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Approximate Solution of Nonlinear Time-Fractional Klein-Gordon Equations Using Yang Transform

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Abstract: The algebras of the symmetry operators for the Klein–Gordon equation are important for a charged test particle, moving in an external electromagnetic field in a space time manifold on the isotropic hydrosulphate. In this paper, we develop an analytical and numerical approach for providing the solution to a class of linear and nonlinear fractional Klein–Gordon equations arising in classical relativistic and quantum mechanics. We study the Yang homotopy perturbation transform method (YHPTM), which is associated with the Yang transform (YT) and the homotopy perturbation method (HPM), where the fractional derivative is taken in a Caputo–Fabrizio (CF) sense. This technique provides the solution very accurately and efficiently in the form of a series with easily computable coefficients. The behavior of the approximate series solution for different fractional-order φ values has been shown graphically. Our numerical investigations indicate that YHPTM is a simple and powerful mathematical tool to deal with the complexity of such problems.

Keywords: fractional Klein–Gordon equation; Yang transform; homotopy perturbation method; series solution



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1. Introduction

Recently, fractional calculus has grown in popularity due to its significant prospective applications in physics and engineering such as biology, mathematics, chemistry, fluid mechanics, physics, and nonlinear optics [1,2]. Fractional partial differential Equations (FPDEs) are a contemporary tool in calculus that can be used to simulate a wide range of classifications in applied sciences and engineering [3–5].

The Klein–Gordon (KG) equation performs a significant role in mathematical physics and many other scientific studies such as quantum field theory, nonlinear optics, and solid-state physic [6–10]. On the other hand, the fractional-order KG equation is derived from the classical KG equation by substituting the time order derivative with the fractional derivative of order φ . The fractional-order KG equation can be illustrated as below

$$D_q^\varphi \vartheta(\epsilon, q) - D_\epsilon^2 \vartheta(\epsilon, q) + a_1 \vartheta(\epsilon, q) + a_2 G(\vartheta(\epsilon, q)) = f(\epsilon, q), \quad (1)$$

with initial conditions

$$\vartheta(\epsilon, 0) = f_1(\epsilon), \quad \vartheta_q(\epsilon, 0) = f_2(\epsilon), \quad (2)$$

where D_q^φ represents the Caputo fractional time derivative, a_1 and a_2 are real constants, $f(\epsilon, q)$, $f_1(\epsilon)$ and $f_2(\epsilon)$ are known as analytical functions, whereas $G(\vartheta(\epsilon, q))$ is a nonlinear, and ϑ is an unknown function of ϵ and q .

Various authors [11–15] have investigated different analytical and numerical strategies to examine the solution to the KG equation but with some restrictions and lacks. Tamsir

and Srivastava [16] used fractional reduced differential transform to obtain the analytical solution of linear and nonlinear KG equation with time-fractional order. Bansu and Kumar [17] used a radial basis approach, and Kurulay [18] applied the homotopy analysis method to evaluate the numerical solution of the space-time fractional KG equation. Later, Khader and Adel [19] applied a hybridization scheme to achieve the solution of the fractional KG equation. Zhmud and Dimitrov [20] developed the fractional integration method, which is based on extrapolation using a series of integrating and differentiating links with a time constant that changes symmetrically from one step to another. In order to obtain the solution of FPDEs, several valuable strategies have been considered, such as the generalized differential transform method [21], the adomian decomposition method [22], the homotopy analysis method [23], the variational iteration method [24], the homotopy perturbation method [25], the Elzaki transform decomposition method [26], the fractional wavelet method [27,28] and the residual power series method [29,30].

In this paper, we present the Yang homotopy perturbation transform method (YHPTM), which is a composition of YT and HPM. The primary objective of this approach is to investigate the approximate solution of fractional KG equations and minimize the computational work that overcomes nonlinear problems easily. Next, this scheme can promptly deal with the nonlinear KG equation. Finally, this method can reduce the range of the computations and generate an approximate solution with elegantly computed expressions, which is its most impressive advantage. The design of this paper is framed as follows. In Section 2, we start with some primary definitions of Caputo–Fabrizio. In Section 3, we formulate the idea of the Yang homotopy perturbation transform method. In Section 4, we perform this scheme on some illustrative examples to show its capability and efficiency. Concluding remarks are given in Section 5.

2. Preliminaries and Concepts

Definition 1. The CF derivative is described as [31]

$${}^{CF}D_q^\varphi \vartheta(\epsilon, q) = \frac{S(\varphi)}{1-\varphi} \int_0^q [Q'(q)K(q, q)]dq, \quad n-1 < \varphi \leq n \quad (3)$$

$S(\varphi)$ is the normalization function with $S(0) = S(1) = 1$, and then, Equation (3) becomes as

$${}^{CF}D_q^\varphi \vartheta(\epsilon, q) = \frac{S(\varphi)}{1-\varphi} \int_0^q [Q(q) - Q(q)]K(q, q) dq, \quad n-1 < \varphi \leq n \quad (4)$$

Definition 2. The fractional CF integral is stated as [32]

$${}^{CF}I_q^\varphi \vartheta(\epsilon, q) = \frac{1-\varphi}{S(\varphi)} Q(q) + \frac{\varphi}{S(\varphi)} \int_0^q Q(q) dq, \quad q \geq 0, \varphi \in (0, 1]. \quad (5)$$

Definition 3. For $S(\varphi) = 1$, the Laplace transform of the CF derivative is [33]

$$L[{}^{CF}D_q^\varphi Q(q)] = \frac{vL[Q(q) - Q(0)]}{v + \varphi(1-v)}. \quad (6)$$

Definition 4. The YT of $Q(q)$ is framed as [34]

$$\mathbb{Y}[Q(q)] = \chi(v) = \int_0^\infty Q(q) e^{-\frac{q}{v}} dq, \quad q > 0 \quad (7)$$

Remarks

The \mathbb{Y} T of some helpful expressions are as follows:

$$\begin{aligned} \mathbb{Y}[1] &= v; \\ \mathbb{Y}[q] &= v^2; \\ \mathbb{Y}[q^i] &= \Gamma(i + 1)v^{i+1}. \end{aligned}$$

Lemma 1. Let the Laplace transform of $Q(q)$ be $F(v)$, and then $\chi(v) = F(1/v)$ [35].

Proof. From Equation (7), we can obtain the Yang transform by putting $q/v = \zeta$ as

$$L[Q(q)] = \int_0^\infty Q(v\zeta)e^\zeta d\zeta, \quad \zeta > 0 \tag{8}$$

since $L[Q(q)] = F(v)$, which implies that

$$F(v) = L[Q(q)] = \int_0^\infty Q(q)e^{-vq} dq. \tag{9}$$

Putting $q = \zeta/v$ in Equation (9), we obtain

$$F(v) = \frac{1}{v} \int_0^\infty Q\left(\frac{\zeta}{v}\right)e^\zeta d\zeta. \tag{10}$$

Thus, from Equation (8), we obtain:

$$F(v) = \chi\left(\frac{1}{v}\right). \tag{11}$$

Furthermore, from Equations (7) and (9), we obtain

$$F\left(\frac{1}{v}\right) = \chi(v). \tag{12}$$

The links between Equations (11) and (12) represent the duality connection among the Laplace and Yang transforms. \square

Lemma 2. Let $Q(q)$ be a function, then \mathbb{Y} T of CF derivatives of $Q(q)$ is [35]

$$\mathbb{Y}[Q(q)] = \frac{\mathbb{Y}[Q(q) - vQ(0)]}{v + \wp(v - 1)}. \tag{13}$$

Proof. The fractional Laplace transform of CF is defined as in Equation (13)

$$L[Q(q)] = \frac{L[vQ(q) - Q(0)]}{v + \wp(1 - v)}. \tag{14}$$

However, we have a correlation among the \mathbb{Y} T and Laplace properties, namely $\chi(v) = F(1/v)$, so put $1/v$ for v in Equation (14), and we obtain

$$\begin{aligned} \mathbb{Y}[Q(q)] &= \frac{\mathbb{Y}\left[\frac{1}{v}Q(q) - Q(0)\right]}{\frac{1}{v} + \wp\left(1 - \frac{1}{v}\right)}, \\ \mathbb{Y}[Q(q)] &= \frac{\mathbb{Y}[Q(q) - vQ(0)]}{1 + \wp(v - 1)}. \end{aligned} \tag{15}$$

Thus, the proof is satisfied. \square

3. Idea of Yang Homotopy Perturbation Transform Method (\mathbb{Y} HPTM)

In this part, we will demonstrate the concept of \mathbb{Y} HPTM. Let us assume a nonlinear fractional-order PDE, such as

$${}^{CF}D_q^\varphi \vartheta(\epsilon, q) + R\vartheta(\epsilon, q) + N\vartheta(\epsilon, q) = g(\epsilon, q), \tag{16}$$

$$\vartheta(\epsilon, 0) = h(\epsilon), \tag{17}$$

where $g(\epsilon, q)$ is called the source function. Applying the \mathbb{Y} T to Equation (16),

$$\begin{aligned} \frac{1}{v^\varphi} \mathbb{Y} [\vartheta(\epsilon, q) - v\vartheta(\epsilon, 0)] &= -\mathbb{Y}[R(\vartheta(\epsilon, q)) + N(\vartheta(\epsilon, q)) + \mathbb{Y}[g(\epsilon, q)]], \\ \mathbb{Y}[\vartheta(\epsilon, q)] &= vh(\epsilon) - v^\varphi [\mathbb{Y}[R(\vartheta(\epsilon, q)) + N(\vartheta(\epsilon, q))]] + \mathbb{Y}[g(\epsilon, q)]. \end{aligned}$$

By using inverse \mathbb{Y} T,

$$\vartheta(\epsilon, q) = \vartheta(\epsilon, 0) - \mathbb{Y}^{-1} \left[v^\varphi [\mathbb{Y}[R(\vartheta(\epsilon, q)) + N(\vartheta(\epsilon, q))]] + \mathbb{Y}[g(\epsilon, q)] \right]. \tag{18}$$

However, HPM is stated as

$$\vartheta(\epsilon, q) = \sum_{i=0}^{\infty} p^i \vartheta_i(\epsilon, q), \tag{19}$$

and

$$N\vartheta(\epsilon, q) = \sum_{i=0}^{\infty} p^i H_i \vartheta_i(\epsilon, q). \tag{20}$$

The following strategy can be operated to acquire the He's polynomials,

$$H_i(\vartheta_0 + \vartheta_1 + \dots + \vartheta_i) = \frac{1}{n!} \frac{\partial^i}{\partial p^i} \left(N \left(\sum_{i=0}^{\infty} p^i \vartheta_i \right) \right)_{p=0} \quad . \quad n = 0, 1, 2, \dots$$

With the help of Equations (19) and (20), we can obtain Equation (18), such as

$$\sum_{i=0}^{\infty} p^i \vartheta_i(\epsilon, q) = \vartheta(\epsilon, 0) - p \mathbb{Y}^{-1} \left[v^\varphi \mathbb{Y} \left\{ R \left(\sum_{i=0}^{\infty} p^i \vartheta_i(\epsilon, q) \right) + \sum_{i=0}^{\infty} p^i H_n \vartheta_i(\epsilon, q) \right\} \right].$$

We can obtain the following terms by evaluating the p components:

$$\begin{aligned}
 p^0 &= \vartheta_0(\epsilon, q) = \vartheta(\epsilon, 0), \\
 p^1 &= \vartheta_1(\epsilon, q) = -\mathbb{Y}^{-1} \left[v^{\wp} \mathbb{Y} \left\{ R\vartheta_0(\epsilon, q) + H_0(\vartheta) \right\} \right], \\
 p^2 &= \vartheta_2(\epsilon, q) = -\mathbb{Y}^{-1} \left[v^{\wp} \mathbb{Y} \left\{ R\vartheta_1(\epsilon, q) + H_1(\vartheta) \right\} \right], \\
 p^3 &= \vartheta_3(\epsilon, q) = -\mathbb{Y}^{-1} \left[v^{\wp} \mathbb{Y} \left\{ R\vartheta_2(\epsilon, q) + H_2(\vartheta) \right\} \right], \\
 &\vdots \\
 p^i &= \vartheta_i(\epsilon, q) = -\mathbb{Y}^{-1} \left[v^{\wp} \mathbb{Y} \left\{ R\vartheta_i(\epsilon, q) + H_i(\vartheta) \right\} \right],
 \end{aligned}
 \tag{21}$$

Thus, we can summarize the set of Equations (21) in the series form, such as

$$\begin{aligned}
 \vartheta(\epsilon, q) &= \vartheta_0(\epsilon, q) + \vartheta_1(\epsilon, q) + \vartheta_2(\epsilon, q) + \dots \\
 \vartheta(\epsilon, q) &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \vartheta_n(\epsilon, q)
 \end{aligned}
 \tag{22}$$

4. Numerical Applications

4.1. Example 1

Consider a linear time-fractional KG problem

$$D_q^{\wp} \vartheta(\epsilon, q) - D_{\epsilon}^2 \vartheta(\epsilon, q) - \vartheta(\epsilon, q) = 0,
 \tag{23}$$

with the initial condition

$$\vartheta(\epsilon, 0) = 1 + \sin(\epsilon).
 \tag{24}$$

Taking $\mathbb{Y}T$ of Equation (23), we obtain

$$\mathbb{Y} \left[\frac{\partial^{\wp} \vartheta}{\partial q^{\wp}} \right] = \mathbb{Y} \left[\frac{\partial^2 \vartheta}{\partial \epsilon^2} + \vartheta \right].$$

Executing the differential property of $\mathbb{Y}T$, we obtain

$$\begin{aligned}
 \frac{1}{v^{\wp}} \mathbb{Y} \left[\vartheta(\epsilon, q) - v\vartheta(\epsilon, 0) \right] &= \mathbb{Y} \left[\frac{\partial^2 \vartheta}{\partial q^2} + \vartheta \right], \\
 \mathbb{Y} \left[\vartheta(\epsilon, q) \right] &= v\vartheta(\epsilon, 0) + v^{\wp} \mathbb{Y} \left[\frac{\partial^2 \vartheta}{\partial q^2} + \vartheta \right].
 \end{aligned}$$

The inverse $\mathbb{Y}T$ indicates

$$\vartheta(\epsilon, q) = \vartheta(\epsilon, 0) + \mathbb{Y}^{-1} \left[v^{\wp} \left\{ \mathbb{Y} \left(\frac{\partial^2 \vartheta}{\partial q^2} + \vartheta \right) \right\} \right].$$

Employing HPM such as

$$\begin{aligned}
 \vartheta(\epsilon, q) &= \vartheta_0 + p\vartheta_1 + p^2\vartheta_2 + \dots, \\
 \sum_{i=0}^{\infty} p^i \vartheta_i(\epsilon, q) &= 1 + \sin(q) + p \left(\mathbb{Y}^{-1} \left[v^{\wp} \left\{ \mathbb{Y} \left(\sum_{i=0}^{\infty} p^i \frac{\partial^2 \vartheta_i}{\partial q^2} + \sum_{i=0}^{\infty} p^i \vartheta_i \right) \right\} \right] \right),
 \end{aligned}$$

on comparing the identical of p , we obtain

$$\begin{aligned}
 p^0 &= \vartheta_0(\epsilon, q) = \vartheta(\epsilon, 0), \\
 p^1 &= \vartheta_1(\epsilon, q) = \mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\frac{\partial^2 \vartheta_0}{\partial q^2} + \vartheta_0 \right) \right\} \right], \\
 p^2 &= \vartheta_2(\epsilon, q) = \mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\frac{\partial^2 \vartheta_1}{\partial q^2} + \vartheta_1 \right) \right\} \right], \\
 p^3 &= \vartheta_3(\epsilon, q) = \mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\frac{\partial^2 \vartheta_2}{\partial q^2} + \vartheta_2 \right) \right\} \right], \\
 &\vdots
 \end{aligned}$$

With help of Equation (24), we gain the iterations successively $\vartheta_i(\epsilon)$, $i = 1, 2, 3, \dots$, as follows:

$$\begin{aligned}
 \vartheta_0(\epsilon, q) &= 1 + \sin(\epsilon), \\
 \vartheta_1(\epsilon, q) &= \frac{1}{\Gamma(1 + \varphi)} q^\varphi, \\
 \\
 \vartheta_2(\epsilon, q) &= \frac{1}{\Gamma(1 + 2\varphi)} q^{2\varphi}, \\
 \vartheta_3(\epsilon, q) &= \frac{1}{\Gamma(1 + 3\varphi)} q^{3\varphi}, \\
 &\vdots \\
 \vartheta_i(\epsilon, q) &= \frac{1}{\Gamma(1 + i\varphi)} q^{i\varphi},
 \end{aligned}$$

Thus, the approximate solution can be obtained by:

$$\begin{aligned}
 \vartheta(\epsilon, q) &= 1 + \sin(\epsilon) + \frac{1}{\Gamma(1 + \varphi)} q^\varphi + \frac{1}{\Gamma(1 + 2\varphi)} q^{2\varphi} + \frac{1}{\Gamma(1 + 3\varphi)} q^{3\varphi} + \dots \quad (25) \\
 &= 1 + \sin(\epsilon) + \sum_{i=0}^{\infty} p^i \frac{q^{i\varphi}}{\Gamma(1 + i\varphi)},
 \end{aligned}$$

which implies the exact solution of Equation (23), In particular, at $\varphi = 1$, we obtain

$$\vartheta(\epsilon, q) = 1 + \sin(\epsilon), \tag{26}$$

which is in full agreement.

Figure 1a–d indicate the physical behavior of the obtained solution at $\epsilon \in [0, 4]$ and $q \in [0, 0.8]$. From these figures, it can be observed that the solution graphs of the problem show the friendly touch with each other. Figure 1a–d demonstrate that the solution achieved by \mathbb{Y} HPTM approaches the precise solution very rapidly with more iterations. In Figure 2, we have plotted the graph of $\vartheta(\epsilon, q)$ at different fractional order of $\varphi = 0.25, 0.50, 0.75, 1$ and $\epsilon \in [0, 2\pi]$ with different values of q .

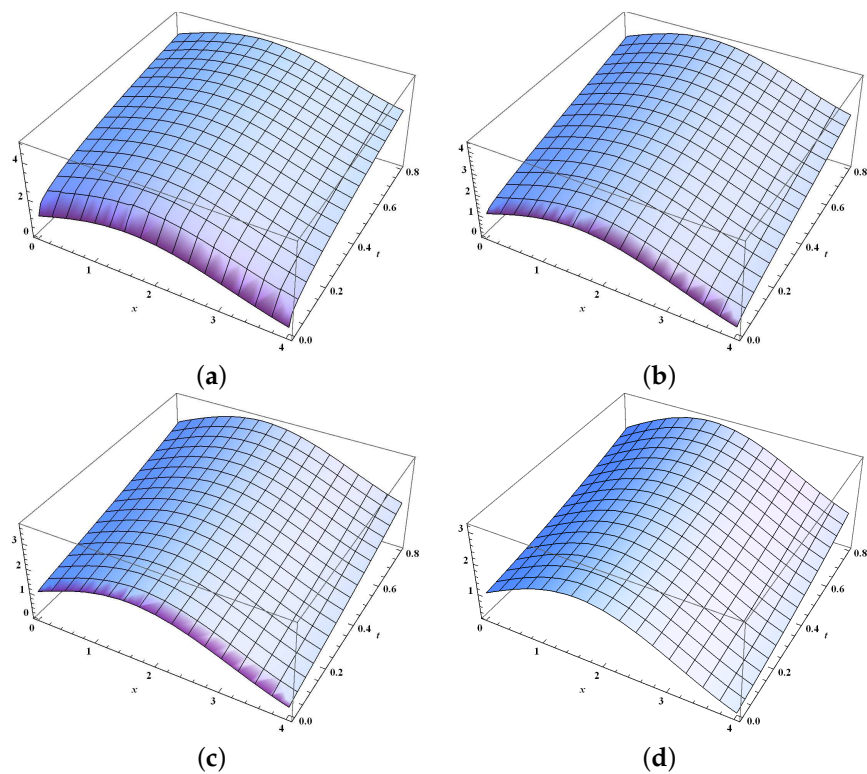


Figure 1. The surface solution of $\vartheta(\epsilon, q)$ with respect to ϵ and q for distinct values of φ : (a) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 0.25$; (b) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 0.50$; (c) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 0.75$; (d) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 1$.

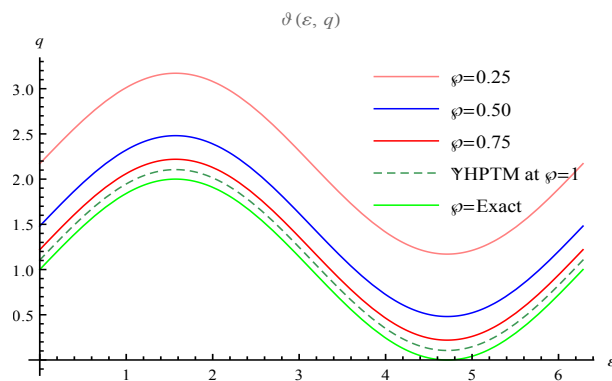


Figure 2. Plot of $\vartheta(\epsilon, q)$ for different values of φ .

4.2. Example 2

Assume a nonlinear time-fractional KG problem

$$D_q^\varphi \vartheta(\epsilon, q) - D_\epsilon^2 \vartheta(\epsilon, q) + \vartheta^2(\epsilon, q) = 0, \tag{27}$$

with the initial condition

$$\vartheta(\epsilon, 0) = 1 + \sin(\epsilon). \tag{28}$$

Taking the Yang transform of Equation (27), we obtain

$$\mathbb{Y} \left[\frac{\partial^\varphi \vartheta}{\partial q^\varphi} \right] = \mathbb{Y} \left[\frac{\partial^2 \vartheta}{\partial \epsilon^2} - \vartheta^2 \right].$$

Executing the differential property of \mathbb{Y} T, we obtain

$$\frac{1}{v^\varphi} \mathbb{Y} \left[\vartheta(\epsilon, q) - v\vartheta(\epsilon, 0) \right] = \mathbb{Y} \left[\frac{\partial^2 \vartheta}{\partial q^2} - \vartheta^2 \right],$$

$$\mathbb{Y} \left[\vartheta(\epsilon, q) \right] = v\vartheta(\epsilon, 0) + v^\varphi \mathbb{Y} \left[\frac{\partial^2 \vartheta}{\partial q^2} - \vartheta^2 \right].$$

The inverse \mathbb{Y} T indicates

$$\vartheta(\epsilon, q) = \vartheta(\epsilon, 0) + \mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\frac{\partial^2 \vartheta}{\partial q^2} - \vartheta^2 \right) \right\} \right].$$

Employing HPM such as

$$\sum_{i=0}^{\infty} p^i \vartheta_i(\epsilon, q) = 1 + \sin(q) + p \left(\mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\sum_{i=0}^{\infty} p^i \frac{\partial^2 \vartheta_i}{\partial q^2} - \sum_{i=0}^{\infty} p^i \vartheta_i^2 \right) \right\} \right] \right),$$

on comparing the identical of p , we obtain

$$p^0 = \vartheta_0(\epsilon, q) = \vartheta(\epsilon, 0),$$

$$p^1 = \vartheta_1(\epsilon, q) = \mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\frac{\partial^2 \vartheta_0}{\partial q^2} - \vartheta_0^2 \right) \right\} \right],$$

$$p^2 = \vartheta_2(\epsilon, q) = \mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\frac{\partial^2 \vartheta_1}{\partial q^2} - 2\vartheta_0\vartheta_1 \right) \right\} \right],$$

$$p^3 = \vartheta_3(\epsilon, q) = \mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\frac{\partial^2 \vartheta_2}{\partial q^2} - \vartheta_1^2 - 2\vartheta_0\vartheta_2 \right) \right\} \right],$$

⋮

With help of Equation (28), we gain the iterations successively $\vartheta_i(\epsilon)$, $i = 1, 2, 3, \dots$, as follows:

$$\vartheta_0(\epsilon, q) = 1 + \sin(\epsilon),$$

$$\vartheta_1(\epsilon, q) = \frac{-q^\varphi}{\Gamma(1 + \varphi)} \left(1 + 3 \sin(\epsilon) + \sin^2(\epsilon) \right),$$

$$\vartheta_2(\epsilon, q) = \frac{q^{2\varphi}}{\Gamma(1 + 2\varphi)} \left(11 \sin(\epsilon) + 12 \sin^2(\epsilon) + 2 \sin^3(\epsilon) \right),$$

$$\vartheta_3(\epsilon, q) = \frac{q^{3\varphi}}{\Gamma(1 + 3\varphi)} \left(18 - 57 \sin(\epsilon) - 160 \sin^2(\epsilon) - 82 \sin^3(\epsilon) - 10 \sin(4\epsilon) \right),$$

⋮

Thus, the approximate solution can be obtained by:

$$\begin{aligned} \vartheta(\epsilon, q) = & 1 + \sin(\epsilon) - \frac{q^\varphi}{\Gamma(1 + \varphi)} \left(1 + 3 \sin(\epsilon) + \sin^2(\epsilon) \right) + \frac{q^{2\varphi}}{\Gamma(1 + 2\varphi)} \left(11 \sin(\epsilon) + 12 \sin^2(\epsilon) + 2 \sin^3(\epsilon) \right) \\ & + \frac{q^{3\varphi}}{\Gamma(1 + 3\varphi)} \left(18 - 57 \sin(\epsilon) - 160 \sin^2(\epsilon) - 82 \sin^3(\epsilon) - 10 \sin(4\epsilon) \right) + \dots \end{aligned}$$

Figure 3a–d indicate the physical behavior of the obtained solution at $\epsilon \in [0, 1]$ and $q \in [0, 1]$. From these figures, it can be observed that with the increase in the value of φ , the approximate solution become close to the exact solution at $\varphi = 1$. In Figure 4, we have plotted the graph of $\vartheta(\epsilon, q)$ with different fractional orders of $\varphi = 0.25, 0.50, 0.75, 1$ at $\epsilon \in [0, 2\pi]$ with different values of q . It is obvious that this approximation can only be employed numerically, even though a closed form solution is not accessible. It can be seen that our approximate solution using \mathbb{Y} HPTM in Table 1 is more significant than that obtained in [36,37].

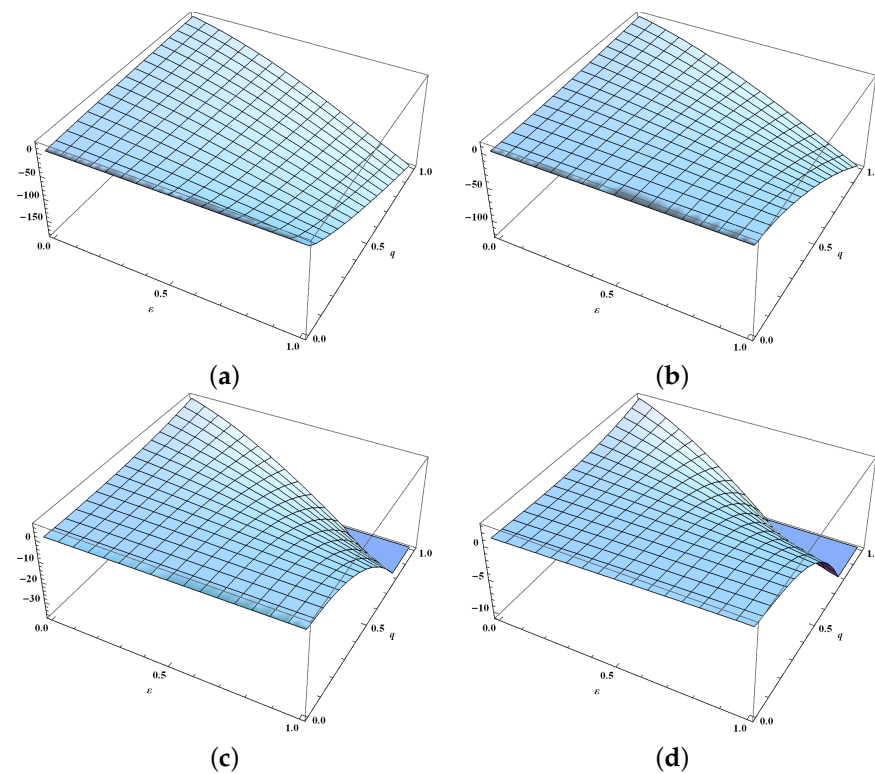


Figure 3. The surface solution of $\vartheta(\epsilon, q)$ with respect to ϵ and q for distinct values of φ : (a) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 0.25$; (b) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 0.50$; (c) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 0.75$; (d) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 1$.

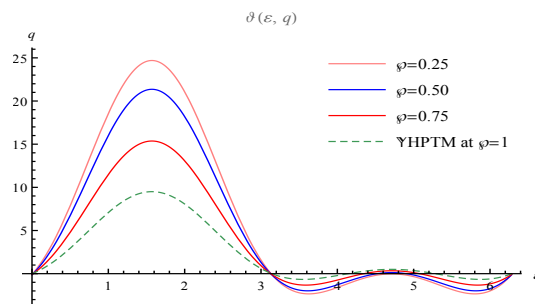


Figure 4. Plot of $\vartheta(\epsilon, q)$ for different values of φ

Table 1. Comparison between the value $\vartheta(\epsilon, q)$ for the solution of the KG equation.

Sr. No.	$q = 0.1$			$q = 0.2$			$q = 0.3$		
	[36]	[37]	YHPTM	[36]	[37]	YHPTM	[36]	[37]	YHPTM
0.0	0.9949999861	0.9950000249	0.903	0.9799991162	0.9800015775	0.824	0.9549900052	0.9550176534	0.781
0.1	1.093291132	1.093291179	0.976100	1.073723730	1.073726319	0.871321	1.073723730	1.073726319	0.792208
0.2	1.190502988	1.190503087	1.04725	1.166134875	1.166138050	0.915126	1.125945576	1.125974851	0.794835
0.3	1.285668610	1.285668848	1.11584	1.256326130	1.256331032	0.955409	1.208114007	1.208147932	0.789972
0.4	1.377844211	1.377844710	1.18132	1.343423788	1.343432104	0.992136	1.287043874	1.287088824	0.778571
0.5	1.466118315	1.466119219	1.24317	1.426594492	1.426608263	1.0252	1.362025218	1.362089477	0.761295
0.6	1.549620480	1.549621939	1.3009	1.505052082	1.505073495	1.05442	1.432404521	1.432497282	0.738476
0.7	1.627529538	1.627531694	1.35406	1.578063673	1.578094808	1.07951	1.497587424	1.497717706	0.710192
0.8	1.699081273	1.699084244	1.40223	1.644954933	1.644997540	1.0023	1.557040327	1.557215916	0.676451
0.9	1.763575490	1.763579356	1.44504	1.705114628	1.705169916	1.11635	1.610291023	1.610517519	0.63744
1.0	1.820382425	1.820387216	1.48219	1.757998450	1.758066925	1.12781	1.656928567	1.657208637	0.593784

4.3. Example 3

Consider another nonlinear time-fractional KG problem

$$D_q^\varphi \vartheta(\epsilon, q) - D_\epsilon^2 \vartheta(\epsilon, q) + \vartheta(\epsilon, q) - \vartheta^3(\epsilon, q) = 0, \tag{29}$$

with the initial condition

$$\vartheta(\epsilon, 0) = -\operatorname{sech}(\epsilon). \tag{30}$$

According to the idea of YHPTM, we can obtain the following relation

$$\sum_{i=0}^\infty p^i \vartheta_i(\epsilon, q) = \operatorname{sech}(\epsilon) + p \left(\mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\sum_{i=0}^\infty p^i \frac{\partial^2 \vartheta_i}{\partial q^2} - \sum_{i=0}^\infty p^i \vartheta_i + \sum_{i=0}^\infty p^i \vartheta_i^2 \right) \right\} \right] \right),$$

when the coefficients of like powers of p are compared, we obtain

$$\begin{aligned} p^0 &= \vartheta_0(\epsilon, q) = \vartheta(\epsilon, 0), \\ p^1 &= \vartheta_1(\epsilon, q) = \mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\frac{\partial^2 \vartheta_0}{\partial q^2} - \vartheta_0 + \vartheta_0^3 \right) \right\} \right], \\ p^2 &= \vartheta_2(\epsilon, q) = \mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\frac{\partial^2 \vartheta_1}{\partial q^2} - \vartheta_1 + 3\vartheta_0^2 \vartheta_1 \right) \right\} \right], \\ p^3 &= \vartheta_3(\epsilon, q) = \mathbb{Y}^{-1} \left[v^\varphi \left\{ \mathbb{Y} \left(\frac{\partial^2 \vartheta_2}{\partial q^2} - \vartheta_2 + 3\vartheta_0 \vartheta_1^2 + 3\vartheta_0^2 \vartheta_2 \right) \right\} \right], \\ &\vdots \end{aligned}$$

with help of Equation (30), we gain the iterations successively $\vartheta_i(\epsilon)$, $i = 1, 2, 3, \dots$, as follows:

$$\begin{aligned} \vartheta_0(\epsilon, q) &= -\operatorname{sech}(\epsilon), \\ \vartheta_1(\epsilon, q) &= -\frac{q^\varphi}{\Gamma(1+\varphi)} \left(2\operatorname{sech}(\epsilon) - 3\operatorname{sech}^3(\epsilon) \right), \\ \vartheta_2(\epsilon, q) &= -\frac{q^{2\varphi}}{\Gamma(1+2\varphi)} \left(3\operatorname{sech}(\epsilon) - 34\operatorname{sech}^3(\epsilon) - 18\operatorname{sech}^5(\epsilon) \right), \\ \vartheta_3(\epsilon, q) &= -\frac{q^{3\varphi}}{\Gamma(1+3\varphi)} \left(64\operatorname{sech}^3(\epsilon) - 288\operatorname{sech}^5(\epsilon) + 240\operatorname{sech}^7(\epsilon) \right), \\ &\vdots \end{aligned}$$

Thus, the approximate solution can be obtained by:

$$\begin{aligned} \vartheta(\epsilon, q) &= -\operatorname{sech}(\epsilon) - \frac{q^\varphi}{\Gamma(1+\varphi)} \left(2\operatorname{sech}(\epsilon) - 3\operatorname{sech}^3(\epsilon) \right) - \frac{q^{2\varphi}}{\Gamma(1+2\varphi)} \left(3\operatorname{sech}(\epsilon) - 34\operatorname{sech}^3(\epsilon) - 18\operatorname{sech}^5(\epsilon) \right) \\ &\quad - \frac{q^{3\varphi}}{\Gamma(1+3\varphi)} \left(64\operatorname{sech}^3(\epsilon) - 288\operatorname{sech}^5(\epsilon) + 240\operatorname{sech}^7(\epsilon) \right) + \dots \end{aligned}$$

Figure 5a–d indicates the physical behavior of the obtained solution at $\epsilon \in [-2, 2]$ and $q \in [0, 0.1]$. From these figures, it can be observed that with increase in the value of φ , the approximate solution graph comes close to the exact exact solution at $\varphi = 1$. In Figure 6, we have plotted the graph of $\vartheta(\epsilon, q)$ at different fractional orders of $\varphi = 0.25, 0.50, 0.75, 1$ and $\epsilon \in [0, 2\pi]$ with different values of q . We compared our graphical results obtained by YHPTM, which converges to the exact solution very rapidly with a small amount of q compared to [38] at $\varphi = 1$.

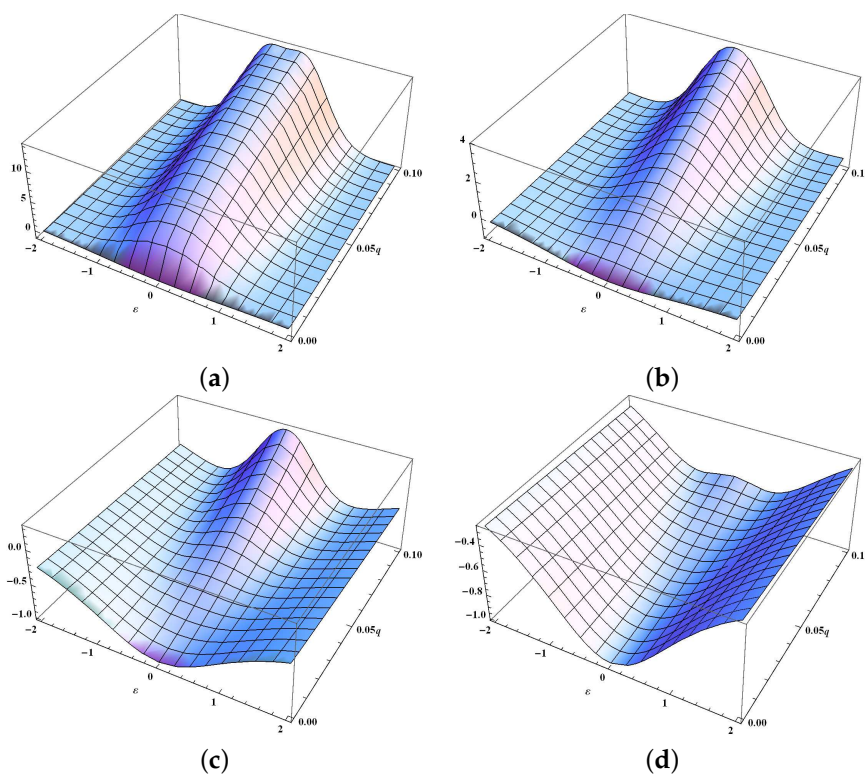


Figure 5. The surface solution of $\vartheta(\epsilon, q)$ with respect to ϵ and q for distinct values of φ . (a) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 0.25$; (b) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 0.50$; (c) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 0.75$; (d) surface solution of $\vartheta(\epsilon, q)$ when $\varphi = 1$.

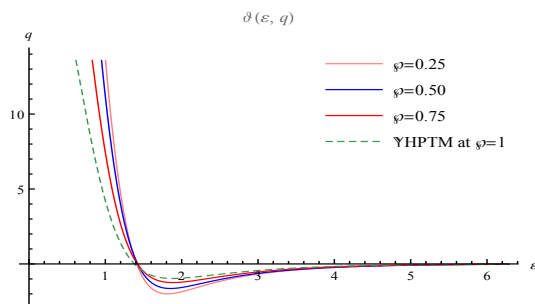


Figure 6. Plot of $\vartheta(\epsilon, q)$ for different values of φ .

5. Conclusions

In this study, YHPTM has been utilized to achieve an approximate solution of nonlinear time-fractional KG equations. We demonstrate some illustrations to show the validity of the method, which reveals that the obtained findings are satisfactory. It is important to note that in order to improve the accuracy, a greater number of iterations with excessive order of p are required. The best advantage of YHPTM is that it generates the approximate solution in a quickly convergent power series form. As a result, this strategy can be adopted to elucidate a wide classification of nonlinear challenges in science and engineering with no need for linearization, discretization or perturbation. The proposed strategy has the privilege of being able to tackle linear and nonlinear time-fractional KG problems simultaneously. Mathematica package 11.0.1 has been operated for the graphical analysis as well as for the computations in this paper. We recommend the readers to consider this problem for the Atangana–Baleanu fractional derivative and many others in the place of the Caputo–Fabrizio operator.

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Abbreviations

The following abbreviations are used in this manuscript:

YHPTM	Yang homotopy perturbation transform method
YT	Yang transform
CF	Caputo–Fabrizio
FPDEs	Fractional partial differential equations
KG	Homotopy perturbation method
HPM	Klein–Gordon

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