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Strong Convergence Theorems for a Finite Family of Enriched Strictly Pseudocontractive Mappings and Φ_T-Enriched Lipschitizian Mappings Using a New Modified Mixed-Type Ishikawa Iteration Scheme with Error

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Abstract: In this paper, we introduce two new classes of mappings known as λ-enriched strictly pseudocontractive mappings and Φ_T-enriched Lipshitizian mappings in the setup of a real Banach space. In addition, a new modified mixed-type Ishikawa iteration scheme was constructed, and it was proved that our iteration method converges strongly to the common fixed points of finite families of the above mappings in the framework of a real uniformly convex Banach space. Moreover, we provided a non-trivial example to support our main result. Our results extend and generalize several results existing in the literature.

Keywords: finite family; enriched strictly pseudocontractive mapping; Φ_T-enriched Lipschitizian self-mapping; modified Ishikawa mixed-type iteration scheme; common fixed point; uniformly convex Banach space; strong convergence

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1. Introduction

Given a structured object Ξ of any sort, a symmetry is a mapping of the object Ξ onto itself such that the structure is preserved. This kind of mapping can occur in many ways: On one hand, if Ξ is a set with no additional structure, a symmetry is a bijective map from the set Ξ to itself, which often results in a permutation group. On the other hand, if object Ξ is a set of points in the plane with its metric structure, a symmetry is a bijection of the set Ξ to itself, which preserves the distance between each pair of points $(s,t) \in \Xi$. In [1], Sain established the idea of left symmetric and right symmetric points in Banach spaces (recall that an element $\hbar \in \Xi$ is known as left symmetric if $\hbar \perp_B \zeta$ implies $\zeta \perp_B \hbar$ for all $\zeta \in \Xi$, whereas an element $\hbar \in \Xi$ is known as right symmetric if $\zeta \perp_B \hbar$ implies $\hbar \perp_B \zeta$ for all $\zeta \in \Xi$. Hence, an element $\hbar \in \Xi$ is a symmetric point if $\hbar$ is both left symmetric and right symmetric).

Let $(\Xi, \|.\|)$ be a normed linear space. For any two elements $\hbar, \zeta$ in Ξ, $\hbar$ is said to be orthogonal to ζ in the sense of Birkhoff–James [2], written $\hbar \perp_B \zeta$, if and only if $\|\hbar\| \leq \|\hbar + \lambda\zeta\|$ for all $\lambda \in \mathbb{R}$. Birkhoff–James orthogonality is related to many important geometric properties of normed linear spaces including strict convexity, uniform convexity and smoothness.

Let C be a nonempty, closed and convex subset of a real Banach space E. If $E^\star$ is a dual of E, then the mapping $J : E \longrightarrow 2^{E^\star}$ defined by the following:

$$J(\hbar) = \{\hbar^\star \in E : \langle \hbar, \hbar^\star \rangle = \|\hbar\|^2, \|\hbar\| = \|\hbar^\star\|\}, \tag{1}$$

is known as normalized duality mapping.

Let $T : C \longrightarrow C$ be a nonlinear mapping. We will denote the set of all fixed points of T by $F(T)$. The set of common fixed points of finite family of mappings $\{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N : C \longrightarrow C$ will be denoted by $\mathcal{F} = \cap_{i=1}^N (F(S_i) \cap F(T_i))$, where $N \in \mathbb{N}$ (the set of natural numbers).

Definition 1. *A self mapping T on C is said to be L-Lipschitzian, if for all $\hbar, \zeta \in C$, there exists a constant $L > 0$ such that the following is the case:*

$$\|T\hbar - T\zeta\| \leq L\|\hbar - \zeta\|, \tag{2}$$

where L is known as Lipschitz constant.

Definition 2. *A mapping T is known as Φ_T-enriched Lipschitizian (or (b, Φ_T)-enriched Lipschitizian) if for all $\hbar, \zeta \in C$, there exists $b \in [0, +\infty)$ and a continuous nondecreasing function $\Phi_T : R^+ \longrightarrow R^+$, with $\Phi(0) = 0$, such that the following is the case.*

$$\|b(\hbar - \zeta) + T\hbar - T\zeta\| \leq (b+1)\Phi_T(\|\hbar - \zeta\|). \tag{3}$$

Remark 1. *In special case, where $b = 0$, then the (b, Φ_T)-enriched Lipschitizian mapping T is known as Φ_T-Lipschitzian; if $b = 0$ and $\Phi_T(r) = Lr$, for $L > 0$, then T is known as Lipschitzian mapping with L as the Lipschitz constant. In particular, if $b = 0$, $\Phi_T(r) = Lr$ and $L = 1$, then the (b, Φ_T)-enriched Lipschitizian mapping T is known as nonexpansive mapping on C.*

Now, if $b > 0$ *and* $\vartheta = \dfrac{1}{b+1}$, *then* $0 < \vartheta < 1$. In this case, inequality (3) becomes the following:

$$\|\Big(\frac{1}{\vartheta} - 1\Big)\hbar + \vartheta T\hbar - \Big(\Big(\frac{1}{\vartheta} - 1\Big)\zeta + \vartheta T\zeta\Big)\| \leq (b+1)\Phi_T(\|\hbar - \zeta\|),$$

and, hence, we obtain the following.

$$\|(1-\vartheta)\hbar + \vartheta T\hbar - ((1-\vartheta)\zeta + \vartheta T\zeta)\| \leq (b+1)\vartheta\Phi_T(\|\hbar - \zeta\|). \tag{4}$$

Inequality (4) can be written as follows:

$$\|T_\vartheta\hbar - T_\vartheta\zeta\| \leq \varrho\Phi_T(\|\hbar - \zeta\|), \tag{5}$$

where $\varrho = (b+1)\vartheta$ *and* $T_\vartheta = (1-\vartheta)I + \vartheta T$. Note that the mapping T_ϑ is Φ_T-Lipschitzian in the sense of Hicks and Kubecek [3].

Remark 2. *Every Lipschitz mapping is automatically Φ_T-Lipschitzian but the converse implications may not be true (see [3] for more details). Moreover, every Φ_T-Lipschitz mapping is a $(0, \Phi_T)$-enriched Lipschitz mapping. Note that if $\Phi_T(r) = r$, then (5) reduces to the following:*

$$\|T_\vartheta\hbar - T_\vartheta\zeta\| \leq \varrho\|\hbar - \zeta\|, \tag{6}$$

and it is known as b-enriched nonexpansive mapping. The concept of b-enriched nonexpansive mapping was established by Berinde [4] as a generalization of an important class of mapping known as nonexpansive mapping. Apart from being an obvious generalization of the contraction mapping (and its connection with monotonicity method), nonexpansive mapping belongs to the first class of nonlinear mapping for which fixed-point theorems were obtained by utilizing geometric properties *instead of the compactness conditions. This class of mapping could also be seen in applications as transition operators for initial value problems of differential inclusion, accretive operators, monotone operators, variational inequality problems and equilibrium problems. Several generalizations of nonexpansive mappings in different directions have been studied by different researchers in the current literature; see, for instance, Refs. [5–13] and the references therein. Note that, in particular that, if Φ_T is not necessarily nondecreasing and satisfies $\Phi_T(r) < r$ for $r > 0$, then T is known as a nonlinear contraction on C.*

Example 1. *Let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by the following.*

$$T\hbar = \sqrt{|\hbar|}, \qquad \text{for all } \hbar \in \mathbb{R}.$$

Consider $\Phi_T(r) = \sqrt{r}, r \geq 0$. Clearly, Φ_T is continuous and nondecreasing. First notice that the mapping T is subadditive. Suppose that $\hbar, \zeta \in \mathbb{R}$. Then, we have the following.

$$\begin{aligned} (T(\hbar+\zeta))^2 &= |\hbar+\zeta| \\ &\leq (\sqrt{|\hbar|}+\sqrt{|\zeta|})^2 \\ &= (T\hbar+T\zeta)^2. \end{aligned}$$

Utilizing the subadditivity of T, we obtain the following.

$$|T\hbar - T\zeta| \leq T(\hbar-\zeta) = \Phi_T(|\hbar-\zeta|).$$

Thus, T is Φ_T-Lipschitizian (or $(0, \Phi_T)$-enriched Lipschitizian) mapping with Φ_T as the Φ_T-function. Now, suppose that T is Lipschitizian with constant $L > 0$. Then, for all $\hbar, \zeta \in \mathbb{R}$ with $\zeta = 0$ and $\hbar \neq 0$, we have $T\hbar \leq L|\hbar|$. Hence, for all $\hbar \neq 0$, $L \geq \dfrac{1}{\sqrt{|\hbar|}}$. Letting $\hbar \to 0$, we obtain a contradiction. Consequently, T is not Lipschitizian.

Definition 3 ([14])**.** *A mapping T is known as (b,k)-enriched strictly pseudocontractive mapping ((b,k)-ESPCM) if for all $\hbar, \zeta \in C$, there exist $b \in [0,+\infty)$ and $k \in [0,1)$ such that the following is the case.*

$$\|b(\hbar-\zeta) + T\hbar - T\zeta\|^2 \leq (b+1)^2\|\hbar-\zeta\|^2 + k\|\hbar-\zeta-(T\hbar-T\zeta)\|^2. \tag{7}$$

Note that if $b = 0$ in inequality (7), we obtain a class of mapping known as k-strictly pseudocontractive mapping, and if $k = 0$, then the inequality (7) reduces to a class of mapping defined by (6). Thus, the class of (b,k)-ESPCM is a superclass of the class of b-enriched nonexpansive mapping and k-strictly pseudocontractive mapping (for more details, see, [14–18]).

Set $b = \dfrac{1}{\vartheta} - 1$, for $\vartheta \in (0,1]$. Then, from inequality (7), we have the following:

$$\|T_\vartheta\hbar - T_\vartheta\zeta\|^2 \leq \|\hbar-\zeta\|^2 + k\|\hbar-\zeta-(T_\vartheta\hbar - T_\vartheta\zeta)\ \|^2, \tag{8}$$

where T_ϑ satisfies the inequality (5). Here, the average operator T_ϑ is k-strictly pseudocontractive mapping. If $k = 1$ in (8), then we have a pseudocontraction. Thus, the class of (b,k)-strictly pseudocontractive mappings is a subclass of the class of b-enriched pseudocontractive mappings.

In a real Banach space, inequality (8) is equivalent to the following:

$$\langle T_\vartheta\hbar - T_\vartheta\zeta, j(\hbar-\zeta)\rangle \leq \|\hbar-\zeta\|^2 - \lambda\|\hbar-\zeta-(T_\vartheta\hbar - T_\vartheta\zeta)\ \|^2, \tag{9}$$

where $\lambda = \dfrac{1}{2}(1-k)$. If I denotes the identity mapping, then inequality (9) can be written in the following form.

$$\langle (I - T_\vartheta)\hbar - (I - T_\vartheta)\zeta, j(\hbar - \zeta)\rangle \geq \lambda \|\hbar - \zeta - (T_\vartheta \hbar - T_\vartheta \zeta)\|^2. \tag{10}$$

Again, the average operator T_ϑ in this setting is still a strict pseudocontraction. The class of (b,k)-enriched strictly pseudocontractive mappings was established in 2019 by Berinde as a generalization of the class of k-strictly pseudocontractive mappings (i.e, a mapping $T : C \longrightarrow C$ such that for all $\hbar, \zeta \in C$ and $k \in [0,1)$, we have $\|T\hbar - T\zeta\|^2 \leq \|\hbar - \zeta\|^2 + k\|\hbar - \zeta - (T\hbar - T\zeta)\|^2$. If $k = 1$, then we have a pseudocontraction. The class of strictly pseudocontractive mappings, defined in the setup of a real Hilbert space, was introduced in 1967 by Browder and Petryshym [19] as a superclass of the class of nonexpansive mappings and a subclass of the class of Lipschitz pseudocontractive mappings. Whereas lipschitz pseudocontractive mappings are generally not continuous, the strictly pseudocontractive mappings inherit Lipschitz properties from their definitions). He proved that if C is a bounded, closed and convex subset of a real Hilbert space and $T : C \longrightarrow C$ is a (b,k)-enriched strictly pseudocontractive mapping, then T has a fixed point. He examined the following theorems.

Theorem 1. *Let C be a bounded closed convex subset of a real Hilbert space and $T : C \longrightarrow C$ is a (b,k)-enriched strictly pseudocontractive demicompact mapping. Then, $F(T) \neq \emptyset$, and for any $\hbar_0 \in C$ and any fixed $0 < \varrho < 1 - k$, the Krasnoselkii iteration sequence given by the following:*

$$\hbar_{n+1} = (1-\varrho)\hbar_n + \varrho T\hbar_n, n \geq 0$$

which converges strongly to a fixed point of the mapping T.

Theorem 2. *Let C is a bounded closed convex subset of a real Hilbert space and $T : C \longrightarrow C$ is a (b,k)-ESPCM for some $0 \leq k < 1$. Then $F(T) \neq \emptyset$, and for any $\hbar_0 \in C$, and any control sequence $\{\mu_n\}_n \geq 1$ such that $k < \mu_n < 1$ and $\sum_{n=1}^{+\infty}(\mu_n - k)(1-\mu_n) = +\infty$, the Krasnoselkii–Mann iteration sequence given by the following:*

$$\hbar_{n+1} = (1 - \lambda\mu_n)\hbar_n + \lambda\mu_n T\hbar_n, n \geq 0,$$

for some $\lambda \in (0,1)$, converges weakly to a fixed point of a mapping T.

Modified Mixed-Type Ishikawa Iteration Scheme

Let E be a real Banach space and K be a nonempty closed and convex subset of E. Let $\{S_i\}_{i=1}^N : C \longrightarrow C$ be a finite family of (b, Φ_S)-enriched L_i-Lipschitizian self mappings and $\{T_i^\vartheta\}_{i=1}^N : C \longrightarrow C$ be a finite family of enriched strictly pseudocontractive self mappings. If $\hbar_0 \in K$, then the new hybrid-type iteration scheme for the above mentioned mappings is as follows:

$$\begin{aligned}
\hbar_1 &= (1 - \mu_0 - \varrho_0)\hbar_0 + \mu_0 T_1 \tau_1 + \varrho_0 u_0, \\
\hbar_2 &= (1 - \mu_1 - \varrho_1)\hbar_1 + \mu_1 T_2 \tau_2 + \varrho_1 u_1, \\
&\vdots \\
\hbar_N &= (1 - \mu_{N-1} - \varrho_{N-1})\hbar_{N-1} + \mu_{N-1} T_N \tau_{N-1} + \varrho_{N-1} u_{N-1},
\end{aligned}$$

with

$$\begin{aligned}
\tau_1 &= (1 - \vartheta_0) S_1 \zeta_1 + \vartheta_0 T_1 \zeta_1, \\
\tau_2 &= (1 - \vartheta_1) S_2 \zeta_2 + \vartheta_1 T_2 \zeta_2, \\
&\vdots \\
\tau_N &= (1 - \vartheta_N) S_N \zeta_N + \vartheta_N T_N \zeta_N,
\end{aligned}$$

where

$$\begin{aligned} \zeta_1 &= (1-\mu'_0-\varrho'_0)\hbar_0+\mu'_0 T_1\rho_1+\varrho'_0 v_0, \\ \zeta_2 &= (1-\mu'_1-\varrho'_1)\hbar_1+\mu'_1 T_2\rho_2+\varrho'_1 v_1, \\ &\vdots \\ \zeta_N &= (1-\mu'_{N-1}-\varrho'_{N-1})\hbar_{N-1}+\mu'_{N-1} T_N\rho_{N-1}+\varrho'_{N-1} v_{N-1}, \end{aligned}$$

with the following being the case.

$$\begin{aligned} \rho_1 &= (1-\vartheta'_0)S_1\hbar_0+\vartheta'_0 T_1\hbar_0, \\ \rho_2 &= (1-\vartheta'_1)S_2\hbar_1+\vartheta'_1 T_2\hbar_1, \\ &\vdots \\ \rho_N &= (1-\vartheta'_N)S_N^2\hbar_N+\vartheta'_N T_N\hbar_N. \end{aligned}$$

The above Hybrid-type iteration sequence can be written in compact form as follows:

$$\begin{cases} \hbar_1 \in K \\ \hbar_{n+1} = (1-\mu_n-\varrho_n)\hbar_n+\mu_n T_i\tau_{n+1}+\varrho_n u_n \\ \zeta_{n+1} = (1-\mu'_n-\varrho'_n)\hbar_n+\mu'_n T_i\rho_{n+1}+\varrho'_n v_n, \end{cases} \tag{11}$$

where

$$\begin{aligned} \tau_{n+1} &= (1-\vartheta_n)S_i\zeta_{n+1}+\vartheta_n T_i\zeta_{n+1}, \\ \rho_{n+1} &= (1-\vartheta'_n)S_i\hbar_n+\vartheta'_n T_i\hbar_n, \end{aligned}$$

also $\{\mu_n\},\{\varrho_n\},\{\vartheta_n\},\{\mu'_n\},\{\varrho'_n\},\{\vartheta'_n\} \in [0,1]$, and $\{u_n\},\{v_n\} \subset K$ are two bounded sequences.

The following well known iteration schemes can be obtained as special cases from inequality (11).

Remark 3.

1. *If $S_i = I$, where I denotes the identity map in K, for all $i = 1,2,\cdots,\mathbb{N}$, $\vartheta_n = \vartheta'_n = 0$ in inequality (11), we have the following:*

$$\begin{cases} \hbar_1 \in K \\ \hbar_{n+1} = (1-\mu_n-\varrho_n)\hbar_n+\mu_n T_i\zeta_{n+1}+\varrho_n u_n \\ \zeta_{n+1} = (1-\mu'_n-\varrho'_n)\hbar_n+\mu'_n T_i\hbar_n+\varrho'_n v_n, \end{cases} \tag{12}$$

   where μ_n, μ'_n, ϱ_n and ϱ'_n are as in inequality (11).

2. *For $i = 1,2,\cdots,N,\ if\ \varrho_n = \varrho'_n = 0$ in inequality (12), we have the following:*

$$\begin{cases} \hbar_1 \in K \\ \hbar_{n+1} = (1-\mu_n)\hbar_n+\mu_n T_i\zeta_{n+1} \\ \zeta_{n+1} = (1-\mu'_n)\hbar_n+\mu'_n T_i\hbar_n, \end{cases} \tag{13}$$

   where μ_n and μ'_n are as stated in inequality (11).

3. *If $T_i = T$ and $\zeta_{n+1} = \zeta_n$ in inequality (13), we obtain the well-known Ishikawa iteration scheme as follows:*

$$\begin{cases} \hbar_1 \in K \\ \hbar_{n+1} = (1-\mu_n)\hbar_n+\mu_n T\zeta_n \\ \zeta_n = (1-\mu'_n)\hbar_n+\mu'_n T_i\hbar_n, \end{cases} \tag{14}$$

   where μ_n and μ'_n are as in inequality (11).

4. *If $\mu'_n = 0$ in (14), we obtain the Mann iteration scheme as discussed below:*
   For an arbitrary $\hbar_1 \in K$, the sequence $\{\hbar_n\}_{n\geq 1}$ is given by the following:

$$\hbar_{n+1} = (1-\mu_n)\hbar_n + \mu_n T\zeta_n, \tag{15}$$

   where μ_n is as in inequality (11).

From (12)–(15), it is clear that the iteration scheme considered in this paper is much more general than several iteration schemes so far employed in obtaining convergence theorems in the current literature.

Motivated and inspired by the results in [4,14,15], our main focus in this manuscript is to examine the new iteration scheme defined by inequality (11), extend the idea of (b,k)-ESPCM from a real Hilbert space to a more general Banach space and from a single (b,k)-ESPCM as considered in [14] to a finite family of λ-enriched strictly pseudocontractive mappings. Furthermore, we shall introduce various strong convergence theorems of the iterative scheme defined by inequality (11) for a mixed-type finite family of λ-enriched strictly pseudocontractive mapping and finite family of Φ_S-enriched L_i-Lipschitizian mapping in the setup of real uniformly convex Banach spaces.

The manuscript is organized as follows: Section 2 is devoted to some preliminary results which will be helpful in examining the main findings of this manuscript are recalled; Theorem 4 and some of its consequences are the subject of Sections 3 and 4 concludes the paper.

2. Preliminaries

For the sake of convenience, we restate the following concepts and results which will be helpful in the prove of our main results.
Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is a function $\delta_E(\varepsilon) : (0,2] \longrightarrow (0,2]$ defined by the following.

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{1}{2}(\hbar+\zeta)\| : \|\hbar\| = 1, \|\zeta\| = 1, \varepsilon = \|\hbar-\zeta\|\}.$$

A Banach space E is uniformly convex if and if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0,2]$.

Lemma 1 ([20])**.** *Let E be a real Banach space. Then, for all $\hbar, \zeta \in E$, $j(\hbar-\zeta) \in J(\hbar-\zeta)$, the following inequality holds.*

$$\|\hbar+\zeta\|^2 \leq \|\hbar\|^2 + 2\langle\zeta, j(\hbar+\zeta)\rangle.$$

Lemma 2 ([21])**.** *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of nonnegative real numbers satisfying the recursive inequality:*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \text{for all } n \geq n_0,$$

where n_0 is some nonnegative integer. If $\sum_{n=1}^{+\infty} b_n < +\infty$ and $\sum_{n=1}^{+\infty} c_n < +\infty$, then $\lim_{n\to+\infty} a_n$ exists.

Lemma 3. *Let $T_\vartheta : C \longrightarrow C$ be an (b,k)-ESPCM. Then T_ϑ is an L-Lipschitizian mapping, where L is a positive constant.*

Proof. By the definition of (b,k)-ESPCM for $b > 0$ *and* $\vartheta = \dfrac{1}{b+1}$, we obtain the following.

$$\begin{aligned}
\langle T_\vartheta\hbar - T_\vartheta\zeta, j(\hbar-\zeta)\rangle &\leq \|\hbar-\zeta\|^2 - \lambda\|\hbar-\zeta-(T_\vartheta\hbar - T_\vartheta\zeta)\|^2 \\
\Rightarrow \langle T_\vartheta\hbar - T_\vartheta\zeta, j(\hbar-\zeta)\rangle - \langle \hbar-\zeta, j(\hbar-\zeta)\rangle &\leq -\lambda\|\hbar-\zeta-(T_\vartheta\hbar - T_\vartheta\zeta)\|^2 \\
\Rightarrow -\langle (I-T_\vartheta)\hbar - (I-T_\vartheta)\zeta, j(\hbar-\zeta)\rangle &\leq -\lambda\|(I-T_\vartheta)\hbar - (I-T_\vartheta)\zeta\|^2 \\
\Rightarrow \langle (I-T_\vartheta)\hbar - (I-T_\vartheta)\zeta, j(\hbar-\zeta)\rangle &\geq \lambda\|(I-T_\vartheta)\hbar - (I-T_\vartheta)\zeta\|^2 \\
\Rightarrow \|\hbar-\zeta\| &\geq \lambda\|(I-T_\vartheta)\hbar - (I-T_\vartheta)\zeta\| \\
&\geq \lambda\|T_\vartheta\hbar - T_\vartheta\zeta\| - \lambda\|\hbar-\zeta\|.
\end{aligned}$$

The last inequality implies the following:

$$\|T_\vartheta\hbar - T_\vartheta\zeta\| \leq L\|\hbar-\zeta\|,$$

where $L = \dfrac{1+\lambda}{\lambda}, 0 < \vartheta < 1$ and $T_\vartheta = (1-\vartheta)I + \vartheta T$ (I denoting the identity map on C). □

Definition 4. *Let E be a uniformly convex Banach space (UCBS) and C be a closed convex subset of E. A mapping $T : C \longrightarrow C$ is known as an asymptotically regular on C if the following is the case:*

$$\|T^{n+1}\hbar = T^n\hbar\| \to 0 \qquad \text{as} \qquad n \to +\infty$$

for all $\hbar \in C$. If T is nonexpansive, then $T_\vartheta = (1-\vartheta)I + \vartheta T$ is asymptotically regular for all $0 < \vartheta < 1$ (see [22,23]). The concept of asymptotic regularity is due to Browder and Petryshyn [24].

Lemma 4 ([22])**.** *Let C be a nonempty bounded closed convex subset of a real Banach space E. If a mapping $T : C \longrightarrow C$ is a nonexpansive and $F(T) \neq \emptyset$, then, for any given $\vartheta \in (0,1)$, the mapping $T_\vartheta = (1-\vartheta)I + \vartheta T$, where I is the identity operator, has the same fixed point as a mapping T and is asymptotically regular.*

Remark 4. *If T is a nonexpansive mapping then the corresponding mapping T_ϑ is also nonexpansive and both have the same fixed point. However, T_ϑ has more felicitous asymptotic behavior than the original mapping (see for details, [22]).*

Definition 5. *Let C be a nonempty bounded closed convex subset of a real Banach space E. A mapping $T : C \longrightarrow C$ is said to be demicompact (see [25]) if for every bounded sequence $\{\hbar_n\}_{n\geq 1}$ in C such that $\hbar_n - T\hbar_n$ converges in C, there exists a convergent subsequence of $\{\hbar_n\}_{n\geq 1}$.*

The results proved in this article generalized the results present in [26–29]. For some more related results, see [30–34].

3. Main Results

In this section, we will provide some fixed point results for (b,k)-enriched strictly pseudocontractive, demicompact and (b,Φ_S)-enriched L_i-Lipschitizian mapping in uniformly convex Banach spaces.

Theorem 3. *Let C be a nonempty bounded closed convex subset of a UCBS and $T_\vartheta : C \longrightarrow C$ be (b,k)-enriched strictly pseudocontractive and demicompact mapping. Let $F(T) \neq \emptyset$, then for any $\hbar_0 \in C$, $\lambda \in [0, \frac{1}{2})$ and $\vartheta, \vartheta', \delta \in (0,1)$, the sequence defined by the following:*

$$\hbar_{n+1} = (1-\delta\vartheta')\hbar_n + \delta\vartheta' T_\vartheta\hbar_n, n \geq 0, \tag{16}$$

converges strongly to a fixed point of a mapping T, where $T_\vartheta = (1-\vartheta)I + \vartheta T$ and I is an identity mapping.

Proof. Using inequality (8), we have the following.

$$\begin{aligned}\langle (I-T_\vartheta)\hbar-(I-T_\vartheta)\zeta, j(\hbar-\zeta)\rangle &\geq \lambda\|\hbar-\zeta-(T_\vartheta\hbar-T_\vartheta\zeta)\|^2\\ &\geq \lambda[\|\hbar-\zeta\|^2-2\langle T_\vartheta\hbar-T_\vartheta\zeta, j(\hbar-\zeta)\rangle+\|T_\vartheta\hbar-T_\vartheta\zeta\|^2]\end{aligned}$$

$$\Rightarrow |\hbar-\zeta|^2-\langle T_\vartheta\hbar-T_\vartheta\zeta, j(\hbar-\zeta)\rangle-\lambda\|\hbar-\zeta\|^2+2\lambda\langle T_\vartheta\hbar-T_\vartheta\zeta, j(\hbar-\zeta)\rangle\geq\|T_\vartheta\hbar-T_\vartheta\zeta\|^2$$

$$\Rightarrow (1-\lambda)\|\hbar-\zeta\|^2-(1-2\lambda)\langle T_\vartheta\hbar-T_\vartheta\zeta, j(\hbar-\zeta)\rangle\geq\|T_\vartheta\hbar-T_\vartheta\zeta\|^2.$$

Therefore, for any $\lambda\in[0,\frac{1}{2})$, the operator $T_\vartheta=(1-\vartheta)I+\vartheta T$ is nonexpansive. Now, consider the sequence $\{\hbar_n\}_{n\geq1}$ defined by the following:

$$\hbar_{n+1}=(1-\delta\vartheta')\hbar_n+\delta\vartheta' T_\vartheta\hbar_n, n\geq 0, \tag{17}$$

where $\delta,\vartheta'\in(0,1)$. It is clear that $\{\hbar_n\}_{n\geq1}\in C$; hence, it is bounded. Set the following.

$$U_{\delta\vartheta}=(1-\delta\vartheta')I+\delta\vartheta' T_\vartheta. \tag{18}$$

Then, by (19) and nonexpansiveness of T_ϑ, it follows that $U_{\delta\vartheta}$ is asymptotically regular.

$$\|\hbar_n-U_{\delta\vartheta}\hbar_n\|\to 0 \quad\text{as}\quad n\to+\infty. \tag{19}$$

Observe the following.

$$\begin{aligned}U_{\delta\vartheta}\hbar-\hbar &= (1-\delta\vartheta')\hbar+\delta\vartheta' T_\vartheta\hbar-\hbar\\ &= \delta\vartheta'(T_\vartheta\hbar-\hbar)\\ &= \delta\vartheta'((1-\vartheta)\hbar+\vartheta T\hbar-\hbar)\\ &= \delta\vartheta'\vartheta(T\hbar-\hbar).\end{aligned}\tag{20}$$

From (19) and (20), we have the following.

$$\|\hbar_n-T\hbar_n\|\to 0 \quad\text{as}\quad n\to+\infty. \tag{21}$$

Since the mapping T is demicompact (by hypothesis), it follows, from (20) that $U_{\delta\vartheta}$ is also demicompact. Since $\{\hbar_n\}_{n\geq1}\in C$ and C is closed and bounded subset of E, it follows that $\{\hbar_n\}_{n\geq1}$ is demicompact. Hence, there exists a subsequence $\{\hbar_{nj}\}_{j\geq1}$ of $\{\hbar_n\}_{n\geq1}$ that converges strongly to a point ℓ, which obviously belongs to C since C is closed. Again, it is clear that $\lim_{n\to+\infty}\|\hbar_{nj}-U_{\delta\vartheta}\hbar_{nj}\|=0$; since $\lim_{j\to+\infty}\|\ell-\hbar_{nj}\|=0$ and $U_{\delta\vartheta}$ are demicompact, $U_{\delta\vartheta}\ell=\ell$. Consequently, using (18), the nonexpansivity of T_ϑ and Lemma 4, it follows that $T_\vartheta\ell=\ell$; that is, $F(U_{\delta\vartheta})=F(T_\vartheta)$.

Following the same argument as above, considering (21) and demicompactness of T, we obtain $Tq=\ell$. Thus, we have the following.

$$F(U_{\delta\vartheta})=F(T_\vartheta)=F(T).$$

Furthermore, using the fact that T_ϑ is nonexpansive, we obtain the following:

$$\begin{aligned}\|\hbar_{n+1}-\ell\| &= \|(1-\delta\vartheta)\hbar_n+\delta\vartheta T_\vartheta\hbar_n-\ell\|\\ &\leq (1-\delta\vartheta)\|\hbar_n-\ell\|+\delta\vartheta\|T_\vartheta\hbar_n-\ell\|\\ &\leq (1-\delta\vartheta)\|\hbar_n-\ell\|+\delta\vartheta\|\hbar_n-\ell\|\\ &= \|\hbar_n-\ell\|,\end{aligned}\tag{22}$$

for any positive integer n. For any $\epsilon>0$, there exists an integer n_0 such that $\|\hbar_{n_0}-\ell\|<\epsilon$, we obtain from (22) that $\|\hbar_n-\ell\|<\epsilon$ for any integer $n\geq n_0$. Therefore, $\{\hbar_n\}_{n\geq1}$ converges strongly to ℓ, a fixed point of a mapping T. □

Example 2. *Let $E = \mathbb{R}^2$ be equipped with the Euclidean norm, and we have the following.*

$$C = \{(\hbar_1, \hbar_2) \in \mathbb{R}^2, \hbar_1, \hbar_2 \geq 0, \hbar_1^2 + \hbar_2^2 \leq 1\}. \tag{23}$$

Define the mapping $T : C \longrightarrow C$ by $T(\hbar, \zeta) = (\frac{\hbar}{2}, \frac{\zeta}{2})$. It is easy to see that E is UCBS and that C is a bounded, closed and convex subset of E. Let $b \in [0, +\infty)$ and $k \in [0, 1)$. Then, for all $\hbar, \zeta \in C$, we have the following.

$$\| b(\hbar - \zeta) + T\hbar - T\zeta \|^2 = \left(\frac{2b+1}{2}\right)^2 \| (\hbar_1, \zeta_1) - (\hbar_2, \zeta_2) \|^2 . \tag{24}$$

Moreover, we have the following.

$$(b+1)^2 \| \hbar - \zeta \|^2 + k \| \hbar - \zeta - (T\hbar - T\zeta) \|^2 = [(b+1)^2 + \frac{k}{4}] \| (\hbar_1, \zeta_1) - (\hbar_2, \zeta_2) \|^2 . \tag{25}$$

From (24) and (25) implies the following.

$$\begin{aligned} \| b(\hbar - \zeta) + T\hbar - T\zeta \|^2 &= \left(\frac{2b+1}{2}\right)^2 \| (\hbar_1, \zeta_1) - (\hbar_2, \zeta_2) \|^2 \\ &\leq [(b+1)^2 + \left(\frac{k}{4}\right)] \| (\hbar_1, \zeta_1) - (\hbar_2, \zeta_2) \| \\ &= (b+1)^2 \| \hbar - \zeta \|^2 + k \| \hbar - \zeta - (T\hbar - T\zeta) \|^2 . \end{aligned}$$

Thus, the mapping T is (b, k)-enriched strictly pseudocontractive mapping. Again, it is not hard to see that T is demicompact. Furthermore, observe that $(0, 0)$ is a unique fixed point of T.

Next, we show that the sequence defined in (16) (Theorem 3) converges strongly to the fixed point of T. Using the fact that $T_\vartheta = (1 - \vartheta)I + \vartheta T$, where I is an identity mapping, and by setting $\hbar_0 = (0.7, 0.7) \in C$ as our initial guess, we proceed as follows.

Fix $\delta = \vartheta = \vartheta' = 0.5$ and define mapping $T : C \longrightarrow C$ by $T\hbar = \dfrac{\hbar}{2}$. Then, for $n = 0$ and $\hbar = \hbar_0$ in (16), we obtain the following.

$$\hbar_1 = (1 - \delta\vartheta')\hbar_0 + \delta\vartheta'[(1 - \vartheta)\hbar_0 + \vartheta T\hbar_0]\hbar_0 = (0.617, 0.617). \tag{26}$$

Again, for $n = 1$ in (16), we obtain the following.

$$\hbar_2 = (1 - \delta\vartheta')\hbar_1 + \delta\vartheta'[(1 - \vartheta)\hbar_1 + \vartheta T\hbar_1]\hbar_1 = (0.534, 0.534). \tag{27}$$

By continuing in this manner, it can be seen that $\hbar_n \to 0$ as $n \to \infty$, and this completes the proof.

Theorem 4. *Let E be a real UCBS and C a nonempty closed convex subset of E. Let $S_i : C \longrightarrow C$ be a finite family (b, Φ_S)-enriched L_i-Lipschitizian self mappings and $T_i^\vartheta : C \longrightarrow C$ a finite family of enriched strictly pseudocontractive self mappings. Let $\{\hbar_n\}$ be a sequence defined by the following:*

$$\begin{cases} \hbar_1 \in K \\ \hbar_{n+1} = (1 - \mu_n - \varrho_n)\hbar_n + \mu_n T_i^\vartheta \tau_{n+1} + \varrho_n u_n \\ \zeta_{n+1} = (1 - \mu'_n - \varrho'_n)\hbar_n + \mu'_n T_i^\vartheta \rho_{n+1} + \varrho'_n v_n, \end{cases} \tag{28}$$

where

$$\tau_{n+1} = (1 - \vartheta_n)S_i\zeta_{n+1} + \vartheta_n T_i^\vartheta \zeta_{n+1}, \rho_{n+1} = (1 - \vartheta'_n)S_i\hbar_n + \vartheta'_n T_i^\vartheta \hbar_n,$$

and $\{\mu_n\}, \{\varrho_n\}, \{\vartheta_n\}, \{\mu'_n\}, \{\varrho'_n\}, \{\vartheta'_n\} \in [0, 1]$ and $\{u_n\}, \{v_n\} \subset K$ are two bounded sequences. Suppose $\mathcal{F} = \cap_{i=1}^{\mathbb{N}}(F(S_i) \cap F(T_i)) \neq \emptyset$. If the following conditions hold:

i. $0 < \xi < \xi_n \leq \eta_n \leq \mu_n < \mu < 1, \sum_{n=1}^{+\infty} \mu_n = +\infty, \sum_{n=1}^{+\infty} \mu_n^2 < +\infty,$ $\sum_{n=1}^{+\infty} \varrho_n < +\infty, \sum_{n=1}^{+\infty} \varrho'_n < +\infty;$

ii. $\varrho_n + \varrho'_n \geq 1, \mu_n + \varrho_n \geq 1, \mu'_n + \varrho'_n \geq 1, \lim_{n\to+\infty} \mu_n^2 \lambda > 0;$

iii. *There exists a constant* L' *such that* $\Phi_t(r) = L'r$, *for some* $L' > 0$.

then the sequence defined by (28) converges strongly to a fixed point $\ell \in \mathcal{F}$.

Proof. Let the following:

$$M = \sup\{\|u_n - \ell\|, |v_n - \ell\|\}$$

and the following be the case.

$$L = \max\{L'_i, L''_i\}.$$

Since $\tau_{n+1} = (1-\vartheta_n)S_i\zeta_{n+1} + \vartheta_n T_i \zeta_{n+1}$, and $\rho_{n+1} = (1-\vartheta'_n)S_i\hbar_n + \vartheta'_n T_i\hbar_n$, we have the following.

$$\begin{aligned}
\|\rho_{n+1} - \ell\| &= \|(1-\vartheta'_n)S_i\hbar_n + \vartheta'_n T_i^{\vartheta}\hbar_n) - \ell\| \\
&\leq (1-\vartheta'_n)\|S_i\hbar_n - \ell\| + \vartheta'_n\|T_i^{\vartheta}\hbar_n - \ell\| \\
&\leq (1-\vartheta'_n)\Phi_S(\|\hbar_n - \ell\|) + \vartheta'_n L''_i\|\hbar_n - \ell\| \\
&\leq (1-\vartheta'_n)L'_i\|\hbar_n - \ell\| + \vartheta'_n L\|\hbar_n - \ell\| \\
&\leq (1-\vartheta'_n)L\|\hbar_n - \ell\| + \vartheta'_n L\|\hbar_n - \ell\| \\
&= L\|\hbar_n - \ell\|.
\end{aligned} \tag{29}$$

Moreover, we have the following.

$$\begin{aligned}
\|\zeta_{n+1} - \ell\| &= \|\xi_n((1-\mu'_n-\varrho'_n)\hbar_n + \mu'_n T_i^{\vartheta}\rho_n + \varrho'_n v_n) - \ell\| \\
&= \|\xi_n(1-\mu'_n-\varrho'_n)(\hbar_n - \ell) + \xi_n\mu'_n(T_i^{\vartheta}\rho_n - \ell) + \xi_n\varrho'_n(v_n - \ell)\| \\
&\leq \xi_n(1-\mu'_n-\varrho'_n)\|\hbar_n - \ell\| + \xi_n\mu'_n\|T_i^{\vartheta}\rho_n - \ell\| + \xi_n\varrho'_n\|v_n - \ell\| \\
&\leq \xi_n(1-\mu'_n-\varrho'_n)\|\hbar_n - \ell\| + \xi_n\mu'_n L''_i\|\rho_n - \ell\| + \xi_n\varrho'_n\|v_n - \ell\| \\
&\leq \xi_n(1-\mu'_n)\|\hbar_n - \ell\| + \xi_n\mu'_n L\|\rho_n - \ell\| + \xi_n\varrho'_n M \\
&= \xi_n(1-\mu'_n)\|\hbar_n - \ell\| + \xi_n\mu'_n L^2\|\hbar_n - \ell\| + \varrho'_n M \qquad \text{by (29)} \\
&\leq \xi_n(1+\mu'_n L^2)\|\hbar_n - \ell\| + \varrho'_n M.
\end{aligned} \tag{30}$$

Furthermore, we have the following.

$$\begin{aligned}
\|\rho_{n+1} - \zeta_{n+1}\| &\leq \|\rho_{n+1} - \ell\| + \|\ell - \zeta_{n+1}\| \\
&\leq L\|\hbar_n - \ell\| + (1+\mu'_n L^2)\|\hbar_n - \ell\| + \varrho'_n M \qquad \text{by (29) and (30)} \\
&= (1 + L + \mu'_n L^2)\|\hbar_n - \ell\| + \varrho'_n M.
\end{aligned} \tag{31}$$

Now, we can write the following.

$$\begin{aligned}
\|\tau_{n+1} - \ell\| &= \|(1-\vartheta_n)S_i\zeta_{n+1} + \vartheta_n T_i^{\vartheta}\zeta_{n+1} - \ell\| \\
&\leq (1-\vartheta_n)\|S_i\zeta_{n+1} - \ell\| + \vartheta_n\|T_i^{\vartheta}\zeta_{n+1} - \ell\| \\
&\leq (1-\vartheta_n)\Phi_S\|\zeta_{n+1} - \ell\| + \vartheta_n L''_i\|\zeta_{n+1} - \ell\| \\
&\leq (1-\vartheta_n)L'\|\zeta_{n+1} - \ell\| + \vartheta_n L''_i\|\zeta_{n+1} - \ell\| \\
&\leq (L' + L''_i)\|\zeta_{n+1} - \ell\| \\
&\leq 2L\|\zeta_{n+1} - \ell\| \\
&\leq 2L[(1+\mu'_n L^2)\|\hbar_n - \ell\| + \varrho'_n M] \qquad \text{by (30)} \\
&= 2L(1+\mu'_n L^2)\|\hbar_n - \ell\| + 2L\varrho'_n M.
\end{aligned} \tag{32}$$

Moreover, we have the following.

$$\begin{aligned}
\|\zeta_{n+1}-\hbar_{n+1}\| &= \|\xi_n((1-\mu_n'-\varrho_n')\hbar_n+\mu_n'T_i^{\vartheta}\rho_{n+1}+\varrho_n'v_{n+1})-\eta_n((1-\mu_n-\varrho_n)\hbar_n \\
&\quad +\mu_nT_i^{\vartheta}\tau_{n+1}+\varrho_nu_n)\| \qquad (33)\\
&\le \|\xi_n((1-\mu_n'-\varrho_n')\hbar_n+\mu_n'T_i^{\vartheta}\rho_{n+1}+\varrho_n'v_n)-\xi_n[(1-\mu_n-\varrho_n)\hbar_n \\
&\quad +\mu_nT_i^{\vartheta}\tau_{n+1}+\varrho_nu_n]\| \\
&= \|\xi_n(1-\mu_n'-\varrho_n')(\hbar_n-\ell)+\xi_n\mu_n'(T_i^{\vartheta}\rho_{n+1}-\ell)+\xi_n\varrho_n'(v_n-\ell) \\
&\quad -\xi_n[(1-\mu_n-\varrho_n)(\hbar_n-\ell)+\mu_n(T_i^{\vartheta}\tau_{n+1}-\ell)+\varrho_n(u_n-\ell)]\| \\
&= \|-\xi_n(\mu_n'+\varrho_n')(\hbar_n-\ell)+\xi_n\mu_n'(T_i^{\vartheta}\rho_{n+1}-\ell)+\xi_n\varrho_n'(v_n-\ell) \\
&\quad -\xi_n[-(\mu_n+\varrho_n)(\hbar_n-\mu\ell)+\mu_n(T_i^{\vartheta}\tau_{n+1}-\ell)+\varrho_n(u_n-\ell)]\| \\
&\le \|\xi_n\mu_n'(T_i^{\vartheta}\rho_{n+1}-\ell)+\xi_n\varrho_n'(v_n-\ell)-\xi_n\mu_n(T_i^{\vartheta}\tau_n-\ell) \\
&\quad -\xi_n\varrho_n(u_n-\ell)\| \\
&\le \xi_n\mu_n'\|T_i^{\vartheta}\rho_{n+1}-\ell\|+\xi_n\varrho_n'\|v_n-\ell\|+\xi_n\mu_n\|T_i^{\vartheta}\tau_{n+1}-\ell\| \\
&\quad +\xi_n\varrho_n\|u_n-\ell\|\|| \\
&\le \xi_n\mu_n'L_i''\|\rho_{n+1}-\ell\|+\xi_n\varrho_n'M+\xi_n\mu_nL_i''\|\tau_{n+1}-\ell\|+\xi_n\varrho_nM \\
&\le \xi_n\mu_n'L\|\rho_{n+1}-\ell\|+\xi_n\mu_nL\|\tau_{n+1}-\ell\|+\xi_n(\varrho_n+\varrho_n')M \\
&\le \xi_n\mu_n'L^2\|\hbar_n-\ell\|+2\xi_n\mu_nL^2(1+\mu_n'L^2)\|\hbar_n-\ell\|+2\xi_n\mu\varrho_n'L^2M \\
&\quad +\xi_n(\varrho_n+\varrho_n')M \qquad \textit{by (29) and (32)} \\
&= \xi_n[\mu_n'+2\mu_n(1+\mu_n'L^2)]L^2\|\hbar_n-\ell\|+\xi_n(\varrho_n+\varrho_n'+2\mu\varrho_n'L^2)M \qquad (34)
\end{aligned}$$

and

$$\begin{aligned}
\|\tau_{n+1}-\hbar_{n+1}\| &= \|(1-\vartheta_n)S_i\zeta_n+\vartheta_nT_i^{\vartheta}\zeta_n-\hbar_{n+1}\| \\
&\le \|S_i\zeta_{n+1}-\zeta_{n+1}\|+\|\zeta_{n+1}-\hbar_{n+1}\|+\vartheta_n\|T_i^{\vartheta}\zeta_{n+1}-\ell\|+\vartheta_n\|\ell-S_i\zeta_{n+1}\| \\
&\le \|S_i\zeta_{n+1}-\ell\|+\|\zeta_{n+1}-\ell\|+\|\zeta_{n+1}-\hbar_{n+1}\|+\vartheta_n\|T_i^{\vartheta}\zeta_{n+1}-\ell\| \\
&\quad +\vartheta_n\|\ell-S_i\zeta_{n+1}\| \\
&\le L_i'\|\zeta_{n+1}-\ell\|+\|\zeta_{n+1}-\mu\ell\|+\|\zeta_{n+1}-\hbar_{n+1}\|+\vartheta_nL_i''\|\zeta_{n+1}-\ell\| \\
&\quad +\vartheta_nL_i'\|\mu\ell-\zeta_{n+1}\| \\
&\le \Phi_S\|\zeta_{n+1}-\ell\|+\|\zeta_{n+1}-\ell\|+\|\zeta_{n+1}-\hbar_{n+1}\|+\vartheta_nL_i''\|\zeta_{n+1}-\ell\|+\vartheta_nL'\|\ell-\zeta_n\| \\
&\le L\|\zeta_{n+1}-\ell\|+\|\zeta_{n+1}-\ell\|+\|\zeta_{n+1}-\hbar_{n+1}\|+\vartheta_nL\|\zeta_{n+1}-\ell\|+\vartheta_nL\|\ell-\zeta_{n+1}\| \\
&= (1+L+2\vartheta_nL)\|\zeta_{n+1}-\ell\|+\|\zeta_{n+1}-\hbar_{n+1}\| \\
&\le \xi_n(1+L+2\vartheta_nL)[(1+\mu_n')L^2\|\hbar_n-\ell\|+\varrho_n'M]+\xi_n[\mu_n'+2\mu_n(1+\mu_n'L^2)]L^2 \\
&\quad \times\|\hbar_n-\ell\|+(\varrho_n+\varrho_n'+2\mu_n\varrho_n'L^2)M] \qquad \textit{by (30) and (34)} \\
&= \xi_n[(1+L+2\vartheta_nL)(1+\mu_n'L^2)+(\mu_n'+2\mu_n(1+\mu_n')L^2))L^2]\|\hbar_n-\mu\ell\|] \\
&\quad +\xi_n[(2+L+2\vartheta_nL+2\mu_nL^2)\varrho_n'+\varrho_n]M.. \qquad (35)
\end{aligned}$$

Now, using Lemma 1, condition (ii) and the fact that T_i^{ϑ} $(i=1,2,\cdots N)$ is strictly pseudocontractive self mapping , we obtain the following.

$$
\begin{aligned}
\|\hbar_{n+1}-\ell\|^2 &= \|\xi_n((1-\mu_n-\varrho_n)\hbar_n+\mu_n T_i^{\vartheta}\tau_{n+1}+\varrho_n u_n)-\ell\|^2 \\
&= \|\xi_n(1-\mu_n-\varrho_n)(\hbar_n-\ell)+\xi_n\mu_n(T_i^{\vartheta}\tau_{n+1}-\ell)+\xi_n\varrho_n(u_n-\ell)\|^2 \\
&\leq \xi_n^2(1-\mu_n-\varrho_n)^2\|\hbar_n-\ell\|^2+2\xi_n\mu_n\langle(T_i^{\vartheta}\tau_{n+1}-\ell) \\
&\quad +\varrho_n(u_n-\ell), j(\hbar_{n+1}-\ell)\rangle \\
&= \xi_n^2(1-\mu_n-\varrho_n)^2\|\hbar_n-\ell\|^2+2\xi_n\mu_n\langle T_i^{\vartheta}\tau_{n+1}-\ell, j(\hbar_{n+1}-\ell)\rangle \\
&\quad +2\xi_n\mu_n\langle\varrho_n u_n-\ell, jx_{n+1}-\ell\rangle \\
&= \xi_n^2(1-\mu_n-\varrho_n)^2\|\hbar_n-\ell\|^2+2\xi_n\mu_n\langle T_i^{\vartheta}\tau_{n+1}-T_i^{\vartheta}\hbar_{n+1} \\
&\quad +(T_i^{\vartheta}\hbar_{n+1}-\ell), j(\hbar_{n+1}-\ell)\rangle+2\xi_n\mu_n\varrho_n\|u_n-\ell\|\|(\hbar_{n+1}-\ell)\| \\
&= \xi_n^2(1-\mu_n-\varrho_n)^2\|\hbar_n-\ell\|^2+2\xi_n\mu_n\langle T_i^{\vartheta}\tau_{n+1}-T_i^{\vartheta}\hbar_{n+1}, j(\hbar_{n+1}-\ell)\rangle \\
&\quad +2\xi_n\mu_n\langle T_i^{\vartheta}\hbar_{n+1}-\ell, j(\hbar_{n+1}-\ell)\rangle+2\xi_n\mu_n\varrho_n\|u_n-\ell\|\|(\hbar_{n+1}-\ell)\| \\
&\leq \xi_n^2(1-\mu_n-\varrho_n)^2\|\hbar_n-\ell\|^2+2\xi_n\mu_n\langle T_i^{\vartheta}\tau_{n+1}-T_i^{\vartheta}\hbar_{n+1}, j(\hbar_{n+1}-\ell)\rangle \\
&\quad +2\xi_n\mu_n(\|\hbar_{n+1}-\ell\|^2-\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2)+2\xi_n\mu_n\varrho_n\|u_n-\ell\|\|(\hbar_{n+1}-\ell)\| \\
&= \xi_n^2(1-\mu_n-\varrho_n)^2\|\hbar_n-\ell\|^2+2\xi_n\mu_n\|T_i^{\vartheta}\tau_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|\|\hbar_{n+1}-\ell\| \\
&\quad +2\xi_n\mu_n\|\hbar_n-\ell\|^2-2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2+2\xi_n\mu_n\varrho_n M\|\hbar_{n+1}-\ell\| \\
&\leq \xi_n^2(1-\mu_n-\varrho_n)^2\|\hbar_n-\ell\|^2+2\xi_n\mu_n L_i''\|\tau_{n+1}-\hbar_{n+1}\|\|\hbar_{n+1}-\ell\| \\
&\quad +2\xi_n\mu_n\|\hbar_{n+1}-\ell\|^2-2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2+2\mu_n\varrho_n M\|\hbar_{n+1}-\ell\| \\
&\leq \xi_n^2(1-\mu_n-\varrho_n)^2\|\hbar_n-\ell\|^2+2\xi_n\mu_n L\|\tau_{n+1}-\hbar_{n+1}\|\|\hbar_{n+1}-\ell\| \\
&\quad +2\xi_n\mu_n\|\hbar_{n+1}-\ell\|^2-2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2 \\
&\quad +2\xi_n\mu_n\varrho_n M\|\hbar_{n+1}-\ell\|. \qquad (36)
\end{aligned}
$$

From (35) and (36), and using the fact that $2ab \leq a^2+b^2$, we have the following.

$$
\begin{aligned}
\|\hbar_{n+1}-\ell\|^2 &\le \xi_n^2(1-\mu_n-\varrho_n)^2\|\hbar_n-\ell\|^2 \\
&\quad +2\xi_n\mu_n L\{[(1+L+2\vartheta_n L)(1+\mu_n' L^2)+(\mu_n'+2\mu_n(1+\mu_n')L^2)L^2]\|\hbar_n-\ell\|] \\
&\quad +[(2+L+2\vartheta_n L+2\mu_n L^2)\varrho_n'+\varrho_n]M\}\|\hbar_{n+1}-\ell\| \\
&\quad +2\xi_n\mu_n\|\hbar_{n+1}-\ell\|^2-2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2+2\xi_n\mu_n\varrho_n M\|\hbar_{n+1}-\ell\| \\
&< (1-(\mu_n+\varrho_n))^2\|\hbar_n-\ell\|^2 \\
&\quad +2\{\{\xi_n\mu_n L[(1+L+2\vartheta_n L)(1+\mu_n' L^2)+(\mu_n'+2\mu_n(1+\mu_n')L^2)L^2]\|\hbar_n-\ell\|] \\
&\quad +[(2+L+2\vartheta_n L+2\mu_n L^2)\varrho_n'+\varrho_n]M\}+2\xi_n\mu_n\varrho_n M\}\|\hbar_{n+1}-\ell\| \\
&\quad +2\|\hbar_{n+1}-\ell\|^2-2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2 \\
&= [1+(\mu_n+\varrho_n)^2]\|\hbar_n-\ell\|^2-2(\mu_n+\varrho_n)\|\hbar_n-\hbar_{n+1}+\hbar_{n+1}-\ell\|^2 \\
&\quad +2\{\{\xi_n\mu_n L[(1+L+2\vartheta_n L)(1+\mu_n' L^2)+(\mu_n'+2\mu_n(1+\mu_n')L^2)L^2]\|\hbar_n-\mu\ell\|] \\
&\quad +[(2+L+2\vartheta_n L+2\mu_n L^2)\varrho_n'+\varrho_n]M\}+2\xi_n\mu_n\varrho_n M\}\|\hbar_{n+1}-\ell\| \\
&\quad +2\|\hbar_{n+1}-\ell\|^2-2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2 \\
&\le [1+(\mu_n+\varrho_n)^2]\|\hbar_n-\ell\|^2-2(\mu_n+\varrho_n)\|\hbar_n-\hbar_{n+1}\|^2-2(\mu_n+\varrho_n)\|\hbar_{n+1}-\ell\|^2 \\
&\quad +2\{\{\xi_n\mu_n L[(1+L+2\vartheta_n L)(1+\mu_n' L^2)+(\mu_n'+2\mu_n(1+\mu_n')L^2)L^2]\|\hbar_n-\ell\|] \\
&\quad +[(2+L+2\vartheta_n L+2\mu_n L^2)\varrho_n'+\varrho_n]M\}+2\xi_n\mu_n\varrho_n M\}\|\hbar_{n+1}-\ell\| \\
&\quad +2\|\hbar_{n+1}-\ell\|^2-2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2 \\
&= [1+(\mu_n+\varrho_n)^2]\|\hbar_n-\ell\|^2-2(\mu_n+\varrho_n)\|\hbar_n-\ell+\ell-\hbar_{n+1}\|^2 \\
&\quad -2(\mu_n+\varrho_n)\|\hbar_{n+1}-\ell\|^2 \\
&\quad +2\{\{\xi_n\mu_n L[(1+L+2\vartheta_n L)(1+\mu_n' L^2)+(\mu_n'+2\mu_n(1+\mu_n')L^2)L^2]\|\hbar_n-\ell\|] \\
&\quad +[(2+L+2\vartheta_n L+2\mu_n L^2)\varrho_n'+\varrho_n]M\}+2\xi_n\mu_n\varrho_n M\}\|\hbar_{n+1}-\ell\| \\
&\quad +2\|\hbar_{n+1}-\ell\|^2-2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2 \\
&\le [1+(\mu_n+\varrho_n)^2]\|\hbar_n-\ell\|^2-2(\mu_n+\varrho_n)\|\ell-\hbar_{n+1}\|^2-2(\mu_n+\varrho_n)\|\hbar_{n+1}-\ell\|^2 \\
&\quad +2\{\{\xi_n\mu_n L[(1+L+2\vartheta_n L)(1+\mu_n' L^2)+(\mu_n'+2\mu_n(1+\mu_n')L^2)L^2]\|\hbar_n-\ell\|] \\
&\quad +[(2+L+2\vartheta_n L+2\mu_n L^2)\varrho_n'+\varrho_n]M\}+2\xi_n\mu_n\varrho_n M\}\|\hbar_{n+1}-\ell\| \\
&\quad +2\|\hbar_{n+1}-\ell\|^2-2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2 \\
&= [1+(\mu_n+\varrho_n)^2]\|\hbar_n-\ell\|^2-4(\mu_n+\varrho_n)\|\hbar_{n+1}-\ell\|^2 \\
&\quad +2\{\{\xi_n\mu_n L[(1+L+2\vartheta_n L)(1+\mu_n' L^2)+(\mu_n'+2\mu_n(1+\mu_n')L^2)L^2]\|\hbar_n-\ell\|] \\
&\quad +[(2+L+2\vartheta_n L+2\mu_n L^2)\varrho_n'+\varrho_n]M\}+2\xi_n\mu_n\varrho_n M\}\|\hbar_{n+1}-\ell\| \\
&\quad +2\|\hbar_{n+1}-\ell\|^2-2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2 \\
&\le [1+(\mu_n+\varrho_n)^2]\|\hbar_n-\ell\|^2-4\|\hbar_{n+1}-\mu\ell\|^2 \\
&\quad +2\{\{\xi_n\mu_n L[(1+L+2\vartheta_n L)(1+\mu_n' L^2)+(\mu_n'+2\mu_n(1+\mu_n')L^2)L^2]\|\hbar_n-\ell\|] \\
&\quad +[(2+L+2\vartheta_n L+2\mu_n L^2)\varrho_n'+\varrho_n]M\}+2\xi_n\mu_n\varrho_n M\}\|\hbar_{n+1}-\ell\| \\
&\quad +2\|\hbar_{n+1}-\ell\|^2-2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2 \\
&= [1+(\mu_n+\varrho_n)^2]\|\hbar_n-\ell\|^2-2\|\hbar_{n+1}-\ell\|^2 \\
&\quad +2\{\{\xi_n\mu_n L[(1+L+2\vartheta_n L)(1+\mu_n' L^2)+(\mu_n'+2\mu_n(1+\mu_n')L^2)L^2]\|\hbar_n-\ell\|] \\
&\quad +[(2+L+2\vartheta_n L+2\mu_n L^2)\varrho_n'+\varrho_n]M\}+2\xi_n\mu_n\varrho_n M\}\|\hbar_{n+1}-\ell\| \\
&\quad -2\xi_n\mu_n\lambda\|\hbar_{n+1}-T_i^{\vartheta}\hbar_{n+1}\|^2. \qquad (37)
\end{aligned}
$$

Let the following be the case.

$$a_n=\|\hbar_n-\mu\ell\|^2,$$

$$\nu_n=\xi_n\mu_n L[(1+L+2\vartheta_n L)(1+\mu_n' L^2)+\xi_n\mu_n L(\mu_n'+2\mu_n L(1+\mu_n' L^2))L^2,$$

and

$$\ell_n = \xi_n \mu_n L[(2 + L + 2\vartheta_n L + 2\mu_n L^2)\varrho_n' + \varrho_n]M + 2\xi_n \mu_n \varrho_n M, \sigma_n = \|\hbar_{n+1} - T_i^\vartheta \hbar_{n+1}\|^2.$$

Using the above information, (37) becomes the following.

$$\begin{aligned} a_{n+1} &\leq (1 + \mu_n^2 + 2\mu_n \varrho_n + \varrho_n^2)a_n - 2a_{n+1} + 2(\nu_n \|\hbar_n - \ell\| + \ell_n)\|\hbar_{n+1} - \ell\| \\ &\quad -2\xi_n \mu_n \lambda \sigma_n. \end{aligned} \tag{38}$$

Again, by using $2ab \leq a^2 + b^2$, (38) becomes the following.

$$\begin{aligned} a_{n+1} &\leq (1 + \mu_n^2 + 2\mu_n \varrho_n + \varrho_n^2)a_n - 2a_{n+1} + (\nu_n \|\hbar_n - \ell\| + \ell_n)^2 + \|\hbar_{n+1} - \ell\|^2 \\ &\quad -2\xi_n \mu_n \lambda \sigma_n \\ &= (1 + \mu_n^2 + 2\mu_n \varrho_n + \varrho_n^2)a_n - 2a_{n+1} + \nu_n^2 \|\hbar_n - \mu\ell\|^2 + 2\nu_n \ell_n \|\hbar_n - \ell\| + \ell_n^2 \\ &\quad +\|\hbar_{n+1} - \ell\|^2 - 2\xi_n \mu_n \lambda \sigma_n \\ &\leq (1 + \mu_n^2 + 2\mu_n \varrho_n + \varrho_n^2)a_n - 2a_{n+1} + \nu_n^2 \|\hbar_n - \mu\ell\|^2 + \nu_n^2 + \ell_n^2 \|\hbar_n - \mu\ell\|^2 + \ell_n^2 \\ &\quad +\|\hbar_{n+1} - \mu\ell\|^2 - 2\xi_n \mu_n \lambda \sigma_n \\ &= (1 + \mu_n^2 + 2\mu_n \varrho_n + \varrho_n^2)a_n - 2a_{n+1} + \nu_n^2 a_n + \nu_n^2 + \ell_n^2 a_n + \ell_n^2 \\ &\quad +a_{n+1} - 2\xi_n \mu_n \lambda \sigma_n \\ &= (1 + \mu_n^2 + 2\mu_n \varrho_n + \varrho_n^2 + \nu_n^2 + \ell_n^2)a_n - a_{n+1} + \nu_n^2 + \ell_n^2 \\ &\quad -2\xi_n \mu_n \lambda \sigma_n \\ &\leq (1 + \mu_n^2 + 2\mu_n \varrho_n + \varrho_n^2 + \nu_n^2 + \ell_n^2)a_n + \nu_n^2 + \ell_n^2 \\ &\quad -2\xi_n \mu_n \lambda \sigma_n \qquad (39) \\ &\leq (1 + \mu_n^2 + 2\mu_n \varrho_n + \varrho_n^2 + \nu_n^2 + \ell_n^2)a_n + \nu_n^2 + \ell_n^2. \qquad (40) \end{aligned}$$

From (40), we have the following.

$$a_{n+1} \leq (1 + g_n)a_n + h_n, \tag{41}$$

where $g_n = \mu_n^2 + 2\mu_n \varrho_n + \varrho_n^2 + \nu_n^2 + \ell_n^2$ and $h_n = \nu_n^2 + \ell_n^2$. By conditions (i) and (iii), we obtain $\sum_{n=1}^{+\infty} g_n < +\infty$ and $\sum_{n=1}^{+\infty} h_n < +\infty$.

Again, from (41) and Lemma 2, we obtain that $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \|\hbar_n - \ell\|$ exists.

Now, we claim that $\{\hbar_n\}_{n\geq 1}$ is a Cauchy sequence in E. To see this, we apply the inequality $e^\theta \geq 1 + \theta$, which holds for all $\theta \geq 0$, in (41) to obtain the following:

$$\begin{aligned} \|\hbar_{n+1} - \ell\| &\leq (1 + g_n)\|\hbar_n - \ell\| + h_n \\ &\leq e^{g_n}\|\hbar_n - \ell\| + h_n, \end{aligned}$$

which, for $m, n \geq 1$, provides the following.

$$\begin{aligned} \|\hbar_{n+m} - \ell\| &\leq e^{g_{n+m-1}}\|\hbar_{n+m-1} - \ell\| + h_{n+m-1} \\ &\leq e^{g_{n+m-1}}[e^{g_{n+m-2}}\|\hbar_{n+m-1} - \xi^\star\| + h_{n+m-2}] + h_{n+m-1} \\ &= e^{g_{n+m-1}+g_{n+m-2}}\|\hbar_{n+m-1} - \ell\| + h_{n+m-2} + h_{n+m-1} \\ &\leq e^{g_{n+m-1}+g_{n+m-2}}\|\hbar_{n+m-1} - \ell\| \\ &\quad +e^{g_{n+m-1}+g_{n+m-2}}(h_{n+m-2} + h_{n+m-1}) \\ &\leq \dots \\ &\leq e^{\sum_{i=1}^{+\infty} g_i}\|\hbar_0 - \ell\| + e^{\sum_{i=1}^{+\infty} g_i}\sum_{i=1}^{n+m-1} h_i. \end{aligned} \tag{42}$$

Set $W = e^{\sum_{i=1}^{+\infty} g_i}$. Then, for any given $\epsilon > 0$, it follows from $\sum_{n=1}^{+\infty} g_n < +\infty$ and $\sum_{n=1}^{+\infty} h_n < +\infty$ that there exists a positive integer n_0 and a point $\ell \in \mathcal{F}$ such that the following is the case.

$$\|\hbar_0 - \ell\| < \frac{\epsilon}{2(1+W)}, \sum_{i=1}^{n+m-1} h_i < \frac{\epsilon}{2W}. \tag{43}$$

Thus, from (42) and (43), we have, for all $m \geq 1$, the following.

$$\begin{aligned} \|\hbar_{n_0+m} - \hbar_{n_0}\| &\leq \|\hbar_{n_0+m} - \ell\| + \|\hbar_{n_0} - \ell\| \\ &\leq W\|\hbar_{n_0} - \ell\| + W\sum_{i=1}^{n+m-1} h_i + \|\hbar_{n_0} - \ell\| \\ &< \epsilon. \end{aligned} \tag{44}$$

Thus, $\{\hbar_n\}_{n\geq 1}$ is a Cauchy sequence in E as claimed. The completeness of E guarantees that $\{\hbar_n\}_{n\geq 1}$ converges strongly to a point $\ell \in E$.

Suppose that $\lim_{n\to+\infty} \hbar_n = \ell$, we need to show that $\ell \in \mathcal{F}$. However, for any given $\epsilon^\star > 0$, there exists a positive integer $N^\star \geq N$ such that the following is the case.

$$\|\hbar_n - \ell\| = d(\hbar_{N^\star}, \ell) \cap d(\hbar_n, F) < \frac{\epsilon^\star}{2(1+L)}.$$

Similarly, there exists $\nu \in F$ such that the following is the case.

$$\|\hbar_n - \nu\| = d(\hbar_{N^\star}, \nu) \cap (\hbar_n, F) < \frac{\epsilon^\star}{2(1+3L)}.$$

Using the above estimates, we have the following.

$$\begin{aligned} \|T_i\ell - \ell\| &= \|T_i\ell - \nu + T_i\hbar_{N^\star} - \nu + \nu - T_i\hbar_{N^\star} + \nu - \hbar_{N^\star} + \hbar_{N^\star} - \ell\| \\ &\leq \|T_i\ell - \nu\| + \|T_i\hbar_{N^\star} - \nu\| + \|\nu - T_i\hbar_{N^\star}\| + \|\nu - \hbar_{N^\star}\| + \|\hbar_{N^\star} - \ell\| \\ &\leq L_i''\|\ell - \nu\| + L_i''\|\hbar_{N^\star} - \nu\| + L_i''\|\nu - \hbar_{N^\star}\| + \|\nu - \hbar_{N^\star}\| + \|\hbar_{N^\star} - \ell\| \\ &\leq L\|\ell - \hbar_{N^\star}\| + 2L\|\hbar_{N^\star} - \nu\| + L\|\nu - \hbar_{N^\star}\| + \|\nu - \hbar_{N^\star}\| + \|\hbar_{N^\star} - \ell\| \\ &= (1+L)\|\ell - \hbar_{N^\star}\| + (1+3L)\|\hbar_{N^\star} - \nu\| \\ &< \epsilon^\star. \end{aligned}$$

Since $\epsilon^\star > 0$ is arbitrary, we obtain the following.

$$T_i\ell = \ell.$$

Again, from the above estimates, we have the following.

$$\begin{aligned} \|S_i\ell - \ell\| &= \|S_i\ell - \nu + S_i\hbar_{N^\star} - \nu + \nu - S_i\hbar_{N^\star} + \nu - \hbar_{N^\star} + \hbar_{N^\star} - \ell\| \\ &\leq \|S_i\ell - \nu\| + \|S_i\hbar_{N^\star} - \nu\| + \|\nu - S_i\hbar_{N^\star}\| + \|\nu - \hbar_{N^\star}\| + \|\hbar_{N^\star} - \ell\| \\ &\leq \Phi_S(\|\ell - \nu\|) + \Phi_S(\|\hbar_{N^\star} - \nu\|) + \Phi_S(\|\nu - \hbar_{N^\star}\|) + \|\nu - \hbar_{N^\star}\| + \|\hbar_{N^\star} - \ell\| \\ &\leq L_i'\|\ell - \nu\| + L_i'\|\hbar_{N^\star} - \nu\| + L_i'\|\nu - \hbar_{N^\star}\| + \|\nu - \hbar_{N^\star}\| + \|\hbar_{N^\star} - \ell\| \\ &\leq L\|\ell - \hbar_{N^\star}\| + 2L\|\hbar_{N^\star} - \nu\| + L\|\nu - \hbar_{N^\star}\| + \|\nu - \hbar_{N^\star}\| + \|\hbar_{N^\star} - \ell\| \\ &= (1+L)\|\ell - \hbar_{N^\star}\| + (1+3L)\|\hbar_{N^\star} - \nu\| \\ &< \epsilon^\star. \end{aligned}$$

Since $\epsilon^\star > 0$ is arbitrary, we obtain the following.

$$S_i\ell = \ell.$$

Consequently, $\ell \in \mathcal{F} = \cap_{i=1}^{\mathbb{N}}(F(S_i) \cap F(T_i))$. This completes the proof. $\square$

Corollary 1. *Let E be a real UCBS and C a nonempty closed convex subset of E. Let $T_i^{\vartheta} : C \longrightarrow C$ be a finite family of enriched strictly pseudocontractive self mappings. Let $\{\hbar_n\}$ be a sequence defined by the following:*

$$\begin{cases} \hbar_1 \in K \\ \hbar_{n+1} = (1 - \mu_n - \varrho_n)\hbar_n + \mu_n T_i^{\vartheta}\tau_{n+1} + \varrho_n u_n \\ \zeta_{n+1} = (1 - \mu'_n - \varrho'_n)\hbar_n + \mu'_n T_i^{\vartheta}\rho_{n+1} + \varrho'_n v_n \end{cases}, \tag{45}$$

where

$$\tau_{n+1} = (1 - \vartheta_n)\zeta_{n+1} + \vartheta_n T_i^{\vartheta}\zeta_{n+1}, \rho_{n+1} = (1 - \vartheta'_n)\hbar_n + \vartheta'_n T_i^{\vartheta}\hbar_n,$$

$\{\mu_n\}, \{\varrho_n\}, \{\vartheta_n\}, \{\mu'_n\}, \{\varrho'_n\}, \{\vartheta'_n\} \in [0,1]$ and $\{u_n\}, \{v_n\} \subset K$ are two bounded sequences. Suppose $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. If the following conditions hold:

i. $0 < \xi < \xi_n \leq \eta_n \leq \mu_n < \mu < 1$,
$\sum_{n=1}^{+\infty} \mu_n = +\infty, \sum_{n=1}^{+\infty} \mu_n^2 < +\infty, \sum_{n=1}^{+\infty} \varrho_n < +\infty, \sum_{n=1}^{+\infty} \varrho'_n < +\infty$;
ii. $\varrho_n + \varrho'_n \geq 1, \mu_n + \varrho_n \geq 1, \mu'_n + \varrho'_n \geq 1, \lim_{n \to +\infty} \mu_n^2 \lambda > 0$;
iii. *There exists a constant L' such that $\Phi_t(r) = L'r$, for some $L' > 0$.*

then, the sequence defined in (45) converges strongly to fixed point $\ell \in \mathcal{F}$.

Proof. Let $S_i = I, i = 1, 2, \cdots, N$, where I is an identity mapping, in (28). Then, the results follows as in the proof of Theorem 4. □

Corollary 2. *Let E be a real UCBS and C be a nonempty closed convex subset of E. Let $T_i^{\vartheta} : C \longrightarrow C$ be finite family of enriched strictly pseudocontractive self mappings. Let $\{\hbar_n\}$ be a sequence defined by the following:*

$$\begin{cases} \hbar_1 \in K \\ \hbar_{n+1} = (1 - \mu_n - \varrho_n)\hbar_n + \mu_n T_i^{\vartheta}\tau_{n+1} + \varrho_n u_n \\ \zeta_{n+1} = (1 - \mu'_n - \varrho'_n)\hbar_n + \mu'_n T_i^{\vartheta}\hbar_n + \varrho'_n v_n, \end{cases} \tag{46}$$

where

$$\tau_{n+1} = (1 - \vartheta_n)\zeta_{n+1} + \vartheta_n T_i^{\vartheta}\zeta_{n+1},$$

$\{\mu_n\}, \{\varrho_n\}, \{\vartheta_n\}, \{\mu'_n\}, \{\varrho'_n\}, \{\vartheta'_n\} \in [0,1]$, and $\{u_n\}, \{v_n\} \subset K$ are two bounded sequences. Suppose $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. If the following conditions hold:

i. $0 < \xi < \xi_n \leq \eta_n \leq \mu_n < \mu < 1$,
$\sum_{n=1}^{+\infty} \mu_n = +\infty, \sum_{n=1}^{+\infty} \mu_n^2 < +\infty, \sum_{n=1}^{+\infty} \varrho_n < +\infty, \sum_{n=1}^{+\infty} \varrho'_n < +\infty$;
ii. $\varrho_n + \varrho'_n \geq 1, \mu_n + \varrho_n \geq 1, \mu'_n + \varrho'_n \geq 1, \lim_{n \to +\infty} \mu_n^2 \lambda > 0$;
iii. *There exists a constant L' such that $\Phi_t(r) = L'r$, for some $L' > 0$.*

then, the sequence defined by (46) converges strongly to fixed point $\ell \in \mathcal{F}$.

Proof. Let $S_i = I, i = 1, 2, \cdots, N$, where I is an identity mapping, and $\vartheta_n = \vartheta'_n = 0$ in (28). Then, the results follows as in the proof of Theorem 4. □

Corollary 3. *Let E be a real Banach space and C a nonempty closed bounded convex subset of E. Let $T^{\vartheta} : C \longrightarrow C$ be two strictly pseudocontractive self mappings. Let $\{\hbar_n\}$ be a sequence defined by the following:*

$$\begin{cases} \hbar_1 \in K \\ \hbar_{n+1} = \xi_n((1 - \mu_n)\hbar_n + \mu_n T_i^{\vartheta}\zeta_n) \\ \zeta_n = \eta_n((1 - \mu'_n)\hbar_n + \mu'_n T_i^{\vartheta}\hbar_n) \end{cases}, \tag{47}$$

where $\{\xi_n\}, \{\eta_n\} \in (0,1), \{\mu_n\}, \{\mu'_n\} \in [0,1]$. Suppose $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. If the following conditions hold:

i. $0 < \xi < \xi_n \leq \eta_n \leq \mu_n < \mu < 1$,
$\sum_{n=1}^{+\infty} \mu_n = +\infty, \sum_{n=1}^{+\infty} \mu_n^2 < +\infty, \sum_{n=1}^{+\infty} \varrho_n < +\infty, \sum_{n=1}^{+\infty} \varrho_n' < +\infty$;
ii. $\varrho_n + \varrho_n' \geq 1, \mu_n + \varrho_n \geq 1, \mu_n' + \varrho_n' \geq 1, \lim_{n\to+\infty} \mu_n^2 \lambda > 0$;
iii. *There exists a constant* L' *such that* $\Phi_t(r) = L'r$, *for some* $L' > 0$.

then, the sequence defined by (47) converges strongly to a fixed point $\mu\ell \in \mathcal{F}$.

Proof. Let $S_i = I$, where I is an identity mapping, and $\vartheta_n = \vartheta_n' = \varrho_n = \varrho_n' = 0$ in (28). Then, the results follows as in the proof of Theorem 4. □

Remark 5. *If T is a k-strictly pseudocontractive self mapping, then the above results still hold very well. Our results generalize the results of Theorem 2 and Corollary 3 in [14] in particular and many other results currently existing in literature.*

4. Conclusions

In this paper, we have introduced and studied (b, k)-ESPCM in the setup of real Banach space. We proved strong convergence theorem (Theorem 4) that extends the remarkable results obtained in [14] from real Hilbert space to a more general UCBS and from one mapping to a finite family of mappings. Moreover, we provided an example that does not only support our main results but also validates the results. The results obtained in this paper extend and improve several convergence theorems in the current literature (for details, see [14,26,28–30] and the references therein).

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