symmetry
MDPI

Article

A Comparative Analysis of Fractional-Order Kaup–Kupershmidt Equation within Different Operators

Nehad Ali Shah [1], Yasser S. Hamed [2], Khadijah M. Abualnaja [2], Jae-Dong Chung [1,*], Rasool Shah [3] and Adnan Khan [3]

1 Department of Mechanical Engineering, Sejong University, Seoul 05006, Korea; nehadali199@yahoo.com
2 Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; yasersalah@tu.edu.sa (Y.S.H.); kh.abualnaja@tu.edu.sa (K.M.A.)
3 Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Pakistan; rasoolshah@awkum.edu.pk (R.S.); adnanmummand@gmail.com (A.K.)
* Correspondence: jdchung@sejong.ac.kr

Abstract: In this paper, we find the solution of the fractional-order Kaup–Kupershmidt (KK) equation by implementing the natural decomposition method with the aid of two different fractional derivatives, namely the Atangana–Baleanu derivative in Caputo manner (ABC) and Caputo–Fabrizio (CF). When investigating capillary gravity waves and nonlinear dispersive waves, the KK equation is extremely important. To demonstrate the accuracy and efficiency of the proposed technique, we study the nonlinear fractional KK equation in three distinct cases. The results are given in the form of a series, which converges quickly. The numerical simulations are presented through tables to illustrate the validity of the suggested technique. Numerical simulations in terms of absolute error are performed to ensure that the proposed methodologies are trustworthy and accurate. The resulting solutions are graphically shown to ensure the applicability and validity of the algorithms under consideration. The results that we obtain confirm that the proposed method is the best tool for handling any nonlinear problems arising in science and technology.

Keywords: Caputo–Fabrizio and Atangana-Baleanu operators; time-fractional Kaup–Kupershmidt equation; natural transform; Adomian decomposition method

Citation: Shah, N.A.; Hamed, Y.S.; Abualnaja, K.M.; Chung, J.-D.; Shah, R.; Khan, A. A Comparative Analysis of Fractional-Order Kaup–Kupershmidt Equation within Different Operators. *Symmetry* **2022**, *14*, 986. https://doi.org/10.3390/sym14050986

Academic Editors: Sergei D. Odintsov and Cemil Tunç

Received: 11 March 2022
Accepted: 9 May 2022
Published: 11 May 2022

1. Introduction

Fractional calculus has grown in popularity over the last three decades, owing to its well-established applications in a wide range of scientific and engineering areas. Many pioneers have demonstrated that fractional-order models can effectively describe complicated phenomena when modified by integer-order models [1,2]. The integer-order derivatives are local in nature, whereas the Caputo fractional derivatives are nonlocal. That is, we can investigate changes in the neighbourhood of a point with the integer-order derivative, but we can analyse changes in the entire interval with the Caputo fractional derivative. Senior mathematicians worked together to establish the basic framework for fractional-order derivatives and integrals, such as Caputo [3], Riemann [4], Liouville [5], Podlubny [6], Miller and Ross [7] and others. Fractional-order calculus theory has been linked to practical projects and it has been applied to signal processing [8], chaos theory [9], human diseases [10,11], electrodynamics [12] and other areas.

Fractional differential equations are becoming more well known nowadays as a result of their numerous applications in science and engineering, such as electrodynamics [13], chaos theory [14], finance [15], fluid and continuum mechanics [16], signal processing [17], biological population models [18] and some others, which are well described by fractional differential equations. The elegance of symmetry analysis is most evident in the study of partial differential equations—more precisely, those derived from finance mathematics. The secret of nature is symmetry, but most observations in nature do not exhibit symmetry.

The phenomenon of spontaneous symmetry breaking is an effective approach to conceal symmetry. Symmetries are classified into two types: finite and infinitesimal. Discrete or continuous symmetries can exist for finite symmetries. Symmetry and time reverse are discrete natural symmetries, whereas space is a continuous transformation. Patterns have captivated mathematicians for centuries. In the nineteenth century, systematic classifications of planar and spatial patterns emerged. Regrettably, solving nonlinear fractional differential equations accurately has proven to be rather challenging [19]. Effective tools are required to solve such problems. As a result, in this article, we will try to use an effective analytic method to obtain a more accurate solution for nonlinear arbitrary-order differential equations. Fractional differential equations can pleasantly and even more precisely analyse a variety of schemes in collaborative areas. In this connection, different techniques have been developed, among which some are as follows: the reduced differential transform method (RDTM) [20], the fractional Adomian decomposition method (FADM) [21], the fractional variational iteration method (FVIM) [22], the Elzaki transform decomposition method (ETDM) [23,24], the iterative Laplace transform method (ILTM) [25], the fractional natural decomposition method (FNDM) [26], the fractional homotopy perturbation method (FHPM) [27] and the Yang transform decomposition method (YTDM) [28]. The main goal of the present paper is to implement the natural decomposition method with the help of two different fractional derivatives to study the fractional-order Kaup–Kupershmidt (KK) equation. Natural decomposition methods avoid round-off errors by not requiring prescriptive assumptions, linearization, discretization or perturbation.

Kaup presented the famous dispersive classical Kaup–Kupershmidt equation [29] in 1980, and Kupershmidt modified it in 1994 [30]. The purpose of this paper is to look at the time-fractional modified Kaup–Kupershmidt (KK) equation. The study of nonlinear dispersive waves and the behaviour of capillary gravity waves is examined using the fractional-order Kaup-Kupershmidt equation. The nonlinear fifth-order evolution equation is of the form:

$$D_\kappa^\gamma \zeta(\varphi,\kappa) + j\zeta\zeta_{\varphi\varphi\varphi} + kp\zeta_\varphi\zeta_{\varphi\varphi} + l\zeta^2\zeta_\varphi + \zeta_{\varphi\varphi\varphi\varphi\varphi} = 0, \tag{1}$$

where j, k and l are constants, and $0 < \gamma \leq 1$ represents the order time-fractional derivative. The above fifth-order nonlinear evolution equation can be transformed into the fifth-order time-fractional Kaup–Kupershmidt equation by changing the values of j, k and l. Thus, by taking $j = -15$, $k = -15$ and $l = 45$, the given equation reduces to

$$D_\kappa^\gamma \zeta(\varphi,\kappa) - 15\zeta\zeta_{\varphi\varphi\varphi} - 15p\zeta_\varphi\zeta_{\varphi\varphi} + 45\zeta^2\zeta_\varphi + \zeta_{\varphi\varphi\varphi\varphi\varphi} = 0, \tag{2}$$

Extensive research has been dedicated in recent years to the investigation of the classical Kaup–Kupershmidt equation. At $p = \frac{5}{2}$, the classical KK equation is integrable [31] and has bilinear representations [32]. For general nonlinear evolution equations, solitary and soliton wave solutions can be obtained by independently applying four different approaches. Ablowitz and Clarkson used the inverse scattering approach in the creation of soliton solutions to investigate nonlinear equations having physical implications [33]. Tam and Hu employed Hirota's approach and used Mathematica to determine the equivalent answer [34]. Musette and Verhoeven reported the fifth-order Kaup–Kupershmidt equation, which was one of the integrable examples of the Henon–Heiles system.

The rest of the paper is organized as follows: in Section 2, some of the suitable definitions related to fractional derivatives and used in our present work are given. For the fractional-order Kaup–Kupershmidt equation, the basic idea of the natural decomposition method with the aid of two different fractional derivatives is presented in Section 3. The convergence phenomenon for the proposed method is presented in Section 4. Section 5 is concerned with the implementation of the suggested technique for the solution of various problems of the fractional-order Kaup–Kupershmidt equation. At the end, a brief conclusion of the whole paper is given.

2. Basic Preliminaries

In this part of the article, we present some basic definitions related to fractional calculus that are further used in our work too.

Definition 1. *For a function $j \in C_v, v \geq -1$, the Riemann–Liouville integral for non-integer order is given as [35]*

$$\begin{aligned} I^{\gamma}j(\vartheta) =& \frac{1}{\Gamma(\gamma)}\int_0^{\vartheta}(\vartheta-\mu)^{\gamma-1}j(\mu)d\mu, \ \ \gamma>0, \ \ \vartheta>0. \\ \text{and} \ \ I^0 j(\vartheta) =& j(\vartheta) \end{aligned} \tag{3}$$

Definition 2. *For a function $j(\vartheta)$, the fractional Caputo derivative is defined as [35]*

$$^{C}D^{\gamma}_{\vartheta}j(\vartheta) = I^{n-\gamma}D^n j(\vartheta) = \frac{1}{n-\gamma}\int_{\vartheta}^{0}(\vartheta-\mu)^{n-\gamma-1}j^n(\mu)d\mu \tag{4}$$

for $n-1<\gamma\leq n, \ \ n \in N, \ \ \vartheta>0, j \in C^n_v, v \geq -1$.

Definition 3. *For a function $j(\vartheta)$, the fractional Caputo–Fabrizio derivative is given as [35]*

$$^{CF}D^{\gamma}_{\vartheta}j(\vartheta) = \frac{F(\gamma)}{1-\gamma}\int_0^{\vartheta}\exp\left(\frac{-\gamma(\vartheta-\mu)}{1-\gamma}\right)D(j(\mu))d\mu, \tag{5}$$

where $0<\gamma<1$ and the normalization function is represented by $F(\gamma)$ with $F(0)=F(1)=1$.

Definition 4. *For a function $j(\vartheta)$, the fractional Atangana–Baleanu Caputo derivative is defined as [35]*

$$^{ABC}D^{\gamma}_{\vartheta}j(\vartheta) = \frac{B(\gamma)}{1-\gamma}\int_0^{\vartheta}E_{\gamma}\left(\frac{-\gamma(\vartheta-\mu)}{1-\gamma}\right)D(j(\mu))d\mu, \tag{6}$$

where $0<\gamma<1$, $B(\gamma)$ represents the normalization function with a similar property as $F(\gamma)$ and $E_{\gamma}(z) = \sum_{m=0}^{\infty}\frac{z^m}{\Gamma(m\gamma+1)}$ represents the Mittag–Leffler function.

Definition 5. *By applying the natural transform, the function $\zeta(\kappa)$ can be rewritten as*

$$\mathcal{N}(\zeta(\kappa)) = \mathcal{V}(\varpi, \upsilon) = \int_{-\infty}^{\infty}e^{-\varpi\kappa}\zeta(\upsilon\kappa)d\kappa, \ \ \varpi, \upsilon \in (-\infty, \infty). \tag{7}$$

Natural transformation of $\zeta(\kappa)$ for $\kappa \in (0,\infty)$ is given as

$$\mathcal{N}(\zeta(\kappa)H(\kappa)) = \mathcal{N}^{+}\zeta(\kappa) = \mathcal{V}^{+}(\varpi, \upsilon) = \int_{-\infty}^{\infty}e^{-\varpi\kappa}\zeta(\upsilon\kappa)d\kappa, \ \ \varpi, \upsilon \in (0, \infty). \tag{8}$$

where $H(\kappa)$ is the Heaviside function.

Definition 6. *On applying the natural inverse transform, the function $\mathcal{V}(\varpi, \upsilon)$ can be written as*

$$\mathcal{N}^{-1}[\mathcal{V}(\varpi, \upsilon)] = \zeta(\kappa), \ \ \forall\kappa \geq 0 \tag{9}$$

Lemma 1. *If the linearity property having natural transformation for $\zeta_1(\kappa)$ is $\zeta_1(\varpi, \upsilon)$ and $\zeta_2(\kappa)$ is $\zeta_2(\varpi, \upsilon)$, then*

$$\mathcal{N}[c_1\zeta_1(\kappa)+c_2\zeta_2(\kappa)] = c_1\mathcal{N}[\zeta_1(\kappa)]+c_2\mathcal{N}[\zeta_2(\kappa)] = c_1\mathcal{V}_1(\varpi, \upsilon)+c_2\mathcal{V}_2(\varpi, \upsilon), \tag{10}$$

where c_1 and c_2 are constants.

Lemma 2. *If the inverse natural transforms of $\mathcal{V}_1(\varpi, \upsilon)$ and $\mathcal{V}_2(\varpi, \upsilon)$ are $\zeta_1(\kappa)$ and $\zeta_2(\kappa)$, respectively, then*

$$\mathcal{N}^{-1}[c_1\mathcal{V}_1(\varpi, \upsilon) + c_2\mathcal{V}_2(\varpi, \upsilon)] = c_1\mathcal{N}^{-1}[\mathcal{V}_1(\varpi, \upsilon)] + c_2\mathcal{N}^{-1}[\mathcal{V}_2(\varpi, \upsilon)] = c_1\zeta_1(\kappa) + c_2\zeta_2(\kappa), \quad (11)$$

where c_1 and c_2 are constants.

Definition 7. *The natural transformation of $D_\kappa^\gamma\zeta(\kappa)$ in the Caputo sense is defined as [35]*

$$\mathcal{N}[{}^{C}D_\kappa^\gamma] = \left(\frac{\varpi}{\upsilon}\right)^\gamma \left(\mathcal{N}[\zeta(\kappa)] - \left(\frac{1}{\varpi}\right)\zeta(0)\right) \quad (12)$$

Definition 8. *The natural transformation of $D_\kappa^\gamma\zeta(\kappa)$ in the Caputo–Fabrizio sense is defined as [35]*

$$\mathcal{N}[{}^{CF}D_\kappa^\gamma] = \frac{1}{1-\gamma+\gamma(\frac{\upsilon}{\varpi})} \left(\mathcal{N}[\zeta(\kappa)] - \left(\frac{1}{\varpi}\right)\zeta(0)\right) \quad (13)$$

Definition 9. *The natural transformation of $D_\kappa^\gamma\zeta(\kappa)$ in the Atangana–Baleanu Caputo sense is defined as [35]*

$$\mathcal{N}[{}^{ABC}D_\kappa^\gamma] = \frac{B(\gamma)}{1-\gamma+\gamma(\frac{\upsilon}{\varpi})^\gamma} \left(\mathcal{N}[\zeta(\kappa)] - \left(\frac{1}{\varpi}\right)\zeta(0)\right) \quad (14)$$

3. Methodology

In this section, we give the general implementation of the natural transform decomposition method with the aid of two different derivatives for solving the given equation [36,37].

$$D_\kappa^\gamma\zeta(\varphi, \kappa) = \mathcal{L}(\zeta(\varphi, \kappa)) + \mathbb{N}(\zeta(\varphi, \kappa)) + h(\varphi, \kappa), \quad (15)$$

with initial condition

$$\zeta(\varphi, 0) = \phi(\varphi), \quad (16)$$

having $\mathcal{L}$ linear term, $\mathbb{N}$ nonlinear term and the source term $h(\varphi, \kappa)$.

3.1. Case I ($NTDM_{CF}$)

By applying the natural transform with the aid of the fractional Caputo–Fabrizio derivative, Equation (1) can be rewritten as

$$\frac{1}{p(\gamma, \upsilon, \varpi)}\left(\mathcal{N}[\zeta(\varphi, \kappa)] - \frac{\phi(\varphi)}{\varpi}\right) = \mathcal{N}\left[\mathcal{L}(\zeta(\varphi, \kappa)) + \mathbb{N}(\zeta(\varphi, \kappa)) + h(\varphi, \kappa)\right], \quad (17)$$

with

$$p(\gamma, \upsilon, \varpi) = 1 - \gamma + \gamma(\frac{\upsilon}{\varpi}). \quad (18)$$

On applying natural inverse transformation, Equation (3) can be presented as

$$\zeta(\varphi, \kappa) = \mathcal{N}^{-1}\left[\frac{\phi(\varphi)}{\varpi} + p(\gamma, \upsilon, \varpi)\mathcal{N}[h(\varphi, \kappa)]\right] + \mathcal{N}^{-1}\left[p(\gamma, \upsilon, \varpi)\mathcal{N}\left(\mathcal{L}(\zeta(\varphi, \kappa)) + \mathbb{N}(\zeta(\varphi, \kappa))\right)\right]. \quad (19)$$

$\mathbb{N}(\zeta(\varphi,\kappa))$ can be decomposed into

$$\mathbb{N}(\zeta(\varphi,\kappa)) = \sum_{i=0}^{\infty} A_i, \tag{20}$$

The series form solution for $\zeta^{CF}(\varphi,\kappa)$ is given as

$$\zeta^{CF}(\varphi,\kappa) = \sum_{i=0}^{\infty} \zeta_i^{CF}(\varphi,\kappa). \tag{21}$$

Substituting Equations (6) and (7) into (5), we get

$$\begin{aligned} \sum_{i=0}^{\infty} \zeta_i(\varphi,\kappa) =& \mathcal{N}^{-1}\left(\frac{\phi(\varphi)}{\varpi} + p(\gamma,v,\varpi)\mathcal{N}[h(\varphi,\kappa)]\right) \\ &+ \mathcal{N}^{-1}\left(p(\gamma,v,\varpi)\mathcal{N}\left[\sum_{i=0}^{\infty}\mathcal{L}(\zeta_i(\varphi,\kappa)) + A_\kappa\right]\right) \end{aligned} \tag{22}$$

From (8), we have

$$\begin{aligned} \zeta_0^{CF}(\varphi,\kappa) =& \mathcal{N}^{-1}\left(\frac{\phi(\varphi)}{\varpi} + p(\gamma,v,\varpi)\mathcal{N}[h(\varphi,\kappa)]\right), \\ \zeta_1^{CF}(\varphi,\kappa) =& \mathcal{N}^{-1}(p(\gamma,v,\varpi)\mathcal{N}[\mathcal{L}(\zeta_0(\varphi,\kappa)) + A_0]), \\ &\vdots \\ \zeta_{l+1}^{CF}(\varphi,\kappa) =& \mathcal{N}^{-1}(p(\gamma,v,\varpi)\mathcal{N}[\mathcal{L}(\zeta_l(\varphi,\kappa)) + A_l]), \;\; l = 1,2,3,\cdots \end{aligned} \tag{23}$$

Finally, we obtain the $NTDM_{CF}$ solution to (1) by putting (23) into (7),

$$\zeta^{CF}(\varphi,\kappa) = \zeta_0^{CF}(\varphi,\kappa) + \zeta_1^{CF}(\varphi,\kappa) + \zeta_2^{CF}(\varphi,\kappa) + \cdots \tag{24}$$

3.2. Case II $(NTDM_{ABC})$

By applying the natural transform with the aid of the fractional Atangana–Baleanu Caputo derivative, Equation (1) can be rewritten as

$$\frac{1}{q(\gamma,v,\varpi)}\left(\mathcal{N}[\zeta(\varphi,\kappa)] - \frac{\phi(\varphi)}{\varpi}\right) = \mathcal{N}\left[\mathcal{L}(\zeta(\varphi,\kappa)) + \mathbb{N}(\zeta(\varphi,\kappa)) + h(\varphi,\kappa)\right], \tag{25}$$

with

$$q(\gamma,v,\varpi) = \frac{1-\gamma+\gamma(\frac{v}{\varpi})^\gamma}{B(\gamma)}. \tag{26}$$

On applying the natural inverse transform, Equation (25) can be presented as

$$\zeta(\varphi,\kappa) = \mathcal{N}^{-1}\left(\frac{\phi(\varphi)}{\varpi} + q(\gamma,v,\varpi)\mathcal{N}[h(\varphi,\kappa)]\right) + \mathcal{N}^{-1}\left[q(\gamma,v,\varpi)\mathcal{N}\left(\mathcal{L}(\zeta(\varphi,\kappa)) + \mathbb{N}(\zeta(\varphi,\kappa))\right)\right]. \tag{27}$$

$\mathbb{N}(\zeta(\varphi,\kappa))$ can be decomposed into

$$\mathbb{N}(\zeta(\varphi,\kappa)) = \sum_{i=0}^{\infty} A_i, \tag{28}$$

The series form solution for $\zeta^{ABC}(\varphi,\kappa)$ is given as

$$\zeta^{ABC}(\varphi,\kappa) = \sum_{i=0}^{\infty} \zeta_i^{ABC}(\varphi,\kappa). \tag{29}$$

Substituting Equations (28) and (29) into (27), we get

$$\begin{aligned}\sum_{i=0}^{\infty} \zeta_i(\varphi,\kappa) =& \mathcal{N}^{-1}\left(\frac{\phi(\varphi)}{\varpi} + q(\gamma,v,\varpi)\mathcal{N}[h(\varphi,\kappa)]\right) \\ &+ \mathcal{N}^{-1}\left(q(\gamma,v,\varpi)\mathcal{N}\left[\sum_{i=0}^{\infty} \mathcal{L}(\zeta_i(\varphi,\kappa)) + A_\kappa\right]\right)\end{aligned} \tag{30}$$

From (8), we have

$$\begin{aligned}\zeta_0^{ABC}(\varphi,\kappa) =& \mathcal{N}^{-1}\left(\frac{\phi(\varphi)}{\varpi} + q(\gamma,v,\varpi)\mathcal{N}[h(\varphi,\kappa)]\right), \\ \zeta_1^{ABC}(\varphi,\kappa) =& \mathcal{N}^{-1}(q(\gamma,v,\varpi)\mathcal{N}[\mathcal{L}(\zeta_0(\varphi,\kappa)) + A_0]), \\ &\vdots \\ \zeta_{l+1}^{ABC}(\varphi,\kappa) =& \mathcal{N}^{-1}(q(\gamma,v,\varpi)\mathcal{N}[\mathcal{L}(\zeta_l(\varphi,\kappa)) + A_l]), \;\; l = 1,2,3,\cdots\end{aligned} \tag{31}$$

Finally, we obtain the $NTDM_{ABC}$ solution to (1) by putting (31) into (29):

$$\zeta^{ABC}(\varphi,\kappa) = \zeta_0^{ABC}(\varphi,\kappa) + \zeta_1^{ABC}(\varphi,\kappa) + \zeta_2^{ABC}(\varphi,\kappa) + \cdots \tag{32}$$

4. Convergence Analysis

The convergence and uniqueness analysis of the $NTDM_{CF}$ and $NTDM_{ABC}$ is discussed here.

Theorem 1. *The result of (1) is unique for* $NTDM_{CF}$ *when* $0 < (\Im_1 + \Im_2)(1-\gamma+\gamma\kappa) < 1$.

Proof. Let $H = (C[J], ||.||)$ with the norm $||\phi(\kappa)|| = max_{\kappa\in J}|\phi(\kappa)|$ as Banach space, with $\forall$ continuous function on J. Let $I : H \to H$ be a nonlinear mapping, where

$$\zeta_{l+1}^C = \zeta_0^C + \mathcal{N}^{-1}[p(\gamma,v,\varpi)\mathcal{N}[\mathcal{L}(\zeta_l(\mu,\kappa)) + \mathbb{N}(\zeta_l(\mu,\kappa))]], \;\; l \geq 0.$$

Suppose that $|\mathcal{L}(\zeta) - \mathcal{L}(\zeta^*)| < \Im_1|\zeta - \zeta^*|$ and $|\mathbb{N}(\zeta) - \mathbb{N}(\zeta^*)| < \Im_2|\zeta - \zeta^*|$, where $\zeta := \zeta(\mu,\kappa)$ and $\zeta^* := \zeta^*(\mu,\kappa)$ are two different function values and $\Im_1,\Im_2$ are Lipschitz constants.

$$\begin{aligned}||I\zeta - I\zeta^*|| \leq& max_{t\in J}|\mathcal{N}^{-1}\Big[p(\gamma,v,\varpi)\mathcal{N}[\mathcal{L}(\zeta) - \mathcal{L}(\zeta^*)] \\ &+ p(\gamma,v,\varpi)\mathcal{N}[\mathbb{N}(\zeta) - \mathbb{N}(\zeta^*)]|\Big] \\ \leq& max_{\kappa\in J}\Big[\Im_1\mathcal{N}^{-1}[p(\gamma,v,\varpi)\mathcal{N}[|\zeta - \zeta^*|]] \\ &+ \Im_2\mathcal{N}^{-1}[p(\gamma,v,\varpi)\mathcal{N}[|\zeta - \zeta^*|]]\Big] \\ \leq& max_{t\in J}(\Im_1 + \Im_2)\Big[\mathcal{N}^{-1}[p(\gamma,v,\varpi)\mathcal{N}|\zeta - \zeta^*|]\Big] \\ \leq& (\Im_1 + \Im_2)\Big[\mathcal{N}^{-1}[p(\gamma,v,\varpi)\mathcal{N}||\zeta - \zeta^*||]\Big] \\ =& (\Im_1 + \Im_2)(1-\gamma+\gamma\kappa)||\zeta - \zeta^*||.\end{aligned} \tag{33}$$

I is a contraction as $0 < (\Im_1 + \Im_2)(1-\gamma+\gamma\kappa) < 1$. From Banach fixed point theorem, the result of (1) is unique. □

Theorem 2. *The result of (1) is unique for* $NTDM_{ABC}$ *when* $0 < (\Im_1 + \Im_2)(1 - \gamma + \gamma\frac{\kappa^\nu}{\Gamma(\nu+1)}) < 1$.

Proof. Let $H = (C[J], ||.||)$ with the norm $||\phi(\kappa)|| = max_{\kappa\in J}|\phi(\kappa)|$ be the Banach space, with $\forall$ continuous function on J. Let $I : H \to H$ be a nonlinear mapping, where

$$\zeta_{l+1}^C = \zeta_0^C + \mathcal{N}^{-1}[p(\gamma, v, \varpi)\mathcal{N}[\mathcal{L}(\zeta_l(\varphi,\kappa)) + \mathbb{N}(\zeta_l(\varphi,\kappa))]], \ \ l \geq 0.$$

Suppose that $|\mathcal{L}(\zeta) - \mathcal{L}(\zeta^*)| < \Im_1|\zeta - \zeta^*|$ and $|\mathbb{N}(\zeta) - \mathbb{N}(\zeta^*)| < \Im_2|\zeta - \zeta^*|$, where $\zeta := \zeta(\mu,\kappa)$ and $\zeta^* := \zeta^*(\mu,\kappa)$ are two different function values and $\Im_1$, $\Im_2$ are Lipschitz constants.

$$\begin{aligned} ||I\zeta - I\zeta^*|| &\leq max_{t\in J}|\mathcal{N}^{-1}\Big[q(\gamma, v, \varpi)\mathcal{N}[\mathcal{L}(\zeta) - \mathcal{L}(\zeta^*)] \\ &+ q(\gamma, v, \varpi)\mathcal{N}[\mathbb{N}(\zeta) - \mathbb{N}(\zeta^*)]|\Big] \\ &\leq max_{t\in J}\Big[\Im_1\mathcal{N}^{-1}[q(\gamma, v, \varpi)\mathcal{N}[|\zeta - \zeta^*|]] \\ &+ \Im_2\mathcal{N}^{-1}[q(\gamma, v, \varpi)\mathcal{N}[|\zeta - \zeta^*|]]\Big] \\ &\leq max_{t\in J}(\Im_1 + \Im_2)\Big[\mathcal{N}^{-1}[q(\gamma, v, \varpi)\mathcal{N}|\zeta - \zeta^*|]\Big] \\ &\leq (\Im_1 + \Im_2)\Big[\mathcal{N}^{-1}[q(\gamma, v, \varpi)\mathcal{N}||\zeta - \zeta^*||]\Big] \\ &= (\Im_1 + \Im_2)(1 - \gamma + \gamma\frac{\kappa^\gamma}{\Gamma\gamma + 1})||\zeta - \zeta^*||. \end{aligned} \tag{34}$$

I is a contraction as $0 < (\Im_1 + \Im_2)(1 - \gamma + \gamma\frac{\kappa^\gamma}{\Gamma\gamma+1}) < 1$. From Banach fixed point theorem, the result of (1) is unique. □

Theorem 3. *The* $NTDM_{CF}$ *result of (1) is convergent.*

Proof. Let $\zeta_m = \sum_{r=0}^m \zeta_r(\varphi,\kappa)$. To show that ζ_m is a Cauchy sequence in H, let

$$\begin{aligned} ||\zeta_m - \zeta_n|| &= max_{\kappa\in J}|\sum_{r=n+1}^m \zeta_r|, \ \ n = 1,2,3,\cdots \\ &\leq max_{\kappa\in J}\left|\mathcal{N}^{-1}\left[p(\gamma, v, \varpi)\mathcal{N}\left[\sum_{r=n+1}^m (\mathcal{L}(\zeta_{r-1}) + \mathbb{N}(\zeta_{r-1}))\right]\right]\right| \\ &= max_{\kappa\in J}\left|\mathcal{N}^{-1}\left[p(\gamma, v, \varpi)\mathcal{N}\left[\sum_{r=n+1}^{m-1} (\mathcal{L}(\zeta_r) + \mathbb{N}(\zeta_r))\right]\right]\right| \\ &\leq max_{\kappa\in J}|\mathcal{N}^{-1}[p(\gamma, v, \varpi)\mathcal{N}[(\mathcal{L}(\zeta_{m-1}) - \mathcal{L}(\zeta_{n-1}) + \mathbb{N}(\zeta_{m-1}) - \mathbb{N}(\zeta_{n-1}))]]| \\ &\leq \Im_1 max_{\kappa\in J}|\mathcal{N}^{-1}[p(\gamma, v, \varpi)\mathcal{N}[(\mathcal{L}(\zeta_{m-1}) - \mathcal{L}(\zeta_{n-1}))]]| \\ &+ \Im_2 max_{\kappa\in J}|\mathcal{N}^{-1}[p(\gamma, v, \varpi)\mathcal{N}[(\mathbb{N}(\zeta_{m-1}) - \mathbb{N}(\zeta_{n-1}))]]| \\ &= (\Im_1 + \Im_2)(1 - \gamma + \gamma\kappa)||\zeta_{m-1} - \zeta_{n-1}|| \end{aligned} \tag{35}$$

Let $m = n + 1$, then

$$||\zeta_{n+1} - \zeta_n|| \leq \Im||\zeta_n - \zeta_{n-1}|| \leq \Im^2||\zeta_{n-1}\zeta_{n-2}|| \leq \cdots \leq \Im^n||\zeta_1 - \zeta_0||, \tag{36}$$

where $\Im = (\Im_1 + \Im_2)(1 - \gamma + \gamma\kappa)$. Similarly, we have

$$\begin{aligned} ||\zeta_m - \zeta_n|| &\leq ||\zeta_{n+1} - \zeta_n|| + ||\zeta_{n+2}\zeta_{n+1}|| + \cdots + ||\zeta_m - \zeta_{m-1}||, \\ &(\Im^n + \Im^{n+1} + \cdots + \Im^{m-1})||\zeta_1 - \zeta_0|| \\ &\leq \Im^n\left(\frac{1 - \Im^{m-n}}{1 - \Im}\right)||\zeta_1||, \end{aligned} \tag{37}$$

As $0 < \Im < 1$, we get $1 - \Im^{m-n} < 1$. Therefore,

$$||\zeta_m - \zeta_n|| \leq \frac{\Im^n}{1-\Im} max_{\kappa \in J} ||\zeta_1||. \tag{38}$$

Since $||\zeta_1|| < \infty$, $||\zeta_m - \zeta_n|| \to 0$ when $n \to \infty$. As a result, ζ_m is a Cauchy sequence in H, implying that the series ζ_m is convergent. □

Theorem 4. *The $NTDM_{ABC}$ result of (1) is convergent.*

Proof. Let $\zeta_m = \sum_{r=0}^{m} \zeta_r(\varphi, \kappa)$. To show that ζ_m is a Cauchy sequence in H, let

$$\begin{aligned}
||\zeta_m - \zeta_n|| &= max_{\kappa \in J} |\sum_{r=n+1}^{m} \zeta_r|, \quad n = 1, 2, 3, \cdots \\
&\leq max_{\kappa \in J} \left| \mathcal{N}^{-1} \left[q(\gamma, v, \varpi) \mathcal{N} \left[\sum_{r=n+1}^{m} (\mathcal{L}(\zeta_{r-1}) + \mathbb{N}(\zeta_{r-1})) \right] \right] \right| \\
&= max_{\kappa \in J} \left| \mathcal{N}^{-1} \left[q(\gamma, v, \varpi) \mathcal{N} \left[\sum_{r=n+1}^{m-1} (\mathcal{L}(\zeta_r) + \mathbb{N}(u_r)) \right] \right] \right| \\
&\leq max_{\kappa \in J} |\mathcal{N}^{-1}[q(\gamma, v, \varpi)\mathcal{N}[(\mathcal{L}(\zeta_{m-1}) - \mathcal{L}(\zeta_{n-1}) + \mathbb{N}(\zeta_{m-1}) - \mathbb{N}(\zeta_{n-1}))]]| \\
&\leq \Im_1 max_{\kappa \in J} |\mathcal{N}^{-1}[q(\gamma, v, \varpi)\mathcal{N}[(\mathcal{L}(\zeta_{m-1}) - \mathcal{L}(\zeta_{n-1}))]]| \\
&+ \Im_2 max_{\kappa \in J} |\mathcal{N}^{-1}[q(\gamma, v, \varpi)\mathcal{N}[(\mathbb{N}(\zeta_{m-1}) - \mathbb{N}(\zeta_{n-1}))]]| \\
&= (\Im_1 + \Im_2)(1 - \gamma + \gamma \frac{\kappa^\gamma}{\Gamma(\gamma+1)}) ||\zeta_{m-1} - \zeta_{n-1}||
\end{aligned} \tag{39}$$

Let $m = n + 1$, then

$$||\zeta_{n+1} - \zeta_n|| \leq \Im ||\zeta_n - \zeta_{n-1}|| \leq \Im^2 ||\zeta_{n-1}\zeta_{n-2}|| \leq \cdots \leq \Im^n ||\zeta_1 - \zeta_0||, \tag{40}$$

where $\Im = (\Im_1 + \Im_2)(1 - \gamma + \gamma \frac{\kappa^\gamma}{\Gamma(\gamma+1)})$. Similarly, we have

$$\begin{aligned}
||\zeta_m - \zeta_n|| &\leq ||\zeta_{n+1} - \zeta_n|| + ||\zeta_{n+2}\zeta_{n+1}|| + \cdots + ||\zeta_m - \zeta_{m-1}||, \\
&(\Im^n + \Im^{n+1} + \cdots + \Im^{m-1}) ||\zeta_1 - \zeta_0|| \\
&\leq \Im^n \left(\frac{1 - \Im^{m-n}}{1 - \Im} \right) ||\zeta_1||,
\end{aligned} \tag{41}$$

As $0 < \Im < 1$, we get $1 - \Im^{m-n} < 1$. Therefore,

$$||\zeta_m - \zeta_n|| \leq \frac{\Im^n}{1-\Im} max_{t \in J} ||\zeta_1||. \tag{42}$$

Since $||\zeta_1|| < \infty$, $||\zeta_m - \zeta_n|| \to 0$ when $n \to \infty$. As a result, ζ_m is a Cauchy sequence in H, implying that the series ζ_m is convergent. □

5. Numerical Examples

In this section, we find the analytical solution of the time-fractional Kaup–Kupershmidt equation.

Example 1. *Consider the time-fractional Kaup–Kupershmidt equation [38]*

$$D_\kappa^\gamma \zeta(\varphi, \kappa) - 15\zeta\zeta_{\varphi\varphi\varphi} - 15p\zeta_\varphi\zeta_{\varphi\varphi} + 45\zeta^2\zeta_\varphi + \zeta_{\varphi\varphi\varphi\varphi\varphi} = 0, \quad 0 < \gamma \leq 1, \tag{43}$$

with initial condition

$$\zeta(\varphi, 0) = \frac{1}{4} w^2 \Upsilon^2 \operatorname{sech}^2(\frac{w\varphi\Upsilon}{2}) + \frac{w^2\Upsilon^2}{12}, \tag{44}$$

Equation (43) can be expressed as follows with the use of the natural transform:

$$\mathcal{N}[D_\kappa^\gamma \zeta(\varphi,\kappa)] = \mathcal{N}\left\{15\zeta\zeta_{\varphi\varphi\varphi}\right\} + \mathcal{N}\left\{15p\zeta_\varphi\zeta_{\varphi\varphi}\right\} - \mathcal{N}\left\{45\zeta^2\zeta_\varphi\right\} - \mathcal{N}\left\{\zeta_{\varphi\varphi\varphi\varphi\varphi}\right\}, \quad (45)$$

Characterize the non-linear operator as

$$\frac{1}{\varpi^\gamma}\mathcal{N}[\zeta(\varphi,\kappa)] - \varpi^{2-\gamma}\zeta(\varphi,0) = \mathcal{N}\left[15\zeta\zeta_{\varphi\varphi\varphi} + 15p\zeta_\varphi\zeta_{\varphi\varphi} - 45\zeta^2\zeta_\varphi - \zeta_{\varphi\varphi\varphi\varphi\varphi}\right], \quad (46)$$

We obtain the following when it comes to simplification:

$$\mathcal{N}[\zeta(\varphi,\kappa)] = \varpi^2\left[\frac{1}{4}w^2\Upsilon^2\,\mathrm{sech}^2(\frac{w\varphi\Upsilon}{2}) + \frac{w^2\Upsilon^2}{12}\right] + \frac{\gamma(\varpi-\gamma(\varpi-\gamma))}{\varpi^2}\mathcal{N}\left[15\zeta\zeta_{\varphi\varphi\varphi} + 15p\zeta_\varphi\zeta_{\varphi\varphi} - 45\zeta^2\zeta_\varphi - \zeta_{\varphi\varphi\varphi\varphi\varphi}\right], \quad (47)$$

Equation (47) can be written as follows with inverse NT:

$$\begin{aligned}\zeta(\varphi,\kappa) = &\left[\frac{1}{4}w^2\Upsilon^2\,\mathrm{sech}^2(\frac{w\varphi\Upsilon}{2}) + \frac{w^2\Upsilon^2}{12}\right]\\ &+\mathcal{N}^{-1}\left[\frac{\gamma(\varpi-\gamma(\varpi-\gamma))}{\varpi^2}\mathcal{N}\left\{15\zeta\zeta_{\varphi\varphi\varphi} + 15p\zeta_\varphi\zeta_{\varphi\varphi} - 45\zeta^2\zeta_\varphi - \zeta_{\varphi\varphi\varphi\varphi\varphi}\right\}\right],\end{aligned} \quad (48)$$

5.1. Implementing NDM_{CF}

The unknown function $\zeta(\varphi,\kappa)$ has a series form solution, which is stated as

$$\zeta(\varphi,\kappa) = \sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa) \quad (49)$$

The nonlinear terms are illustrated by using Adomian polynomials $\zeta\zeta_{\varphi\varphi\varphi} = \sum_{l=0}^{\infty}\mathcal{A}_l$, $\zeta_\varphi\zeta_{\varphi\varphi} = \sum_{l=0}^{\infty}\mathcal{B}_l$ and $\zeta^2\zeta_\varphi = \sum_{l=0}^{\infty}\mathcal{C}_l$ Thus, Equation (48) can be expressed with the help of the following terms

$$\begin{aligned}\sum_{l=0}^{\infty}\zeta_{l+1}(\varphi,\kappa) = &\frac{1}{4}w^2\Upsilon^2\,\mathrm{sech}^2(\frac{w\varphi\Upsilon}{2}) + \frac{w^2\Upsilon^2}{12}\\ &+\mathcal{N}^{-1}\left[\frac{\gamma(\varpi-\gamma(\varpi-\gamma))}{\varpi^2}\mathcal{N}\left\{15\sum_{l=0}^{\infty}\mathcal{A}_l + 15\sum_{l=0}^{\infty}\mathcal{B}_l - 45\sum_{l=0}^{\infty}\mathcal{C}_l - \sum_{l=0}^{\infty}\zeta_{l\varphi\varphi\varphi\varphi\varphi}\right\}\right],\end{aligned} \quad (50)$$

When both sides of Equation (50) are compared, we obtain

$$\zeta_0(\varphi,\kappa) = \frac{1}{4}w^2\Upsilon^2\,\mathrm{sech}^2(\frac{w\varphi\Upsilon}{2}) + \frac{w^2\Upsilon^2}{12},$$

$$\begin{aligned}\zeta_1(\varphi,\kappa) = &-\left(-\frac{1}{512}w^7\Upsilon^7(3843+480p-4(209+60p)cosh(w\varphi\Upsilon)+cosh(2w\varphi\Upsilon))sech^6\left(\frac{w\varphi\Upsilon}{2}\right)\right.\\ &\left.tanh\left(\frac{w\varphi\Upsilon}{2}\right)\right)(\gamma(\kappa-1)+1),\end{aligned} \quad (51)$$

$$\zeta_2(\varphi,\kappa) = \frac{w^{12}\Upsilon^{12}}{524288}(-3947228724 - 733469760p - 20736000p^2 + 6(777305099 + 148082560p + 4358400p^2) \\ cosh(w\varphi\Upsilon) - 48(18859301 + 3850520p + 124800p^2)cosh(2w\varphi\Upsilon) + 46313277cosh(3w\varphi\Upsilon) + 10287360p \\ cosh(3w\varphi\Upsilon) + 345600p^2cosh(3w\varphi\Upsilon) - 305756cosh(4w\varphi\Upsilon) - 87360pcosh(4w\varphi\Upsilon) + cosh(5w\varphi\Upsilon)) \\ sech^{12}\left(\frac{w\varphi\Upsilon}{2}\right)\left((1-\gamma)^2 + 2\gamma(1-\gamma)\kappa + \frac{\gamma^2\kappa^2}{2}\right), \tag{52}$$

Using the same procedure, we can easily find the remaining ζ_l components for $(l \geq 3)$. Following this, we define series form solutions as

$$\zeta(\varphi,\kappa) = \sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa) = \zeta_0(\varphi,\kappa) + \zeta_1(\varphi,\kappa) + \zeta_2(\varphi,\kappa) + \cdots,$$

$$\zeta(\varphi,\kappa) = \frac{1}{4}w^2\Upsilon^2\,\mathrm{sech}^2(\frac{w\varphi\Upsilon}{2}) + \frac{w^2\Upsilon^2}{12} - \Bigg(- \frac{1}{512}w^7\Upsilon^7(3843 + 480p - 4(209 + 60p)cosh(w\varphi\Upsilon) + cosh(2w\varphi\Upsilon)) \\ sech^6\left(\frac{w\varphi\Upsilon}{2}\right)tanh\left(\frac{w\varphi\Upsilon}{2}\right)\Bigg)(\gamma(\kappa-1)+1) + \frac{w^{12}\Upsilon^{12}}{524288}(-3947228724 - 733469760p - 20736000p^2 + \\ 6(777305099 + 148082560p + 4358400p^2)\cosh(w\varphi\Upsilon) - 48(18859301 + 3850520p + 124800p^2)cosh(2w\varphi\Upsilon) \\ + 46313277cosh(3w\varphi\Upsilon) + 10287360p\cosh(3w\varphi\Upsilon) + 345600p^2cosh(3w\varphi\Upsilon) - 305756cosh(4w\varphi\Upsilon) \\ - 87360pcosh(4w\varphi\Upsilon) + cosh(5w\varphi\Upsilon))sech^{12}\left(\frac{w\varphi\Upsilon}{2}\right)\left((1-\gamma)^2 + 2\gamma(1-\gamma)\kappa + \frac{\gamma^2\kappa^2}{2}\right) + \cdots. \tag{53}$$

5.2. Implementing NDM_{ABC}

The unknown function $\zeta(\varphi,\kappa)$ has a series form solution, which is stated as

$$\zeta(\varphi,\kappa) = \sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa) \tag{54}$$

The nonlinear terms are illustrated by using Adomian polynomials $\zeta\zeta_{\varphi\varphi\varphi} = \sum_{l=0}^{\infty}\mathcal{A}_l$, $\zeta_\varphi\zeta_{\varphi\varphi} = \sum_{l=0}^{\infty}\mathcal{B}_l$ and $\zeta^2\zeta_\varphi = \sum_{l=0}^{\infty}\mathcal{C}_l$. Thus, Equation (48) can be expressed with the help of the following terms:

$$\sum_{l=0}^{\infty}\zeta_{l+1}(\varphi,\kappa) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\varphi}{2}\right) \\ + \mathcal{N}^{-1}\left[\frac{\upsilon^\gamma(\varpi^\gamma + \gamma(\upsilon^\gamma - \varpi^\gamma))}{\varpi^{2\gamma}}\mathcal{N}\left\{15\sum_{l=0}^{\infty}\mathcal{A}_l + 15\sum_{l=0}^{\infty}\mathcal{B}_l - 45\sum_{l=0}^{\infty}\mathcal{C}_l - \sum_{l=0}^{\infty}\zeta_{l\varphi\varphi\varphi\varphi\varphi}\right\}\right], \tag{55}$$

When both sides of Equation (55) are compared, we obtain

$$\zeta_0(\varphi,\kappa) = \frac{1}{4}w^2\Upsilon^2\,\mathrm{sech}^2(\frac{w\varphi\Upsilon}{2}) + \frac{w^2\Upsilon^2}{12},$$

$$\zeta_1(\varphi,\kappa) = -\Bigg(- \frac{1}{512}w^7\Upsilon^7(3843 + 480p - 4(209 + 60p)\cosh(w\varphi\Upsilon) + \cosh(2w\varphi\Upsilon))sech^6\left(\frac{w\varphi\Upsilon}{2}\right) \\ \tanh\left(\frac{w\varphi\Upsilon}{2}\right)\Bigg)\left(1 - \gamma + \frac{\gamma\kappa^\gamma}{\Gamma(\gamma+1)}\right), \tag{56}$$

$$\zeta_2(\varphi,\kappa) = \frac{w^{12}\Upsilon^{12}}{524288}(-3947228724 - 733469760p - 20736000p^2 + 6(777305099 + 148082560p + 4358400p^2) \cosh(w\varphi\Upsilon) - 48(18859301 + 3850520p + 124800p^2)\cosh(2w\varphi\Upsilon) + 46313277cosh(3w\varphi\Upsilon) + 10287360p \cosh(3w\varphi\Upsilon) + 345600p^2\cosh(3w\varphi\Upsilon) - 305756\cosh(4w\varphi\Upsilon) - 87360p\cosh(4w\varphi\Upsilon) + \cosh(5w\varphi\Upsilon)) \operatorname{sech}^{12}\left(\frac{w\varphi\Upsilon}{2}\right)\left[\frac{\gamma^2\kappa^{2\gamma}}{\Gamma(2\gamma+1)} + 2\gamma(1-\gamma)\frac{\kappa^\gamma}{\Gamma(\gamma+1)} + (1-\gamma)^2\right], \tag{57}$$

Using the same procedure, we can easily find the remaining ζ_l components for $(l \geq 3)$. Following this, we define series form solutions as

$$\zeta(\varphi,\kappa) = \sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa) = \zeta_0(\varphi,\kappa) + \zeta_1(\varphi,\kappa) + \zeta_2(\varphi,\kappa) + \cdots,$$

$$\zeta(\varphi,\kappa) = \frac{1}{4}w^2\Upsilon^2\operatorname{sech}^2(\frac{w\varphi\Upsilon}{2}) + \frac{w^2\Upsilon^2}{12} - \left(-\frac{1}{512}w^7\Upsilon^7(3843 + 480p - 4(209+60p)cosh(w\varphi\Upsilon) + cosh(2w\varphi\Upsilon))\right. sech^6\left(\frac{w\varphi\Upsilon}{2}\right)tanh\left(\left.\frac{w\varphi\Upsilon}{2}\right)\right)\left(1 - \gamma + \frac{\gamma\kappa^\gamma}{\Gamma(\gamma+1)}\right) + \frac{w^{12}\Upsilon^{12}}{524288}(-3947228724 - 733469760p - 20736000p^2 + 6(777305099 + 148082560p + 4358400p^2)\cosh(w\varphi\Upsilon) - 48(18859301 + 3850520p + 124800p^2)cosh(2w\varphi\Upsilon) + 46313277cosh(3w\varphi\Upsilon) + 10287360p\cosh(3w\varphi\Upsilon) + 345600p^2cosh(3w\varphi\Upsilon) - 305756cosh(4w\varphi\Upsilon) - 87360pcosh(4w\varphi\Upsilon) + cosh(5w\varphi\Upsilon))sech^{12}\left(\frac{w\varphi\Upsilon}{2}\right)\left[\frac{\gamma^2\kappa^{2\gamma}}{\Gamma(2\gamma+1)} + 2\gamma(1-\gamma)\frac{\kappa^\gamma}{\Gamma(\gamma+1)} + (1-\gamma)^2\right] + \cdots. \tag{58}$$

We obtain the exact solution if we set $\gamma = 1$

$$\zeta(\varphi,\kappa) = \frac{1}{4}w^2\Upsilon^2\operatorname{sech}^2\left(\frac{\Upsilon}{2}\left(\frac{-w^5(-8\Upsilon^2\ell + 16\ell^2 + \Upsilon^4)}{16} + w\varphi\right)\frac{w^2\Upsilon^2}{12}\right), \tag{59}$$

Example 2. *Consider the nonlinear time-fractional Kaup–Kupershmidt equation [38]*

$$D_\kappa^\gamma\zeta(\varphi,\kappa) - 15\zeta\zeta_{\varphi\varphi\varphi} - 15p\zeta_\varphi\zeta_{\varphi\varphi} + 45\zeta^2\zeta_\varphi + \zeta_{\varphi\varphi\varphi\varphi\varphi} = 0, \quad 0 < \gamma \leq 1, \tag{60}$$

with initial condition

$$\zeta(\varphi,0) = \frac{4}{3}c - \frac{4}{p}c\operatorname{sech}^2(\sqrt{c\varphi}) \tag{61}$$

Equation (60) can be expressed as follows with the use of the natural transform:

$$\mathcal{N}[D_\kappa^\gamma\zeta(\varphi,\kappa)] = \mathcal{N}\left\{15\zeta\zeta_{\varphi\varphi\varphi}\right\} + \mathcal{N}\left\{15p\zeta_\varphi\zeta_{\varphi\varphi}\right\} - \mathcal{N}\left\{45\zeta^2\zeta_\varphi\right\} - \mathcal{N}\left\{\zeta_{\varphi\varphi\varphi\varphi\varphi}\right\}, \tag{62}$$

Characterize the nonlinear operator as

$$\frac{1}{\varpi^\gamma}\mathcal{N}[\zeta(\varphi,\kappa)] - \varpi^{2-\gamma}\zeta(\varphi,0) = \mathcal{N}\left[15\zeta\zeta_{\varphi\varphi\varphi} + 15p\zeta_\varphi\zeta_{\varphi\varphi} - 45\zeta^2\zeta_\varphi - \zeta_{\varphi\varphi\varphi\varphi\varphi}\right], \tag{63}$$

We obtain the following when it comes to simplification:

$$\mathcal{N}[\zeta(\varphi,\kappa)] = \varpi^2\left[\frac{4}{3}c - \frac{4}{p}c\operatorname{sech}^2(\sqrt{c\varphi})\right] + \frac{\gamma(\varpi - \gamma(\varpi-\gamma))}{\varpi^2}\mathcal{N}\left[15\zeta\zeta_{\varphi\varphi\varphi} + 15p\zeta_\varphi\zeta_{\varphi\varphi} - 45\zeta^2\zeta_\varphi - \zeta_{\varphi\varphi\varphi\varphi\varphi}\right], \tag{64}$$

Equation (64) can be written as follows with inverse NT:

$$\zeta(\varphi,\kappa) = \left[\frac{4}{3}c - \frac{4}{p}c\,\mathrm{sech}^2(\sqrt{c\varphi})\right] + \mathcal{N}^{-1}\left[\frac{\gamma(\varpi-\gamma(\varpi-\gamma))}{\varpi^2}\mathcal{N}\left\{15\zeta\zeta_{\varphi\varphi\varphi} + 15p\zeta_{\varphi}\zeta_{\varphi\varphi} - 45\zeta^2\zeta_{\varphi} - \zeta_{\varphi\varphi\varphi\varphi\varphi}\right\}\right], \quad (65)$$

5.3. Applying NDM_{CF}

The unknown function $\zeta(\varphi,\kappa)$ has a series form solution, which is stated as

$$\zeta(\varphi,\kappa) = \sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa) \quad (66)$$

The nonlinear terms are illustrated by using Adomian polynomials $\zeta\zeta_{\varphi\varphi\varphi} = \sum_{l=0}^{\infty}\mathcal{A}_l$, $\zeta_{\varphi}\zeta_{\varphi\varphi} = \sum_{l=0}^{\infty}\mathcal{B}_l$ and $\zeta^2\zeta_{\varphi} = \sum_{l=0}^{\infty}\mathcal{C}_l$. Thus, Equation (65) can be expressed with the help of the following terms:

$$\sum_{l=0}^{\infty}\zeta_{l+1}(\varphi,\kappa) = \frac{4}{3}c - \frac{4}{p}c\,\mathrm{sech}^2(\sqrt{c\varphi}) + \mathcal{N}^{-1}\left[\frac{\gamma(\varpi-\gamma(\varpi-\gamma))}{\varpi^2}\mathcal{N}\left\{15\sum_{l=0}^{\infty}\mathcal{A}_l + 15\sum_{l=0}^{\infty}\mathcal{B}_l - 45\sum_{l=0}^{\infty}\mathcal{C}_l - \sum_{l=0}^{\infty}\zeta_{l\varphi\varphi\varphi\varphi\varphi}\right\}\right], \quad (67)$$

When both sides of Equation (67) are compared, we obtain

$$\zeta_0(\varphi,\kappa) = \frac{4}{3}c - \frac{4}{p}\,\mathrm{csech}^2(\sqrt{c\varphi}),$$

$$\zeta_1(\varphi,\kappa) = -\frac{16c^{\frac{7}{2}}}{p^3}(360 - 420p + 63p^2 + 4p(-15+16p)\cosh(2\sqrt{cx}) + p^2\cosh(4\sqrt{cx}))\,\mathrm{sech}^6(\sqrt{cx})\tanh(\sqrt{cx})(\gamma(\kappa-1)+1)$$

$$\zeta_2(\varphi,\kappa) = \frac{16c^6\,\mathrm{sech}^{12}(\sqrt{c\varphi})}{p^5}(-3110400 + 14515200p - 26369280p^2 + 15270480p^3 - 306084p^4 - 6(-432000 + 2217600p - 4451160p^2 + 2656400p^3 + 9181p^4)\cosh(2\sqrt{c\varphi}) + 48p(14400 - 60780p + 41590p^2 + 4789p^3)\cosh(4\sqrt{c\varphi}) + 79920p^2\cosh(6\sqrt{c\varphi}) - 59040p^3\cosh(6\sqrt{c\varphi}) - 20883p^4\cosh(6\sqrt{c\varphi}) - 240p^3\cosh(8\sqrt{c\varphi}) + 244p^4\cosh(8\sqrt{c\varphi}) + p^4\cosh(10\sqrt{c\varphi}))\left((1-\gamma)^2 + 2\gamma(1-\gamma)\kappa + \frac{\gamma^2\kappa^2}{2}\right),$$

Using the same procedure, we can easily find the remaining ζ_l components for $(l \geq 3)$. Following this, we define series form solutions as

$$\zeta(\varphi,\kappa)=\sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa)=\zeta_0(\varphi,\kappa)+\zeta_1(\varphi,\kappa)+\zeta_2(\varphi,\kappa)+\cdots,$$

$$\begin{aligned}\zeta(\varphi,\kappa)=&\frac{4}{3}c-\frac{4}{p}c\,\mathrm{sech}^2(\sqrt{c\varphi})-\frac{16c^{\frac{7}{2}}}{p^3}(360-420p+63p^2+4p(-15+16p)\cosh(2\sqrt{cx})+p^2\cosh(4\sqrt{cx}))\\&\mathrm{sech}^6(\sqrt{cx})\tanh(\sqrt{cx})(\gamma(\kappa-1)+1)\frac{16c^6\,\mathrm{sech}^{12}(\sqrt{c\varphi})}{p^5}(-3110400+14515200p-26369280p^2+\\&15270480p^3-306084p^4-6(-432000+2217600p-4451160p^2+2656400p^3+9181p^4)\cosh(2\sqrt{c\varphi})\\&+48p(14400-60780p+41590p^2+4789p^3)\cosh(4\sqrt{c\varphi})+79920p^2\cosh(6\sqrt{c\varphi})-59040p^3\\&\cosh(6\sqrt{c\varphi})-20883p^4\cosh(6\sqrt{c\varphi})-240p^3\cosh(8\sqrt{c\varphi})+244p^4\cosh(8\sqrt{c\varphi})\\&+p^4\cosh(10\sqrt{c\varphi}))\left((1-\gamma)^2+2\gamma(1-\gamma)\kappa+\frac{\gamma^2\kappa^2}{2}\right)+\cdots,\end{aligned}\tag{68}$$

5.4. Applying NDM_{ABC}

The unknown function $\zeta(\varphi,\kappa)$ has a series form solution, which is stated as

$$\zeta(\varphi,\kappa)=\sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa)\tag{69}$$

The nonlinear terms are illustrated by using Adomian polynomials $\zeta\zeta_{\varphi\varphi\varphi}=\sum_{l=0}^{\infty}\mathcal{A}_l$, $\zeta_\varphi\zeta_{\varphi\varphi}=\sum_{l=0}^{\infty}\mathcal{B}_l$ and $\zeta^2\zeta_\varphi=\sum_{l=0}^{\infty}\mathcal{C}_l$. Thus, Equation (65) can be expressed with the help of the following terms:

$$\begin{aligned}\sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa)=&\frac{4}{3}c-\frac{4}{p}c\,\mathrm{sech}^2(\sqrt{c\varphi})\\&+\mathcal{N}^{-1}\left[\frac{\upsilon^\gamma(\varpi^\gamma+\gamma(\upsilon^\gamma-\varpi^\gamma))}{\varpi^{2\gamma}}\mathcal{N}\left\{15\sum_{l=0}^{\infty}\mathcal{A}_l+15\sum_{l=0}^{\infty}\mathcal{B}_l-45\sum_{l=0}^{\infty}\mathcal{C}_l-\sum_{l=0}^{\infty}\zeta_{l\varphi\varphi\varphi\varphi\varphi}\right\}\right],\end{aligned}\tag{70}$$

When both sides of Equation (70) are compared, we obtain

$$\zeta_0(\varphi,\kappa)=\frac{4}{3}c-\frac{4}{p}c\,\mathrm{sech}^2(\sqrt{c\varphi}),$$

$$\begin{aligned}\zeta_1(\varphi,\kappa)=&-\frac{16c^{\frac{7}{2}}}{p^3}(360-420p+63p^2+4p(-15+16p)\cosh(2\sqrt{cx})+p^2\cosh(4\sqrt{cx}))\,\mathrm{sech}^6(\sqrt{cx})\\&\tanh(\sqrt{cx})\left(1-\gamma+\frac{\gamma\kappa^\gamma}{\Gamma(\gamma+1)}\right),\end{aligned}$$

$$\begin{aligned}\zeta_2(\varphi,\kappa)=&\frac{16c^6\,\mathrm{sech}^{12}(\sqrt{c\varphi})}{p^5}(-3110400+14515200p-26369280p^2+\\&15270480p^3-306084p^4-6(-432000+2217600p-4451160p^2+2656400p^3+9181p^4)\cosh(2\sqrt{c\varphi})\\&+48p(14400-60780p+41590p^2+4789p^3)\cosh(4\sqrt{c\varphi})+79920p^2\cosh(6\sqrt{c\varphi})-59040p^3\\&\cosh(6\sqrt{c\varphi})-20883p^4\cosh(6\sqrt{c\varphi})-240p^3\cosh(8\sqrt{c\varphi})+244p^4\cosh(8\sqrt{c\varphi})\\&+p^4\cosh(10\sqrt{c\varphi}))\left[\frac{\gamma^2\kappa^{2\gamma}}{\Gamma(2\gamma+1)}+2\gamma(1-\gamma)\frac{\kappa^\gamma}{\Gamma(\gamma+1)}+(1-\gamma)^2\right]\end{aligned}$$

Using the same procedure, we can easily find the remaining ζ_l components for $(l \geq 3)$. Following this, we define series form solutions as

$$\zeta(\varphi,\kappa) = \sum_{l=0}^{\infty} \zeta_l(\varphi,\kappa) = \zeta_0(\varphi,\kappa) + \zeta_1(\varphi,\kappa) + \zeta_2(\varphi,\kappa) + \cdots,$$

$$\begin{aligned}\zeta(\varphi,\kappa) = {}& \frac{4}{3}c - \frac{4}{p}c\,\mathrm{sech}^2(\sqrt{c\varphi}) - \frac{16c^{\frac{7}{2}}}{p^3}(360 - 420p + 63p^2 + 4p(-15+16p)\cosh(2\sqrt{cx}) + p^2\cosh(4\sqrt{cx})) \\ & \mathrm{sech}^6(\sqrt{cx})\tanh(\sqrt{cx})\left(1 - \gamma + \frac{\gamma\kappa^\gamma}{\Gamma(\gamma+1)}\right)\frac{16c^6\,\mathrm{sech}^{12}(\sqrt{c\varphi})}{p^5}(-3110400 + 14515200p - 26369280p^2 + \\ & 15270480p^3 - 306084p^4 - 6(-432000 + 2217600p - 4451160p^2 + 2656400p^3 + 9181p^4)\cosh(2\sqrt{c\varphi}) \\ & + 48p(14400 - 60780p + 41590p^2 + 4789p^3)\cosh(4\sqrt{c\varphi}) + 79920p^2\cosh(6\sqrt{c\varphi}) - 59040p^3 \\ & \cosh(6\sqrt{c\varphi}) - 20883p^4\cosh(6\sqrt{c\varphi}) - 240p^3\cosh(8\sqrt{c\varphi}) + 244p^4\cosh(8\sqrt{c\varphi}) \\ & + p^4\cosh(10\sqrt{c\varphi}))\left[\frac{\gamma^2\kappa^{2\gamma}}{\Gamma(2\gamma+1)} + 2\gamma(1-\gamma)\frac{\kappa^\gamma}{\Gamma(\gamma+1)} + (1-\gamma)^2\right] + \cdots,\end{aligned} \tag{71}$$

We achieve the exact solution if we set $\gamma = 1$

$$\zeta(\varphi,\kappa) = \frac{4}{3}c - \frac{4}{p}c\,\mathrm{sech}^2(\sqrt{c} + (\varphi + 8(3c^2 - 5pc)\kappa)). \tag{72}$$

Example 3. *Consider the nonlinear time-fractional Kaup–Kupershmidt equation [38]*

$$D_\kappa^\gamma \zeta(\varphi,\kappa) = 5\zeta\zeta_{\varphi\varphi\varphi} + \frac{25}{2}\zeta_\varphi\zeta_{\varphi\varphi} + 5\zeta^2\zeta_\varphi + \zeta_{\varphi\varphi\varphi\varphi\varphi}, \quad 0 < \gamma \leq 1, \tag{73}$$

with initial condition

$$\zeta(\varphi,0) = -2k^2 + \frac{24k^2}{1+e^{k\varphi}} - \frac{24k^2}{(1+e^{k\varphi})^2} \tag{74}$$

Equation (73) can be expressed as follows with the use of the natural transform:

$$\mathcal{N}[D_\kappa^\gamma \zeta(\varphi,\kappa)] = \mathcal{N}\left\{5\zeta\zeta_{\varphi\varphi\varphi}\right\} + \mathcal{N}\left\{\frac{25}{2}\zeta_\varphi\zeta_{\varphi\varphi}\right\} + \mathcal{N}\left\{5\zeta^2\zeta_\varphi\right\} + \mathcal{N}\left\{\zeta_{\varphi\varphi\varphi\varphi\varphi}\right\}, \tag{75}$$

Characterize the nonlinear operator as

$$\frac{1}{\varpi^\gamma}\mathcal{N}[\zeta(\varphi,\kappa)] - \varpi^{2-\gamma}\zeta(\varphi,0) = \mathcal{N}\left[5\zeta\zeta_{\varphi\varphi\varphi} + \frac{25}{2}\zeta_\varphi\zeta_{\varphi\varphi} + 5\zeta^2\zeta_\varphi + \zeta_{\varphi\varphi\varphi\varphi\varphi}\right], \tag{76}$$

We obtain the following when it comes to simplification:

$$\mathcal{N}[\zeta(\varphi,\kappa)] = \varpi^2\left[-2k^2 + \frac{24k^2}{1+e^{k\varphi}} - \frac{24k^2}{(1+e^{k\varphi})^2}\right] + \frac{\gamma(\varpi - \gamma(\varpi-\gamma))}{\varpi^2}\mathcal{N}\left[5\zeta\zeta_{\varphi\varphi\varphi} + \frac{25}{2}\zeta_\varphi\zeta_{\varphi\varphi} + 5\zeta^2\zeta_\varphi + \zeta_{\varphi\varphi\varphi\varphi\varphi}\right], \tag{77}$$

Equation (77) can be written as follows with inverse NT

$$\begin{aligned}\zeta(\varphi,\kappa) = {}& \left[-2k^2 + \frac{24k^2}{1+e^{k\varphi}} - \frac{24k^2}{(1+e^{k\varphi})^2}\right] \\ & + \mathcal{N}^{-1}\left[\frac{\gamma(\varpi - \gamma(\varpi-\gamma))}{\varpi^2}\mathcal{N}\left\{5\zeta\zeta_{\varphi\varphi\varphi} + \frac{25}{2}\zeta_\varphi\zeta_{\varphi\varphi} + 5\zeta^2\zeta_\varphi + \zeta_{\varphi\varphi\varphi\varphi\varphi}\right\}\right],\end{aligned} \tag{78}$$

5.5. Applying NDM_{CF}

The unknown function $\zeta(\varphi,\kappa)$ has a series form solution, which is stated as

$$\zeta(\varphi,\kappa)=\sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa) \tag{79}$$

The nonlinear terms are illustrated by using Adomian polynomials $\zeta\zeta_{\varphi\varphi\varphi}=\sum_{l=0}^{\infty}\mathcal{A}_l$, $\zeta_\varphi\zeta_{\varphi\varphi}=\sum_{l=0}^{\infty}\mathcal{B}_l$ and $\zeta^2\zeta_\varphi=\sum_{l=0}^{\infty}\mathcal{C}_l$. Thus, Equation (78) can be expressed with the help of the following terms:

$$\begin{aligned}\sum_{l=0}^{\infty}\zeta_{l+1}(\varphi,\kappa)&=-2k^2+\frac{24k^2}{1+e^{k\varphi}}-\frac{24k^2}{(1+e^{k\varphi})^2}\\&+\mathcal{N}^{-1}\left[\frac{\gamma(\varpi-\gamma(\varpi-\gamma))}{\varpi^2}\mathcal{N}\left\{5\sum_{l=0}^{\infty}\mathcal{A}_l+\frac{25}{2}\sum_{l=0}^{\infty}\mathcal{B}_l+5\sum_{l=0}^{\infty}\mathcal{C}_l+\sum_{l=0}^{\infty}\zeta_{l\varphi\varphi\varphi\varphi\varphi}\right\}\right],\end{aligned} \tag{80}$$

When both sides of Equation (80) are compared, we obtain

$$\zeta_0(\varphi,\kappa)=-2k^2+\frac{24k^2}{1+e^{k\varphi}}-\frac{24k^2}{(1+e^{k\varphi})^2},$$

$$\zeta_1(\varphi,\kappa)=-\left(\frac{264e^{k\varphi}(-1+e^{k\varphi})k^7}{(1+e^{k\varphi})^3}\right)(\gamma(\kappa-1)+1)$$

$$\zeta_2(\varphi,\kappa)=2904e^{k\varphi}(\frac{264e^{k\varphi}(1-4e^{k\varphi}+e^{2k\varphi})k^{12}}{(1+e^{k\varphi})^4})\left((1-\gamma)^2+2\gamma(1-\gamma)\kappa+\frac{\gamma^2\kappa^2}{2}\right),$$

Using the same procedure, we can easily find the remaining ζ_l components for $(l\geq 3)$. Following this, we define series form solutions as

$$\begin{aligned}\zeta(\varphi,\kappa)&=\sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa)=\zeta_0(\varphi,\kappa)+\zeta_1(\varphi,\kappa)+\zeta_2(\varphi,\kappa)+\cdots,\\ \zeta(\varphi,\kappa)&=-2k^2+\frac{24k^2}{1+e^{k\varphi}}-\frac{24k^2}{(1+e^{k\varphi})^2}-\left(\frac{264e^{k\varphi}(-1+e^{k\varphi})k^7}{(1+e^{k\varphi})^3}\right)(\gamma(\kappa-1)+1)+\\&2904e^{k\varphi}(\frac{264e^{k\varphi}(1-4e^{k\varphi}+e^{2k\varphi})k^{12}}{(1+e^{k\varphi})^4})\left((1-\gamma)^2+2\gamma(1-\gamma)\kappa+\frac{\gamma^2\kappa^2}{2}\right)+\cdots,\end{aligned} \tag{81}$$

5.6. Applying NDM_{ABC}

The unknown function $\zeta(\varphi,\kappa)$ has a series form solution, which is stated as

$$\zeta(\varphi,\kappa)=\sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa) \tag{82}$$

The nonlinear terms are illustrated by using Adomian polynomials $\zeta\zeta_{\varphi\varphi\varphi}=\sum_{l=0}^{\infty}\mathcal{A}_l$, $\zeta_\varphi\zeta_{\varphi\varphi}=\sum_{l=0}^{\infty}\mathcal{B}_l$ and $\zeta^2\zeta_\varphi=\sum_{l=0}^{\infty}\mathcal{C}_l$. Thus, Equation (78) can be expressed with the help of the following terms:

$$\begin{aligned}\sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa)&=-2k^2+\frac{24k^2}{1+e^{k\varphi}}-\frac{24k^2}{(1+e^{k\varphi})^2}\\&+\mathcal{N}^{-1}\left[\frac{\upsilon^\gamma(\varpi^\gamma+\gamma(\upsilon^\gamma-\varpi^\gamma))}{\varpi^{2\gamma}}\mathcal{N}\left\{5\sum_{l=0}^{\infty}\mathcal{A}_l+\frac{25}{2}\sum_{l=0}^{\infty}\mathcal{B}_l+5\sum_{l=0}^{\infty}\mathcal{C}_l+\sum_{l=0}^{\infty}\zeta_{l\varphi\varphi\varphi\varphi\varphi}\right\}\right],\end{aligned} \tag{83}$$

When both sides of Equation (83) are compared, we obtain

$$\zeta_0(\varphi,\kappa) = -2k^2 + \frac{24k^2}{1+e^{k\varphi}} - \frac{24k^2}{(1+e^{k\varphi})^2},$$

$$\zeta_1(\varphi,\kappa) = -\left(\frac{264e^{k\varphi}(-1+e^{k\varphi})k^7}{(1+e^{k\varphi})^3}\right)\left(1-\gamma+\frac{\gamma\kappa^\gamma}{\Gamma(\gamma+1)}\right),$$

$$\zeta_2(\varphi,\kappa) = 2904e^{k\varphi}\left(\frac{264e^{k\varphi}(1-4e^{k\varphi}+e^{2k\varphi})k^{12}}{(1+e^{k\varphi})^4}\right)\left[\frac{\gamma^2\kappa^{2\gamma}}{\Gamma(2\gamma+1)} + 2\gamma(1-\gamma)\frac{\kappa^\gamma}{\Gamma(\gamma+1)} + (1-\gamma)^2\right]$$

Using the same procedure, we can easily find the remaining ζ_l components for $(l \geq 3)$. Following this, we define series form solutions as

$$\zeta(\varphi,\kappa) = \sum_{l=0}^{\infty}\zeta_l(\varphi,\kappa) = \zeta_0(\varphi,\kappa) + \zeta_1(\varphi,\kappa) + \zeta_2(\varphi,\kappa) + \cdots,$$

$$\zeta(\varphi,\kappa) = -2k^2 + \frac{24k^2}{1+e^{k\varphi}} - \frac{24k^2}{(1+e^{k\varphi})^2} - \left(\frac{264e^{k\varphi}(-1+e^{k\varphi})k^7}{(1+e^{k\varphi})^3}\right)\left(1-\gamma+\frac{\gamma\kappa^\gamma}{\Gamma(\gamma+1)}\right) + \tag{84}$$

$$2904e^{k\varphi}\left(\frac{264e^{k\varphi}(1-4e^{k\varphi}+e^{2k\varphi})k^{12}}{(1+e^{k\varphi})^4}\right)\left[\frac{\gamma^2\kappa^{2\gamma}}{\Gamma(2\gamma+1)} + 2\gamma(1-\gamma)\frac{\kappa^\gamma}{\Gamma(\gamma+1)} + (1-\gamma)^2\right] + \cdots,$$

We achieve the exact solution if we set $\gamma = 1$

$$\zeta(\varphi,\kappa) = -2k^2 + \frac{24k^2}{1+e^{k\varphi+11k^5t}} - \frac{24k^2}{(1+e^{k\varphi+11k^5t})^2}. \tag{85}$$

6. Results and Discussion

We find the solution of fractional-order Kaup-Kupershmidt (KK) equation by implementing Natural decomposition method with the aid of two different fractional derivatives. Figure 1 exhibits the nature of the exact and proposed method solution while Figure 2 shows nature of the absolute error of example 1 at $\gamma = 1$. Figure 3 exhibits the nature of the exact and proposed method solution whereas Figures 4 and 5 shows the nature of the proposed method solution at different fractional orders. Figure 6 exhibits the nature of the exact and proposed method solution whereas Figures 7 and 8 shows the nature of the proposed method solution at different fractional orders.

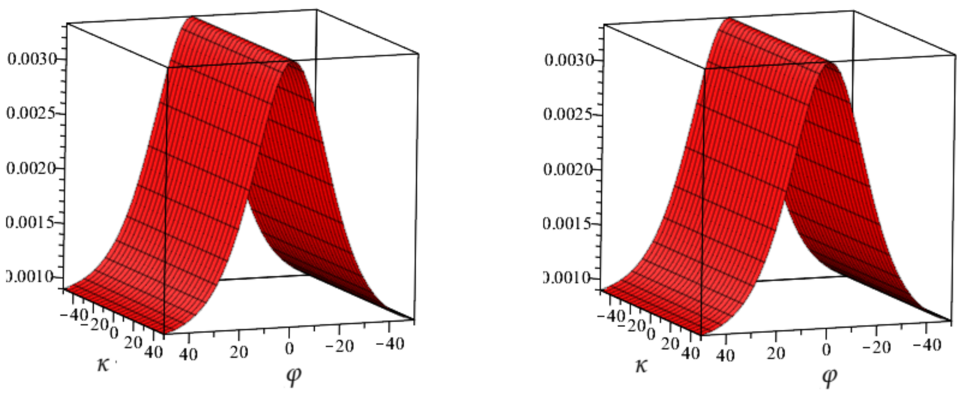

Figure 1. Nature of the exact and proposed method solution of example 1 at $\gamma = 1$.

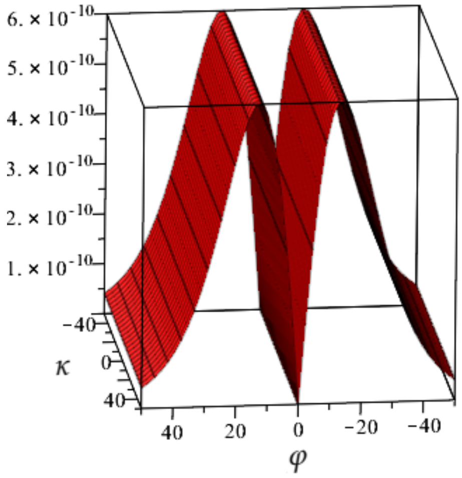

Figure 2. Nature of the absolute error of example 1.

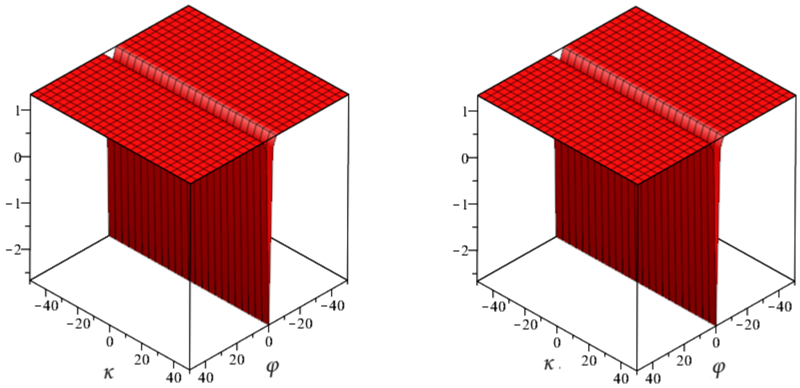

Figure 3. Nature of the exact and proposed method solution of example 2 at at $\gamma = 1$.

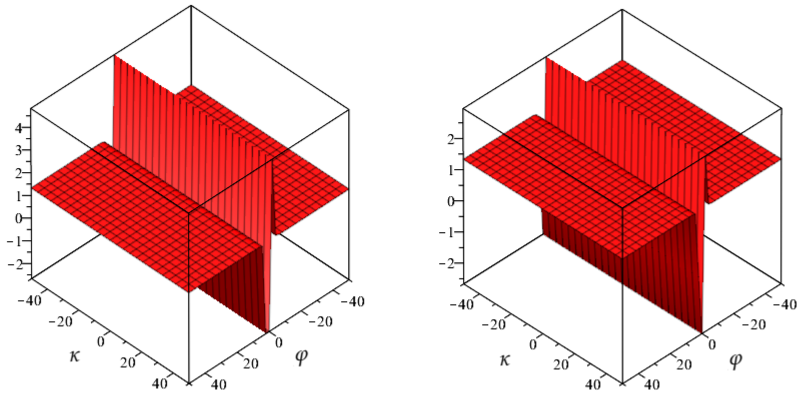

Figure 4. Nature of the proposed method solution of example 2 at $\gamma = 0.8$, 0.6.

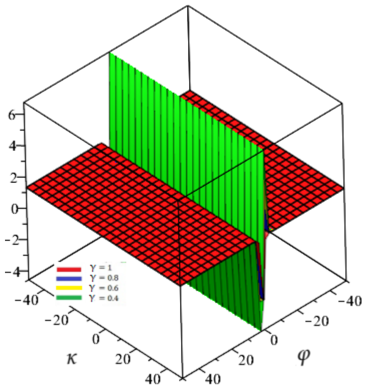

Figure 5. Nature of the proposed method solution at various orders of γ for example 2.

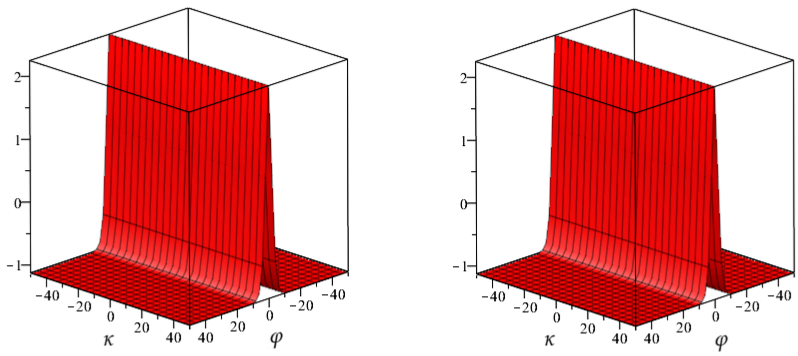

Figure 6. Nature of the exact and proposed method solution of example 3 at $\gamma = 1$.

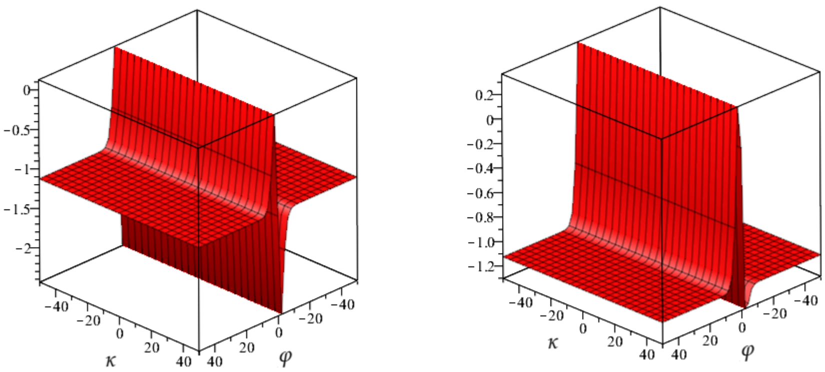

Figure 7. Nature of the proposed method solution of example 3 at $\gamma = 0.8, 0.6$.

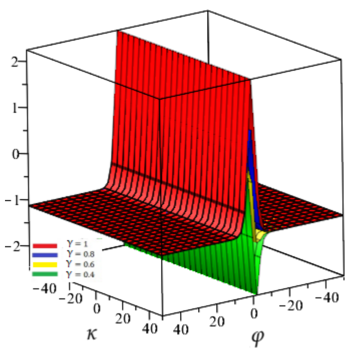

Figure 8. Nature of the proposed method solution at various orders of γ for example 3.

7. Conclusions

In this paper, we find the solution of the time-fractional Kaup–Kupershmidt equation by means of the natural decomposition method with the aid of two different fractional derivatives. To demonstrate the validity of the proposed method, we study the time-fractional KK equation in three different cases. The results that we obtain by implementing the proposed methods show that our results are in good agreement with the exact solution. The results shown in Tables 1–7 are suitable when compared with other techniques such as the two-dimensional Legendre multiwavelet method, optimal homotopy analysis transform method (OHAM) and q-homotopy analysis transform method (q-HATM). Finally, we can conclude that the suggested method is sufficiently consistent and can be used to examine a wide range of fractional-order nonlinear mathematical models that enable us to understand the behaviour of highly nonlinear complicated phenomena in related fields of science and engineering.

Table 1. Comparison at different fractional order of γ on the basis of error for example 1.

κ	φ	$\gamma = 0.4$	$\gamma = 0.6$	$\gamma = 0.8$	$\gamma = 1(NTDM_{CF})$	$\gamma = 1(NTDM_{ABC})$
	0.2	$7.7794000000 \times 10^{-8}$	$5.9046000000 \times 10^{-8}$	$3.1881000000 \times 10^{-8}$	$1.5379000000 \times 10^{-8}$	$1.5379000000 \times 10^{-8}$
	0.4	$1.5668400000 \times 10^{-7}$	$1.1893800000 \times 10^{-7}$	$6.4232000000 \times 10^{-8}$	$3.0990000000 \times 10^{-8}$	$3.0990000000 \times 10^{-8}$
0.1	0.6	$2.3529200000 \times 10^{-7}$	$1.7864200000 \times 10^{-7}$	$9.6509000000 \times 10^{-8}$	$4.6575000000 \times 10^{-8}$	$4.6575000000 \times 10^{-8}$
	0.8	$3.1347800000 \times 10^{-7}$	$2.3807000000 \times 10^{-7}$	$1.2867500000 \times 10^{-7}$	$6.2123000000 \times 10^{-8}$	$6.2123000000 \times 10^{-8}$
	1	$3.9110400000 \times 10^{-7}$	$2.9712500000 \times 10^{-7}$	$1.6069300000 \times 10^{-7}$	$7.7622000000 \times 10^{-8}$	$7.7622000000 \times 10^{-8}$
	0.2	$1.5316700000 \times 10^{-7}$	$1.1624400000 \times 10^{-7}$	$6.2757000000 \times 10^{-8}$	$3.0269000000 \times 10^{-8}$	$3.0269000000 \times 10^{-8}$
	0.4	$3.1100100000 \times 10^{-7}$	$2.3605500000 \times 10^{-7}$	$1.2746600000 \times 10^{-7}$	$6.1491000000 \times 10^{-8}$	$6.1491000000 \times 10^{-8}$
0.2	0.6	$4.6827700000 \times 10^{-7}$	$3.5549900000 \times 10^{-7}$	$1.9202900000 \times 10^{-7}$	$9.2664000000 \times 10^{-8}$	$9.2664000000 \times 10^{-8}$
	0.8	$6.2471400000 \times 10^{-7}$	$4.7438600000 \times 10^{-7}$	$2.5637000000 \times 10^{-7}$	$1.2376100000 \times 10^{-7}$	$1.2376100000 \times 10^{-7}$
	1	$7.8003300000 \times 10^{-7}$	$5.9253400000 \times 10^{-7}$	$3.2041800000 \times 10^{-7}$	$1.5476200000 \times 10^{-7}$	$1.5476200000 \times 10^{-7}$
	0.2	$2.2606900000 \times 10^{-7}$	$1.7156700000 \times 10^{-7}$	$9.2620000000 \times 10^{-8}$	$4.4671000000 \times 10^{-8}$	$4.4671000000 \times 10^{-8}$
	0.4	$4.6285600000 \times 10^{-7}$	$3.5130300000 \times 10^{-7}$	$1.8968900000 \times 10^{-7}$	$9.1506000000 \times 10^{-8}$	$9.1506000000 \times 10^{-8}$
0.3	0.6	$6.9881100000 \times 10^{-7}$	$5.3049100000 \times 10^{-7}$	$2.8654200000 \times 10^{-7}$	$1.3826500000 \times 10^{-7}$	$1.3826500000 \times 10^{-7}$
	0.8	$9.3351200000 \times 10^{-7}$	$7.0884900000 \times 10^{-7}$	$3.8306300000 \times 10^{-7}$	$1.8491500000 \times 10^{-7}$	$1.8491500000 \times 10^{-7}$
	1	$1.1665440000 \times 10^{-6}$	$8.8610300000 \times 10^{-7}$	$4.7914600000 \times 10^{-7}$	$2.3141700000 \times 10^{-7}$	$2.3141700000 \times 10^{-7}$
	0.2	$2.9650200000 \times 10^{-7}$	$2.2501400000 \times 10^{-7}$	$1.2147100000 \times 10^{-7}$	$5.8586000000 \times 10^{-8}$	$5.8586000000 \times 10^{-8}$
	0.4	$6.1224700000 \times 10^{-7}$	$4.6468100000 \times 10^{-7}$	$2.5090400000 \times 10^{-7}$	$1.2103200000 \times 10^{-7}$	$1.2103200000 \times 10^{-7}$
0.4	0.6	$9.2689100000 \times 10^{-7}$	$7.0362200000 \times 10^{-7}$	$3.8004700000 \times 10^{-7}$	$1.8338100000 \times 10^{-7}$	$1.8338100000 \times 10^{-7}$
	0.8	$1.2398730000 \times 10^{-6}$	$9.4146100000 \times 10^{-7}$	$5.0875300000 \times 10^{-7}$	$2.4558200000 \times 10^{-7}$	$2.4558200000 \times 10^{-7}$
	1	$1.5506360000 \times 10^{-6}$	$1.1778340000 \times 10^{-6}$	$6.3687500000 \times 10^{-7}$	$3.0758800000 \times 10^{-7}$	$3.0758800000 \times 10^{-7}$
	0.2	$3.6446500000 \times 10^{-7}$	$2.7658900000 \times 10^{-7}$	$1.4931100000 \times 10^{-7}$	$7.2012000000 \times 10^{-8}$	$7.2012000000 \times 10^{-8}$
	0.4	$7.5917400000 \times 10^{-7}$	$5.7618900000 \times 10^{-7}$	$3.1110700000 \times 10^{-7}$	$1.5007100000 \times 10^{-7}$	$1.5007100000 \times 10^{-7}$
0.5	0.6	$1.1525180000 \times 10^{-6}$	$8.7488900000 \times 10^{-7}$	$4.7254400000 \times 10^{-7}$	$2.2800900000 \times 10^{-7}$	$2.2800900000 \times 10^{-7}$
	0.8	$1.5437950000 \times 10^{-6}$	$1.1722200000 \times 10^{-6}$	$6.3343900000 \times 10^{-7}$	$3.0576500000 \times 10^{-7}$	$3.0576500000 \times 10^{-7}$
	1	$1.9323090000 \times 10^{-6}$	$1.4677220000 \times 10^{-6}$	$7.9360600000 \times 10^{-7}$	$3.8327700000 \times 10^{-7}$	$3.8327700000 \times 10^{-7}$

Table 2. Comparison of absolute error among Legendre Multiwavelet [39], $OHAM$ [39], $q-HATM$ [38], NDM_{CF} and NDM_{ABC} for example 1 at $w = 1, \ell = 0, \Upsilon = 0.1, \gamma = 1$ and $\kappa = 0.1$.

φ	$\lvert Legendre\ Multiwelet \rvert$	$\lvert OHAM \rvert$	$\lvert q-HATM \rvert$	$\lvert NTDM_{CF} \rvert$	$\lvert NTDM_{ABC} \rvert$
0.1	3.5268×10^{-10}	3.4968×10^{-10}	3.1482×10^{-10}	$7.5000000000 \times 10^{-13}$	$7.5000000000 \times 10^{-13}$
0.2	7.0308×10^{-10}	7.2934×10^{-6}	6.3101×10^{-10}	$1.5400000000 \times 10^{-12}$	$1.5400000000 \times 10^{-12}$
0.3	1.0532×10^{-9}	2.6793×10^{-5}	9.4682×10^{-10}	$2.3200000000 \times 10^{-12}$	$2.3200000000 \times 10^{-12}$
0.4	1.4028×10^{-9}	5.8103×10^{-5}	1.2620×10^{-9}	$3.1000000000 \times 10^{-12}$	$3.1000000000 \times 10^{-12}$
0.5	1.7520×10^{-9}	1.0061×10^{-4}	1.5765×10^{-9}	$3.8800000000 \times 10^{-12}$	$3.8800000000 \times 10^{-12}$

Table 3. Comparison of absolute error among Legendre Multiwavelet [39], *OHAM* [39], $q - HATM$ [38], NDM_{CF} and NDM_{ABC} for example 1 at $w = 1, \ell = 0, \Upsilon = 0.1, \gamma = 0.75$ and $\kappa = 0.1$.

φ	$\lvert Legendre\ Multiwelet \rvert$	$\lvert OHAM \rvert$	$\lvert q - HATM \rvert$	$\lvert NTDM_{CF} \rvert$	$\lvert NTDM_{ABC} \rvert$
0.1	6.7734×10^{-10}	6.7141×10^{-10}	6.0478×10^{-10}	$1.4700000000 \times 10^{-12}$	$1.4700000000 \times 10^{-12}$
0.2	1.3533×10^{-9}	7.2899×10^{-6}	1.2165×10^{-10}	$3.0200000000 \times 10^{-12}$	$3.0200000000 \times 10^{-12}$
0.3	2.0287×10^{-9}	2.6785×10^{-5}	1.8276×10^{-10}	$4.5900000000 \times 10^{-12}$	$4.5900000000 \times 10^{-12}$
0.4	2.7033×10^{-9}	5.8094×10^{-5}	2.4376×10^{-9}	$6.1500000000 \times 10^{-12}$	$6.1500000000 \times 10^{-12}$
0.5	3.3768×10^{-9}	1.0060×10^{-4}	3.0461×10^{-9}	$7.7100000000 \times 10^{-12}$	$7.7100000000 \times 10^{-12}$

Table 4. Comparison of absolute error among Legendre Multiwavelet [39], *OHAM* [39], $q - HATM$ [38], NDM_{CF} and NDM_{ABC} for example 1 at $w = 1, \ell = 0, \Upsilon = 0.1, \gamma = 0.5$ and $\kappa = 0.1$.

φ	$\lvert Legendre\ Multiwelet \rvert$	$\lvert OHAM \rvert$	$\lvert q - HATM \rvert$	$\lvert NTDM_{CF} \rvert$	$\lvert NTDM_{ABC} \rvert$
0.1	1.2348×10^{-9}	1.2175×10^{-9}	1.0979×10^{-9}	$2.1300000000 \times 10^{-12}$	$2.1300000000 \times 10^{-12}$
0.2	2.4789×10^{-9}	7.2836×10^{-6}	2.2262×10^{-9}	$1.5400000000 \times 10^{-12}$	$4.4700000000 \times 10^{-12}$
0.3	3.7221×10^{-9}	2.6773×10^{-5}	3.3531×10^{-9}	$6.8100000000 \times 10^{-12}$	$6.8100000000 \times 10^{-12}$
0.4	4.9638×10^{-9}	5.8078×10^{-5}	4.4781×10^{-9}	$9.1600000000 \times 10^{-12}$	$9.1600000000 \times 10^{-12}$
0.5	6.2035×10^{-9}	1.0058×10^{-4}	5.6004×10^{-9}	$1.1500000000 \times 10^{-11}$	$1.1500000000 \times 10^{-11}$

Table 5. Comparison at different fractional order of γ on the basis of error for example 2.

κ	φ	$\gamma = 0.4$	$\gamma = 0.6$	$\gamma = 0.8$	$\gamma = 1(NTDM_{CF})$	$\gamma = 1(NTDM_{ABC})$
	0.2	$5.2120000000 \times 10^{-7}$	$3.6017600000 \times 10^{-7}$	$2.0943200000 \times 10^{-7}$	$6.4513000000 \times 10^{-8}$	$6.4513000000 \times 10^{-8}$
	0.4	$1.0384330000 \times 10^{-6}$	$7.1776700000 \times 10^{-7}$	$4.1757400000 \times 10^{-7}$	$1.2893800000 \times 10^{-7}$	$1.2893800000 \times 10^{-7}$
0.1	0.6	$1.5474640000 \times 10^{-6}$	$1.0698980000 \times 10^{-6}$	$6.2282300000 \times 10^{-7}$	$1.9293900000 \times 10^{-7}$	$1.9293900000 \times 10^{-7}$
	0.8	$2.0443500000 \times 10^{-6}$	$1.4139470000 \times 10^{-6}$	$8.2379300000 \times 10^{-7}$	$2.5631800000 \times 10^{-7}$	$2.5631800000 \times 10^{-7}$
	1	$2.5253100000 \times 10^{-6}$	$1.7473930000 \times 10^{-6}$	$1.0191440000 \times 10^{-6}$	$3.1886900000 \times 10^{-7}$	$3.1886900000 \times 10^{-7}$
	0.2	$5.8984400000 \times 10^{-7}$	$4.2773700000 \times 10^{-7}$	$2.7455400000 \times 10^{-7}$	$1.2876600000 \times 10^{-7}$	$1.2876600000 \times 10^{-7}$
	0.4	$1.1759660000 \times 10^{-6}$	$8.5314400000 \times 10^{-7}$	$5.4809200000 \times 10^{-7}$	$2.5761600000 \times 10^{-7}$	$2.5761600000 \times 10^{-7}$
0.2	0.6	$1.7533840000 \times 10^{-6}$	$1.2726080000 \times 10^{-6}$	$8.1829600000 \times 10^{-7}$	$3.8562800000 \times 10^{-7}$	$3.8562800000 \times 10^{-7}$
	0.8	$2.3179040000 \times 10^{-6}$	$1.6832640000 \times 10^{-6}$	$1.0835570000 \times 10^{-6}$	$5.1239700000 \times 10^{-7}$	$5.1239700000 \times 10^{-7}$
	1	$2.8655420000 \times 10^{-6}$	$2.0823970000 \times 10^{-6}$	$1.3423590000 \times 10^{-6}$	$6.3748800000 \times 10^{-7}$	$6.3748800000 \times 10^{-7}$
	0.2	$6.5642700000 \times 10^{-7}$	$4.9400400000 \times 10^{-7}$	$3.3914500000 \times 10^{-7}$	$1.9276800000 \times 10^{-7}$	$1.9276800000 \times 10^{-7}$
	0.4	$1.3096920000 \times 10^{-6}$	$9.8624000000 \times 10^{-7}$	$6.7785000000 \times 10^{-7}$	$3.8605500000 \times 10^{-7}$	$3.8605500000 \times 10^{-7}$
0.3	0.6	$1.9537820000 \times 10^{-6}$	$1.4720680000 \times 10^{-6}$	$1.0127840000 \times 10^{-6}$	$5.7806700000 \times 10^{-7}$	$5.7806700000 \times 10^{-7}$
	0.8	$2.5842940000 \times 10^{-6}$	$1.9484160000 \times 10^{-6}$	$1.3421460000 \times 10^{-6}$	$7.6821500000 \times 10^{-7}$	$7.6821500000 \times 10^{-7}$
	1	$3.1969960000 \times 10^{-6}$	$2.4123230000 \times 10^{-6}$	$1.6641870000 \times 10^{-6}$	$9.5586800000 \times 10^{-7}$	$9.5586800000 \times 10^{-7}$
	0.2	$7.2188300000 \times 10^{-7}$	$5.5944200000 \times 10^{-7}$	$4.0328700000 \times 10^{-7}$	$2.5652100000 \times 10^{-7}$	$2.5652100000 \times 10^{-7}$
	0.4	$1.4414910000 \times 10^{-6}$	$1.1180020000 \times 10^{-6}$	$8.0703200000 \times 10^{-7}$	$5.1423300000 \times 10^{-7}$	$5.1423300000 \times 10^{-7}$
0.4	0.6	$2.1514870000 \times 10^{-6}$	$1.6697170000 \times 10^{-6}$	$1.2065920000 \times 10^{-6}$	$7.7025600000 \times 10^{-7}$	$7.7025600000 \times 10^{-7}$
	0.8	$2.8472250000 \times 10^{-6}$	$2.2112730000 \times 10^{-6}$	$1.5999330000 \times 10^{-6}$	$1.0237930000 \times 10^{-6}$	$1.0237930000 \times 10^{-6}$
	1	$3.5242520000 \times 10^{-6}$	$2.7394880000 \times 10^{-6}$	$1.9850950000 \times 10^{-6}$	$1.2740070000 \times 10^{-6}$	$1.2740070000 \times 10^{-6}$
	0.2	$7.8654200000 \times 10^{-7}$	$6.2423400000 \times 10^{-7}$	$4.6702100000 \times 10^{-7}$	$3.2001400000 \times 10^{-7}$	$3.2001400000 \times 10^{-7}$
	0.4	$1.5720280000 \times 10^{-6}$	$1.2488040000 \times 10^{-6}$	$9.3572800000 \times 10^{-7}$	$6.4215100000 \times 10^{-7}$	$6.4215100000 \times 10^{-7}$
0.5	0.6	$2.3474550000 \times 10^{-6}$	$1.8660800000 \times 10^{-6}$	$1.3998180000 \times 10^{-6}$	$9.6219500000 \times 10^{-7}$	$9.6219500000 \times 10^{-7}$
	0.8	$3.1079660000 \times 10^{-6}$	$2.4725350000 \times 10^{-6}$	$1.8570540000 \times 10^{-6}$	$1.2791210000 \times 10^{-6}$	$1.2791210000 \times 10^{-6}$
	1	$3.8489010000 \times 10^{-6}$	$3.0647790000 \times 10^{-6}$	$2.3052760000 \times 10^{-6}$	$1.5918960000 \times 10^{-6}$	$1.5918960000 \times 10^{-6}$

Table 6. Comparison at different fractional order of γ on the basis of error for example 3.

κ	φ	$\gamma = 0.4$	$\gamma = 0.6$	$\gamma = 0.8$	$\gamma = 1(NTDM_{CF})$	$\gamma = 1(NTDM_{ABC})$
0.1	0.2	$6.4600000000 \times 10^{-10}$	$4.8300000000 \times 10^{-10}$	$2.7900000000 \times 10^{-10}$	$6.7000000000 \times 10^{-11}$	$6.7000000000 \times 10^{-11}$
	0.4	$6.4300000000 \times 10^{-10}$	$4.8100000000 \times 10^{-10}$	$2.7700000000 \times 10^{-10}$	$6.6000000000 \times 10^{-11}$	$6.6000000000 \times 10^{-11}$
	0.6	$6.4400000000 \times 10^{-10}$	$4.8200000000 \times 10^{-10}$	$2.8000000000 \times 10^{-10}$	$6.9000000000 \times 10^{-11}$	$6.9000000000 \times 10^{-11}$
	0.8	$6.4500000000 \times 10^{-10}$	$4.8400000000 \times 10^{-10}$	$2.8200000000 \times 10^{-10}$	$7.1000000000 \times 10^{-11}$	$7.1000000000 \times 10^{-11}$
	1	$6.4100000000 \times 10^{-10}$	$4.8000000000 \times 10^{-10}$	$2.7900000000 \times 10^{-10}$	$6.9000000000 \times 10^{-11}$	$6.9000000000 \times 10^{-11}$
0.2	0.2	$6.7100000000 \times 10^{-10}$	$5.4000000000 \times 10^{-10}$	$3.5400000000 \times 10^{-10}$	$1.4300000000 \times 10^{-10}$	$1.4300000000 \times 10^{-10}$
	0.4	$6.6700000000 \times 10^{-10}$	$5.3700000000 \times 10^{-10}$	$3.5100000000 \times 10^{-10}$	$1.4000000000 \times 10^{-10}$	$1.4000000000 \times 10^{-10}$
	0.6	$6.5800000000 \times 10^{-10}$	$5.2800000000 \times 10^{-10}$	$3.4300000000 \times 10^{-10}$	$1.3300000000 \times 10^{-10}$	$1.3300000000 \times 10^{-10}$
	0.8	$6.6300000000 \times 10^{-10}$	$5.3300000000 \times 10^{-10}$	$3.4900000000 \times 10^{-10}$	$1.4000000000 \times 10^{-10}$	$1.4000000000 \times 10^{-10}$
	1	$6.5700000000 \times 10^{-10}$	$5.2800000000 \times 10^{-10}$	$3.4400000000 \times 10^{-10}$	$1.3500000000 \times 10^{-10}$	$1.3500000000 \times 10^{-10}$
0.3	0.2	$6.8100000000 \times 10^{-10}$	$5.7600000000 \times 10^{-10}$	$4.1200000000 \times 10^{-10}$	$2.1000000000 \times 10^{-10}$	$2.1000000000 \times 10^{-10}$
	0.4	$6.8400000000 \times 10^{-10}$	$5.8000000000 \times 10^{-10}$	$4.1700000000 \times 10^{-10}$	$2.1500000000 \times 10^{-10}$	$2.1500000000 \times 10^{-10}$
	0.6	$6.7400000000 \times 10^{-10}$	$5.7100000000 \times 10^{-10}$	$4.0700000000 \times 10^{-10}$	$2.0700000000 \times 10^{-10}$	$2.0700000000 \times 10^{-10}$
	0.8	$6.7700000000 \times 10^{-10}$	$5.7400000000 \times 10^{-10}$	$4.1100000000 \times 10^{-10}$	$2.1100000000 \times 10^{-10}$	$2.1100000000 \times 10^{-10}$
	1	$6.6500000000 \times 10^{-10}$	$5.6200000000 \times 10^{-10}$	$4.0000000000 \times 10^{-10}$	$2.0000000000 \times 10^{-10}$	$2.0000000000 \times 10^{-10}$
0.4	0.2	$6.9600000000 \times 10^{-10}$	$6.1600000000 \times 10^{-10}$	$4.7400000000 \times 10^{-10}$	$2.8700000000 \times 10^{-10}$	$2.8700000000 \times 10^{-10}$
	0.4	$6.8900000000 \times 10^{-10}$	$6.0900000000 \times 10^{-10}$	$4.6800000000 \times 10^{-10}$	$2.8100000000 \times 10^{-10}$	$2.8100000000 \times 10^{-10}$
	0.6	$6.8600000000 \times 10^{-10}$	$6.0600000000 \times 10^{-10}$	$4.6500000000 \times 10^{-10}$	$2.7900000000 \times 10^{-10}$	$2.7900000000 \times 10^{-10}$
	0.8	$6.8900000000 \times 10^{-10}$	$6.0900000000 \times 10^{-10}$	$4.6900000000 \times 10^{-10}$	$2.8300000000 \times 10^{-10}$	$2.8300000000 \times 10^{-10}$
	1	$6.7700000000 \times 10^{-10}$	$5.9700000000 \times 10^{-10}$	$4.5700000000 \times 10^{-10}$	$2.7200000000 \times 10^{-10}$	$2.7200000000 \times 10^{-10}$
0.5	0.2	$6.9800000000 \times 10^{-10}$	$6.3800000000 \times 10^{-10}$	$5.2000000000 \times 10^{-10}$	$3.5100000000 \times 10^{-10}$	$3.5100000000 \times 10^{-10}$
	0.4	$7.0100000000 \times 10^{-10}$	$6.4200000000 \times 10^{-10}$	$5.2400000000 \times 10^{-10}$	$3.5600000000 \times 10^{-10}$	$3.5600000000 \times 10^{-10}$
	0.6	$6.9700000000 \times 10^{-10}$	$6.3800000000 \times 10^{-10}$	$5.2100000000 \times 10^{-10}$	$3.5300000000 \times 10^{-10}$	$3.5300000000 \times 10^{-10}$
	0.8	$6.9300000000 \times 10^{-10}$	$6.3400000000 \times 10^{-10}$	$5.1700000000 \times 10^{-10}$	$3.4900000000 \times 10^{-10}$	$3.4900000000 \times 10^{-10}$
	1	$6.9000000000 \times 10^{-10}$	$6.3100000000 \times 10^{-10}$	$5.1500000000 \times 10^{-10}$	$3.4700000000 \times 10^{-10}$	$3.4700000000 \times 10^{-10}$

Table 7. Comparison of absolute error among $q - HATM$ [38], NDM_{CF} and NDM_{ABC} for example 3 at $k = 0.25$.

κ	φ	$\lvert q - HATM \rvert$	$\lvert NTDM_{CF} \rvert$	$\lvert NTDM_{ABC} \rvert$
0.25	1	7.0832×10^{-13}	$2.0000000000 \times 10^{-13}$	$2.0000000000 \times 10^{-13}$
	2	4.4031×10^{-13}	$1.0000000000 \times 10^{-13}$	$1.0000000000 \times 10^{-13}$
	3	1.1304×10^{-13}	$1.0000000000 \times 10^{-13}$	$1.0000000000 \times 10^{-13}$
	4	1.6642×10^{-13}	$1.0000000000 \times 10^{-13}$	$1.0000000000 \times 10^{-13}$
	5	3.3639×10^{-13}	$1.0000000000 \times 10^{-13}$	$1.0000000000 \times 10^{-13}$

Author Contributions: Formal analysis, N.A.S.; Methodology, R.S.; Project administration, Y.S.H.; Software, A.K.; Supervision, J.-D.C.; Validation, K.M.A.; Writing—original draft, R.S., and A.K.; Writing—review & editing, Y.S.H. and K.M.A. Also, N.A.S. and R.S. have equal contribution. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data were used to support this study.

Acknowledgments: This research was supported by Taif University Researchers Supporting Project Number (TURSP-2020/217), Taif University, Taif, Saudi Arabia. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2017R1D1A1B05030422).

Conflicts of Interest: The authors declare that they have no competing interest.

References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
2. Baleanu, D.; Guvenc, Z.B.; Tenreiro Machado, J.A. *New Trends in Nanotechnology and Fractional Calculus Applications*; Springer: New York, NY, USA, 2010.
3. Caputo, M. *Elasticita e Dissipazione*; Zanichelli: Bologna, Italy, 1969.
4. Riemann, G.F.B. *Versuch Einer Allgemeinen Auffassung der Integration und Differentiation*; Gesammelte Mathematische Werke: Leipzig, Germany, 1896.
5. Liouville, J. Memoire sur quelques questions de geometrie et de mecanique, et sur un nouveaugenre de calcul pour resoudre ces questions. *J. Ecole Polytech.* **1832**, *13*, 1–69.
6. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
7. Miller, K.S.; Ross, B. *An Introduction to Fractional Calculus and Fractional Differential Equations*; Wiley: New York, NY, USA, 1993.
8. Li, Y.; Liub, F.; Turner, I.W.; Li, T. Time-fractional diffusion equation for signal smoothing. *Appl. Math Comput.* **2018**, *326*, 108–116. [CrossRef]
9. Lin, W. Global existence theory and chaos control of fractional differential equations. *J. Math Anal. Appl.* **2007**, *332*, 709–726. [CrossRef]
10. Veeresha, P.; Prakasha, D.G.; Baskonus, H.M. Solving smoking epidemic model of fractional order using a modified homotopy analysis transform method. *Math. Sci.* **2019**, *13*, 115–128. [CrossRef]
11. Prakasha, D.G.; Veeresha, P.; Baskonus, H.M. Analysis of the dynamics of hepatitis E virus using the Atangana-Baleanu fractional derivative. *Eur. Phys. J. Plus.* **2019**, *134*, 1–11. [CrossRef]
12. Shah, N.; Alyousef, H.; El-Tantawy, S.; Shah, R.; Chung, J. Analytical Investigation of Fractional-Order Korteweg–De-Vries-Type Equations under Atangana–Baleanu–Caputo Operator: Modeling Nonlinear Waves in a Plasma and Fluid. *Symmetry* **2022**, *14*, 739. [CrossRef]
13. Sunthrayuth, P.; Zidan, A.; Yao, S.; Shah, R.; Inc, M. The Comparative Study for Solving Fractional-Order Fornberg–Whitham Equation via ρ-Laplace Transform. *Symmetry* **2021**, *13*, 784. [CrossRef]
14. Baleanu, D.; Wu, G.C.; Zeng, S.D. Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations. *Chaos Solitons Fractals* **2017**, *102*, 99–105. [CrossRef]
15. Scalas, E.; Gorenflo, R.; Mainardi, F. Fractional calculus and continuous-time finance. *Physica A* **2000**, *284*, 376–384. [CrossRef]
16. Drapaka, C.S.; Sivaloganathan, S. A fractional model of continuum mechanics. *J. Elst.* **2012**, *107*, 105–123. [CrossRef]
17. Cruz-Duarte, J.M.; Rosales-Garcia, J.; Correa-Cely, C.R.; Garcia-Perez, A.; Avina-Cervantes, J.G. A closed form expression for the Gaussian-based Caputo-Fabrizio fractional derivative for signal processing applications. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *61*, 138–148. [CrossRef]
18. Singh, B.K. A novel approach for numeric study of 2D biological population model. *Cogent Math.* **2016**, *3*, 1–15. [CrossRef]
19. El-Sayed, A.; Hamdallah, E.; Ba-Ali, M. Qualitative Study for a Delay Quadratic Functional Integro-Differential Equation of Arbitrary (Fractional) Orders. *Symmetry* **2022**, *14*, 784. [CrossRef]
20. Keskin, Y.; Oturanc, G. Reduced differential transform method for partial differential equations. *Int. J. Nonlinear Sci. Numer. Simul.* **2009**, *10*, 741–750. [CrossRef]
21. Momani, S.; Odibat, Z. Analytical solution of a time-fractional Navier-Stokes equation by Adomian decomposition method. *Appl. Math. Comput.* **2006**, *177*, 488–494. [CrossRef]
22. Wu, G.C. A fractional variational iteration method for solving fractional nonlinear differential equations. *Comput. Math. Appl.* **2011**, *61*, 2186–2190. [CrossRef]
23. Khan, H.; Khan, A.; Kumam, P.; Baleanu, D.; Arif, M. An approximate analytical solution of the Navier-Stokes equations within Caputo operator and Elzaki transform decomposition method. *Adv. Differ. Equ.* **2020**, *2011*, 1–23. [CrossRef]
24. Nonlaopon, K.; Alsharif, A.M.; Zidan, A.M.; Khan, A.; Hamed, Y.S.; Shah, R. Numerical investigation of fractional-order Swif-Hohenberg equations via a Novel transform. *Symmetry* **2021**, *13*, 1263. [CrossRef]
25. Khan, H.; Khan, A.; Al-Qurashi, M.; Shah, R.; Baleanu, D. Modified modelling for heat like equations within Caputo operator. *Energies* **2020**, *13*, 2002. [CrossRef]
26. Aljahdaly, N.; Agarwal, R.; Shah, R.; Botmart, T. Analysis of the Time Fractional-Order Coupled Burgers Equations with Non-Singular Kernel Operators. *Mathematics* **2021**, *9*, 2326. [CrossRef]
27. Qin, Y.; Khan, A.; Ali, I.; Al Qurashi, M.; Khan, H.; Shah, R.; Baleanu, D. An efficient analytical approach for the solution of certain fractional-order dynamical systems. *Energies* **2020**, *13*, 2725. [CrossRef]
28. Alaoui, M.K.; Fayyaz, R.; Khan, A.; Shah, R.; Abdo, M.S. Analytical Investigation of Noyes-Field Model for Time-Fractional Belousov-Zhabotinsky Reaction. *Complexity* **2021**, *2021*, 3248376. [CrossRef]
29. Kaup, D.J. On the inverse scattering for cubic eigenvalue problems of the class equations. *Stud. Appl. Math.* **1980**, *62*, 183–195. [CrossRef]
30. Kupershmidt, B.A. A super Korteweg-de-Vries equations: An integrable system. *Phys. Lett. A* **1994**, *102*, 213–218. [CrossRef]
31. Fan, E. Uniformly constructing a series of explicit exact solutions to nonlinear equations in mathematical physics. *Chaos Solitons Fractals* **2003**, *16*, 819–839. [CrossRef]

32. Inc, M. On numerical soliton solution of the Kaup-Kupershmidt equation and convergence analysis of the decomposition method. *Appl. Math. Comput.* **2006**, *172*, 72–85.
33. Rashid, S.; Khalid, A.; Sultana, S.; Hammouch, Z.; Shah, R.; Alsharif, A. A Novel Analytical View of Time-Fractional Korteweg-De Vries Equations via a New Integral Transform. *Symmetry* **2021**, *13*, 1254. [CrossRef]
34. Tam, H.W.; Hu, X.B. Two integrable differential-difference equations exhibiting soliton solutions of the Kaup-Kupershimdt equation type. *Phys. Lett. A* **2000**, *272*, 174–183. [CrossRef]
35. Zhou, M.X.; Kanth, A.S.V.; Aruna, K.; Raghavendar, K.; Rezazadeh, H.; Inc, M.; Aly, A.A. Numerical Solutions of Time Fractional Zakharov-Kuznetsov Equation via Natural Transform Decomposition Method with Nonsingular Kernel Derivatives. *J. Funct. Spaces* **2021**, *2021*, 9884027. [CrossRef]
36. Adomian, G. A new approach to nonlinear partial differential equations. *J. Math. Anal. Appl.* **1984**, *102*, 420–434. [CrossRef]
37. Shah, R.; Khan, H.; Baleanu, D.; Kumam, P.; Arif, M. The analytical investigation of time-fractional multi-dimensional Navier–Stokes equation. *Alex. Eng. J.* **2020**, *59*, 2941–2956. [CrossRef]
38. Prakasha, D.G.; Malagi, N.S.; Veeresha, P.; Prasannakumara, B.C. An efficient computational technique for time-fractional Kaup-Kupershmidt equation. *Numer. Methods Partial. Differ. Equ.* **2021**, *37*, 1299–1316. [CrossRef]
39. Gupta, A.K.; Ray, S.S. The comparison of two reliable methods for accurate solution of time-fractional Kaup-Kupershmidt equation arising in capillary gravity waves. *Math. Methods Appl. Sci.* **2016**, *39*, 583–592. [CrossRef]