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Generating Soft Topologies via Soft Set Operators

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Abstract: As daily problems involve a great deal of data and ambiguity, it has become vital to build new mathematical ways to cope with them, and soft set theory is the greatest tool for doing so. As a result, we study methods of generating soft topologies through several soft set operators. A soft topology is known to be determined by the system of special soft sets, which are called soft open (dually soft closed) sets. The relationship between specific types of soft topologies and their classical topologies (known as parametric topologies) is linked to the idea of symmetry. Under this symmetry, we can study the behaviors and properties of classical topological concepts via soft settings and vice versa. In this paper, we show that soft topological spaces can be characterized by soft closure, soft interior, soft boundary, soft exterior, soft derived set, or co-derived set operators. All of the soft topologies that result from such operators are equivalent, as well as being identical to their classical counterparts under enriched (extended) conditions. Moreover, some of the soft topologies are the systems of all fixed points of specific soft operators. Multiple examples are presented to show the implementation of these operators. Some of the examples show that, by removing any axiom, we will miss the uniqueness of the resulting soft topology.

Keywords: soft topology; soft closure operator; soft interior operator; soft boundary operator; soft exterior operator; soft derived set operator; co-derived set operator; fixed point

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1. Introduction

Most real-world problems in engineering, medical science, economics, the environment, and other fields are full of uncertainty. Soft set theory was proposed by Molodtsov [1], in 1999, as a mathematical instrument for dealing with uncertainty. This theory is free of the obstacles associated with previous theories including fuzzy set theory [2], rough set theory [3], and so on. The nature of parameter sets related to soft sets, in particular, provides a uniform framework for modeling uncertain data. This has resulted in the rapid development of soft set theory in a short period of time, as well as diverse applications of soft sets in real life.

The mathematical area of topology known as general topology is concerned with the core set-theoretic principles and procedures. The Kuratowski closure axioms [4] are a collection of axioms that can be used to establish a topological structure on a set in topology and related disciplines of mathematics. They are the same as the more widely used open set concept. The closure system of axioms is significant in domain theory, and also has real applications (see [5] for more details).

Influenced by the standard postulates of traditional topological space, Shabir and Naz [6], and Çağman et al. [7], separately, established another branch of topology known as "soft topology", which is a mixture of soft set theory and topology. It focuses on the development of the system of all soft sets. The study in [6,7], in particular, was essential in building the subject of soft topology. Despite the fact that many studies followed their directions and many ideas appeared in soft contexts (see, [8–11]), significant contributions can indeed be made. Hence, we generalize the Kuratowski closure system together with five other operators in soft settings. The role of these operators is to characterize soft topologies over a domain set. Some operators and bioperators have been explored via soft topologies [12]. The symmetry between soft topology and its parametric topologies was investigated by Al-shami and Kočinac [13]. Under an extended soft topology, they proved that $\mathrm{Int}(F,\sum) = (\mathrm{Int}(F),\sum)$ and $\mathrm{Cl}(F,\sum) = (\mathrm{Cl}(F),\sum)$, which obviously shows the symmetry of soft interior and closure operators with their classical topological counterparts. This symmetry between soft topology and its parametric topologies has been recently investigated for some kinds of soft separation axioms as illustrated in [14,15]

It is well known that soft topologies form a family of parametric classical topologies. In contrast, producing soft topologies from classical topologies was studied in some published literature, such as [8,13,16]. Investigation of the methods of producing soft topologies is among the most important and interesting notions concerning soft topologies because they are employed to build various classes of soft topological spaces, as well as providing a new environment to discuss and characterize topological concepts, such as the compactness, connectedness, and separation axioms. This matter motivated us to generate new types of soft topologies using different types of soft operators, which represent a rich area for discussion of topological concepts and researching their characterizations.

The body of the paper is structured as follows: In Section 2, we present an overview of the literature on soft set theory and soft topology. More precisely, the main properties of soft closure, soft interior, soft boundary, soft exterior and soft derived set of a soft set with respect to underlying soft topological space are considered. Section 3 focuses on the concepts of soft set operators and their implications for characterizing soft topologies over domain sets. Section 4 illustrates that, by omitting an axiom from the stated operators, we can still attain a soft topology but lose uniqueness. We end our paper, in Section 5, with a brief discussion and conclusions.

2. Preliminaries

Let X be a domain set and Σ be a set of parameters. An ordered pair $(F,\Sigma) = \{(e, F(e)) : e \in \Sigma\}$ is said to be a soft set over X, where $F : \Sigma \to 2^X$ is a set-valued mapping. The set of all soft sets on X parameterized by Σ is identified by $S_\Sigma(X)$. A soft set $(X,\Sigma) - (F,\Sigma)$ (or simply $(F,\Sigma)'$) is the complement of (F,Σ), where $F' : \Sigma \to \mathcal{P}(X)$ is given by $F'(e) = X - F(e)$ for each $e \in \Sigma$. A soft set $(F,\Sigma) \in S_\Sigma(X)$ is called a null soft set, denoted by Φ, if $F(e) = \emptyset$ for each $e \in \Sigma$, it is called an absolute soft set, denoted by $\widetilde{X}$, if $F(e) = X$ for each $e \in \Sigma$. Evidently, $\widetilde{X}' = \Phi$ and $\Phi' = \widetilde{X}$. A soft point [6] is a soft set (F,Σ) over X in which $F(e) = \{x\}$ for each $e \in \Sigma$, where $x \in X$, and is denoted by $(\{x\},\Sigma)$. It is said that a soft point $(\{x\},\Sigma)$ is in (F,Σ) (briefly, $x \in (F,\Sigma)$) if $x \in F(e)$ for each $e \in \Sigma$. On the other hand, $x \notin (F,\Sigma)$ if $x \notin F(e)$ for some $e \in \Sigma$. This implies that if $(\{x\},\Sigma)\widetilde{\cap}(F,\Sigma) = \Phi$, then $x \notin (F,\Sigma)$. A soft element [17,18], denoted by x_e, is a soft set (F,Σ) over X in which $F(e) = \{x\}$ and $x(e') = \emptyset$ for each $e' \in \Sigma$ with $e' \neq e$, where $e \in \Sigma$ and $x \in X$. An argument $x_e \in (F,\Sigma)$ means that $x \in F(e)$. It is said that (A,Σ_1) is a soft subset of (B,Σ_2) (written by $(A,\Sigma_1)\widetilde{\subseteq}(B,\Sigma_2)$, [19]) if $\Sigma_1 \subseteq \Sigma_2$ and $A(e) \subseteq B(e)$ for each $e \in \Sigma_1$, and $(A,\Sigma_1) = (B,\Sigma_2)$ if $(A,\Sigma_1)\widetilde{\subseteq}(B,\Sigma_2)$ and $(B,\Sigma_2)\widetilde{\subseteq}(A,\Sigma_1)$. The union of soft sets $(A,\Sigma),(B,\Sigma)$ is represented by $(F,\Sigma) = (A,\Sigma)\widetilde{\cup}(B,\Sigma)$, where $F(e) = A(e) \cup B(e)$ for each $e \in \Sigma$, and the intersection of soft sets $(A,\Sigma),(B,\Sigma)$ is given by $(F,\Sigma) = (A,\Sigma)\widetilde{\cap}(B,\Sigma)$, where $F(e) = A(e) \cap B(e)$ for each $e \in \Sigma$, (see, [20]).

Definition 1 ([6])**.** *A collection $\mathcal{T}$ of $S_\Sigma(X)$ is said to be a soft topology on X if it has the next postulates:*

(T.1) $\Phi, \widetilde{X} \in \mathcal{T}$.

(T.2) *If $(F_1,\Sigma),(F_2,\Sigma) \in \mathcal{T}$, then $(F_1,\Sigma)\widetilde{\cap}(F_2,\Sigma) \in \mathcal{T}$.*

(T.3) *If* $\{(F_i,\Sigma) : i \in I\}\widetilde{\subseteq}\mathcal{T}$, *then* $\widetilde{\cup}_{i\in I}(F_i,\Sigma) \in \mathcal{T}$.

Terminologically, we call $(X,\mathcal{T},\Sigma)$ *a soft topological space on* X. *The elements of* $\mathcal{T}$ *are called soft open sets. The complements of every soft open (or elements of* $\mathcal{T}'$*) are called soft closed sets. The lattice of all soft topologies on* X *is referred to* $T_\Sigma(X)$, *(see, [21]).*

Definition 2 ([6])**.** *Let* $(B,\Sigma) \in S_\Sigma(X)$ *and* $\mathcal{T} \in T_\Sigma(X)$. *The soft closure of* (B,Σ) *is*

$$\mathrm{Cl}(B,\Sigma) := \widetilde{\bigcap}\{(F,\Sigma) : (B,\Sigma)\widetilde{\subseteq}(F,\Sigma), (F,\Sigma) \in \mathcal{T}'\}.$$

Lemma 1 ([6], Theorem 1)**.** *Let* $(F,\Sigma),(G,\Sigma) \in S_\Sigma(X)$ *and* $\mathcal{T} \in T_\Sigma(X)$. *The following properties are valid:*

1. $\mathrm{Cl}(\Phi) = \Phi, \mathrm{Cl}(\widetilde{X}) = \widetilde{X}$.
2. $(F,\Sigma)\widetilde{\subseteq}\,\mathrm{Cl}(F,\Sigma)$.
3. $\mathrm{Cl}(F,\Sigma) = \mathrm{Cl}(\mathrm{Cl}(F,\Sigma))$.
4. $(F,\Sigma)\widetilde{\subseteq}(G,\Sigma) \implies \mathrm{Cl}(F,\Sigma)\widetilde{\subseteq}\,\mathrm{Cl}(F,\Sigma)$.
5. $(F,\Sigma) \in \mathcal{T}' \iff (F,\Sigma) = \mathrm{Cl}(F,\Sigma)$.
6. $\mathrm{Cl}\left((F,\Sigma)\widetilde{\cup}(G,\Sigma)\right) = \mathrm{Cl}(F,\Sigma)\widetilde{\cup}\,\mathrm{Cl}(G,\Sigma)$.
7. $\mathrm{Cl}\left((F,\Sigma)\widetilde{\cap}(G,\Sigma)\right)\widetilde{\subseteq}\,\mathrm{Cl}(F,\Sigma)\widetilde{\cap}\,\mathrm{Cl}(G,\Sigma)$.

Definition 3 ([22])**.** *Let* $(B,\Sigma) \in S_\Sigma(X)$ *and* $\mathcal{T} \in T_\Sigma(X)$. *The soft interior of* (B,Σ) *is*

$$\mathrm{Int}(B,\Sigma) := \widetilde{\bigcup}\{(F,\Sigma) : (F,\Sigma)\widetilde{\subseteq}(B,\Sigma), (F,\Sigma) \in \mathcal{T}\}.$$

Lemma 2 ([22], Theorem 2)**.** *Let* $(F,\Sigma),(G,\Sigma) \in S_\Sigma(X)$ *and* $\mathcal{T} \in T_\Sigma(X)$. *The following properties are valid:*

1. $\mathrm{Int}(\Phi) = \Phi, \mathrm{Int}(\widetilde{X}) = \widetilde{X}$.
2. $\mathrm{Int}(F,\Sigma)\widetilde{\subseteq}(F,\Sigma)$.
3. $\mathrm{Int}(F,\Sigma) = \mathrm{Int}(\mathrm{Int}(F,\Sigma))$.
4. $(F,\Sigma)\widetilde{\subseteq}(G,\Sigma) \implies \mathrm{Int}(F,\Sigma)\widetilde{\subseteq}\,\mathrm{Int}(F,\Sigma)$.
5. $(F,\Sigma) \in \mathcal{T} \iff (F,\Sigma) = \mathrm{Int}(F,\Sigma)$.
6. $\mathrm{Int}\left(F,\Sigma)\widetilde{\cap}(G,\Sigma)\right) = \mathrm{Int}(F,\Sigma)\widetilde{\cap}\,\mathrm{Int}(G,\Sigma)$.
7. $\mathrm{Int}(F,\Sigma)\widetilde{\cap}\,\mathrm{Int}(G,\Sigma)\widetilde{\subseteq}\,\mathrm{Int}\left(F,\Sigma)\widetilde{\cup}(G,\Sigma)\right)$.

Definition 4 ([22])**.** *Let* $(B,\Sigma) \in S_\Sigma(X)$ *and* $\mathcal{T} \in T_\Sigma(X)$. *The soft exterior of* (B,Σ) *is* $\mathrm{Ext}(B,\Sigma) := \mathrm{Int}((B,\Sigma)')$.

Lemma 3. *Let* $(F,\Sigma),(G,\Sigma) \in S_\Sigma(X)$ *and* $\mathcal{T} \in T_\Sigma(X)$. *The following properties are valid:*

1. $\mathrm{Ext}(\Phi) = \widetilde{X}, \mathrm{Ext}(\widetilde{X}) = \Phi$.
2. $\mathrm{Ext}(F,\Sigma)\widetilde{\subseteq}(F,\Sigma)'$.
3. $\mathrm{Ext}(F,\Sigma) = \mathrm{Ext}([\mathrm{Ext}(F,\Sigma)]')$.
4. $(F,\Sigma)\widetilde{\subseteq}(G,\Sigma) \implies \mathrm{Ext}(G,\Sigma)\widetilde{\subseteq}\,\mathrm{Ext}(F,\Sigma)$.
5. $\mathrm{Int}(F,\Sigma)\widetilde{\subseteq}\,\mathrm{Ext}(\mathrm{Ext}(F,\Sigma))$.
6. $\mathrm{Ext}\left((F,\Sigma)\widetilde{\cup}(G,\Sigma)\right) = \mathrm{Ext}(F,\Sigma)\widetilde{\cap}\,\mathrm{Ext}(G,\Sigma)$.

Proof. Theorems 6 & 8 in [23] and Theorem 4 in [22]. □

Definition 5 ([22])**.** *Let* $(B,\Sigma) \in S_\Sigma(X)$ *and* $\mathcal{T} \in T_\Sigma(X)$. *The soft boundary of* (B,Σ) *is* $\mathrm{Bd}(B,\Sigma) := \mathrm{Cl}(B,\Sigma)\widetilde{\cap}\,\mathrm{Cl}((B,\Sigma)')$.

Lemma 4. *Let* $(F,\Sigma),(G,\Sigma) \in S_\Sigma(X)$ *and* $\mathcal{T} \in T_\Sigma(X)$. *The following properties are valid:*

1. $\mathrm{Bd}(\Phi) = \Phi, \mathrm{Bd}(\widetilde{X}) = \widetilde{X}$.
2. $\mathrm{Bd}(F,\Sigma) = \mathrm{Cl}(F,\Sigma) - \mathrm{Int}(F,\Sigma)$.

3. $\operatorname{Int}(F,\Sigma) = (F,\Sigma) - \operatorname{Bd}(F,\Sigma)$.
4. $\operatorname{Bd}(\operatorname{Int}(F,\Sigma)) \widetilde{\subseteq} \operatorname{Bd}(F,\Sigma)$.
5. $\operatorname{Bd}(\operatorname{Cl}(F,\Sigma)) \widetilde{\subseteq} \operatorname{Bd}(F,\Sigma)$.
6. $\operatorname{Bd}(\operatorname{Bd}(F,\Sigma)) \widetilde{\subseteq} \operatorname{Bd}(F,\Sigma)$.
7. $\operatorname{Bd}\left((F,\Sigma) \widetilde{\cup} (G,\Sigma)\right) \widetilde{\subseteq} \operatorname{Bd}(F,\Sigma) \widetilde{\cup} \operatorname{Bd}(G,\Sigma)$.
8. $\operatorname{Bd}\left((F,\Sigma) \widetilde{\cap} (G,\Sigma)\right) \widetilde{\subseteq} \operatorname{Bd}(F,\Sigma) \widetilde{\cup} \operatorname{Bd}(G,\Sigma)$.

Proof. Theorems 7 & 10 in [23] and Theorem 17 in [7]. □

Lemma 5. *Let $(F,\Sigma) \in S_\Sigma(X)$ and $\mathcal{T} \in T_\Sigma(X)$. The following properties are valid:*

1. $(F,\Sigma) \in \mathcal{T} \Longleftrightarrow (F,\Sigma) \widetilde{\cap} \operatorname{Bd}(F,\Sigma) = \Phi$.
2. $(F,\Sigma) \in \mathcal{T} \widetilde{\cap} \mathcal{T}' \Longleftrightarrow \operatorname{Bd}(F,\Sigma) = \Phi$.
3. $(F,\Sigma) \in \mathcal{T}' \Longleftrightarrow \operatorname{Bd}(F,\Sigma) \widetilde{\subseteq} (F,\Sigma)'$.

Proof. Theorems 6 & 11 in [22]. □

Definition 6 ([7])**.** *Let $(B,\Sigma) \in S_\Sigma(X)$ and $\mathcal{T} \in T_\Sigma(X)$. A soft point x_e in $\widetilde{X}$ is called a soft limit point of (B,Σ) if $(G,\Sigma) \widetilde{\cap} (B,\Sigma) - \{x_e\} \neq \Phi$ for all $(G,\Sigma) \in \mathcal{T}$ with $x_e \in (G,\Sigma)$. The set of all soft limit points is symbolized by* $\operatorname{Dr}(B,\Sigma)$.

Lemma 6. *Let $(F,\Sigma), (G,\Sigma) \in S_\Sigma(X)$ and $\mathcal{T} \in T_\Sigma(X)$. The following properties are valid:*

1. $\operatorname{Cl}(F,\Sigma) = (F,\Sigma) \widetilde{\cup} \operatorname{Dr}(F,\Sigma)$.
2. $x_e \notin \operatorname{Dr}(\{x_e\},\Sigma)$
3. $\operatorname{Dr}(F,\Sigma) \widetilde{\subseteq} \operatorname{Cl}(F,\Sigma)$.
4. $(F,\Sigma) \widetilde{\subseteq} (G,\Sigma) \implies \operatorname{Dr}(F,\Sigma) \widetilde{\subseteq} \operatorname{Dr}(G,\Sigma)$.
5. $\operatorname{Dr}\left[(F,\Sigma) \widetilde{\cap} (G,\Sigma)\right] \widetilde{\subseteq} \operatorname{Dr}(F,\Sigma) \widetilde{\cap} \operatorname{Dr}(G,\Sigma)$.
6. $\operatorname{Dr}\left[(F,\Sigma) \widetilde{\cup} (G,\Sigma)\right] = \operatorname{Dr}(F,\Sigma) \widetilde{\cup} \operatorname{Dr}(G,\Sigma)$.

Proof. Theorem 15 in [7]. □

It should be highlighted that several of the properties in the preceding lemmas are new, and their proofs are common; consequently, they have been disregarded.

3. Soft Operators and the Soft Topologies Generated by Them

The following two lemmas are presented before we begin with the definitions of soft operators:

Lemma 7. *For $(F,\Sigma) \in S_\Sigma(X)$ and $\mathcal{T} \in T_\Sigma(X)$, the following are equivalent:*

1. $(F,\Sigma) \in \mathcal{T}'$.
2. $\operatorname{Cl}(F,\Sigma) = (F,\Sigma)$.
3. $\operatorname{Dr}(F,\Sigma) \widetilde{\subseteq} (F,\Sigma)$.
4. $\operatorname{Bd}(F,\Sigma) \widetilde{\subseteq} (F,\Sigma)$.

Proof. (1) $\implies$ (2) Lemma 1 (5).

(2) $\implies$ (3) Since $(F,\Sigma) = \operatorname{Cl}(F,\Sigma) = (F,\Sigma) \widetilde{\cup} \operatorname{Dr}(F,\Sigma)$, so $\operatorname{Dr}(F,\Sigma) \widetilde{\subseteq} (F,\Sigma)$.

(3) $\implies$ (4) By (3), $\operatorname{Cl}(F,\Sigma) = \operatorname{Dr}(F,\Sigma) \widetilde{\cup} (F,\Sigma) \widetilde{\subseteq} (F,\Sigma)$. Then, by Lemma 4 (2), $\operatorname{Bd}(F,\Sigma) = (F,\Sigma) - \operatorname{Int}(F,\Sigma) \widetilde{\subseteq} (F,\Sigma)$. Thus (4) holds.

(4) $\implies$ (1) Since $\operatorname{Int}(F,\Sigma) \widetilde{\subseteq} (F,\Sigma)$ and $\operatorname{Bd}(F,\Sigma) \widetilde{\subseteq} (F,\Sigma)$, from Lemma 4 (2), we get $\operatorname{Cl}(F,\Sigma) = \operatorname{Bd}(F,\Sigma) \widetilde{\cup} \operatorname{Int}(F,\Sigma)$ and so $\operatorname{Cl}(F,\Sigma) \widetilde{\subseteq} (F,\Sigma)$. Thus, $(F,\Sigma) \in \mathcal{T}'$. □

Lemma 8. *For $\{(F_i,\Sigma) : i \in I\} \widetilde{\subseteq} S_\Sigma(X)$, where I is any index, and $\mathcal{T} \in T_\Sigma(X)$, the following are valid:*

1. $\operatorname{Int}\left(\widetilde{\bigcap}_{i \in I} (F_i,\Sigma)\right) \widetilde{\subseteq} \widetilde{\bigcap}_{i \in I} \operatorname{Int}(F_i,\Sigma)$.

2. $\widetilde{\bigcup}_{i\in I}\operatorname{Int}(F_i,\Sigma)\widetilde{\subseteq}\operatorname{Int}\left(\widetilde{\bigcup}_{i\in I}(F_i,\Sigma)\right)$.
3. $\operatorname{Cl}\left(\widetilde{\bigcap}_{i\in I}(F_i,\Sigma)\right)\widetilde{\subseteq}\widetilde{\bigcap}_{i\in I}\operatorname{Cl}(F_i,\Sigma)$.
4. $\widetilde{\bigcup}_{i\in I}\operatorname{Cl}(F_i,\Sigma)\widetilde{\subseteq}\operatorname{Cl}\left(\widetilde{\bigcup}_{i\in I}(F_i,\Sigma)\right)$.
5. $\operatorname{Dr}\left(\widetilde{\bigcap}_{i\in I}(F_i,\Sigma)\right)\widetilde{\subseteq}\widetilde{\bigcap}_{i\in I}\operatorname{Dr}(F_i,\Sigma)$.
6. $\widetilde{\bigcup}_{i\in I}\operatorname{Dr}(F_i,\Sigma)\widetilde{\subseteq}\operatorname{Dr}\left(\widetilde{\bigcup}_{i\in I}(F_i,\Sigma)\right)$.

Proof. It is deduced from the lemmas presented in Section 2. □

Definition 7 (Soft Closure Operator)**.** *A mapping $c : S_\Sigma(X) \to S_\Sigma(X)$ is said to be a soft closure operator on X if it has the following properties for every $(F,\Sigma),(G,\Sigma)\in S_\Sigma(X)$:*

(C.1) $c(\Phi)=\Phi$.

(C.2) $(F,\Sigma)\widetilde{\subseteq}c(F,\Sigma)$.

(C.3) $c(c(F,\Sigma))=c(F,\Sigma)$.

(C.4) $c\big((F,\Sigma)\widetilde{\cup}(G,\Sigma)\big)=c(F,\Sigma)\widetilde{\cup}c(G,\Sigma)$.

The following result illustrates how a closure operator derives a soft topology and concludes that the operator is a soft topological closure in this topology.

Before stating our theorem, we note that the monotonicity of c follows from (C.4). That is, if $(F,\Sigma)\widetilde{\subseteq}(G,\Sigma)$, then $c(F,\Sigma)\widetilde{\subseteq}c(G,\Sigma)$. Suppose $(F,\Sigma)\widetilde{\subseteq}(G,\Sigma)$, then $(G,\Sigma)=(F,\Sigma)\widetilde{\cup}\big((G,\Sigma)-(F,\Sigma)\big)$. By axiom (C.4), $c(G,\Sigma)=c(F,\Sigma)\widetilde{\cup}c\big((G,\Sigma)-(F,\Sigma)\big)$, and thus, $c(F,\Sigma)\widetilde{\subseteq}c(G,\Sigma)$.

Theorem 1. *Let c be a soft closure operator on X, and let $\mathcal{C}=\{(B,\Sigma):(B,\Sigma)\widetilde{\subseteq}\widetilde{X}, c(B,\Sigma)=(B,\Sigma)\}$. The system $\mathcal{T}=\{(F,\Sigma):(F,\Sigma)'\in\mathcal{C}\}$ is the unique soft topology on X having the property that $c(F,\Sigma)=\operatorname{Cl}(F,\Sigma)$ for every $(F,\Sigma)\in S_\Sigma(X)$, and Range$(c)=\mathcal{C}$.*

Proof. (T.1) Since $\Phi\in\mathcal{C}$, by (C.1), so $\widetilde{X}\in\mathcal{T}$. By (C.2), we have $\widetilde{X}\widetilde{\subseteq}c(\widetilde{X})$ which follows that $c(\widetilde{X})=\widetilde{X}$. Therefore $\widetilde{X}\in\mathcal{C}$, and so $\Phi\in\mathcal{T}$.

(T.2) Given $(F,\Sigma),(G,\Sigma)\in\mathcal{T}$, then $(F,\Sigma)',(G,\Sigma)'\in\mathcal{C}$. Therefore, $c\big((F,\Sigma)'\big)=(F,\Sigma)'$ and $c\big((G,\Sigma)'\big)=(G,\Sigma)'$. By (C.4),

$$c\big[(F,\Sigma)'\widetilde{\bigcup}(G,\Sigma)'\big]=c\big[(F,\Sigma)'\big]\widetilde{\bigcup}c\big[(G,\Sigma)'\big]=(F,\Sigma)'\widetilde{\bigcup}(G,\Sigma)'.$$

This implies that $(F,\Sigma)'\widetilde{\cup}(G,\Sigma)'\in\mathcal{C}$. But, $(F,\Sigma)\widetilde{\cap}(G,\Sigma)=\big[(F,\Sigma)'\widetilde{\cup}(G,\Sigma)'\big]'$, hence $(F,\Sigma)\widetilde{\cap}(G,\Sigma)\in\mathcal{T}$.

(T.3) Assume $\{(F_i,\Sigma):i\in I\}\widetilde{\subseteq}\mathcal{T}$. Then $\{(F_i,\Sigma)':i\in I\}\widetilde{\subseteq}\mathcal{C}$ and for every i, we have that $c\big[(F_i,\Sigma)'\big]=(F_i,\Sigma)'$. Set $(R,\Sigma)=\widetilde{\bigcap}_{i\in I}(F_i,\Sigma)'$. Then $(R,\Sigma)\widetilde{\subseteq}(F_i,\Sigma)'$ for every i, and so $c(R,\Sigma)\widetilde{\subseteq}c\big[(F_i,\Sigma)'\big]=(F_i,\Sigma)'$ for every i. Therefore, $c(R,\Sigma)\widetilde{\subseteq}c\big[(F_i,\Sigma)'\big]=\widetilde{\bigcap}_{i\in I}(F_i,\Sigma)'=(R,\Sigma)$, and by (C.2), $(R,\Sigma)\widetilde{\subseteq}c(R,\Sigma)$. Hence, $(R,\Sigma)=c(R,\Sigma)$, which implies that $(R,\Sigma)=\widetilde{\bigcap}_{i\in I}(F_i,\Sigma)'\in\mathcal{C}$. But, $\widetilde{\bigcup}_{i\in I}(F_i,\Sigma)=\big[\widetilde{\bigcap}_{i\in I}(F_i,\Sigma)'\big]'$, thus $\widetilde{\bigcup}_{i\in I}(F_i,\Sigma)\in\mathcal{T}$. This proves that $\mathcal{T}$ is a soft topology on X.

We now show that $c(F,\Sigma)=\operatorname{Cl}(F,\Sigma)$ for every $(F,\Sigma)\in S_\Sigma(X)$. Recalling that the closure of a set with respect to $\mathcal{T}$ is defined as $\operatorname{Cl}(G,\Sigma)=\widetilde{\bigcap}\{(F,\Sigma):(F,\Sigma)\in\mathcal{C},(G,\Sigma)\widetilde{\subseteq}(F,\Sigma)\}$. Since, by (C.2) and (C.3), $(G,\Sigma)\widetilde{\subseteq}c(G,\Sigma)$, $c\big[c(G,\Sigma)\big]=c(G,\Sigma)$, then $c(G,\Sigma)\in\mathcal{C}$ and so $c(G,\Sigma)\in\{(F,\Sigma):(F,\Sigma)\in\mathcal{C},(G,\Sigma)\widetilde{\subseteq}(F,\Sigma)\}$. Hence $\operatorname{Cl}(G,\Sigma)\widetilde{\subseteq}c(G,\Sigma)$. For the converse of the inclusion, since $(G,\Sigma)\widetilde{\subseteq}\operatorname{Cl}(G,\Sigma)$ and $\operatorname{Cl}(G,\Sigma)$ is soft closed, then $c(G,\Sigma)\widetilde{\subseteq}c\big[\operatorname{Cl}(G,\Sigma)\big]=\operatorname{Cl}(G,\Sigma)$. Thus, $c(G,\Sigma)=\operatorname{Cl}(G,\Sigma)$.

The last claim directly follows. □

Notice that the set of all fixed points of c constitutes a soft closed topological system (c.f. Lemma 7).

Definition 8 (Soft Interior Operator)**.** *A mapping* $i : S_\Sigma(X) \to S_\Sigma(X)$ *is said to be a soft interior operator on* X *if it has the next postulates for every* $(F,\Sigma),(G,\Sigma) \in S_\Sigma(X)$*:*

(I.1) $i(\widetilde{X}) = \widetilde{X}$.

(I.2) $i(F,\Sigma)\widetilde{\subseteq}(F,\Sigma)$.

(I.3) $i(i(F,\Sigma)) = i(F,\Sigma)$.

(I.4) $i\big((F,\Sigma)\widetilde{\cap}(G,\Sigma)\big) = i(F,\Sigma)\widetilde{\cap}i(G,\Sigma)$.

Here, we note that if $(F,\Sigma)\widetilde{\subseteq}(G,\Sigma)$, then $(F,\Sigma) = (F,\Sigma)\widetilde{\cap}(G,\Sigma)$. By (I.4), $i(F,\Sigma) = i(F,\Sigma)\widetilde{\cap}i(G,\Sigma)$ implies that $i(F,\Sigma)\widetilde{\subseteq}i(G,\Sigma)$.

The next result demonstrates that a soft interior operator produces a soft topology on X which is the set of all fixed points of i:

Theorem 2. *Let* i *be a soft interior operator on* X*, and let* $\mathcal{T} = \{(F,\Sigma) : (F,\Sigma) \in S_\Sigma(X), i(F,\Sigma) = (F,\Sigma)\}$*. Then* $\mathcal{T}$ *is the unique soft topology on* X *having the property that* $i(F,\Sigma) = \mathrm{Int}(F,\Sigma)$ *for every* $(F,\Sigma) \in S_\Sigma(X)$*, and* $Range(i) = \mathcal{T}$*.*

Proof. (T.1) Indeed, $\widetilde{X} \in \mathcal{T}$ because $i(\widetilde{X}) = \widetilde{X}$, see (I.1). From (I.2), $i(\Phi)\widetilde{\subseteq}\Phi$ and $\Phi\widetilde{\subseteq}i(\Phi)$ always hold, then $i(\Phi) = \Phi$. Thus $\Phi \in \mathcal{T}$.

(T.2) Assume $(F,\Sigma),(G,\Sigma) \in \mathcal{T}$. Then $i(F,\Sigma) = (F,\Sigma)$, $i(G,\Sigma) = (G,\Sigma)$. By (I.3), we have

$$i\big[(F,\Sigma)\widetilde{\bigcap}(G,\Sigma)\big] = i(F,\Sigma)\widetilde{\bigcap}i(G,\Sigma) = (F,\Sigma)\widetilde{\bigcap}(G,\Sigma).$$

Therefore, $(F,\Sigma)\widetilde{\cap}(G,\Sigma) \in \mathcal{T}$.

(T.3) If $\{(F_j,\Sigma) : j \in J\}\widetilde{\subseteq}\mathcal{T}$, then $i(F_j,\Sigma) = (F_j,\Sigma)$ for every j. By (I.2), one can get $i\big[\widetilde{\bigcup}_{j\in J}(F_j,\Sigma)\big]\widetilde{\subseteq}\widetilde{\bigcup}_{j\in J}(F_j,\Sigma)$. It remains to prove that $\widetilde{\bigcup}_{j\in J}(F_j,\Sigma)\widetilde{\subseteq}i\big[\widetilde{\bigcup}_{j\in J}(F_j,\Sigma)\big]$. By Lemma 8 (2), $\widetilde{\bigcup}_{j\in J}i\big[(F_j,\Sigma)\big]\widetilde{\subseteq}i\big[\widetilde{\bigcup}_{j\in J}(F_j,\Sigma)\big]$. But, for every j, $i(F_j,\Sigma) = (F_j,\Sigma)$, then

$$\widetilde{\bigcup}_{j\in J}(F_j,\Sigma) = \widetilde{\bigcup}_{j\in J}i\big[(F_j,\Sigma)\big]\widetilde{\subseteq}i\big[\widetilde{\bigcup}_{j\in J}(F_j,\Sigma)\big].$$

Thus, $i\big[\widetilde{\bigcup}_{j\in J}(F_j,\Sigma)\big] = \widetilde{\bigcup}_{j\in J}(F_j,\Sigma)$ implies $\widetilde{\bigcup}_{j\in J}(F_j,\Sigma) \in \mathcal{T}$. Hence $\mathcal{T}$ is a soft topology.

The soft interior with respect to $(X,\mathcal{T},\Sigma)$ is given by $\mathrm{Int}(G,\Sigma) = \widetilde{\bigcup}\{(F,\Sigma) : (F,\Sigma) \in \mathcal{T}, (F.\Sigma)\widetilde{\subseteq}(G,\Sigma)\}$. By (I.3), $i(i(G,\Sigma)) = i(G,\Sigma)$, and $i(G,\Sigma)\widetilde{\subseteq}(G,\Sigma)$ always, so $i(G,\Sigma) \in \{(F,\Sigma) : (F,\Sigma) \in \mathcal{T}, (F,\Sigma)\widetilde{\subseteq}(G,\Sigma)\}$. Hence, $i(G,\Sigma)\widetilde{\subseteq}\,\mathrm{Int}(G,\Sigma)$. On the other hand, since $\mathrm{Int}(G,\Sigma)\widetilde{\subseteq}\,(G,\Sigma)$ and $\mathrm{Int}(G,\Sigma) \in \mathcal{T}$, then $i\big[\mathrm{Int}(G,\Sigma)\big]\widetilde{\subseteq}i(G,\Sigma)$ and $i\big[\mathrm{Int}(G,\Sigma)\big] = \mathrm{Int}(G,\Sigma)$. Therefore, $\mathrm{Int}(G,\Sigma)\widetilde{\subseteq}i(G,\Sigma)$ and hence $\mathrm{Int}(G,\Sigma) = i(G,\Sigma)$.

The range of i can be concluded from Lemma 2 (5). □

Definition 9 (Soft Boundary Operator, I)**.** *A mapping* $b : S_\Sigma(X) \to S_\Sigma(X)$ *is said to be a soft boundary operator on* X *if it has the next postulates for every* $(F,\Sigma),(G,\Sigma) \in S_\Sigma(X)$*:*

(B.1) $b(\Phi) = \Phi$.

(B.2) $b(F,\Sigma) = b((F,\Sigma)')$.

(B.3) $b(b(F,\Sigma))\widetilde{\subseteq}b(F,\Sigma)$.

(B.4) $(F,\Sigma)\widetilde{\cap}(G,\Sigma)\widetilde{\cap}b\big[(F,\Sigma)\widetilde{\cap}(G,\Sigma)\big] = (F,\Sigma)\widetilde{\cap}(G,\Sigma)\widetilde{\cap}\big[b(F,\Sigma)\widetilde{\cup}b(G,\Sigma)\big]$.

Lemma 9. *Let* $(F,\Sigma) \in S_\Sigma(X)$ *and* $\mathcal{T} \in T_\Sigma(X)$*. If* $b(F,\Sigma)\widetilde{\cap}(F,\Sigma) = \Phi$*, then* $b(G,\Sigma)\widetilde{\cap}(F,\Sigma) = \Phi$ *for every* (G,Σ) *with* $(F,\Sigma)\widetilde{\subseteq}(G,\Sigma)$*.*

Proof. Since $(F,\Sigma)\widetilde{\subseteq}(G,\Sigma)$, then $(F,\Sigma) = (F,\Sigma)\widetilde{\cap}(G,\Sigma)$. Put this in (B.4) implies

$$(F,\Sigma)\widetilde{\bigcap}b(F,\Sigma) = (F,\Sigma)\widetilde{\bigcap}\big[b(F,\Sigma)\widetilde{\bigcup}b(G,\Sigma)\big] = \big[(F,\Sigma)\widetilde{\bigcap}b(F,\Sigma)\big]\widetilde{\bigcup}\big[(F,\Sigma)\widetilde{\bigcap}b(G,\Sigma)\big].$$

By assumption, $(F,\Sigma)\widetilde{\cap} b(F,\Sigma) = \Phi$, hence $(F,\Sigma)\widetilde{\cap} b(G,\Sigma) = \Phi$. □

The following result shows that a boundary operator yields a soft topology on X:

Theorem 3. *Let b be a soft boundary operator on X that satisfies the axioms in Definition 9, and let $\mathcal{T} = \{(F,\Sigma) : (F,\Sigma) \in S_\Sigma(X), b(F,\Sigma)\widetilde{\cap}(F,\Sigma) = \Phi\}$. Then $\mathcal{T}$ is the unique soft topology on X having the property that $b(F,\Sigma) = \mathrm{Bd}(F,\Sigma)$ for every $(F,\Sigma) \in S_\Sigma(X)$, and Range$(b) = \mathcal{T}$.*

Proof. (T.1) By (B.1), $b(\Phi) = \Phi$ implies $\Phi \in \mathcal{T}$. Furthermore, applying (B.1), (B.2), we obtain $b(\widetilde{X})\widetilde{\cap}\widetilde{X} = b(\widetilde{X}')\widetilde{\cap}\widetilde{X} = \Phi\widetilde{\cap}\widetilde{X} = \Phi$. Thus, $\widetilde{X} \in \mathcal{T}$.

(T.2) Suppose $(F,\Sigma), (G,\Sigma) \in \mathcal{T}$. Then $b(F,\Sigma)\widetilde{\cap}(F,\Sigma) = \Phi$ and $b(G,\Sigma)\widetilde{\cap}(G,\Sigma) = \Phi$. Now, consider (B.4),

$$\begin{aligned}
(F,\Sigma)\widetilde{\cap}(G,\Sigma)\widetilde{\cap} b\big[(F,\Sigma)\widetilde{\cap}(G,\Sigma)\big] &= (F,\Sigma)\widetilde{\cap}(G,\Sigma)\widetilde{\cap}\big[b(F,\Sigma)\widetilde{\cup} b(G,\Sigma)\big] \\
&= \big[(F,\Sigma)\widetilde{\cap}(G,\Sigma)\widetilde{\cap} b(F,\Sigma)\big]\widetilde{\cup}\big[(F,\Sigma)\widetilde{\cap}(G,\Sigma)\widetilde{\cap} b(G,\Sigma)\big] \\
&= \big[((F,\Sigma)\widetilde{\cap} b(F,\Sigma)\widetilde{\cap}(G,\Sigma)\big]\widetilde{\cup}\big[(F,\Sigma)\widetilde{\cap}((G,\Sigma)\widetilde{\cap} b(G,\Sigma))\big] \\
&= (\Phi\widetilde{\cap}(G,\Sigma))\widetilde{\cup}((F,\Sigma)\widetilde{\cap}\Phi) \\
&= \Phi.
\end{aligned}$$

(T.3) Assume $\{(F_i,\Sigma) : i \in I\}\widetilde{\subseteq}\mathcal{T}$. We need to prove that $b\big(\widetilde{\bigcup}_{i\in I}(F_i,\Sigma)\big)\widetilde{\bigcap}\big(\widetilde{\bigcup}_{i\in I}(F_i,\Sigma)\big) = \Phi$. Since, for every i, $(F_i,\Sigma) \in \mathcal{T}$, then $b(F_i,\Sigma)\widetilde{\cap}(F_i,\Sigma) = \Phi$. Since $(F_i,\Sigma)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(F_i,\Sigma)$ for every i, so by Lemma 9, $b\big(\widetilde{\bigcup}_{i\in I}(F_i,\Sigma)\big)\widetilde{\bigcap}(F_i,\Sigma) = \Phi$ for every i. Therefore,

$$b(\widetilde{\bigcup}_{i\in I}(F_i,\Sigma))\widetilde{\bigcap}(\widetilde{\bigcup}_{i\in I}(F_i,\Sigma)) = \widetilde{\bigcup}_{i\in I}[(\widetilde{\bigcup}_{i\in I}(F_i,\Sigma))\widetilde{\bigcap}(F_i,\Sigma)] = \Phi.$$

Thus, $\widetilde{\bigcup}_{i\in I}(F_i,\Sigma) \in \mathcal{T}$ and hence $\mathcal{T}$ is a soft topology on X.

Now, we examine that $b(H,\Sigma) = \mathrm{Bd}(H,\Sigma)$ for every $(H,\Sigma) \in S_\Sigma(X)$. We start by showing that for every (H,Σ), $(H,\Sigma)\widetilde{\cup} b(H,\Sigma)$ is a soft closed set including (H,Σ). By Lemma 5 (1), a soft set (F,Σ) is closed if and only if $b\big((F,\Sigma)'\big)\widetilde{\cap}\big((F,\Sigma)'\big) = \Phi$ if and only $b(F,\Sigma)\widetilde{\cap}\big((F,\Sigma)'\big) = \Phi$ (by (B.3)). Therefore, we must check that

$$b\big((H,\Sigma)\widetilde{\cup} b(H,\Sigma)\big)\widetilde{\cap}\big((H,\Sigma)\widetilde{\cup} b(H,\Sigma)\big)' = \Phi.$$

Set $(F,\Sigma) = (H,\Sigma)'$, $(G,\Sigma) = \big(b(H,\Sigma)\big)'$, and substitute them in (B.4), yields

$$(H,\Sigma)'\widetilde{\cap}(b(H,\Sigma))'\widetilde{\cap} b[(H,\Sigma)'\widetilde{\cap}(b(H,\Sigma))'] = (H,\Sigma)'\widetilde{\cap}(b(H,\Sigma))'\widetilde{\cap}[b((H,\Sigma)')\widetilde{\cup} b((b(H,\Sigma))')].$$

Applying (B.2) and after some computations, we get

$$\big((H,\Sigma)\widetilde{\cup} b(H,\Sigma)\big)'\widetilde{\cap} b\big((H,\Sigma)\widetilde{\cup} b(H,\Sigma)\big) = (H,\Sigma)'\widetilde{\cap}\big(b(H,\Sigma)\big)'\widetilde{\cap}\big[b(H,\Sigma)\widetilde{\cup} b(b(H,\Sigma))\big].$$

By (B.3), $b(b(H,\Sigma))\widetilde{\subseteq} b(H,\Sigma)$, so $b(H,\Sigma)\widetilde{\cup} b(b(H,\Sigma)) = b(H,\Sigma)$, and thus

$$\big((H,\Sigma)\widetilde{\cup} b(H,\Sigma)\big)'\widetilde{\cap} b\big((H,\Sigma)\widetilde{\cup} b(H,\Sigma)\big) = \big((H,\Sigma)\widetilde{\cup} b(H,\Sigma)\big)'\widetilde{\cap} b(H,\Sigma) = \Phi.$$

This proves that $(H,\Sigma)\widetilde{\bigcup} b(H,\Sigma)$ is soft closed.

For showing $b(H,\Sigma)\widetilde{\subseteq}\mathrm{Bd}(H,\Sigma)$, we shall recall that $\mathrm{Bd}(H,\Sigma) = \mathrm{Cl}(H,\Sigma)\widetilde{\cap}\mathrm{Cl}((H,\Sigma)')$. Now,

$$\mathrm{Bd}(H,\Sigma) = \mathrm{Cl}(H,\Sigma)\widetilde{\cap}\mathrm{Cl}((H,\Sigma)')\widetilde{\subseteq}((H,\Sigma)\widetilde{\cup} b(H,\Sigma))\widetilde{\cap}((H,\Sigma)'\widetilde{\cup} b((H,\Sigma)')) = b(H,\Sigma).$$

For the other direction, we have to prove that $(H,\Sigma)\widetilde{\cup} b(H,\Sigma)\widetilde{\subseteq}\mathrm{Cl}(H,\Sigma)$ reduces to $b(H,\Sigma)\widetilde{\subseteq}\mathrm{Cl}(H,\Sigma)$. If $b(H,\Sigma)\widetilde{\nsubseteq}\mathrm{Cl}(H,\Sigma)$, then $b(H,\Sigma)\widetilde{\cap}\big(\mathrm{Cl}(H,\Sigma)\big)' \neq \Phi$. Since $\mathrm{Cl}(H,\Sigma)$

is soft closed, then $b(\mathrm{Cl}(H,\Sigma))\widetilde{\cap}\big(\mathrm{Cl}(H,\Sigma)\big)' = \Phi$. We apply (B.4) for $(F,\Sigma) = (H,\Sigma)'$, $(G,\Sigma) = \big(\mathrm{Cl}(H,\Sigma)\big)'$, and get

$$(K,\Sigma)\widetilde{\cap}b(K,\Sigma) = (K,\Sigma)\widetilde{\cap}\big((H,\Sigma)'\big)\widetilde{\cap}b\big((\mathrm{Cl}(H,\Sigma))'\big),$$

where $(K,\Sigma) = (H,\Sigma)'\widetilde{\cap}\big(\mathrm{Cl}(H,\Sigma)\big)'$. This turns into

$$((H,\Sigma)\widetilde{\cup}\,\mathrm{Cl}(H,\Sigma))'\widetilde{\cap}b((H,\Sigma)\widetilde{\cup}\,\mathrm{Cl}(H,\Sigma)) = ((H,\Sigma)\widetilde{\cup}\,\mathrm{Cl}(H,\Sigma))\widetilde{\cap}(b(H,\Sigma)\widetilde{\cup}b((\mathrm{Cl}(H,\Sigma))')).$$

Since $(H,\Sigma)\widetilde{\cup}\,\mathrm{Cl}(H,\Sigma) = \mathrm{Cl}(H,\Sigma)$, therefore

$$\begin{aligned}(H,\Sigma)'\widetilde{\cap}b\big(\mathrm{Cl}(H,\Sigma)\big) &= (H,\Sigma)'\widetilde{\cap}\big(b(H,\Sigma)\widetilde{\cup}b(\mathrm{Cl}(H,\Sigma))\big)\\ &= \big((H,\Sigma)'\widetilde{\cap}b(H,\Sigma)\big)\widetilde{\cup}\big((H,\Sigma)'\widetilde{\cap}b(\mathrm{Cl}(H,\Sigma))\big).\end{aligned}$$

But $(H,\Sigma)'\widetilde{\cap}b\big(\mathrm{Cl}(H,\Sigma)\big) = \Phi$ implies that $(H,\Sigma)'\widetilde{\cap}b(H,\Sigma) = \Phi$, a contradiction. Hence $b(H,\Sigma) = \mathrm{Bd}(H,\Sigma)$ for every $(H,\Sigma) \in S_\Sigma(X)$.

The last claim follows from Lemma 5 (1). □

Definition 10 (Soft Boundary Operator, II)**.** *A mapping $b : S_\Sigma(X) \to S_\Sigma(X)$ is said to be a soft boundary operator on X if it has the next postulates for every $(F,\Sigma),(G,\Sigma) \in S_\Sigma(X)$:*

(B′.1) $b(\Phi) = \Phi$.
(B′.2) $b(F,\Sigma) = b((F,\Sigma)')$.
(B′.3) $b(b(F,\Sigma))\widetilde{\subseteq}b(F,\Sigma)$.
(B′.4) $(F,\Sigma)\widetilde{\subseteq}(G,\Sigma)$ *implies* $b(F,\Sigma)\widetilde{\subseteq}(G,\Sigma)\widetilde{\cup}b(G,\Sigma)$.
(B′.5) $b\big((F,\Sigma)\widetilde{\cup}(G,\Sigma)\big) = b(F,\Sigma)\widetilde{\cup}b(G,\Sigma)$.

The following result shows that the boundary operator defined above provides a soft topology on X:

Theorem 4. *Let b be a soft boundary operator on X that satisfies the axioms in Definition 10, and let $c(F,\Sigma) = (F,\Sigma)\widetilde{\cup}b(F,\Sigma)$. The system $\mathcal{T} = \{(H,\Sigma) : (H,\Sigma) \in S_\Sigma(X), c((H,\Sigma)') = (H,\Sigma)'\}$ is the unique soft topology on X such that $b(F,\Sigma) = \mathrm{Bd}(F,\Sigma)$ for every $(F,\Sigma) \in S_\Sigma(X)$.*

Proof. We begin by illustrating that the operator $c(F,\Sigma)$ fulfills the axioms stated in Definition 7.

(C.1) $c(\Phi) = \Phi\widetilde{\cup}b(\Phi) = \Phi$.
(C.2) $(F,\Sigma)\widetilde{\subseteq}(F,\Sigma)\widetilde{\cup}b(F,\Sigma) = c(F,\Sigma)$.
(C.3)

$$\begin{aligned}c(c(F,\Sigma)) &= c\big[(F,\Sigma)\widetilde{\bigcup}b(F,\Sigma)\big]\\ &= \big[(F,\Sigma)\widetilde{\cup}b(F,\Sigma)\big]\widetilde{\bigcup}b\big[(F,\Sigma)\widetilde{\cup}b(F,\Sigma)\big]\\ &\widetilde{\subseteq} \big[(F,\Sigma)\widetilde{\cup}b(F,\Sigma)\big]\widetilde{\bigcup}\big[b(F,\Sigma)\widetilde{\cup}b(b(F,\Sigma))\big]\\ &= \big[(F,\Sigma)\widetilde{\cup}b(F,\Sigma)\big]\widetilde{\bigcup}b(F,\Sigma) = c(F,\Sigma).\end{aligned}$$

(C.4) The first direction can be computed as:

$$\begin{aligned}c\big[(F,\Sigma)\widetilde{\cup}(G,\Sigma)\big] &= ((F,\Sigma)\widetilde{\cup}(G,\Sigma))\widetilde{\bigcup}b((F,\Sigma)\widetilde{\cup}(G,\Sigma))\\ &= ((F,\Sigma)\widetilde{\cup}(G,\Sigma))\widetilde{\bigcup}(b(F,\Sigma)\widetilde{\cup}b(G,\Sigma))\\ &= ((F,\Sigma)\widetilde{\cup}b(F,\Sigma))\widetilde{\bigcup}((G,\Sigma)\widetilde{\cup}b(G,\Sigma))\\ &= c(F,\Sigma)\widetilde{\cup}c(G,\Sigma).\end{aligned}$$

Following the similar steps established in the proof of Theorem 1, one can show that $\mathcal{T} = \{(H,\Sigma) : (H,\Sigma) \in S_\Sigma(X), c((H,\Sigma)') = (H,\Sigma)'\}$ is the unique soft topology on X. In which for every $(F,\Sigma) \in S_\Sigma(X)$, the soft boundary of (F,Σ) is

$$\begin{aligned} c(F,\Sigma)\widetilde{\cap}(F,\Sigma)' &= [(F,\Sigma)\widetilde{\cup}b(F,\Sigma)]\widetilde{\bigcap}[(F,\Sigma)'\widetilde{\cup}b((F,\Sigma)')] \\ &= [(F,\Sigma)\widetilde{\cup}b(F,\Sigma)]\widetilde{\bigcap}[(F,\Sigma)'\widetilde{\cup}b(F,\Sigma)] \\ &= b(F,\Sigma). \end{aligned}$$

□

Remark 1. *If $\mathcal{J} = \{(H,\Sigma) : (H,\Sigma) \in S_\Sigma(X), b((H,\Sigma)')\widetilde{\subseteq}(H,\Sigma)'\}$, then $\mathcal{J}$ also produces another soft topology on X, but $\mathcal{J} = \mathcal{T}$. Other systems can be provided and the resulting soft topologies are either dual or equivalent.*

Definition 11 (Soft Boundary Operator, III)**.** *A mapping $b : S_\Sigma(X) \to S_\Sigma(X)$ is said to be a soft boundary operator on X if it has the following postulates for every $(F,\Sigma), (G,\Sigma) \in S_\Sigma(X)$:*

(B*.1) $b(\Phi) = \Phi$.

(B*.2) $b(F,\Sigma) = b((F,\Sigma)')$.

(B*.3) $b\big((F,\Sigma)\widetilde{\cup}b(F,\Sigma)\big)\widetilde{\subseteq}b(F,\Sigma)$.

(B*.4) $(F,\Sigma)\widetilde{\subseteq}(G,\Sigma)$ *implies* $b(F,\Sigma)\widetilde{\subseteq}(G,\Sigma)\widetilde{\cup}b(G,\Sigma)$.

(B*.5) $b\big((F,\Sigma)\widetilde{\cup}(G,\Sigma)\big) = b(F,\Sigma)\widetilde{\cup}b(G,\Sigma)$.

We now show that the set of axioms mentioned in Definitions 10 and 11 are equivalent. We need only to work on (B′.3) and (B*.3).

Lemma 10. *(1) If (B′.3) and (B′.5) hold, then (B*.3) holds.*
(2) If (B.2), (B*.3), and (B*.4) hold, then (B′.3) holds.*

Proof. (1) By (B′.5), $b\big((F,\Sigma)\widetilde{\bigcup}b(F,\Sigma)\big)\widetilde{\subseteq}b(F,\Sigma)\widetilde{\bigcup}b(b(F,\Sigma))$. Since $b(b(F,\Sigma))\widetilde{\subseteq}b(F,\Sigma)$, so

$$b\big((F,\Sigma)\widetilde{\bigcup}b(F,\Sigma)\big)\widetilde{\subseteq}b(F,\Sigma)$$

follows, i.e., (B*.3).

(2) Since for every $(F,\Sigma) \in S_\Sigma(X)$, $b(F,\Sigma)\widetilde{\subseteq}(F,\Sigma)\widetilde{\bigcup}b(F,\Sigma)$. By (B*.3) and (B*.4),

$$b(b(F,\Sigma))\widetilde{\subseteq}[(F,\Sigma)\widetilde{\bigcup}b(F,\Sigma)]\widetilde{\bigcup}b[(F,\Sigma)\widetilde{\bigcup}b(F,\Sigma)] = (F,\Sigma)\widetilde{\bigcup}b(F,\Sigma). \tag{1}$$

For the complement $(F,\Sigma)'$ of (F,Σ), we also have $b(b((F,\Sigma)'))\widetilde{\subseteq}(F,\Sigma)'\widetilde{\cup}b((F,\Sigma)')$. Apply (B*.2) and get

$$b(b((F,\Sigma)'))\widetilde{\subseteq}(F,\Sigma)'\widetilde{\bigcup}b((F,\Sigma)). \tag{2}$$

From (1) and (2), we obtain

$$\begin{aligned} b(b((F,\Sigma)')) &\widetilde{\subseteq} (F,\Sigma)\widetilde{\bigcup}b(F,\Sigma)\widetilde{\bigcup}(F,\Sigma)'\widetilde{\bigcup}b((F,\Sigma)) \\ &= [(F,\Sigma)\widetilde{\bigcap}(F,\Sigma)']\widetilde{\bigcup}b(F,\Sigma) \\ &= \Phi\widetilde{\bigcup}b(F,\Sigma) = b(F,\Sigma). \end{aligned}$$

Thus (B′.3) holds. □

Definition 12 (Soft Exterior Operator)**.** *A mapping $e : S_\Sigma(X) \to S_\Sigma(X)$ is said to be a soft exterior operator on X if it has the following postulates for every $(F,\Sigma), (G,\Sigma) \in S_\Sigma(X)$:*

(E.1) $e(\Phi) = \widetilde{X}$.

(E.2) $(F,\Sigma)\widetilde{\cap}e(F,\Sigma) = \Phi$.

(E.3) $e((e(F,\Sigma))') = e(F,\Sigma)$.

(E.4) $e(F,\Sigma)\widetilde{\cap}e(G,\Sigma) = e\big[(F,\Sigma)\widetilde{\cup}(G,\Sigma)\big]$.

Theorem 5. *Let e be a soft exterior operator on X. The system $\mathcal{T} = \{(H,\Sigma) : (H,\Sigma) \in S_\Sigma(X), e((H,\Sigma)') = (H,\Sigma)\}$ is the unique soft topology on X such that $e(F,\Sigma) = \mathrm{Ext}(F,\Sigma)$ for every $(F,\Sigma) \in S_\Sigma(X)$.*

Proof. Theorem 2 will finish the proof whenever we show that e fulfills the interior operator axioms.

(I.1) $i(\widetilde{X}) = e(\widetilde{X}') = e(\Phi) = \widetilde{X}$ (by (E.1)).

(I.2) $i(F,\Sigma) = e((F,\Sigma)')\widetilde{\subseteq}(F,\Sigma)$ (by (E.2)).

(I.3) $i(i(F,\Sigma)) = i\big[e((F,\Sigma)')\big] = e\big(\big[e((F,\Sigma)')\big]'\big) = e((F,\Sigma)') = i(F,\Sigma)$ (by (E.3)).

(I.4) $i\big[(F,\Sigma)\widetilde{\cap}(G,\Sigma)\big] = e\big(\big[(F,\Sigma)\widetilde{\cap}(G,\Sigma)\big]'\big) = e\big[(F,\Sigma)'\widetilde{\cup}(G,\Sigma)'\big] = e((F,\Sigma)')\widetilde{\cap}e\big((G,\Sigma)'\big) = i(F,\Sigma)\widetilde{\cap}i(G,\Sigma)$.

Therefore, $\mathcal{T} = \{(H,\Sigma) : (H,\Sigma) \in S_\Sigma(X), e((H,\Sigma)') = (H,\Sigma)\}$ is the unique soft topology on X such that $e(F,\Sigma) = \mathrm{Ext}(F,\Sigma)$ for every $(F,\Sigma) \in S_\Sigma(X)$. □

Definition 13 (Soft Derived Set Operator)**.** *A mapping $d : S_\Sigma(X) \to S_\Sigma(X)$ is said to be a soft derived set operator on X if it satisfies the following axioms for every $(F,\Sigma), (G,\Sigma) \in S_\Sigma(X)$:*

(D.1) $d(\Phi) = \Phi$.

(D.2) $x_e \in d(F,\Sigma) \Longleftrightarrow x_e \in d\big((F,\Sigma) - \{x_e\}\big)$.

(D.3) $d\big[(F,\Sigma)\widetilde{\cup}d(F,\Sigma)\big]\widetilde{\subseteq}(F,\Sigma)\widetilde{\cup}d(F,\Sigma)$.

(D.4) $d\big[(F,\Sigma)\widetilde{\cup}(G,\Sigma)\big] = d(F,\Sigma)\widetilde{\cup}d(G,\Sigma)$.

Remark 2. *If $(F,\Sigma)\widetilde{\subseteq}(G,\Sigma)$, then, $d(F,\Sigma)\widetilde{\subseteq}(G,\Sigma)\widetilde{\cup}d(G,\Sigma)$ (by (D.4)).*

Lemma 11. *If an operator c^* is defined by $c^*(F,\Sigma) = (F,\Sigma)\widetilde{\cup}d(F,\Sigma)$ for every $(F,\Sigma) \in S_\Sigma(X)$, then $c^* = c$ (c is a soft closure operator, see Definition 7).*

Proof. (C.1) Since $d(\Phi) = \Phi$, so $c^*(\Phi) = \Phi$.

(C.2) Since $c^*(F,\Sigma) = (F,\Sigma)\widetilde{\cup}d(F,\Sigma)$, then $(F,\Sigma)\widetilde{\subseteq}c^*(F,\Sigma)$.

(C.3) $c^*(c^*(F,\Sigma)) = \big[(F,\Sigma)\widetilde{\cup}d(F,\Sigma)\big]\widetilde{\cup}d\big[(F,\Sigma)\widetilde{\cup}d(F,\Sigma)\big] = (F,\Sigma)\widetilde{\cup}d(F,\Sigma) = c^*(F,\Sigma)$ (by (D.3)).

(C.4) $c^*\big[(F,\Sigma)\widetilde{\cup}(G,\Sigma)\big] = \big[(F,\Sigma)\widetilde{\cup}(G,\Sigma)\big]\widetilde{\cup}d\big[(F,\Sigma)\widetilde{\cup}(G,\Sigma)\big]$, by (D.4) and some simplification, we have $c^*\big[(F,\Sigma)\widetilde{\cup}(G,\Sigma)\big] = \big[(F,\Sigma)\widetilde{\cup}d(F,\Sigma)\big]\widetilde{\cup}\big[(G,\Sigma)\widetilde{\cup}d(G,\Sigma)\big] = c^*(F,\Sigma)\widetilde{\cup}c^*(G,\Sigma)$. Thus, $c^* = c$. □

Theorem 6. *Let d be a soft derived set operator on X, and let $\mathcal{C} = \{(C,\Sigma) : (C,\Sigma) \in S_\Sigma(X), d(C,\Sigma)\widetilde{\subseteq}(C,\Sigma)\}$. The system $\mathcal{T} = \{(F,\Sigma) : (F,\Sigma)' \in \mathcal{C}\}$ is the unique soft topology on X having the property that $d(F,\Sigma) = \mathrm{Dr}(F,\Sigma)$ for every $(F,\Sigma) \in S_\Sigma(X)$.*

Proof. (T.1) By (D.1), $\Phi \in \mathcal{C}$, so $\widetilde{X} \in \mathcal{T}$. By definition of d, $d(\widetilde{X})\widetilde{\subseteq}\widetilde{X}$, then $\widetilde{X} \in \mathcal{C}$. Thus, $\Phi \in \mathcal{T}$.

(T.2) Let $(F,\Sigma), (G,\Sigma) \in \mathcal{T}$. Then $(F,\Sigma)', (G,\Sigma)' \in \mathcal{C}$. Therefore, $d((F,\Sigma)')\widetilde{\subseteq}(F,\Sigma)'$ and $d((G,\Sigma)')\widetilde{\subseteq}(G,\Sigma)'$, and so $d((F,\Sigma)')\widetilde{\cup}d((G,\Sigma)')\widetilde{\subseteq}(F,\Sigma)'\widetilde{\cup}(G,\Sigma)'$. Apply (D.4) and obtain $d\big[(F,\Sigma)')\widetilde{\cup}(G,\Sigma)'\big]\widetilde{\subseteq}(F,\Sigma)'\widetilde{\cup}(G,\Sigma)'$. Thus, $(F,\Sigma)')\widetilde{\cup}(G,\Sigma)' \in \mathcal{C}$ implies $(F,\Sigma)\widetilde{\cap}(G,\Sigma) \in \mathcal{T}$.

(T.3) Let $\{(F_i,\Sigma) : i \in I\}\widetilde{\subseteq}\mathcal{T}$. Then, for every i, $(F_i,\Sigma)' \in \mathcal{C}$. Therefore, $d(F_i,\Sigma)')\widetilde{\subseteq}(F_i,\Sigma)'$ for every i. Since $\widetilde{\cap}_{i\in I}(F_i,\Sigma)'\widetilde{\subseteq}(F_i,\Sigma)'$ for any i, by Remark 2, $d\big[\widetilde{\cap}_{i\in I}(F_i,\Sigma)'\big]\widetilde{\subseteq}d((F_i,\Sigma)')$

$\widetilde{\cup}(F_i,\Sigma)' = (F_i,\Sigma)'$. Thus, $d\left[\widetilde{\cap}_{i\in I}(F_i,\Sigma)'\right]\widetilde{\subseteq}\widetilde{\cap}_{i\in I}(F_i,\Sigma)'$. This shows that $\widetilde{\cap}_{i\in I}(F_i,\Sigma)' \in \mathcal{C}$ and consequently, $\widetilde{\cup}_{i\in I}(F_i,\Sigma) \in \mathcal{T}$.

The uniqueness of $\mathcal{T}$ follows from Theorem 1 and Lemma 2.

Now, by Theorem 1, $\mathrm{Cl}(F,\Sigma) = c(F,\Sigma)$. By Lemma 6 (1), $\mathrm{Cl}(F,\Sigma) = (F,\Sigma)\widetilde{\cup}\,\mathrm{Dr}(F,\Sigma)$. Then $(F,\Sigma)\widetilde{\cup}d(F,\Sigma) = (F,\Sigma)\widetilde{\cup}\,\mathrm{Dr}(F,\Sigma)$ for every $(F,\Sigma) \in S_\Sigma(X)$. More precisely, $((F,\Sigma)-\{x_e\})\widetilde{\cup}\,\mathrm{Dr}((F,\Sigma)-\{x_e\}) = ((F,\Sigma)-\{x_e\})\widetilde{\cup}d((F,\Sigma)-\{x_e\})$, for every $x_e \in \widetilde{X}$. This means that $x_e \in \mathrm{Dr}((F,\Sigma)-\{x_e\}) \Longleftrightarrow x_e \in d((F,\Sigma)-\{x_e\})$. By (D.2), $x_e \in d(F,\Sigma) \Longleftrightarrow x_e \in d\big((F,\Sigma)-x_e\big)$. By Lemma 6 (2), $x_e \in \mathrm{Dr}(F,\Sigma) \Longleftrightarrow x_e \in \mathrm{Dr}\big((F,\Sigma)-x_e\big)$. Hence, $x_e \in d(F,\Sigma) \Longleftrightarrow x_e \in \mathrm{Dr}(F,\Sigma)$, and so $d(F,\Sigma) = \mathrm{Dr}(F,\Sigma)$. □

Definition 14 (Soft Coderived Set Operator)**.** *A mapping $cd : S_\Sigma(X) \to S_\Sigma(X)$ is said to be a soft co-derived set operator on X if it satisfies the following axioms for every $(F,\Sigma),(G,\Sigma) \in S_\Sigma(X)$:*

(D.1) $cd(\widetilde{X}) = \widetilde{X}$.

(D.2) $x_e \in cd(F,\Sigma) \Longleftrightarrow x_e \in cd\big((F,\Sigma)\widetilde{\cup}\{x_e\}\big)$.

(D.3) $(F,\Sigma)\widetilde{\cap}cd(F,\Sigma)\widetilde{\subseteq}cd\big[(F,\Sigma)\widetilde{\cap}cd(F,\Sigma)\big]$.

(D.4) $cd\big[(F,\Sigma)\widetilde{\cap}(G,\Sigma)\big] = cd(F,\Sigma)\widetilde{\cap}cd(G,\Sigma)$.

Remark 3. *Similar to the soft derived set operator, the soft co-derived operator can be used to define a unique soft topology on X that meets the axioms stated above. The proposed soft topology is $\mathcal{T}^* = \{(F,\Sigma) : (F,\Sigma) \in S_\Sigma(X), (F,\Sigma)\widetilde{\subseteq}cd(F,\Sigma)\}$, which is the dual of soft topology $\mathcal{T}$ constructed in Theorem 6.*

4. Some Examples

In this section, we show that, by removing an axiom from the operators defined earlier, we may still obtain a soft topology, but we miss some properties.

Example 1. *Let $X = \{1,2,3\}$ and let E be any parameters set. Define a soft operator $\alpha : S_\Sigma(X) \to S_\Sigma(X)$ by:*

$$\alpha(F,\Sigma) = \begin{cases} \Phi & \text{if } (F,\Sigma) = \Phi, \\ (\{1,2\},\Sigma) & \text{if } (F,\Sigma) = (\{1\},\Sigma) \text{ or } (\{2\},\Sigma) \text{ or } (\{1,2\},\Sigma), \\ (\{1,3\},\Sigma) & \text{if } (F,\Sigma) = (\{3\},\Sigma), \\ \widetilde{X} & \text{if } (F,\Sigma) = (\{1,3\},\Sigma) \text{ or } (\{2,3\},\Sigma) \text{ or } \widetilde{X}. \end{cases}$$

We can simply verify that α meets all soft closure operator axioms except (C.3). On the other hand, a soft topology formed by α (in Theorem 1) is $\mathcal{T} = \{\Phi,(\{3\},\Sigma),\widetilde{X}\}$. Therefore, $\mathrm{Cl}(\{3\},\Sigma) = \widetilde{X}$, but $\alpha(\{3\},\Sigma) = (\{1,3\},\Sigma)$.

Example 2. *Let X be a set and let E be a parameters set. Define a soft operator $\beta : S_\Sigma(X) \to S_\Sigma(X)$ by:*

$$\beta(F,\Sigma) = (F,\Sigma) \text{ for every } (F,\Sigma) \in S_\Sigma(X).$$

Then β satisfies all soft derived set operator axioms except (D.2), and β generates the soft discrete topology. Therefore, every soft subset of $\widetilde{X}$ includes all of its soft limits points, but this is not the case for all soft topology. That x_e may not be a soft limit point of $\{x_e\}$.

The next examples explain how these soft operators naturally generate soft topologies.

Example 3. *Let X be a set and E be a parameters set. Define a soft operator $\gamma : S_\Sigma(X) \to S_\Sigma(X)$ by:*

$$\gamma(F,\Sigma) = \begin{cases} \Phi & \text{if } (F,\Sigma) = \Phi, \\ \widetilde{X} & \text{if } (F,\Sigma) \neq \Phi. \end{cases}$$

It is clear that $\gamma(\Phi) = \Phi$ and for any soft set (F,Σ) we have $(F,\Sigma)\widetilde{\subseteq}\gamma(F,\Sigma) = \widetilde{X}$ and $\gamma(\gamma(F,\Sigma)) = \widetilde{X} = \gamma(F,\Sigma)$. Also, $\gamma\big((F,\Sigma)\widetilde{\cup}(G,\Sigma)\big) = \gamma(F,\Sigma)\widetilde{\cup}\gamma(G,\Sigma)$ for any soft sets (F,Σ) and (G,Σ). Thus, γ satisfies the axioms in Definition 7, and it forms the soft indiscrete topology $\mathcal{T} = \{\Phi, \widetilde{X}\}$. On the other hand, γ does not satisfy (I.4) of the soft interior operator given in Definition 8.

Example 4. *Let X be an infinite set and let E be a parameters set. Define a soft operator λ: $S_\Sigma(X) \to S_\Sigma(X)$ by:*

$$\lambda(F,\Sigma) = \begin{cases} (F,\Sigma) & \text{if } (F,\Sigma) \text{ is finite,} \\ \widetilde{X} & \text{if } (F,\Sigma) \text{ is infinite.} \end{cases}$$

One can verify that all axioms in Definition 7. By Theorem 1 and Lemma 1, every finite soft set is soft closed together with $\widetilde{X}$. Accordingly, Φ and all $(F,\Sigma) \in S_\Sigma(X)$ such that $(F,\Sigma)'$ is finite are soft open. Therefore, the obtained $\mathcal{T}$ is the soft co-finite topology on X.

Example 5. *Let X be a set and E be a parameters set. For a fixed $x_e \in \widetilde{X}$, we define a soft operator $\rho : S_\Sigma(X) \to S_\Sigma(X)$ as follows:*

$$\rho(F,\Sigma) = \begin{cases} \Phi & \text{if } x_e \notin (F,\Sigma), \\ (F,\Sigma) & \text{if } x_e \in (F,\Sigma). \end{cases}$$

Then ρ meets all the axioms in Definition 8. Therefore, for a soft point x_e, ρ forms a soft topology $\mathcal{T} = \{(F,\Sigma) \in S_\Sigma(X) : x_e \in (F,\Sigma)$ or $(F,\Sigma) = \Phi\}$ or dually $\mathcal{T}^* = \{(F,\Sigma) \in S_\Sigma(X) : x_e \notin (F,\Sigma)$ or $(F,\Sigma) = \widetilde{X}\}$. The soft topologies $\mathcal{T}, \mathcal{T}^*$ are, respectively, called included point soft topology and excluded point soft topology on X.

5. Conclusions

In this paper, we have considered a closure operator as an extension of the set of axioms that postulates a topological system. Five other axiomatized set operators have been introduced: soft interior operator, soft boundary operators, soft exterior operator, soft derived set operator, and soft co-derived set operator. Three different versions of soft boundary operators have been defined under which all generated soft topologies are equivalent. The interactions between these six soft operators are comparable to their relationships in a soft topological space, so topological descriptions, which are based on the linked connections between these soft operators and their corresponding fixed points, can be enlarged to the general setting of an arbitrary closure system. As a result, our axiomatized soft set operators in closure systems throw fresh light on the interaction between soft topology and fixed point theory. Multiple examples have been proposed that show the implementations of these operators. Some of the examples show that by weakening a set of axioms of any soft operator, the uniqueness of the resulting soft topologies will be dropped.

We remarked that all the obtained results are valid when implementing the concept of soft elements except for the last two soft operators. The reason is $\mathrm{Cl}(F,\Sigma) \neq (F,\Sigma)\widetilde{\cup}\,\mathrm{Dr}(F,\Sigma)$ for soft points, (see, Example 3.21 in [24]). Even more, if we assume $\mathrm{Cl}(F,\Sigma)$ to be the set of all soft point x for which every soft open set including x intersects (F,Σ), we may not get a unique soft topology (see, Example 3.14 in [24]).

To complete this line of research, we plan to characterize new kinds of soft separation axioms using some soft topological operators. In addition, we will reveal the relationships between these soft operators and their counterparts via classical topologies.

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