\beta(u_{2n}, u) \geq 1$. We conclude that the hypothesis (4) of Theorem 1 holds for $X = Y$. Consequently, T has a unique fixed point in $Y = A \cup B$, say u. Since $u \in A$ implies that $u = Tu \in B$ and $u \in B$ implies $u = Tu \in A$, then $u \in A \cap B$. $\square$

2.2. Fixed Point Results for Graphic (α, β)-Admissible Generalized Geraghty F-Contractions

In this subsection, we present a fixed point result for graphic (α, β)-admissible generalized Geraghty F-contractions in the setting of partial metric spaces endowed with a directed graph.

Consistent with Jachymisk [30], let (Ω, ϱ) be a partial space and $\triangle = \{(x, x) : x \in \Omega\}$. Let $G = (V(G), E(G))$ be a directed graph, where $V(G)$ stands for the set of vertices which coincides with Ω and $E(G)$ the set of edges contains all loops, that is, $E(G) \supseteq \triangle$. Assume that G has no parallel edges. The graph G can be converted to a weighted graph by assigning to each edge a weight equal to the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length $n (n \in \mathbb{N})$ is a sequence $\{u_i\}_{i=0}^{n}$ of $n + 1$ vertices such that $u_0 = x, u_n = y$ and $(u_{i-1}, u_i) \in E(G)$. A graph G is said to be connected if there exists a path between any two vertices. Recently, several results have appeared concerning sufficient conditions for a certain contractive mapping to admit a fixed point in the underlying space endowed with a graph. The first result in this direction was initiated by Jachymski [30].

Definition 12 ([30])**.** *Let (Ω, ϱ) be a partial metric space endowed with a graph G and T be a self-mapping defined on Ω. We say T is a G-contraction if T preserves edge of G, that is, for all $x, y \in \Omega$,*

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G),$$

and T decreases weights of edges of G in the following way: there exists $\alpha \in (0, 1)$ such that, for all $x, y \in \Omega$,

$$(x, y) \in E(G) \Rightarrow \varrho(Tx, Ty) \leq \alpha \varrho(x, y).$$

Definition 13 ([30])**.** *Let (Ω, ϱ) be a partial metric space endowed with a graph G and T be a self-mapping defined on Ω. We say T is a G-continuous if, for any $u \in \Omega$ and a sequence $\{u_n\}$ with $u_n \to u$ as $n \to \infty$, $(u_n, u_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ implies $Tu_n \to Tu$ as $n \to \infty$.*

Definition 14. *Let $G = (V(G), E(G))$ be a connected graph with $V(G) = \Omega$. We say graph G is said to satisfy the property (P^*), if a connected T-Picard sequence $\{u_n\}$ converges to u in Ω implies that there exists $n_0 \in \mathbb{N}$ such that $(u_n, u) \in E(G)$ or $(u, u_n) \in E(G)$ for all $n > n_0$.*

Definition 15. *Let (Ω, ϱ) be a partial metric space endowed with a graph G and T be a self-mapping defined on Ω. We say T is a graphic generalized Geraghty $\mathcal{F}$-contractions, if there exist $\mathcal{F} \in \triangle_{\mathcal{F}}, \theta \in \Theta$ and $\phi \in \Phi$ such that, for all $x, y \in X$ and $s \geq 1$ with $\varrho(Tx, Ty) > 0$,*

$$\mathcal{F}(p(Tx, Ty)) \leq \theta(\phi(M(x, y)))\mathcal{F}(N(x, y)), \tag{25}$$

where $M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\}$,
$N(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty)\}$.

Theorem 4. *Let (Ω, ϱ) be a complete partial metric space endowed with a graph G and T be a self-mapping defined on Ω satisfying the following conditions:*
(i) *T preserves the edge of G;*
(ii) *there exists $u_0 \in X$ such that $(u_0, Tu_0) \in E(G)$;*
(iii) *T is G-continuous or G satisfies property (P^*);*
(iv) *T is a graphic generalized Geraghty $\mathcal{F}$-contractions.*

Proof. Define a mapping $\alpha : \Omega \times \Omega \to [0, \infty)$ as $\alpha(u, v) = \begin{cases} 1, & \text{if} \quad (u, v) \in G, \\ 0, & \text{otherwise.} \end{cases}$

Now, we show that T is an α-admissible mapping. Suppose that $\alpha(u, v) \geq 1$. Therefore, we have $(u, v) \in E(G)$. From condition (i), we have $(Tu, Tv) \in E(G)$. Thus, $\alpha(Tu, Tv) \geq 1$

and T is an α-admissible mapping. Hence, from the definitions of α and graphic generalized Geraghty F-contractions, we have

$$\alpha(u,v)\mathcal{F}(p(Tu,Tv)) \leq \theta(\phi(M(u,v)))\mathcal{F}(N(u,v)),$$

where $M(u,v) = \max\{\varrho(u,v), \varrho(u,Tu), \varrho(v,Tv), \frac{\varrho(u,Tv)+\varrho(v,Tu)}{2}\}$, $N(u,v) = \max\{\varrho(u,v), \varrho(u,Tu), \varrho(v,Tv)\}$.

Due to condition (ii), there exists $u_0 \in X$ such that $(u_0, Tu_0) \in E(G)$ and $\alpha(u_0, Tu_0) \geq 1$.

Suppose that $\{u_n\}$ is a sequence in Ω such that $u_n \to u$ as $n \to \infty$ and $(u_n, Tu_n) \in E(G)$ for all $n \in \mathbb{N}$. Thus, from condition (iii), we have that T is continuous or T satisfies an α-admissible property.

Therefore, all conditions of Theorem 2 hold true and T has a fixed point. □

Now, we present an example to support Theorem 4 as follows.

Example 5. *Let $\Omega = \{a,b,c\}$ be endowed with the function $\chi : \Omega \times \Omega \to [0,\infty)$ defined by*

$$\chi(a,b) = \chi(b,a) = \frac{3}{10}, \quad \chi(b,c) = \chi(c,b) = \frac{4}{10}, \quad \chi(a,c) = \chi(c,a) = \frac{2}{10}$$
$$\chi(a,a) = \frac{1}{30}, \quad \chi(b,b) = \frac{1}{20}, \quad \chi(c,c) = \frac{1}{10}.$$

It is easy to check that χ is a partial metric.

Define a function $\theta \in \Theta, \theta : (0,\infty) \to [0,1)$ by $\theta(x) = \begin{cases} e^{-\frac{x}{4}}, & 0 < x, \\ 0, & x = 0. \end{cases}$. Now, define a mapping $T : X \to X$ by

$$T(a) = T(c) = a, T(b) = c.$$

In addition, define two functions $\mathcal{F} \in \Delta_{\mathcal{F}}$ and $\phi \in \Phi$ by $\mathcal{F}(t) = \ln t$, for all $t > 0$ and $\phi(s) = s$, for all $s \geq 0$.

Suppose that G is a direct graph such that $V(G) = \Omega$ and $E(G) = \{(x,y) : x,y \in \{a,b,c\}\}$. It is easy to show that T preserves edges in G and T is G-continuous. Moreover, there exists $u_0 = a \in X$ such that $(a, Ta) = (a,a) \in E(G)$. Without loss of generality, let $x,y \in \Omega$ with $x \neq y$.

Next, we will show that condition (iv) in Theorem 4 holds. Consider the following cases:

Case I. If $x = a, y = b$, then we have

$$\begin{aligned} \mathcal{F}(\chi(Ta,Tb)) &\leq \theta(\phi(M(a,b)))\mathcal{F}(N(a,b)) \\ \ln(\chi(a,c)) &\leq e^{-\frac{M(a,b)}{4}} \ln(N(a,b)) \\ \ln(0.2) &\leq e^{-0.1} \ln(0.4). \end{aligned}$$

Case II. If $x = a, y = c$, then we have

$$\begin{aligned} \mathcal{F}(\chi(Ta,Tc)) &\leq \theta(\phi(M(a,c)))\mathcal{F}(N(a,c)) \\ \ln(\chi(a,a)) &\leq e^{-\frac{M(a,c)}{4}} \ln(N(a,c)) \\ \ln(\frac{1}{30}) &\leq e^{-0.05} \ln(0.2). \end{aligned}$$

Case III. If $x = b, y = c$, then we have

$$\begin{aligned}\mathcal{F}(p(Tb, Tc)) &\leq \theta(\phi(M(b,c)))\mathcal{F}(N(b,c)) \\ \ln(\chi(c,a)) &\leq e^{-\frac{M(b,c)}{4}} \ln(N(b,c)) \\ \ln(0.2) &\leq e^{-0.1} \ln(0.4).\end{aligned}$$

Figure 10 represents the graph with all the possible cases. Therefore, all the conditions of Theorem 4 are satisfied and a is a fixed point of T.

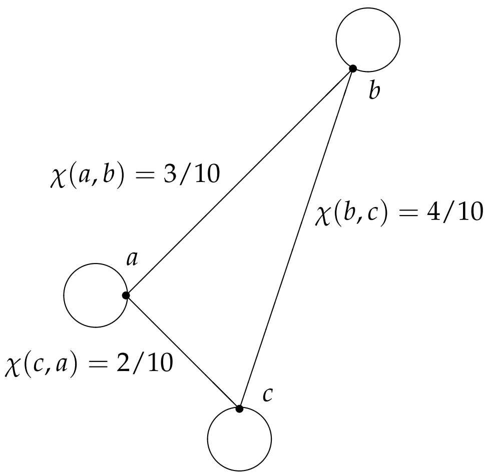

Figure 10. The graph G defined in Example 5.

3. An Application to the First Order Periodic Boundary Value Problem

In this section, we will examine the solution of the following first order periodic boundary value problem:

$$\begin{cases} \nu'(t) = \vartheta(t, \nu(t)), \quad t \in [0, T]; \\ \nu(0) = \nu(T), \end{cases} \tag{26}$$

where $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function for $T > 0$, and $\eta > 0$ be any real number such that $\eta > T$. Then, the following integral equation is equivalent to the preceding problem:

$$\nu(t) = \int_0^T G(t,s)[\vartheta(s, \nu(s)) + \eta\nu(s)]ds, \tag{27}$$

where $G(t,s)$ is a Green's function, defined by

$$G(t,s) = \begin{cases} \frac{e^{\eta(T+s-t)}}{e^{\eta T}-1}, \quad 0 \leq s \leq t \leq T; \\ \frac{e^{\eta(s-t)}}{e^{\eta T}-1}, \quad 0 \leq t \leq s \leq T. \end{cases} \tag{28}$$

Let $\Delta = C([0, T], \mathbb{R})$ be a set of all real valued continuous functions on $[0, T]$ and $\varrho_b : X \times X \to [0, \infty)$ be defined by

$$\varrho_b(\nu, \omega) = \max_{t \in [0,T]} |\nu(t) - \omega(t)|^2, \tag{29}$$

for all $\nu, \omega \in \Delta$. Obviously, (X, ϱ_b) is a complete partial b-metric space. Define the map $f : \Delta \to \Delta$ by

$$f(\omega(t)) = \int_0^T G(t,s)[\vartheta(s, \omega(s)) + \eta\omega(s)]ds, \quad t \in [0, T].$$

Then, ω is a solution of (27) if and only if it is a fixed point of f.

Theorem 5. *Assume there exist real numbers $\eta, T > 0$ such that $\eta > T$, then, for any $x(t), y(t) \in \Delta$,*

$$|\vartheta(t, x(t)), \eta x(t) - \vartheta(t, y(t)) - \eta y(t)| \leq \frac{\eta}{2^{\frac{\epsilon}{2}+1}} \sqrt{|x(t) - y(t)|^2}, \tag{30}$$

where $t \in [0, T]$. Then, the differential equation (26) has a solution.

Proof. Observe that $(C([0, T], \mathbb{R}), \varrho_b)$ is a complete partial b-metric space defined in (29)). For $\nu(t), \omega(t) \in \Delta$, we have that

$$\begin{aligned}
|f(\nu(t)) - f(\omega(t))| &= |\int_0^T G(t,s)[\vartheta(s,\nu(s)) + \eta\nu(s)]ds - \int_0^T G(t,s)[\vartheta(s,\omega(s)) + \eta\omega(s)]ds| \\
&\leq \int_0^T G(t,s)|\vartheta(s,\nu(s)) + \eta\nu(s) - \vartheta(s,\omega(s)) - \eta\omega(s)|ds \\
&\leq \max_{t\in[0,T]} |\vartheta(t,\nu(t)) + \eta\nu(t) - \vartheta(t,\omega(t)) - \eta\omega(t)| \int_0^T G(t,s)ds \\
&\leq \frac{\eta}{2^{\frac{\epsilon}{2}+1}} \max_{t\in[0,T]} \sqrt{|\nu(t) - \omega(t)|^2} \int_0^T G(t,s)ds \\
&\leq \frac{\eta}{2^{\frac{\epsilon}{2}+1}} \max_{t\in[0,T]} \sqrt{|\nu(t) - \omega(t)|^2} [\int_0^T \frac{e^{\eta(T+s-t)}}{e^{\eta T} - 1} ds + \int_0^T \frac{e^{\eta(s-t)}}{e^{\eta T} - 1}]ds \\
&= \frac{\eta}{2^{\frac{\epsilon}{2}+1}} \max_{t\in[0,T]} \sqrt{|\nu(t) - \omega(t)|^2} [\frac{1}{\eta(e^{\eta T} - 1)}(e^{\eta(2T-t)} - e^{-\eta t}] \\
&\leq \frac{\eta}{2^{\frac{\epsilon}{2}+1}} \max_{t\in[0,T]} \sqrt{|\nu(t) - \omega(t)|^2},
\end{aligned}$$

which yields

$$2^{\frac{\epsilon}{2}} \max_{t\in[0,T]} |f(\nu(t)) - f(\omega(t))|^2 \leq \frac{\eta}{2} \max_{t\in[0,T]} |\nu(t) - \omega(t)|^2,$$

or

$$\frac{1}{\eta} 2^{\frac{\epsilon}{2}} \varrho_b(f(\nu), f(\omega)) \leq \frac{1}{2} \varrho_b(\nu, \omega) \leq \frac{1}{2} N_s(\nu, \omega).$$

This implies that

$$\frac{1}{\eta} 2^{\frac{\epsilon}{2}} \varrho_b(f(\nu), f(\omega)) \leq \frac{1}{2} N_s(\nu, \omega),$$

where $N_s(\nu, \omega) = \max\{\varrho_b(\nu, \omega), \varrho_b(\nu, f(\nu)), \varrho_b(\omega, f(\omega))\}$.

Since $\mathcal{F}$ is increasing, we have that

$$\frac{1}{\eta} \mathcal{F}(2^{\frac{\epsilon}{2}} \varrho_b(f(\nu), f(\omega))) \leq \frac{1}{2} \mathcal{F}(N_s(\nu, \omega)).$$

Taking $s = 2^{\frac{\epsilon}{2}}$, we have that

$$\frac{1}{\eta} \mathcal{F}(s\varrho_b(f(\nu), f(\omega))) \leq \frac{1}{2} \mathcal{F}(N_s(\nu, \omega)).$$

Putting $\theta(t) = \frac{1}{2}$, together with $\phi \in \Phi$, we can deduce that $\theta(\phi(M_s(\nu, \omega))) = \frac{1}{2}$, where

$$M_s(\nu, \omega) = \max\{\varrho_b(\nu, \omega), \varrho_b(\nu, f(\nu)), \varrho_b(\omega, f(\omega)), \frac{\varrho_b(\nu, f(\omega)) + \varrho_b(\omega, f(\nu))}{2s}\}.$$

In addition, by letting $\eta \in (0, 1)$, we claim that $\alpha(\nu, \omega)\beta(\nu, \omega) = \frac{1}{\eta}$.

From the fact of (α, β)-admissibility, we obtain that

$$\alpha(\nu,\omega)\beta(\nu,\omega)\mathcal{F}(s\varrho_b(f(\nu), f(\omega))) \leq \theta(\phi(M_s(\nu,\omega)))F(N_s(\nu,\omega)).$$

Hence, all the conditions of Theorem 1 are satisfied which implies that f has a fixed point, that is, the integral Equation (27) has a solution. □

4. Conclusions and Future Work

In this paper, we generalized Geraghty contractions by introducing (α, β)-admissible generalized Geraghty F-contractions and establishing the corresponding fixed point theorem in partial b-metric spaces. In addition, we extended our main result to a class of graphic generalized contractions called graphic generalized Geraghty F-contractions. An application to a first order periodic boundary value problem was presented. On the other hand, there are a lot of studies on the non-unique fixed points (or called fixed figure) in the literature (for example, see [31] and the references therein). Let (X, d) be a metric space, T be a self-mapping of X, and $Fix(T) = \{x \in X : Tx = x\}$ be the fixed point set of T. A circle/disc contained in the set $Fix(T)$ is called the fixed-circle/fixed-disc of T (for more details, see [32,33]). At this point, some future directions of our study appear as the following:

Exploring the concept of (α, β)-admissible generalized Geraghty F-contractions,

(1) some new fixed figure results for such contractions can be investigated;
(2) some new common fixed point (resp. coincidence point) results can be examined for the cases where the set $Fix(T)$ is not a singleton.

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