symmetry

MDPI

Article

A Note on q-analogue of Degenerate Catalan Numbers Associated with p-adic Integral on $\mathbb{Z}_p$

Waseem A. Khan

Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia; wkhan1@pmu.edu.sa

Abstract: In this paper, we introduce q-analogues of degenerate Catalan numbers and polynomials with the help of a fermionic p-adic q-integrals on $\mathbb{Z}_p$ and establish some new connections with the degenerate Stirling numbers of the first and second kinds. Furthermore, we also find a few new identities and results of this type of polynomials and numbers.

Keywords: q-Catalan numbers; q-analogue of degenerate Catalan polynomials and numbers; Stirling numbers

MSC: 11B83; 11B73; 05A19

check for updates

Citation: Khan, W.A. A Note on q-analogue of Degenerate Catalan Numbers Associated with p-adic Integral on $\mathbb{Z}_p$. *Symmetry* **2022**, *14*, 1119. https://doi.org/10.3390/sym14061119

Academic Editor: Serkan Araci

Received: 2 May 2022
Accepted: 20 May 2022
Published: 29 May 2022

1. Introduction

In recent years, some researchers have used generating functions to study these combinatorial numbers and polynomials to obtain the various identities associated with those numbers and polynomials. There is a study of degenerate versions of special numbers and polynomials with the degenerate Bernoulli and Euler polynomials by Carlitz [1,2]. The examination of numerous degenerate versions of special numbers and polynomials is to apply to differential equations, identification of symmetry, and possibility principle, in addition to a few mathematics and combinatorial components [3,4]. Catalan numbers and polynomials [5] were derived by utilizing umbral calculus strategies. The own family of linear differential equations springing up from the generating feature of Catalan numbers had been considered in [6] to be able to derive a few express identities involving Catalan numbers. In [4], w-Catalan polynomials had been introduced as a generalization of Catalan polynomials and plenty of symmetric identities in three variables associated with the w-Catalan polynomials and analogues of alternating strength sums have been received by p-adic fermionic integrals. In [7], the authors constructed producing features for new instructions of Catalan-type numbers and polynomials. Using those capabilities and their purposeful equations, they gave diverse new identities and polynomials, and different classes of special numbers, polynomials, and features. In [8], authors studied the q-analogues of the Catalan numbers and polynomials with the assist of fermionic p-adic q-integrals on $\mathbb{Z}_p$ and derived explicit expressions and a few identities for the ones numbers. Moreover, they deduced specific expressions of C_n, q, as a rational characteristic in terms of q-Euler numbers and Stirling numbers of the primary type, as a fermionic p-adic q-integrals on $\mathbb{Z}_p$, and regarding (q,λ)-Changhee numbers. Indeed, the authors constructed polynomial extension of the q-analogues of Catalan numbers, particularly the q-analogues of Catalan polynomials $C_{n,q}(x)$ and derived explicit expressions in phrases of Catalan numbers and Stirling numbers of the primary kind and of q-Euler polynomials and Stirling numbers of the first kind.

Let p be a fixed odd prime number and $\mathbb{Z}_p$, the ring of p-adic integers, $\mathbb{Q}_p$ and $\mathbb{C}_p$, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The $|\ .\ |_p$ is p-adic norm and $|\ p\ |_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$

with $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-analogue of ξ is defined through $[\xi]_q = \frac{1-q^{\xi}}{1-q}$. Note that $\lim_{q\to 1}[\xi]_q = \xi$. Let f be a uniformly differentiable function on $\mathbb{Z}_p$. The fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by (see [9–11])

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(\xi)d\mu_{-q}(\xi) = \lim_{N\to\infty}\sum_{\xi=0}^{p^N-1} f(\xi)\mu_{-q}(\xi + p^N\mathbb{Z}_p),$$

$$= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{\xi=0}^{p^N-1} f(\xi)(-q)^{\xi}. \tag{1}$$

Let $f_1(\xi) = f(\xi+1)$. By (1), we acquire

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{2}$$

The Euler numbers are defined by

$$\frac{2}{e^z+1} = \sum_{j=0}^{\infty} E_j \frac{z^j}{j!}. \tag{3}$$

From (2), the q-analogues of Euler numbers are given by (see [12])

$$\int_{\mathbb{Z}_p} \xi^j d\mu_{-q}(\xi) = \frac{[2]_q}{qe^z+1} = \sum_{j=0}^{\infty} E_{j,q}\frac{z^j}{j!}. \tag{4}$$

Note that $\lim_{q\to 1} E_{j,q} = E_j, (j \geq 0)$.
The q-Changhee numbers are given by (see [8])

$$\frac{[2]_q}{[2]_q + z} = \sum_{j=0}^{\infty} Ch_{j,q}\frac{z^j}{j!}. \tag{5}$$

Let $\lambda \in \mathbb{Z}_p$, the λ-Changhee polynomials are defined by (see [5,8])

$$\frac{2}{(1+z)^{\lambda}+1}(1+z)^{\lambda\xi} = \sum_{j=0}^{\infty} Ch_{j,\lambda}(\xi)\frac{z^j}{j!}, \tag{6}$$

At the point $\xi = 0$, $Ch_{j,\lambda} = Ch_{j,\lambda}(0)$ are called the λ-Changhee numbers.
For $j \geq 0$, the Stirling numbers of the first kind are given by

$$(\xi)_j = \sum_{l=0}^{j} S_1(j,l)\xi^l, \tag{7}$$

where $(\xi)_0 = 1$, and $(\xi)_j = \xi(\xi-1)\cdots(\xi-j+1), (j \geq 1)$. From (7), we get (see [13,14])

$$\frac{1}{r!}(\log(1+z))^r = \sum_{j=r}^{\infty} S_1(j,r)\frac{z^j}{j!}, \quad (r \geq 0). \tag{8}$$

For $j \geq 0$, the Stirling numbers of the second kind are given by

$$\xi^j = \sum_{l=0}^{j} S_2(j,l)(\xi)_l. \tag{9}$$

From (9), we get (see [8,14])

$$\frac{1}{r!}(e^z-1)^r=\sum_{j=r}^{\infty}S_2(j,r)\frac{z^j}{j!}. \tag{10}$$

The Catalan polynomials are defined by (see [5,6])

$$\begin{aligned}\int_{\mathbb{Z}_p}(1-4z)^{\frac{\xi+\eta}{2}}d\mu_{-1}(\eta)&=\frac{2}{1+\sqrt{1-4z}}(1-4z)^{\frac{\xi}{2}}\\ &=\sum_{j=0}^{\infty}C_j(\xi)z^j.\end{aligned} \tag{11}$$

At the point $\xi=0$, $C_j=C_j(0)$ are called the Catalan numbers.
Thus, by (11), we have

$$C_j(\xi)=\sum_{l=0}^{j}\sum_{j=0}^{l}\left(\frac{\xi}{2}\right)^j S_1(l,j)(-4)^l\frac{C_{j-l}}{l!}.$$

The $\frac{1}{2}$-Changhee polynomials are given by (see [5])

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+z)^{\frac{\xi+\eta}{2}}d\mu_{-1}(\eta)&=\frac{2}{1+\sqrt{1+z}}\sqrt{(1+z)^{\xi}}\\ &=\sum_{j=0}^{\infty}Ch_{j,\frac{1}{2}}(\xi)\frac{z^j}{j!}.\end{aligned} \tag{12}$$

In the case $\xi=0$, $Ch_{j,\frac{1}{2}}=Ch_{j,\frac{1}{2}}(0)$ are called the $\frac{1}{2}$-Changhee numbers.
We note that

$$C_j(\xi)=\frac{(-1)^j}{j!}Ch_{j,\frac{1}{2}}(\xi)2^{2j}.$$

The q-analogues of Catalan polynomials which are given by (see [8])

$$\begin{aligned}\int_{\mathbb{Z}_p}(1-4z)^{\frac{\xi+\eta}{2}}d\mu_{-q}(\eta)&=\frac{[2]_q}{1+q\sqrt{1-4z}}(1-4z)^{\frac{\xi}{2}}\\ &=\sum_{j=0}^{\infty}C_{j,q}(\xi)z^j.\end{aligned} \tag{13}$$

When $\xi=0$, $C_{j,q}=C_{j,q}(0)$ are called the q-Catalan numbers.
The degenerate q-Daehee polynomials are defined by (see [15])

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\log(1+\lambda z)^{\frac{1}{\lambda}})^{\xi+\eta}d\mu_q(\eta)&=\frac{q-1+\frac{q-1}{\log q}\log(1+\log(1+\lambda z)^{\frac{1}{\lambda}})}{q-1+q\log(1+\lambda z)^{\frac{1}{\lambda}}}(1+\log(1+\lambda z)^{\frac{1}{\lambda}})^{\xi}\\ &=\sum_{j=0}^{\infty}D_{j,q}(\xi|\lambda)\frac{z^j}{j!}.\end{aligned} \tag{14}$$

In the case when $\xi=0$, $D_{j,q}(\lambda)=D_{j,q}(0|\lambda)$ are called the degenerate q-Daehee numbers.

Note that

$$\lim_{\lambda\to 0}D_{j,q}(\xi|\lambda)=D_{j,q}(\xi),(j\geq 0).$$

Motivated by the works of Kim and Kim et al. [8,16], we introduce the q-analogues of degenerate Catalan numbers $C_{j,q,\lambda}$ with the help of a fermionic p-adic q-integrals on

$\mathbb{Z}_p$ and obtain several explicit expressions and some identities for those numbers. Also, we express the q-analogues of degenerate Catalan numbers $C_{j,q,\lambda}$, in terms of the q-Euler numbers, q-analogues of degenerate Catalan numbers, and Stirling numbers of the first and second kinds. Furthermore, we obtain the explicit expression of the q-analogues of degenerate Catalan polynomials $C_{j,q,\lambda}(\xi)$, which involves the q-analogue of $\frac{1}{2}$-Changhee polynomials $Ch_{j,\frac{1}{2},q}(\xi)$.

2. q-analogues of Degenerate Catalan Numbers Associated with p-adic Integral on $\mathbb{Z}_p$

In this section, we consider q-analogues of degenerate Catalan numbers by using the fermionic p-adic q-integrals on $\mathbb{Z}_p$. Here, the function $\log(1+\lambda z)^{\frac{1}{\lambda}}$ is called the degenerate function of z. Let us start the following definition as.

For $\lambda, z, q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $|\lambda z| < p^{-\frac{1}{p-1}}$. Let us take $f(\xi) = (1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}$ in (2). Then, we have

$$\int_{\mathbb{Z}_p}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}d\mu_{-q}(\xi) = \frac{[2]_q}{q\sqrt{1-4\log(1+\lambda z)^{\frac{1}{\lambda}}}+1}$$
$$= \frac{[2]_q}{1-q^2+4q^2\log(1+\lambda z)^{\frac{1}{\lambda}}}(1-q\sqrt{1-4\log(1+\lambda z)^{\frac{1}{\lambda}}}). \quad (15)$$

Now, we define the q-analogues of degenerate Catalan numbers which are given by the generating function

$$\frac{[2]_q}{1-q^2+4q^2\log(1+\lambda z)^{\frac{1}{\lambda}}}(1-q\sqrt{1-4\log(1+\lambda z)^{\frac{1}{\lambda}}}) = \sum_{j=0}^{\infty} C_{j,\lambda,q}z^j. \quad (16)$$

From (13) and (16), we note that

$$\sum_{j=0}^{\infty}\lim_{\lambda\to 0} C_{j,\lambda,q}z^j = \frac{[2]_q}{q\sqrt{1-4z}+1} = \sum_{j=0}^{\infty} C_{j,q}z^j. \quad (17)$$

Thus, by (17), we get

$$\lim_{\lambda\to 0} C_{j,\lambda,q} = C_{j,q}, \quad (j \geq 0).$$

Theorem 1. *For $j \geq 0$, we have*

$$C_{j,\lambda,q} = \sum_{s=0}^{j} E_{s,q}2^s\lambda^{j-s}S_1(j,s)\frac{(-1)^s}{j!}.$$

Proof. From (4) and (16), we have

$$\int_{\mathbb{Z}_p}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}d\mu_{-q}(\xi) = \sum_{s=0}^{\infty}\int_{\mathbb{Z}_p}\xi^s d\mu_{-q}(\xi)\frac{1}{2^s}\frac{1}{s!}(-4\log(1+\lambda z)^{\frac{1}{\lambda}})^s$$
$$= \sum_{s=0}^{\infty}E_{s,q}(-2)^s\lambda^{j-s}\sum_{j=s}^{\infty}S_1(j,s)\frac{z^j}{j!}$$
$$= \sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}E_{s,q}2^s(-1)^s\lambda^{j-s}S_1(j,s)\right)\frac{z^j}{j!}. \quad (18)$$

By (16) and (18), we obtain the result. □

Theorem 2. *For $j \geq 0$, we have*

$$C_{j,q} = \sum_{s=0}^{j} C_{s,\lambda,q} \lambda^{j-s} s! S_2(j,s) \frac{s!}{j!}.$$

Proof. By replacing z by $\frac{1}{\lambda}(e^{\lambda z}-1)$ in (16), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} (1-4z)^{\frac{\xi}{2}} d\mu_{-q}(\xi) &= \sum_{s=0}^{\infty} C_{s,\lambda,q} s! \frac{(\frac{1}{\lambda}(e^{\lambda z}-1))^s}{s!} \\ &= \sum_{s=0}^{\infty} C_{s,\lambda,q} \lambda^{-s} s! \sum_{j=s}^{\infty} S_2(j,s) \lambda^j \frac{z^j}{j!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{s=0}^{j} C_{s,\lambda,q} \lambda^{j-s} s! S_2(j,s) \right) \frac{z^j}{j!}. \end{aligned} \tag{19}$$

On the other hand,

$$\int_{\mathbb{Z}_p} (1-4z)^{\frac{\xi}{2}} d\mu_{-q}(\xi) = \frac{[2]_q}{q\sqrt{1-4z}+1} = \sum_{j=0}^{\infty} C_{j,q} z^j. \tag{20}$$

Therefore, by (19) and (20), we get the result. □

Theorem 3. *For $j \geq 0$, we have*

$$C_{j,\lambda,q} = \sum_{s=0}^{j} C_{s,q} 2^{2s} \lambda^{j-s} S_1(j,s) \frac{(-1)^s}{s!}.$$

Proof. From (16), we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p} (1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}} d\mu_{-q}(\xi) &= \sum_{s=0}^{\infty} (-1)^s 4^s \int_{\mathbb{Z}_p} \binom{\frac{\xi}{2}}{s} d\mu_{-q}(\xi) \frac{[\log(1+\lambda z)^{\frac{1}{\lambda}}]^s}{s!} \\ &= \sum_{s=0}^{\infty} C_{s,q} (-1)^s 4^s \lambda^{j-s} \sum_{j=s}^{\infty} S_1(j,s) \frac{z^j}{j!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{s=0}^{j} C_{s,q} (-1)^s 4^s \lambda^{j-s} S_1(j,s) \right) \frac{z^j}{j!}. \end{aligned} \tag{21}$$

In (16) and (21), we obtain the result. □

Theorem 4. *For $j \geq 0$, we have*

$$C_{j,\lambda,q} + q \sum_{k=0}^{j} \sum_{s=0}^{k} \lambda^{k-s} 2^s S_1(k,s) C_{j-k,\lambda,q} \frac{(-1)^s}{k!} = \begin{cases} [2]_q, & \text{if } j=0 \\ 0, & \text{if } j>0. \end{cases}$$

Proof. First, we note that

$$\begin{aligned}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{1}{2}} &= \sum_{s=0}^{\infty} \left(\frac{1}{2}\right)^s (-1)^s 4^s \frac{[\log(1+\lambda z)^{\frac{1}{\lambda}}]^s}{s!} \\ &= \sum_{s=0}^{\infty} \left(\frac{1}{2}\right)^s \lambda^{-s} (-1)^s 4^s \sum_{j=s}^{\infty} S_1(j,s) \frac{\lambda^j z^j}{j!} \end{aligned}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}\left(\frac{1}{2}\right)_s \lambda^{j-s}(-1)^s 4^s S_1(j,s)\right)\frac{z^j}{j!}. \tag{22}$$

By (16) and (22), we get

$$[2]_q = \left(\sum_{j=0}^{\infty} C_{j,\lambda,q} z^j\right)\left(q(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{1}{2}}+1\right)$$

$$= \sum_{j=0}^{\infty} C_{j,\lambda,q} z^j + q\left(\sum_{j=0}^{\infty} C_{j,\lambda,q} z^j\right)\left(\sum_{k=0}^{\infty}\left(\sum_{s=0}^{k}\left(\frac{1}{2}\right)_s \lambda^{k-s}(-1)^s 4^s S_1(k,s)\right)\frac{z^k}{k!}\right)$$

$$= \sum_{j=0}^{\infty} C_{j,\lambda,q} z^j + q\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{s=0}^{k} \lambda^{k-s}(-1)^s 2^s S_1(k,s) C_{j-k,\lambda,q}\frac{1}{k!}\right) z^j. \tag{23}$$

In view of (23), we obtain the result. □

Theorem 5. *For $j \geq 0$, we have*

$$C_{j,\lambda,q} = \sum_{s=0}^{j} C_{s,q} \lambda^{j-s} s! S_1(j,s)\frac{s!}{j!}.$$

Proof. By replacing z by $\log(1+\lambda z)^{\frac{1}{\lambda}}$ in (13), we get

$$\frac{[2]_q}{q\sqrt{1-4\log(1+\lambda z)^{\frac{1}{\lambda}}}+1} = \sum_{s=0}^{\infty} C_{s,q} s! \frac{[\log(1+\lambda z)^{\frac{1}{\lambda}}]^s}{s!}$$

$$= \sum_{s=0}^{\infty} C_{s,q} \lambda^{-s} s! \frac{(\log(1+\lambda z))^s}{s!}$$

$$= \sum_{s=0}^{\infty} C_{s,q} \lambda^{-s} s! \sum_{j=s}^{\infty} S_1(j,s)\frac{\lambda^j z^j}{j!}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{s=0}^{j} C_{s,q} \lambda^{j-s} s! S_1(j,s)\right)\frac{z^j}{j!}. \tag{24}$$

Therefore, by way of (16) and (24), we get the result. □

Now, we observe that

$$(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}} = \sum_{s=0}^{\infty}\left(\frac{\xi}{2}\right)_s (-1)^s 4^s \frac{[\log(1+\lambda z)^{\frac{1}{\lambda}}]^s}{s!}$$

$$= \sum_{s=0}^{\infty}\left(\frac{\xi}{2}\right)_s \lambda^{-s}(-1)^s 4^s \sum_{j=s}^{\infty} S_1(j,s)\frac{\lambda^j z^j}{j!}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}\left(\frac{\xi}{2}\right)_s \lambda^{j-s}(-1)^s 4^s S_1(j,s)\right)\frac{z^j}{j!}. \tag{25}$$

Now, we consider the q-analogues of degenerate Catalan polynomials which are given by the generating function to be

$$\int_{\mathbb{Z}_p}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi+\eta}{2}} d\mu_{-q}(\eta) = \frac{[2]_q}{q\sqrt{1-4\log(1+\lambda z)^{\frac{1}{\lambda}}}+1}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}$$

$$= \sum_{j=0}^{\infty} C_{j,\lambda,q}(\xi) z^j. \tag{26}$$

When $\xi = 0$, $C_{j,\lambda,q} = C_{j,\lambda,q}(0)$ are called the q-analogues of degenerate Catalan numbers.

Theorem 6. *For $j \geq 0$, we have*

$$C_{j,\lambda,q}(x) = \sum_{l=0}^{j} \sum_{s=0}^{l} \binom{\frac{\xi}{2}}{s} (-1)^s 2^{2s} \lambda^{l-s} S_1(l,s) C_{j-l,\lambda,q} \frac{s!}{l!}.$$

Proof. From (26), we note that

$$\frac{[2]_q}{q\sqrt{1-4\log(1+\lambda z)^{\frac{1}{\lambda}}}+1}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}$$
$$= \sum_{j=0}^{\infty} C_{j,\lambda,q} z^j \sum_{s=0}^{\infty} \binom{\frac{\xi}{2}}{s} (-1)^s 2^{2s} s! \frac{(\log(1+\lambda z)^{\frac{1}{\lambda}})^s}{s!}$$
$$= \sum_{j=0}^{\infty} C_{j,\lambda,q} z^j \sum_{s=0}^{\infty} \binom{\frac{\xi}{2}}{s} (-1)^s 2^{2s} \lambda^{-s} s! \sum_{l=s}^{\infty} S_1(l,s) \frac{\lambda^l z^l}{l!}$$
$$= \sum_{j=0}^{\infty} C_{j,\lambda,q} z^j \sum_{l=0}^{\infty} \sum_{s=0}^{l} \binom{\frac{\xi}{2}}{s} (-1)^s 2^{2s} \lambda^{l-s} s! S_1(l,s) \frac{z^l}{l!}$$
$$= \sum_{j=0}^{\infty} \left(\sum_{l=0}^{j} \sum_{m=0}^{l} \binom{\frac{\xi}{2}}{s} (-1)^s 2^{2s} \lambda^{l-s} S_1(l,s) C_{j-l,\lambda,q} \frac{s!}{l!} \right) z^j. \tag{27}$$

By (26) and (27), we get the result. □

Theorem 7. *For $j \geq 0$, we have*

$$C_{j,\lambda,q}(\xi) = \sum_{k=0}^{j} \sum_{s=0}^{k} \left(\frac{\xi}{2}\right)_s \lambda^{k-s} (-1)^s 4^s S_1(k,s) C_{j-k,\lambda,q} \frac{1}{k!}.$$

Proof. From (26), we see that

$$\sum_{j=0}^{\infty} C_{j,\lambda,q}(\xi) z^j = \frac{[2]_q}{q\sqrt{1-4\log(1+\lambda z)^{\frac{1}{\lambda}}}+1}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}$$
$$= \left(\sum_{j=0}^{\infty} C_{j,\lambda,q} z^j \right) \left(\sum_{k=0}^{\infty} \left(\sum_{s=0}^{k} \left(\frac{\xi}{2}\right)_s \lambda^{k-s} (-1)^s 4^s S_1(k,s) \right) \frac{z^k}{k!} \right)$$
$$= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \sum_{s=0}^{k} \left(\frac{\xi}{2}\right)_s \lambda^{k-s} S_1(k,s) (-1)^s 4^s C_{j-k,\lambda,q} \frac{1}{k!} \right). \tag{28}$$

By (28), we obtain the result. □

Theorem 8. *For $j \geq 0$, we have*

$$C_{j,q}(\xi) = \sum_{s=0}^{j} C_{s,\lambda,q}(\xi) \lambda^{j-s} S_1(j,s) \frac{s!}{j!}.$$

Proof. By replacing z by $\frac{1}{\lambda}e^{\lambda z}-1$ in (26), we have

$$\int_{\mathbb{Z}_p}(1-4z)^{\frac{\xi+\eta}{2}}d\mu_{-q}(\eta)=\frac{[2]_q}{q\sqrt{1-4z}+1}\sqrt{(1-4z)^{\xi}}=\sum_{j=0}^{\infty}C_{j,q}(\xi)z^j. \tag{29}$$

On the other hand,

$$\begin{aligned}\sum_{s=0}^{\infty}C_{s,\lambda,q}(\xi)s!\frac{(\frac{1}{\lambda}e^{\lambda z}-1)^s}{s!}&=\sum_{s=0}^{\infty}C_{s,\lambda,q}(\xi)s!\lambda^{-s}\sum_{j=s}^{\infty}S_1(j,s)\frac{\lambda^j z^j}{j!}\\&=\sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}C_{s,\lambda,q}(\xi)\lambda^{j-s}S_1(j,s)s!\right)\frac{z^j}{j!}.\end{aligned} \tag{30}$$

In (29) and (30), we get the result. □

Theorem 9. *For $j\geq 0$, we have*

$$Ch_{j,\frac{1}{2},q}(\xi)=(-1)^j\sum_{s=0}^{j}C_{s,\lambda,q}(\xi)s!\lambda^{j-s}s!4^jS_2(j,s).$$

Proof. On replacing z by $\frac{e^{-\frac{\lambda z}{4}}-1}{\lambda}$ in (16), we have

$$\int_{\mathbb{Z}_p}(1+z)^{\frac{\xi+\eta}{2}}d\mu_{-q}(\eta)=\frac{[2]_q}{q\sqrt{1+z}+1}\sqrt{(1+z)^{\xi}}=\sum_{j=0}^{\infty}Ch_{j,\frac{1}{2},q}(\xi)\frac{z^j}{j!}. \tag{31}$$

On the other hand,

$$\begin{aligned}\sum_{s=0}^{\infty}C_{s,\lambda,q}(\xi)s!\frac{(\frac{e^{-\frac{\lambda z}{4}}-1}{\lambda})^s}{s!}&=\sum_{s=0}^{\infty}C_{s,\lambda,q}(\xi)s!\lambda^{-s}\sum_{j=s}^{\infty}S_2(j,s)\frac{(-4)^j\lambda^j z^j}{j!}\\&=\sum_{j=0}^{\infty}\sum_{s=0}^{j}C_{s,\lambda,q}(\xi)s!\lambda^{j-s}(-1)^j4^jS_2(j,s)\frac{z^j}{j!}.\end{aligned} \tag{32}$$

In (31) and (32), we obtain the result. □

From (2), we note that (see [12])

$$\sum_{j=0}^{\infty}\int_{\mathbb{Z}_p}(\xi+\eta)^jd\mu_{-q}(\eta)\frac{z^j}{j!}=\int_{\mathbb{Z}_p}e^{(\xi+\eta)z}d\mu_{-q}(\eta)=\frac{[2]_q}{qe^z+1}e^{\xi z}=\sum_{j=0}^{\infty}E_{j,q}(\xi)\frac{z^j}{j!}, \tag{33}$$

where

$$E_{j,q}(\xi)=\sum_{j=0}^{k}\binom{j}{k}E_{k,q}\xi^{j-k}=\int_{\mathbb{Z}_p}(\xi+\eta)^jd\mu_{-q}(\eta)$$

are the q-Euler polynomials.

Theorem 10. *For $j\geq 0$, we have*

$$C_{j,\lambda,q}(\xi)=\sum_{s=0}^{j}2^s(-1)^sE_{s,q}(\xi)\lambda^{j-s}S_1(j,s)\frac{1}{j!}.$$

Proof. From (26) and (33), we have

$$\frac{[2]_q}{q\sqrt{1-4\log(1+\lambda z)^{\frac{1}{\lambda}}}+1}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}} = \int_{\mathbb{Z}_p}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi+\eta}{2}} d\mu_{-q}(\eta)$$
$$= \sum_{s=0}^{\infty} 2^{-s}\frac{1}{s!}(-4\log(1+\lambda z)^{\frac{1}{\lambda}})^s \int_{\mathbb{Z}_p}(\xi+\eta)^s d\mu_{-q}(\eta)$$
$$= \sum_{s=0}^{\infty} 2^s(-1)^s E_{s,q}(\xi)\lambda^{-s}\sum_{j=s}^{\infty} S_1(j,s)\frac{\lambda^j z^j}{j!}$$
$$= \sum_{j=0}^{\infty}\left(\sum_{s=0}^{j} 2^s(-1)^s E_{s,q}(\xi)\lambda^{j-s}S_1(j,s)\frac{1}{j!}\right)z^j. \quad (34)$$

By (26) and (34), we get the result. □

3. Conclusions

In this paper, we introduced the q-analogues of degenerate Catalan numbers $C_{j,q,\lambda}$ with the help of a p-adic q-integrals on $\mathbb{Z}_p$ and obtained several explicit expressions and some identities for those numbers. In more detail, we expressed the q-analogues of degenerate Catalan numbers $C_{j,q,\lambda}$, in terms of the Euler numbers, q-analogues of degenerate Catalan numbers, and Stirling numbers of the first and second kinds. After that, we obtained the explicit expression of the q-analogues of degenerate Catalan polynomials $C_{j,q,\lambda}(\xi)$ which involves the q-analogue of $\frac{1}{2}$-Changhee polynomials $Ch_{j,\frac{1}{2},q}(\xi)$. It is one of future projects to continue to study the degenerate twisted Catalan numbers and q-analogues of degenerate twisted Catalan numbers along the the line of the direction of degenerate Catalan-type paper.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data were used to support the study.

Acknowledgments: The author Waseem A. Khan thanks Prince Mohammad Bin Fahd University, Saudi Arabia for providing facilities and support.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers. *Utilitas Math.* **1979**, *15*, 51–88.
2. Carlitz, L. A degenerate Staudt-Clausen theorem. *Arch. Math.* **1956**, *7*, 28–33. [CrossRef]
3. Haroon, H.; Khan, W.A. Degenerate Bernoulli numbers and polynomials associated with Hermite polynomials. *Commun. Korean Math. Soc.* **2018**, *33*, 651–669.
4. Kim, D.S.; Kim, T. Triple symmetric identities for w-Catalan polynomials. *J. Korean Math. Soc.* **2017**, *54*, 1243–1264.
5. Kim, T. A note on Catalan numbers associated with p-adic integral on Z_p. *Proc. Jangjeon Math. Soc.* **2016**, *19*, 493–501.
6. Kim, D.S.; Kim, T. A new approach to Catalan numbers using differential equations. *Russ. J. Math. Phys.* **2017**, *24*, 465–475. [CrossRef]
7. Kucukoglu, I.; Simsek, B.; Simsek, Y. New classes of Catalan-type numbers and polynomials with their applications related to p-adic integrals and computational algorithms. *Turk. J. Math.* **2020**, *44*, 2337–2355. [CrossRef]
8. Kim, T.; Kim, D.S.; Kwon, J. A note on q-analogues of Catalan numbers arising from fermionic p-adic q-integral on $\mathbb{Z}_p$. *Adv. Stud. Contemp. Math.* **2021**, *31*, 279–286.
9. Araci, S.; Acikgoz, M.; Kilicman, A. Extended p-adic q-invariant integrals on Z_p associated with applications of umbral calculus. *Adv. Differ. Equ.* **2013**, *2013*, 14. [CrossRef]
10. Duran, U.; Acikgoz, M. On applications for Mahler expansion associated with p-adic q-integrals. *Int. J. Number Theory* **2019**, *15*, 67–84. [CrossRef]
11. Kim, T. q-Volkenborn integration. *Russ. J. Math. Phys.* **2002**, *9*, 288–299.

12. Cangul, I.N.; Kurt, V.; Simsek, Y.; Pak, H.K.; Rim, S.-H. An invariant p-adic q-integral associated with q-Euler numbers and polynomials. *J. Nonlinear Math. Phys.* **2007**, *14*, 8–14. [CrossRef]
13. Kim, T.; Kim, D.S. Some identities of Catalan-Daehee polynomials arising from umbral calculus. *Appl. Comput. Math.* **2017**, *16*, 177–189.
14. Khan, W.A. A note on q-analogues of degenerate Catalan-Daehee numbers and polynomials. *J. Math.* **2022**, *2022*, 9486880. [CrossRef]
15. Jeong, J.; Kang, D.-J.; Rim, S.H. On degenerate q-Daehee numbers and polynomials. *Int. J. Math. Anal*. **2015**, *9*, 2157–2170. [CrossRef]
16. Kim, T. On the degenerate Cauchy numbers and polynomials. *Proc. Jangjeon Math. Soc.* **2015**, *18*, 307–312.