Article

Another Method for Proving Certain Reduction Formulas for the Humbert Function ψ_2 Due to Brychkov et al. with an Application

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Abstract: Recently, Brychkov et al. established several new and interesting reduction formulas for the Humbert functions (the confluent hypergeometric functions of two variables). The primary objective of this study was to provide an alternative and simple approach for proving four reduction formulas for the Humbert function ψ_2. We construct intriguing series comprising the product of two confluent hypergeometric functions as an application. Numerous intriguing new and previously known outcomes are also achieved as specific instances of our primary discoveries. It is well-known that the hypergeometric functions in one and two variables and their confluent forms occur naturally in a wide variety of problems in applied mathematics, statistics, operations research, physics (theoretical and mathematical) and engineering mathematics, so the results established in this paper may be potentially useful in the above fields. Symmetry arises spontaneously in the abovementioned functions.

Keywords: hypergeometric function; confluent hypergeometric function; Humbert functions; Appell's functions; reduction formula; integral representation

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1. Introduction

The generalized hyper geometric function $_pF_q$ is defined by [1,2]:

$$_pF_q\left[\begin{array}{ccc} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!} \tag{1}$$

where $(a)_n$ denotes the well known Pochhammer symbol (or the shifted or the raised factorial, since $(1)_n = n!$) defined for any complex number $(a \neq 0)$ by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\ldots(a+n-1) & ; n \in \mathbb{N} \\ 1 & ; n = 0 \end{cases} \tag{2}$$

It is interesting to mention here that symmetry occurs in the numerator parameters $a_1, a_2, \ldots, a_p$ and also symmetry occurs in the denominator parameters $b_1, b_2, \ldots, b_q$ of the generalized hypergeometric function

$$
{}_pF_q \left[\begin{array}{ccc} a_1, & \ldots, & a_p \\ & & \\ b_1, & \ldots, & b_q \end{array} \; ; \; z \right].
$$

That means that every reordering of the numerator parameters $a_1, a_2, \ldots, a_p$ of the generalized hypergeometric function provides the same function, and every reordering of the denominator parameters $b_1, b_2, \ldots, b_q$ of the generalized hypergeometric function produces the same function.

Additionally, the Humbert function ψ_2 (confluent form of the well-known Appell function of two variables) is defined by the following series [3,4]:

$$
\psi_2(\alpha_1; \beta_1, \beta_2; z_1, z_2) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha_1)_{k+m} z_1^k z_2^m}{(\beta_1)_k (\beta_2)_m m! k!}, \tag{3}
$$

which converges absolutely at any $z_1, z_2 \in \mathbb{C}$.

For some elementary relations between the Humbert function ψ_2 and the generalized hypergeometric function, we refer to [5,6]. Additionally, it is to be noted here that the symmetry occurs naturally in the denominator parameters β_1 and β_2.

The integral representation of the Humbert function ψ_2 is given by:

$$
\psi_2(a; c, c'; x, y) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-t} t^{a-1} \, {}_0F_1 \left[\begin{array}{c} - \\ c \end{array} \; ; xt \right] {}_0F_1 \left[\begin{array}{c} - \\ c' \end{array} \; ; yt \right] dt. \tag{4}
$$

The following expression for the Humbert function ψ_2 is given by Makano [7]:

$$
\psi_2(a; c, c'; \omega, z) = \sum_{m=0}^{\infty} \frac{(a)_m \omega^m}{(c)_m m!} \, {}_2F_1 \left[\begin{array}{cc} -m, & -m - c + 1 \\ & \\ c' & \end{array} \; ; \frac{z}{\omega} \right]. \tag{5}
$$

On the other hand, the classical Kummer summation theorem for the series ${}_2F_1$ is given by [1,2]:

$$
{}_2F_1 \left[\begin{array}{cc} a, & b \\ & \\ 1 + a - b & \end{array} \; ; -1 \right] = \frac{\Gamma\left(1 + \frac{1}{2}a\right) \Gamma(1 + a - b)}{\Gamma(1 + a) \Gamma\left(1 + \frac{1}{2}a - b\right)}. \tag{6}
$$

By employing (6), Bailey [8] obtained the following two interesting results involving the product of two generalized hypergeometric functions viz.

$$
{}_0F_1 \left[\begin{array}{c} - \\ c \end{array} \; ; x \right] {}_0F_1 \left[\begin{array}{c} - \\ c \end{array} \; ; -x \right] = {}_0F_3 \left[\begin{array}{ccc} & - & \\ c, & \frac{1}{2}c, & \frac{1}{2}c + \frac{1}{2} \end{array} \; ; -\frac{x^2}{4} \right], \tag{7}
$$

and

$$
{}_0F_1 \left[\begin{array}{c} - \\ c \end{array} \; ; x \right] {}_0F_1 \left[\begin{array}{c} - \\ 2 - c \end{array} \; ; -x \right] = {}_0F_3 \left[\begin{array}{ccc} & - & \\ \frac{1}{2}, & \frac{1}{2}c + \frac{1}{2}, & \frac{3}{2} - \frac{1}{2}c \end{array} \; ; -\frac{x^2}{4} \right]
$$

$$
+ \frac{2(1-c)x}{c(2-c)} \, {}_0F_3 \left[\begin{array}{ccc} & - & \\ \frac{3}{2}, & \frac{1}{2}c + 1, & 2 - \frac{1}{2}c \end{array} \; ; -\frac{x^2}{4} \right]. \tag{8}
$$

In 2011, Rakha and Rathie [9] established the following generalizations of the classical Kummer summation theorem (6) in the most general form for any $i \in \mathbb{Z}_0$ viz.

$$
{}_2F_1\left[\begin{array}{cc} a, & b \\ 1 + a - b + i \end{array} ; -1 \right] = \frac{2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(b - i)\Gamma(1 + a - b + i)}{\Gamma(b)\Gamma\left(\frac{1}{2}a - b + \frac{1}{2}i + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}a - b + \frac{1}{2}i + 1\right)}
$$
$$
\times \sum_{r=0}^{i}\binom{i}{r}(-1)^r \frac{\Gamma\left(\frac{1}{2}a - b + \frac{1}{2}i + \frac{1}{2}r + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a - \frac{1}{2}i + \frac{1}{2}r + \frac{1}{2}\right)}, \tag{9}
$$

and

$$
{}_2F_1\left[\begin{array}{cc} a, & b \\ 1 + a - b - i \end{array} ; -1 \right] = \frac{2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(1 + a - b + i)}{\Gamma\left(\frac{1}{2}a - b + \frac{1}{2}i + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}a - b + \frac{1}{2}i + 1\right)}
$$
$$
\times \sum_{r=0}^{i}\binom{i}{r}(-1)^r \frac{\Gamma\left(\frac{1}{2}a - b - \frac{1}{2}i + \frac{1}{2}r + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a - \frac{1}{2}i + \frac{1}{2}r + \frac{1}{2}\right)}. \tag{10}
$$

Clearly, for $i = 0$, the results (9) and (10) reduce to (6).

Very recently, by employing (9) and (10), Rathie et al. [10] established the following general results involving product of generalized hypergeometric functions viz.

$$
{}_0F_1\left[\begin{array}{c} - \\ c \end{array} ; x \right] {}_0F_1\left[\begin{array}{c} - \\ c + i \end{array} ; -x \right]
$$
$$
= k_1 \sum_{r=0}^{i}\binom{i}{r}(-1)^r \frac{\Gamma\left(c + \frac{1}{2}i + \frac{1}{2}r - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}r - \frac{1}{2}i\right)}
$$
$$
\times {}_2F_5\left[\begin{array}{c} c + \frac{1}{2}r + \frac{1}{2}i - \frac{1}{2}, \quad \frac{1}{2} - \frac{1}{2}r + \frac{1}{2}i \\ \frac{1}{2}, \quad \frac{1}{2}c + \frac{1}{2}i, \quad \frac{1}{2}c + \frac{1}{2}i + \frac{1}{2}, \quad c + \frac{1}{2}i - \frac{1}{2}, \quad c + \frac{1}{2}i \end{array} ; -\frac{x^2}{4} \right]
$$
$$
+ k_2 x \sum_{r=0}^{i}\binom{i}{r}(-1)^r \frac{\Gamma\left(c + \frac{1}{2}i + \frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}r - \frac{1}{2}i\right)}
$$
$$
\times {}_2F_5\left[\begin{array}{c} c + \frac{1}{2}r + \frac{1}{2}i, \quad 1 - \frac{1}{2}r + \frac{1}{2}i \\ \frac{3}{2}, \quad \frac{1}{2}c + \frac{1}{2}i + \frac{1}{2}, \quad \frac{1}{2}c + \frac{1}{2}i + 1, \quad c + \frac{1}{2}i, \quad c + \frac{1}{2}i + \frac{1}{2} \end{array} ; -\frac{x^2}{4} \right], \tag{11}
$$

where

$$
k_1 = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c + i)\Gamma(1 - c - i)}{\Gamma(1 - c)\Gamma\left(c + \frac{1}{2}i\right)\Gamma\left(c + \frac{1}{2} - \frac{1}{2}\right)},
$$

and

$$
k_2 = \frac{-2\Gamma\left(\frac{1}{2}\right)\Gamma(c + i)\Gamma(-c - i)x}{\Gamma(1 - c)\Gamma\left(c + \frac{1}{2}i\right)\Gamma\left(c + \frac{1}{2} + \frac{1}{2}\right)},
$$

$$
{}_0F_1 \left[\begin{array}{c} - \\ c \end{array} ; x \right] {}_0F_1 \left[\begin{array}{c} - \\ c-i \end{array} ; -x \right]
$$

$$
= k_3 \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(c + \frac{1}{2}r - \frac{1}{2}i - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}r - \frac{1}{2}i + \frac{1}{2}\right)}
$$

$$
\times {}_2F_5 \left[\begin{array}{c} c + \frac{1}{2}r - \frac{1}{2}i - \frac{1}{2}, \quad \frac{1}{2} - \frac{1}{2}r + \frac{1}{2}i \\[2mm] \frac{1}{2}, \qquad \frac{1}{2}c, \qquad \frac{1}{2}c + \frac{1}{2}, \quad c - \frac{1}{2}i, \quad c - \frac{1}{2}i - \frac{1}{2} \end{array} ; -\frac{x^2}{4} \right]
$$

$$
+ k_4 x \sum_{r=0}^{i} \binom{i}{r} (-1)^r \frac{\Gamma\left(c + \frac{1}{2}r - \frac{1}{2}i\right)}{\Gamma\left(\frac{1}{2}r - \frac{1}{2}i\right)}
$$

$$
\times {}_2F_5 \left[\begin{array}{c} c + \frac{1}{2}r - \frac{1}{2}i, \quad 1 - \frac{1}{2}r + \frac{1}{2}i \\[2mm] \frac{3}{2}, \qquad \frac{1}{2}c + \frac{1}{2}, \qquad \frac{1}{2}c + 1, \quad c - \frac{1}{2}i, \quad c - \frac{1}{2}i + \frac{1}{2} \end{array} ; -\frac{x^2}{4} \right], \tag{12}
$$

where

$$
k_3 = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c-i)}{\Gamma\left(c - \frac{1}{2}i\right)\Gamma\left(c - \frac{1}{2}i - \frac{1}{2}\right)},
$$

and

$$
k_4 = \frac{2\Gamma\left(\frac{1}{2}\right)\Gamma(c-i)}{c\Gamma\left(c - \frac{1}{2}i\right)\Gamma\left(c - \frac{1}{2}i + \frac{1}{2}\right)}.
$$

$$
{}_0F_1 \left[\begin{array}{c} - \\ c \end{array} ; x \right] {}_0F_1 \left[\begin{array}{c} - \\ 2-c+i \end{array} ; -x \right]
$$

$$
= \frac{(-2)^i}{i!} \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(1 - \frac{1}{2}c + \frac{1}{2}r + \frac{1}{2}i\right)}{\Gamma\left(1 - \frac{1}{2}c + \frac{1}{2}r - \frac{1}{2}i\right)}
$$

$$
\times {}_3F_6 \left[\begin{array}{c} 1, \quad 1 - \frac{1}{2}c + \frac{1}{2}i + \frac{1}{2}r, \quad \frac{1}{2}c + \frac{1}{2}i - \frac{1}{2}r \\[2mm] \frac{1}{2}c, \quad \frac{1}{2}c + \frac{1}{2}, \qquad \frac{1}{2}i + \frac{1}{2}, \quad \frac{1}{2}i + 1, \quad 1 - \frac{1}{2}c + \frac{1}{2}i, \quad \frac{3}{2} - \frac{1}{2}c + \frac{1}{2}i \end{array} ; -\frac{x^2}{4} \right]
$$

$$
+ \frac{4(-2)^i x}{(i+1)!c(2-c+i)} \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(-\frac{1}{2}c + \frac{1}{2}r + \frac{1}{2}i + \frac{3}{2}\right)}{\Gamma\left(-\frac{1}{2}c + \frac{1}{2}r - \frac{1}{2}i + \frac{1}{2}\right)}
$$

$$
\times {}_3F_6 \left[\begin{array}{c} 1, \quad -\frac{1}{2}c + \frac{1}{2}i + \frac{1}{2}r + \frac{3}{2}, \quad \frac{1}{2}c + \frac{1}{2}i - \frac{1}{2}r + \frac{1}{2} \\[2mm] \frac{1}{2}c + \frac{1}{2}, \qquad \frac{1}{2}c + 1, \qquad \frac{1}{2}i + 1, \qquad \frac{1}{2}i + \frac{3}{2}, \quad \frac{3}{2} - \frac{1}{2}c + \frac{1}{2}i, \quad 2 - \frac{1}{2}c + \frac{1}{2}i \end{array} ; -\frac{x^2}{4} \right], \tag{13}
$$

$$
{}_0F_1\left[\begin{array}{c} - \\ c \end{array}; x\right] {}_0F_1\left[\begin{array}{c} - \\ 2-c-i \end{array}; -x\right]
$$

$$
= (-2)^i \sum_{r=0}^{i} \binom{i}{r}
$$

$$
\times\ {}_2F_5\left[\begin{array}{c} \tfrac{1}{2}c+\tfrac{1}{2}i-\tfrac{1}{2}r,\quad 1-\tfrac{1}{2}c-\tfrac{1}{2}i+\tfrac{1}{2}r \\[4pt] \tfrac{1}{2},\qquad\qquad \tfrac{1}{2}c,\qquad \tfrac{1}{2}c+\tfrac{1}{2},\quad 1-\tfrac{1}{2}c-\tfrac{1}{2}i,\quad \tfrac{3}{2}-\tfrac{1}{2}c-\tfrac{1}{2}i \end{array}; -\tfrac{x^2}{4}\right]
$$

$$
+\ \frac{(2)^{1-i}x}{c(2-c-i)}\sum_{r=0}^{i}\binom{i}{r}(1+r-i-c)
$$

$$
\times\ {}_2F_5\left[\begin{array}{c} \tfrac{3}{2}-\tfrac{1}{2}c-\tfrac{1}{2}i+\tfrac{1}{2}r,\quad \tfrac{1}{2}+\tfrac{1}{2}c+\tfrac{1}{2}i-\tfrac{1}{2}r \\[4pt] \tfrac{3}{2},\qquad\qquad \tfrac{1}{2}c+\tfrac{1}{2},\qquad \tfrac{1}{2}c+1,\quad \tfrac{3}{2}-\tfrac{1}{2}c-\tfrac{1}{2}i,\quad 2-\tfrac{1}{2}c-\tfrac{1}{2}i \end{array}; -\tfrac{x^2}{4}\right]. \tag{14}
$$

Remark 1. *For $i=0$, the results (11) or (12) reduce to (7) and the results (13) or (14) reduce to (8).*

In 2017, by employing result (5) and with the help of (9) and (10), Brychkov, et al. [5] established the following four interesting and general reduction formulas for the Humbert function ψ_2.

1. For $c \neq 0, -1, -2, \ldots$ and $n = 0, 1, 2, \ldots$, the following result holds true:

$$
\psi_2(a; c, c+n; z, -z) = \frac{(-1)^n 2^{2c+n-2}\Gamma(c)}{\Gamma(2c+n)} \sum_{k=0}^{n}(-1)^k \binom{n}{k}
$$

$$
\times \left\{ \frac{(2c+n-1)\Gamma\left(c+\tfrac{k+n-1}{2}\right)}{\Gamma\left(\tfrac{k-n+1}{2}\right)}\ {}_4F_5\left[\begin{array}{c} \tfrac{a}{2},\quad \tfrac{a+1}{2},\quad \tfrac{n-k+1}{2},\quad \tfrac{n+k-1}{2} \\[4pt] \tfrac{1}{2},\quad \tfrac{c+n}{2},\quad \tfrac{c+n+1}{2},\quad c+\tfrac{n-1}{2},\quad c+\tfrac{n}{2} \end{array}; -z^2\right] \right.
$$

$$
\left. +\frac{4az\Gamma\left(c+\tfrac{k+n}{2}\right)}{(c+n)\Gamma\left(\tfrac{k-n}{2}\right)}\ {}_4F_5\left[\begin{array}{c} \tfrac{a+1}{2},\quad \tfrac{a}{2}+1,\quad \tfrac{n-k}{2}+1,\quad c+\tfrac{n+k}{2} \\[4pt] \tfrac{3}{2},\quad \tfrac{c+n+1}{2},\quad \tfrac{c+n}{2}+1,\quad c+\tfrac{n}{2},\quad c+\tfrac{n+1}{2} \end{array}; -z^2\right] \right\}
$$

$$
= \Delta_1. \tag{15}
$$

2. For $c, c-n \neq 0, -1, -2, \ldots$ and $n = 0, 1, 2, \ldots$, the following result holds true:

$$
\psi_2(a; c, c-n; z, -z) = \frac{2^{2c-n-2}\Gamma(c-n)}{\Gamma(2c-n)} \sum_{k=0}^{n}\binom{n}{k}
$$

$$
\times \left\{ \frac{(2c-n-1)\Gamma\left(c+\tfrac{k-n-1}{2}\right)}{\Gamma\left(\tfrac{k-n+1}{2}\right)}\ {}_4F_5\left[\begin{array}{c} \tfrac{a}{2},\quad \tfrac{a+1}{2},\quad \tfrac{n-k+1}{2},\quad \tfrac{k-n-1}{2} \\[4pt] \tfrac{1}{2},\quad \tfrac{c}{2},\quad \tfrac{c+1}{2},\quad c-\tfrac{n}{2},\quad c+\tfrac{n+1}{2} \end{array}; -z^2\right] \right.
$$

$$
+\frac{4az\Gamma\left(c+\tfrac{k-n}{2}\right)}{c\,\Gamma\left(\tfrac{k-n}{2}\right)}
$$

$$
\left. {}_4F_5\left[\begin{array}{c} \tfrac{a+1}{2},\quad \tfrac{a}{2}+1,\quad \tfrac{n-k}{2}+1,\quad c+\tfrac{k-n}{2} \\[4pt] \tfrac{3}{2},\quad \tfrac{c+1}{2},\quad \tfrac{c}{2}+1,\quad c-\tfrac{n}{2},\quad c-\tfrac{n-1}{2} \end{array}; -z^2\right] \right\}
$$

$$
= \Delta_2 \tag{16}
$$

3. For $c \neq 0, -1, -2, \ldots$ and $n = 0, 1, 2, \ldots$, the following result holds true:

$$\psi_2(a; c, 2 - c + n; z, -z) = \frac{(-2)^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

$$\times \left\{ \frac{(2c + n - 1)\Gamma\left(\frac{k-c+n}{2} + 1\right)}{\Gamma\left(\frac{k-c-n}{2} + 1\right)} \right.$$

$${}_5F_6 \left[\begin{array}{cccccc} 1, & \frac{a}{2}, & \frac{a+1}{2}, & \frac{c+n-k}{2}, & \frac{n-c+k}{2} + 1 \\ \frac{n+1}{2}, & \frac{n}{2} + 1, & \frac{c}{2}, & \frac{c+1}{2}, & \frac{n-c}{2} + 1, & \frac{n-c+3}{2} \end{array} ; -z^2 \right]$$

$$+ \frac{4az\Gamma\left(\frac{n-c+k+3}{2}\right)}{c(n+1)(2-c+n)\Gamma\left(\frac{k-c-n+1}{2}\right)}$$

$$\times {}_5F_6 \left[\begin{array}{cccccc} 1, & \frac{a+1}{2}, & \frac{a}{2} + 1, & \frac{c+n-k+1}{2}, & \frac{n-c+k+3}{2} \\ \frac{n}{2} + 1, & \frac{n+3}{2}, & \frac{c+1}{2}, & \frac{c}{2} + 1, & \frac{n-c+3}{2}, & \frac{n-c}{2} + 2 \end{array} ; -z^2 \right] \right\}$$

$$= \Delta_3. \tag{17}$$

4. For $c, 2 - c - n \neq 0, -1, -2, \ldots$ and $n = 0, 1, 2, \ldots$, the following result holds true:

$$\psi_2(a; c, 2 - c - n; z, -z) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k}$$

$$\left\{ {}_4F_5 \left[\begin{array}{cccc} \frac{a}{2}, & \frac{a+1}{2}, & \frac{c+n-k}{2}, & \frac{k-n-c}{2} + 1 \\ \frac{1}{2}, & \frac{c}{2}, & \frac{c+1}{2}, & 1 - \frac{c+n}{2}, & \frac{3-c-n}{2} \end{array} ; -z^2 \right] \right.$$

$$+ \frac{2a(c - k + n - 1)z}{c(c + n - 2)} {}_4F_5 \left[\begin{array}{cccc} \frac{a+1}{2}, & \frac{a}{2} + 1, & \frac{c+n-k+1}{2}, & \frac{k-n-c+3}{2} \\ \frac{3}{2}, & \frac{c+1}{2}, & \frac{c}{2} + 1, & \frac{3-c-n}{2}, & 2 - \frac{c+n}{2} \end{array} ; -z^2 \right] \right\}$$

$$= \Delta_4. \tag{18}$$

If $c = 0$, the result (15) or (16) reduces to:

$$\psi_2(a; c, c; x, -x) = {}_2F_3 \left[\begin{array}{cc} \frac{a}{2}, & \frac{a+1}{2} \\ c, & \frac{c}{2}, & \frac{c+1}{2} \end{array} ; -x^2 \right]. \tag{19}$$

Moreover, the following interesting series containing the product of a confluent hypergeometric function available in the literature [2] can be obtained by (19):

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!(b)_k} x^{2k} {}_1F_1 \left[\begin{array}{c} a + k \\ b + k \end{array} ; x \right] {}_1F_1 \left[\begin{array}{c} a + k \\ b + k \end{array} ; -x \right] = {}_2F_3 \left[\begin{array}{cc} \frac{a}{2}, & \frac{a+1}{2} \\ b, & \frac{b}{2}, & \frac{b+1}{2} \end{array} ; -x^2 \right]. \tag{20}$$

The rest of the paper is organized as follows. In Section 2, we provide another method for proving the results (15) to (18). As an application of these results, in Section 3, we shall establish the explicit expressions of the following general series identities containing product of confluent hypergeometric functions of the form for any $i \in \mathbb{Z}_0$ viz.

(a)

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!(b)_k(b + i)_k} x^{2k} {}_1F_1 \left[\begin{array}{c} a + k \\ b + k \end{array} ; x \right] {}_1F_1 \left[\begin{array}{c} a + k \\ b + k + i \end{array} ; -x \right]$$

(b)

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!(b)_k(b-i)_k} x^{2k} \,_1F_1 \left[\begin{array}{c} a+k \\ b+k \end{array} ; x \right] \,_1F_1 \left[\begin{array}{c} a+k \\ b+k-i \end{array} ; -x \right]$$

(c)

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!(b)_k(2-b+i)_k} x^{2k} \,_1F_1 \left[\begin{array}{c} a+k \\ b+k \end{array} ; x \right] \,_1F_1 \left[\begin{array}{c} a+k \\ b-2+k+i \end{array} ; -x \right]$$

(d)

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!(b)_k(2-b-i)_k} x^{2k} \,_1F_1 \left[\begin{array}{c} a+k \\ b+k \end{array} ; x \right] \,_1F_1 \left[\begin{array}{c} a+k \\ b-2+k-i \end{array} ; -x \right]$$

And, in Section 4, we shall discuss several interesting special cases of our main findings.

2. Derivations of the Results (15) to (18) by Another Method

In this section, we shall establish results (15) to (18) by another method. In order to reach result (15), we proceed as follows. In result (15), we replace x by xt and then multiply both sides by $e^{-t}t^{a-1}$ and, interesting with respect to t in the interval $[0, \infty)$, we have:

$$\int_0^{\infty} e^{-t}t^{a-1} \,_0F_1 \left[\begin{array}{c} - \\ c \end{array} ; xt \right] \,_0F_1 \left[\begin{array}{c} - \\ c+i \end{array} ; -xt \right] dt$$

$$= k_1 \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(c+\frac{1}{2}r+\frac{1}{2}i-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}r-\frac{1}{2}i\right)} \int_0^{\infty} e^{-t}t^{a-1}$$

$$\times \,_2F_5 \left[\begin{array}{c} c+\frac{1}{2}r+\frac{1}{2}i-\frac{1}{2}, \quad \frac{1}{2}-\frac{1}{2}r+\frac{1}{2}i \\ \frac{1}{2}, \qquad \frac{1}{2}c+\frac{1}{2}i, \quad \frac{1}{2}c+\frac{1}{2}i+\frac{1}{2}, \quad c+\frac{1}{2}i-\frac{1}{2}, \quad c+\frac{1}{2}i \end{array} ; -\frac{x^2t^2}{4} \right] dt$$

$$+ k_2 x \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(c+\frac{1}{2}r+\frac{1}{2}i\right)}{\Gamma\left(\frac{1}{2}r-\frac{1}{2}i\right)} \int_0^{\infty} e^{-t}t^{a}$$

$$\times \,_2F_5 \left[\begin{array}{c} c+\frac{1}{2}r+\frac{1}{2}i, \quad 1-\frac{1}{2}r+\frac{1}{2}i \\ \frac{3}{2}, \qquad \frac{1}{2}c+\frac{1}{2}i+\frac{1}{2}, \quad \frac{1}{2}c+\frac{1}{2}i+1, \quad c+\frac{1}{2}i, \quad c+\frac{1}{2}i+\frac{1}{2} \end{array} ; -\frac{x^2t^2}{4} \right] dt \qquad (21)$$

Now, if we denote the left-hand side of (21) by S_1, then:

$$S_1 = \int_0^{\infty} e^{-t}t^{a-1} \,_0F_1 \left[\begin{array}{c} - \\ c \end{array} ; xt \right] \,_0F_1 \left[\begin{array}{c} - \\ c+i \end{array} ; -xt \right] dt$$

Expressing both $_0F_1$ as series, we change the order of integration and series (which is easily seen to be justified due to the uniform convergence of the series in the interval $[0, \infty)$; evaluating the gamma integral and making use of identity (2) and finally summing up the double series, we have:

$$S_1 = \Gamma(a) \, \psi_2(a; c, c+i; x, -x). \qquad (22)$$

Next, if we denote the right-hand side of (21) by S_2, and then we have:

$$S_2 = AI_1 + BI_2, \tag{23}$$

where

$$A = k_1 \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(c + \frac{1}{2}r + \frac{1}{2}i - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}r - \frac{1}{2}i\right)},$$

$$B = k_2 x \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(c + \frac{1}{2}r + \frac{1}{2}i\right)}{\Gamma\left(\frac{1}{2}r - \frac{1}{2}i\right)},$$

$$I_1 = \int_0^\infty e^{-t} t^{a-1}$$
$$\times \; {}_2F_5 \left[\begin{array}{cccc} c + \frac{1}{2}r + \frac{1}{2}i - \frac{1}{2}, & \frac{1}{2} - \frac{1}{2}r + \frac{1}{2}i & & \\[4pt] \frac{1}{2}, & \frac{1}{2}c + \frac{1}{2}i, & \frac{1}{2}c + \frac{1}{2}i + \frac{1}{2}, \quad c + \frac{1}{2}i - \frac{1}{2}, \quad c + \frac{1}{2}i \end{array} ; -\frac{x^2 t^2}{4} \right] dt,$$

and

$$I_2 = \int_0^\infty e^{-t} t^{a}$$
$$\times \; {}_2F_5 \left[\begin{array}{cccc} c + \frac{1}{2}r + \frac{1}{2}i, & 1 - \frac{1}{2}r + \frac{1}{2}i & & \\[4pt] \frac{3}{2}, & \frac{1}{2}c + \frac{1}{2}i + \frac{1}{2}, & \frac{1}{2}c + \frac{1}{2}i + 1, \quad c + \frac{1}{2}i, \quad c + \frac{1}{2}i + \frac{1}{2} \end{array} ; -\frac{x^2 t^2}{4} \right] dt$$

Evaluation of I_1: Expressing ${}_2F_1$ as a series, changing the order of integration and summation, evaluating the gamma integral and using the identities (2) and

$$(a)_{2n} = 2^{2n} \left(\frac{1}{2}a\right)_n \left(\frac{1}{2}a + \frac{1}{2}\right)_n$$

then, after some simplification, summing up the series, we obtain:

$$I_1 = \Gamma(a) \; {}_4F_5 \left[\begin{array}{ccccc} \frac{1}{2}a, & \frac{1}{2}a + \frac{1}{2}, & c + \frac{1}{2}r + \frac{1}{2}i - \frac{1}{2}, & \frac{1}{2} - \frac{1}{2}r + \frac{1}{2}i \\[4pt] \frac{1}{2}, & \frac{1}{2}c + \frac{1}{2}i, & \frac{1}{2}c + \frac{1}{2}i + \frac{1}{2}, & c + \frac{1}{2}i - \frac{1}{2}, \quad c + \frac{1}{2}i \end{array} ; -x^2 \right]$$

Evaluation of I_2: Proceeding on similar lines as in the case of evaluation of I_1, it is not difficult to see that:

$$I_2 = a\Gamma(a) \; {}_4F_5 \left[\begin{array}{ccccc} \frac{1}{2}a + \frac{1}{2}, & \frac{1}{2}a + 1, & c + \frac{1}{2}i + \frac{1}{2}, & \frac{1}{2} - \frac{1}{2}r + \frac{1}{2}i \\[4pt] \frac{3}{2}, & \frac{1}{2}c + \frac{1}{2}i + \frac{1}{2}, & \frac{1}{2}c + \frac{1}{2}i + 1, & c + \frac{1}{2}i, \quad c + \frac{1}{2}i + \frac{1}{2} \end{array} ; -x^2 \right]$$

Finally, upon substituting the expressions for A, B, I_1 and I_2 in (23) and equating (22) and (23), we obtain our first result, (15). In exactly the same manner, results (16) to (18) can be established. We, however, prefer to omit the details. We conclude this section by remarking that the application of results (15) to (18) is given in the next section.

3. General Series Identities Containing the Product of Confluent Hypergeometric Functions

In this section, we shall establish the following four general series identities containing the product of confluent hypergeometric functions asserted in the following theorem.

Theorem 1. *For any $i \in \mathbb{Z}_0$, the following results hold true.*

$$\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k}{k!(b)_k(b+i)_k}x^{2k}\,{}_1F_1\left[\begin{array}{c}a+k\\b+k\end{array};x\right]\,{}_1F_1\left[\begin{array}{c}a+k\\b+k+i\end{array};-x\right]=\Delta_1,\qquad(24)$$

where Δ_1 is the same as the right-hand side of (15).

$$\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k}{k!(b)_k(b-i)_k}x^{2k}\,{}_1F_1\left[\begin{array}{c}a+k\\b+k\end{array};x\right]\,{}_1F_1\left[\begin{array}{c}a+k\\b+k-i\end{array};-x\right]=\Delta_2,\qquad(25)$$

where Δ_2 is the same as the right-hand side of (16).

$$\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k}{k!(b)_k(2-b+i)_k}x^{2k}\,{}_1F_1\left[\begin{array}{c}a+k\\b+k\end{array};x\right]\,{}_1F_1\left[\begin{array}{c}a+k\\2-b+k+i\end{array};-x\right]=\Delta_3,\quad(26)$$

where Δ_3 is the same as the right-hand side of (17).

$$\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k}{k!(b)_k(2-b-i)_k}x^{2k}\,{}_1F_1\left[\begin{array}{c}a+k\\b+k\end{array};x\right]\,{}_1F_1\left[\begin{array}{c}a+k\\2-b+k-i\end{array};-x\right]=\Delta_4,\quad(27)$$

where Δ_4 is the same as the right-hand side of (18).

Proof. In order to establish the first general series identity (24) asserted in the theorem, we proceed as follows. Denoting the left-hand side of (24) by S, we have:

$$S=\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k}{k!(b)_k(b+i)_k}x^{2k}\,{}_1F_1\left[\begin{array}{c}a+k\\b+k\end{array};x\right]\,{}_1F_1\left[\begin{array}{c}a+k\\b+k+i\end{array};-x\right]$$

Expressing both confluent hypergeometric functions as series, we have after some arrangement:

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+n}(a)_k(a+k)_m(a)_k(a+k)_n x^{2k+m+n}}{(a)_k k!(b)_k(b+k)_m(b+i)_k(b+k+i)_n m!n!}$$

using the identity

$$(a)_k(a+k)_m=(a)_{k+m},$$

we have:

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+n}(a)_{k+m}(a)_{k+n}x^{2k+m+n}}{(a)_k k!(b)_{k+m}(b+i)_{k+n}m!n!}$$

Now replacing m by $m-k$ and n by $n-k$ and using the result [[9], equ.(1), p.56]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(n,k)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k),$$

we have

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{min(m,n)}\frac{(-1)^n(a)_m(a)_n x^{m+n}}{(a)_k k!(b)_m(b+i)_n(m-k)!(n-k)!}$$

Now if we use the identities

$$(\alpha)_{m-n} = \frac{(-1)^n (\alpha)_m}{(1 - \alpha - m)_n},$$

and

$$(m - n)! = \frac{(-1)^n m!}{(-m)_n},$$

we have, after some simplification:

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (a)_m (a)_n x^{m+n}}{(b)_m (b+i)_n m! n!} \sum_{k=0}^{min(m,n)} \frac{(-m)_k (-n)_k}{(a)_k k!}$$

Summing up the inner series, we have:

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (a)_m (a)_n x^{m+n}}{(b)_m (b+i)_n m! n!} \, {}_2F_1 \left[\begin{array}{cc} -m, & -n \\ & \\ a \end{array} \; ; 1 \right].$$

Using Gauss's summation theorem

$$ {}_2F_1 \left[\begin{array}{cc} a, & b \\ & \\ c \end{array} \; ; 1 \right] = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)},$$

provided $Re(c - a - b) > 0$, and using identity (2), we have:

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (a)_{m+n} (a)_n x^{m+n}}{(b)_m (b+i)_n m! n!}$$

and, using definition (3), we have:

$$S = \psi_2(a; c, c + i; x, -x)$$

Finally, using result (15), we easily arrive at the right-hand side of our first general series identity (24).

In exactly the same manner, the other series identities (25) to (27) can be established. □

4. Corollaries

In this section, we shall mention some of the very interesting special cases of our four general series identities (24) to (27).

Corollary 1. *In (24) or (25), if we set $i = 0$, we obtain a known series identity (20) available in the literature.*

Corollary 2. *In (24), if we take $i = 1, 2$; we obtain the following results:*

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!(b)_k(b+1)_k} x^{2k} \, {}_1F_1 \left[\begin{array}{c} a + k \\ b + k \end{array} \; ; x \right] {}_1F_1 \left[\begin{array}{c} a + k \\ b + k + 1 \end{array} \; ; -x \right]$$

$$= {}_2F_3 \left[\begin{array}{c} \frac{a}{2}, \quad \frac{a+1}{2} \\ b, \quad \frac{b}{2} + \frac{1}{2}, \quad \frac{b}{2} + 1 \end{array} \; ; -x^2 \right] + \frac{ax}{b(b+1)} \, {}_2F_3 \left[\begin{array}{c} \frac{a+1}{2}, \quad \frac{a}{2} + 1 \\ b + 1, \quad \frac{b}{2} + 1, \quad \frac{b+3}{2} \end{array} \; ; -x^2 \right],$$

and

$$\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k}{k!(b)_k(b+2)_k}x^{2k} {}_1F_1\left[\begin{array}{c}a+k\\b+k\end{array};x\right] {}_1F_1\left[\begin{array}{c}a+k\\b+k+2\end{array};-x\right]$$

$$= {}_2F_3\left[\begin{array}{c}\frac{a}{2},\quad\frac{a+1}{2}\\b+1,\quad\frac{b}{2}+1,\quad\frac{b}{2}+\frac{3}{2}\end{array};-x^2\right]+\frac{2ax}{b(b+2)}{}_2F_3\left[\begin{array}{c}\frac{a+1}{2},\quad\frac{a}{2}+1\\b+1,\quad\frac{b}{2}+\frac{3}{2},\quad\frac{b}{2}+2\end{array};-x^2\right].$$

Similarly, other results can be obtained.

Corollary 3. *In (25), if we take $i = 1, 2$; we obtain the following results:*

$$\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k}{k!(b)_k(b-1)_k}x^{2k} {}_1F_1\left[\begin{array}{c}a+k\\b+k\end{array};x\right] {}_1F_1\left[\begin{array}{c}a+k\\b+k-1\end{array};-x\right]$$

$$= {}_2F_3\left[\begin{array}{c}\frac{a}{2},\quad\frac{a+1}{2}\\b-1,\quad\frac{b}{2},\quad\frac{b}{2}+\frac{1}{2}\end{array};-x^2\right]-\frac{ax}{b(b-1)}{}_2F_3\left[\begin{array}{c}\frac{a+1}{2},\quad\frac{a}{2}+1\\b,\quad\frac{b}{2},\quad\frac{b}{2}+\frac{1}{2}\end{array};-x^2\right],$$

and

$$\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k}{k!(b)_k(b-2)_k}x^{2k} {}_1F_1\left[\begin{array}{c}a+k\\b+k\end{array};x\right] {}_1F_1\left[\begin{array}{c}a+k\\b+k-2\end{array};-x\right]$$

$$= {}_2F_3\left[\begin{array}{c}\frac{a}{2},\quad\frac{a+1}{2}\\b-1,\quad\frac{b}{2},\quad\frac{b}{2}+\frac{1}{2}\end{array};-x^2\right]-\frac{2ax}{b(b-2)}{}_2F_3\left[\begin{array}{c}\frac{a+1}{2},\quad\frac{a}{2}+1\\b-1,\quad\frac{b}{2}+\frac{1}{2},\quad\frac{b}{2}+1\end{array};-x^2\right].$$

Similarly, other results can be obtained.

Corollary 4. *In (26) or (27), if we set $i = 0$, we obtain the following results:*

$$\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k}{k!(b)_k(2-b)_k}x^{2k} {}_1F_1\left[\begin{array}{c}a+k\\b+k\end{array};x\right] {}_1F_1\left[\begin{array}{c}a+k\\2-b+k\end{array};-x\right]$$

$$= {}_2F_3\left[\begin{array}{c}\frac{a}{2},\quad\frac{a+1}{2}\\\frac{1}{2},\quad\frac{b}{2}+\frac{1}{2},\quad\frac{3}{2}-\frac{1}{2}b\end{array};-x^2\right]+\frac{2ax(1-b)}{b(2-b)}{}_2F_3\left[\begin{array}{c}\frac{a+1}{2},\quad\frac{a}{2}+1\\\frac{3}{2},\quad\frac{b}{2}+1,\quad2-\frac{b}{2}\end{array};-x^2\right].$$

Corollary 5. *In (26), if we take $i = 1, 2$, we obtain the following results:*

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!(b)_k(3-b)_k} x^{2k} \, {}_1F_1\left[\begin{array}{c} a+k \\ b+k \end{array}; x\right] {}_1F_1\left[\begin{array}{c} a+k \\ 3-b+k \end{array}; -x\right]$$

$$= (2-b) \, {}_2F_3\left[\begin{array}{cc} \frac{a}{2}, & \frac{a+1}{2} \\ \frac{3}{2}, \, \frac{b}{2}+\frac{1}{2}, \, \frac{3}{2}-\frac{1}{2}b \end{array}; -x^2\right] - (1-b) \, {}_2F_3\left[\begin{array}{cc} \frac{a}{2}, & \frac{a}{2}+\frac{1}{2} \\ \frac{3}{2}, \quad \frac{b}{2}, \quad 2-\frac{b}{2} \end{array}; -x^2\right]$$

$$+ \frac{a(2-b)x}{(3-b)} \, {}_3F_4\left[\begin{array}{ccc} 1, & \frac{a}{2}+\frac{1}{2}, & \frac{a}{2}+1 \\ 2, & \frac{3}{2}, & \frac{b}{2}+\frac{1}{2}, \, \frac{5}{2}-\frac{b}{2} \end{array}; -x^2\right]$$

$$+ \frac{a(1-b)x}{b} \, {}_3F_4\left[\begin{array}{ccc} 1, & \frac{a}{2}+\frac{1}{2}, & \frac{a}{2}+1 \\ 2, & \frac{3}{2}, & \frac{b}{2}+1, \, 2-\frac{b}{2} \end{array}; -x^2\right],$$

and

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!(b)_k(4-b)_k} x^{2k} \, {}_1F_1\left[\begin{array}{c} a+k \\ b+k \end{array}; x\right] {}_1F_1\left[\begin{array}{c} a+k \\ 4-b+k \end{array}; -x\right]$$

$$= (2-b)^2 \, {}_3F_4\left[\begin{array}{ccc} 1, & \frac{a}{2}, & \frac{a+1}{2} \\ 2, & \frac{3}{2}, \, \frac{b}{2}+\frac{1}{2}, \, \frac{5}{2}-\frac{1}{2}b \end{array}; -x^2\right]$$

$$- (1-b(3-b)) \, {}_3F_4\left[\begin{array}{ccc} 1, & \frac{a}{2}, & \frac{a+1}{2} \\ 2, & \frac{3}{2}, & \frac{b}{2}, \, 2-\frac{1}{2}b \end{array}; -x^2\right]$$

$$+ \frac{2(1-b)(2-b)(3-b)ax}{3b(4-b)} \, {}_3F_4\left[\begin{array}{ccc} 1, & \frac{a}{2}+\frac{1}{2}, & \frac{a}{2}+1 \\ 2, & \frac{5}{2}, & \frac{b}{2}+1, \, 3-\frac{b}{2} \end{array}; -x^2\right]$$

$$+ \frac{2a(2-b)x}{b} \, {}_3F_4\left[\begin{array}{ccc} 1, & \frac{a}{2}+\frac{1}{2}, & \frac{a}{2}+1 \\ 2, & \frac{5}{2}, & \frac{b}{2}+\frac{1}{2}, \, \frac{5}{2}-\frac{b}{2} \end{array}; -x^2\right].$$

Similarly, other results can be obtained.

Corollary 6. *In (27), if we take $i = 1, 2$, we obtain the following results:*

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!(b)_k(1-b)_k} x^{2k} \, {}_1F_1\left[\begin{array}{c} a+k \\ b+k \end{array}; x\right] {}_1F_1\left[\begin{array}{c} a+k \\ 1-b+k \end{array}; -x\right]$$

$$= \frac{1}{2} \, {}_2F_3\left[\begin{array}{cc} \frac{a}{2}, & \frac{a+1}{2} \\ \frac{1}{2}, \, \frac{b}{2}, \, 1-\frac{b}{2} \end{array}; -x^2\right] + \frac{1}{2} \, {}_2F_3\left[\begin{array}{cc} \frac{a}{2}, & \frac{a}{2}+\frac{1}{2} \\ \frac{1}{2}, \, \frac{b}{2}+\frac{1}{2}, \, \frac{1}{2}-\frac{b}{2} \end{array}; -x^2\right]$$

$$- \frac{ax}{1-b} \, {}_2F_3\left[\begin{array}{cc} \frac{a}{2}+\frac{1}{2}, & \frac{a}{2}+1 \\ \frac{3}{2}, \, \frac{b}{2}+\frac{1}{2}, \, \frac{3}{2}-\frac{b}{2} \end{array}; -x^2\right] + \frac{ax}{b} \, {}_2F_3\left[\begin{array}{cc} \frac{a}{2}+\frac{1}{2}, & \frac{a}{2}+1 \\ \frac{3}{2}, \, \frac{b}{2}+1, \, 1-\frac{b}{2} \end{array}; -x^2\right],$$

and

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!(b)_k(-b)_k} x^{2k} \, {}_1F_1 \left[\begin{array}{c} a+k \\ b+k \end{array} ; x \right] \, {}_1F_1 \left[\begin{array}{c} a+k \\ -b+k \end{array} ; -x \right]$$

$$= \frac{1}{2} \, {}_2F_3 \left[\begin{array}{ccc} \frac{a}{2}, & \frac{a+1}{2} & \\ \frac{1}{2}, & \frac{b}{2}+\frac{1}{2}, & \frac{1}{2}-\frac{b}{2} \end{array} ; -x^2 \right] + \frac{1}{2} \, {}_2F_3 \left[\begin{array}{ccc} \frac{a}{2}, & \frac{a}{2}+\frac{1}{2} & \\ \frac{1}{2}, & \frac{b}{2}, & -\frac{b}{2} \end{array} ; -x^2 \right]$$

$$+ \frac{ax}{b} \, {}_2F_3 \left[\begin{array}{ccc} \frac{a}{2}+\frac{1}{2}, & \frac{a}{2}+1 & \\ \frac{3}{2}, & \frac{b}{2}+\frac{1}{2}, & \frac{1}{2}-\frac{b}{2} \end{array} ; -x^2 \right] + \frac{ax}{b} \, {}_2F_3 \left[\begin{array}{ccc} \frac{a}{2}+\frac{1}{2}, & \frac{a}{2}+1 & \\ \frac{3}{2}, & \frac{b}{2}+1, & 1-\frac{b}{2} \end{array} ; -x^2 \right].$$

Similarly, other results can be obtained.

5. Concluding Remark

In the beginning of the paper, we have provided another method for the derivation of the four reduction formulas for the Humbert functions obtained recently by Brychkov et al. Our method of derivation is simpler than the method given by Brychkov et al. Next, we applied these results to obtain four general results for the series involving the product of two confluent hypergeometric functions. In the end we mentioned known as well as new special cases of our main findings. The hypergeometric functions in one and two variables and their confluent forms occur naturally in a wide variety of problems in applied mathematics, statistics, operations research, theoretical and mathematical physics and engineering mathematics, as well as applications in such diverse fields as mechanics of deformable media, communications engineering, perturbation theory, theory of heat conduction, integral equations, theory of Lie algebra and Lie groups, decision theory, theory of elasticity and statistical distributions theory. Therefore, the results established in this paper may be potentially useful in the abovementioned areas.

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