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Sequence Spaces and Spectrum of q-Difference Operator of Second Order

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Abstract: The sequence spaces $\ell_p(\nabla_q^2)$ $(0 \leq p < \infty)$ and $\ell_\infty(\nabla_q^2)$ are introduced by using the q-difference operator ∇_q^2 of the second order. Apart from studying some basic properties of these spaces, we construct the basis and obtain the α-, β- and γ-duals of these spaces. Besides some matrix classes involving q-difference sequence spaces, $\ell_p(\nabla_q^2)$ and $\ell_\infty(\nabla_q^2)$ are characterized. The final section is devoted to classifying the spectrum of the q-difference operator ∇_q^2 over the space ℓ_1 of absolutely summable sequences.

Keywords: q-difference operator; sequence spaces; duals; matrix transformation; spectrum

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1. Introduction and Preliminaries

A linear subspace of ω, i.e., the set of all real- or complex-valued sequences, is known as a sequence space. The sets ℓ_p, ℓ_∞, c, c_0, bs and cs are standard notations for the sequence spaces of absolutely p-summable sequences, bounded sequences, convergent sequences, null sequences, bounded series, and convergent series, respectively. A BK-space is a Banach space possessing continuous coordinates. The fact that ℓ_p $(1 \leq p < \infty)$ and ℓ_∞ are BK-spaces normed by $\|\mathfrak{f}\|_{\ell_p} = \left(\sum_r |\mathfrak{f}_r|^p\right)^{1/p}$ and $\|\mathfrak{f}\|_{\ell_\infty} = \sup_r |\mathfrak{f}_r|$, respectively, are well known. It follows from the choice of $0 \leq p < 1$ that ℓ_p is a complete p-normed space due to p-norm $\|\mathfrak{f}\|_{\ell_p} = \sum_r |\mathfrak{f}_r|^p$. For simplicity, we utilize the notations $\sum_r$ and $\sup_r$ in place of $\sum_{r=0}^\infty$ and sup over $r \in \mathbb{N}$(the set of natural numbers).

Let $\mathfrak{X}, \mathfrak{Y} \subset \omega$ and the notation H_r mean the rth row of an infinite matrix $H = (h_{rt})_{r,t\in\mathbb{N}}$ of real (or complex) entries. Let $\mathfrak{f} \in \omega$ then, its H-transform denoted by $H\mathfrak{f}$, be given by the sequence $\mathfrak{g} = (\mathfrak{g}_r)$ defined by $\mathfrak{g}_r = (H\mathfrak{f})_r = \sum_t h_{rt}\mathfrak{f}_t$, given that the infinite sum $\sum_t h_{rt}\mathfrak{f}_t$ converges for each $r \in \mathbb{N}$. We take the convention that the matrix $H \in (\mathfrak{X}, \mathfrak{Y})$ if $\mathfrak{g} \in \mathfrak{Y}$ for all $\mathfrak{f} \in \mathfrak{X}$. The domain of the matrix H in $\mathfrak{X}$ is defined by the set $\mathfrak{X}_H := \{\mathfrak{f} = (\mathfrak{f}_r) \in \omega : H\mathfrak{f} \in \mathfrak{X}\}$, which is also a sequence space. Additionally, H is known as a triangle if $h_{rr} \neq 0$ and $h_{rt} = 0$ for all $r < t$. Moreover, when $\mathfrak{X}$ is a BK-space and H is a triangle, then $\mathfrak{X}_H$ is also a BK-space normed by $\|\mathfrak{f}\|_{\mathfrak{X}_H} = \|H\mathfrak{f}\|_{\mathfrak{X}}$. Thus, by using this technique, several authors have constructed new sequence spaces that are generated from special matrices. We refer to the monograph [1] wherein the author described various aspects of summability, including the construction of BK-spaces as domains of some special matrices.

1.1. Difference Sequence Spaces

The forward difference operator Δ and the backward difference operator ∇ are, respectively, defined by $(\Delta\mathfrak{f})_r = \mathfrak{f}_r - \mathfrak{f}_{r+1}$ and $(\nabla\mathfrak{f})_r = \mathfrak{f}_r - \mathfrak{f}_{r-1}$ for all $k \in \mathbb{N}$. These operators play a significant role in the field of theory of sequence spaces and summability. For instance, the sequence $(\mathfrak{f}_r) = (r)_{r=1}^\infty$ is divergent, but the sequence $((\Delta\mathfrak{f})_r) = (1)$ is convergent. Kızmaz introduced the difference sequence spaces $\ell_\infty(\Delta) = (\ell_\infty)_\Delta$, $c(\Delta) = c_\Delta$ and $c_0(\Delta) = (c_0)_\Delta$.

The operators Δ and ∇ were later generalized to the forward and the backward difference operators of the second order Δ^2 and ∇^2, respectively, defined by $(\Delta^2\mathfrak{f})_r = (\Delta\mathfrak{f})_r - (\Delta\mathfrak{f})_{r+1}$ and $(\nabla^2\mathfrak{f})_r = (\nabla\mathfrak{f})_r - (\nabla\mathfrak{f})_{r-1}$, (cf. [2,3]). Since then, several generalizations of the difference operators Δ and ∇ were contributed in the literature. The few of the celebrated generalized difference operators are $\mathfrak{B}(a,b)$ [4,5], $\mathfrak{B}(a,b,c)$ [6], Δ^m [7], ∇^m [8], $\mathfrak{B}^{(m)}$ [9] and $\mathfrak{B}_v^{(m)}$ [10] defined by

$$(\mathfrak{B}(a,b)\mathfrak{f})_r = a\mathfrak{f}_r + b\mathfrak{f}_{r-1};\ (\mathfrak{B}(a,b,c)\mathfrak{f})_r = a\mathfrak{f}_r + b\mathfrak{f}_{r-1} + c\mathfrak{f}_{r-2};\ (\Delta^m\mathfrak{f})_r = (\Delta(\Delta^{m-1}\mathfrak{f}))_r;$$
$$(\nabla^m\mathfrak{f})_r = (\nabla(\nabla^{m-1}\mathfrak{f}))_r;\ (\mathfrak{B}^{(m)}\mathfrak{f})_r = \sum_{t=0}^{m}\binom{m}{t}a^{m-t}b^t\mathfrak{f}_{r-t};$$
$$(\mathfrak{B}_v^{(m)}\mathfrak{f})_r = \sum_{t=0}^{m}\binom{m}{t}a^{m-t}b^t v_{r-t}\mathfrak{f}_{r-t};$$

respectively. One may also refer to these papers [9,11–15] for the relevant studies.

1.2. *q*-Analog

The theory of the q-analog plays a significant role in various fields of mathematical, physical and engineering sciences. Due to its vast applications in diverse field of mathematics, several studies related to q-calculus can be traced in the literature. Initially, Jackson [16] gave the application of q-calculus while introducing q-analog of classical derivative and integral operators. Since then, studies on q-analogs of well-known mathematical notions have taken a rapid pace, and studies involving q-analogs of hypergeometric functions, algebras, approximation theory, combinatorics, difference and integral equations, etc., have been researched.

Throughout this article, we assume that $q \in (0,1)$. The following notions and definitions are very familiar in the field of q-calculus.

The q-number (cf. [17]) is defined by

$$[t]_q = \begin{cases} \sum_{v=0}^{t-1} q^v & (t = 1,2,3,\ldots), \\ 0 & (t = 0). \end{cases}$$

One may notice that, when $q \to 1^-$, then $[t]_q = t$.

The q-binomial coefficient is given as

$$\binom{r}{t}_q = \begin{cases} \frac{[r]_q!}{[r-t]_q![t]_q!} & (r \geq t), \\ 0 & (t > r), \end{cases}$$

where the notation $[t]_q!$ is known as the q-factorial of t and is given as

$$[t]_q! = \begin{cases} \prod_{v=1}^{t} [v]_q & (t = 1,2,3,\ldots), \\ 1 & (t = 0). \end{cases}$$

We strictly refer to [17,18] for basic terminologies in q-calculus.

1.3. *q*-Sequence Spaces and Motivation

The construction of sequence spaces by using q-calculus was realized very recently in the literature. The following q-analogs (or (p,q)-analogs) $C^q = (c^q_{rt})$, $C(q) = (\tilde{c}_{rt})$, $\nabla^2_q = (\delta^{2;q}_{rt})$ and $E(p,q) = (e^{p,q}_{rt})$ of the Cesàro matrix, Catalan matrix, difference matrix of the second order and Euler matrix, respectively, can be found in [19–22]:

$$c_{rt}^{q} = \begin{cases} \dfrac{q^t}{[r+1]_q} & , \quad 0 \leq t \leq r, \\ 0 & , \quad t > r, \end{cases}$$

$$\tilde{c}_{rt} = \begin{cases} q^t \frac{c_t(q) c_{r-t}(q)}{c_{r+1}(q)} & , \quad 0 \leq t \leq r, \\ 0 & , \quad t > r, \end{cases}$$

$$\delta_{rt}^{2;q} = \begin{cases} (-1)^{r-t} q^{\binom{r-t}{2}} \binom{2}{r-t}_q & , \quad 0 \leq t \leq r, \\ 0 & , \quad t > r, \end{cases}$$

$$e_{rt}^{a,b}(p,q) = \begin{cases} \frac{1}{(a \oplus b)_{p,q}^r} \binom{r}{t}_{p,q} p^{\binom{r-t}{2}} q^{\binom{t}{2}} a^t b^{r-t} & , \quad 0 \leq t \leq r, \\ 0 & , \quad t > r, \end{cases}$$

where $c_r(q)$ is the rth q-Catalan number. Demiriz and Şahin [23] and Yaying et al. [24] studied the q-analogs of Cesàro sequence spaces $X_p^q = (\ell_p)_{C^q}$, $X_0^q = (c_0)_{C^q}$, $X_c^q = c_{C^q}$ and $X_\infty^q = (\ell_\infty)_{C^q}$. Additionally, q-analogs of Catalan sequence spaces $c_0(C(q)) = (c_0)_{C(q)}$ and $c(C(q)) = c_{C(q)}$ (cf. [22]) were contributed recently to the literature. Moreover, Yaying et al. [20] studied the (p',q)-analog $e_p^{a,b}(p',q) = (\ell_p)_{E(p',q)}$ and $e_\infty^{a,b}(p',q) = (\ell_\infty)_{E(p',q)}$ of Euler sequence spaces. Bustoz and Gordillo [25] introduced the m-th order q-difference operator ∇_q^m defined as follows:

$$(\nabla_q^m \mathfrak{f})_r = \sum_{v=0}^{m} (-1)^v \binom{m}{v}_q q^{\binom{v}{2}} \mathfrak{f}_{m+r-v}.$$

By following this theory of the q-difference operator, quite recently, Yaying et al. [21] introduced q-difference sequence spaces of the second order $c_0(\nabla_q^2) = (c_0)_{\nabla_q^2}$ and $c(\nabla_q^2) = c_{\nabla_q^2}$ and obtained the spectral analysis of ∇_q^2 over the space c_0.

One can observe from the above discussion that the widely studied spaces c_0, c, ℓ_∞ and ℓ_p have been modified by various authors with the use of difference operators, as well as investigating the relations of the aforesaid spaces. Most recently, involving q-calculus and the difference operators, the authors of [21] presented a modification of c_0 and c, which were denoted by $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$. In the next section, we present the generalization of ℓ_∞ and ℓ_p by defining $\ell_p(\nabla_q^2)$ and $\ell_\infty(\nabla_q^2)$, which will fill the gap of further research in this direction.

Obviously, $c \subseteq \ell_\infty$. Previous work had limitations in that we could not demonstrate the relationship between c and ℓ_∞ as well as further research related to ℓ_∞ and ℓ_p in the quantum sense, but one can study these types of work after the present studies.

Motivated by the above studies, in particular [8,21], we construct the domains $(\ell_p)_{\nabla_q^2}$ and $(\ell_\infty)_{\nabla_q^2}$. Additionally, the spectral analysis of the operator ∇_q^2 over the space ℓ_1 is also obtained.

2. $\ell_p(\nabla_q^2)$ and $\ell_\infty(\nabla_q^2)$

In this section, the q-difference sequence spaces $\ell_p(\nabla_q^2)$ and $\ell_\infty(\nabla_q^2)$ are presented, inclusion relations are obtained, and the basis of the space $\ell_p(\nabla_q^2)$ is determined.

Yaying et al. [21,26] defined the difference operator $\nabla_q^2 : \omega \to \omega$ by

$$(\nabla_q^2 \mathfrak{f})_r = \mathfrak{f}_r - (1+q)\mathfrak{f}_{r-1} + q\mathfrak{f}_{r-2},$$

where $r \in \mathbb{N}$ and $\mathfrak{f}_r = 0$ for $r < 0$. Equivalently,

$$\nabla_q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -(1+q) & 1 & 0 & 0 & \cdots \\ q & -(1+q) & 1 & 0 & \cdots \\ 0 & q & -(1+q) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

It is evident that $\nabla_q^2 = \nabla^2$ when $q \to 1^-$. Additionally, unlike its ordinary form, $\nabla_q^2 \neq \nabla_q \circ \nabla_q$. In fact

$$(\nabla_q^2 \mathfrak{f})_r = (\nabla_q \mathfrak{f})_r - q(\nabla_q \mathfrak{f})_{r-1}.$$

The inverse $\nabla_q^{-2} = \left((\nabla_q^{-2})_{rt}\right)$ of the operator ∇_q^2 is obtained as (cf. [21])

$$(\nabla_q^{-2})_{rt} = \begin{cases} \binom{r-t+1}{r-t}_q & , \quad 0 \leq t \leq r, \\ 0 & , \quad t > r. \end{cases}$$

Define the q-difference sequence spaces $\ell_p(\nabla_q^2)$ and $\ell_\infty(\nabla_q^2)$ by

$$\begin{aligned} \ell_p(\nabla_q^2) &:= \left\{\mathfrak{f} \in \omega : \mathfrak{g} = \nabla_q^2 \mathfrak{f} \in \ell_p\right\}, \\ \ell_\infty(\nabla_q^2) &:= \left\{\mathfrak{f} \in \omega : \mathfrak{g} = \nabla_q^2 \mathfrak{f} \in \ell_\infty\right\}. \end{aligned}$$

These spaces can also be illustrated in the notation of the matrix domain as follows:

$$\ell_p(\nabla_q^2) = (\ell_p)_{\nabla_q^2} \text{ and } \ell_\infty(\nabla_q^2) = (\ell_\infty)_{\nabla_q^2}.$$

It is clear from the above definition of the sequence spaces $\ell_p(\nabla_q^2)$ and $\ell_\infty(\nabla_q^2)$ that the sequence $\mathfrak{g} = \nabla_q^2 \mathfrak{f} = (\mathfrak{g}_r)$ defined by

$$\mathfrak{g}_r = (\nabla_q^2 \mathfrak{f})_r = \sum_{t=0}^{r} (-1)^t q^{\binom{t}{2}} \binom{2}{t}_q \mathfrak{f}_{r-t} = \mathfrak{f}_r - (1+q)\mathfrak{f}_{r-1} + q\mathfrak{f}_{r-2} \ (r \in \mathbb{N}) \tag{1}$$

is the ∇_q^2-transform of the sequence $\mathfrak{f} = (\mathfrak{f}_r)$. Moreover, by using (1), we notice that

$$\mathfrak{f}_r = \sum_{v=0}^{r} \binom{r-v+1}{r-v}_q \mathfrak{g}_v \tag{2}$$

for each $r \in \mathbb{N}$. Here onward, the sequences $\mathfrak{f}$ and $\mathfrak{g}$ are related by (1) (or by (2)).

For $q = 1$, the space $\ell_p(\nabla_q^2)$ becomes $\ell_p(\nabla^m)$ $(m = 2)$ due to Altay [27], and $\ell_\infty(\nabla_q^2)$ becomes $\ell_\infty(\nabla^m)$ $(m = 2)$ due to Malkowsky and Parashar [28]. We notice that $\nabla_q^1 = \nabla$ [25], so it is not meaningful to work on $\ell_\infty(\nabla_q^1)$ [29], but the studies involving the difference operator ∇^2 is stronger than ∇. Based on these facts, we conclude that the spaces $\ell_\infty(\nabla_q^2)$ and $\ell_p(\nabla_q^2)$ are stronger than $\ell_\infty(\nabla^2)$ (and so $\ell_\infty(\nabla)$) and $\ell_p(\nabla^2)$ (and so $\ell_p(\nabla)$), respectively, and hence, our results too.

We recall that a sequence space $\mathfrak{X}$ is symmetric (cf. [30]) if $\mathfrak{f}_{\pi(r)} \in \mathfrak{X}$ whenever $(\mathfrak{f}_r) \in \mathfrak{X}$, where $\pi(r)$ is a permutation on $\mathbb{N}_0$. We consider the sequence $(\mathfrak{f}_r) = (r)_{r \in \mathbb{N}_0}$, then $(\mathfrak{f}_r) \in \ell_\infty(\nabla_q^2)$. Now, we consider the rearranged sequence

$$(\mathfrak{f}'_r) = (\mathfrak{f}_0, \mathfrak{f}_1, \mathfrak{f}_3, \mathfrak{f}_2, \mathfrak{f}_8, \mathfrak{f}_4, \mathfrak{f}_{15}, \mathfrak{f}_5, \mathfrak{f}_{24}, \mathfrak{f}_6, \mathfrak{f}_{35}, \mathfrak{f}_7, \mathfrak{f}_{48}, \mathfrak{f}_9, \ldots).$$

Then, $(\mathfrak{f}'_r) \notin \ell_\infty(\nabla_q^2)$. Consequently, $\ell_\infty(\nabla_q^2)$ is not a symmetric space.

Now, we state our first result:

Proposition 1.

1. $\ell_p(\nabla_q^2)$ $(0 \leq p < 1)$ *is a complete p-normed space due to the p-norm*

$$\|\mathfrak{f}\|_{\ell_p(\nabla_q^2)} = \|\mathfrak{g}\|_{\ell_p} = \sum_r |\mathfrak{f}_r - (1+q)\mathfrak{f}_{r-1} + q\mathfrak{f}_{r-2}|^p.$$

2. $\ell_p(\nabla_q^2)$ $(1 \leq p < \infty)$ *is a BK-space normed by*

$$\|\mathfrak{f}\|_{\ell_p(\nabla_q^2)} = \|\mathfrak{g}\|_{\ell_p} = \left(\sum_r |\mathfrak{f}_r - (1+q)\mathfrak{f}_{r-1} + q\mathfrak{f}_{r-2}|^p\right)^{1/p}.$$

3. $\ell_\infty(\nabla_q^2)$ *is a BK-space normed by*

$$\|\mathfrak{f}\|_{\ell_\infty(\nabla_q^2)} = \|\mathfrak{g}\|_{\ell_\infty} = \sup_r |\mathfrak{f}_r - (1+q)\mathfrak{f}_{r-1} + q\mathfrak{f}_{r-2}|.$$

Proposition 2. $\ell_p(\nabla_q^2) \cong \ell_p$ *and* $\ell_\infty(\nabla_q^2) \cong \ell_\infty$.

Proof. The result is proved for the space $\ell_p(\nabla_q^2)$. Since the q-difference operator ∇_q^2 is a triangular matrix, its inverse exists. This immediately implies that the mapping τ defined by

$$\begin{array}{rccl} \tau & : & \ell_p(\nabla_q^2) & \longrightarrow & \ell_p \\ & & \mathfrak{f} & \longmapsto & \tau\mathfrak{f} = \mathfrak{g} = \nabla_q^2\mathfrak{f}. \end{array}$$

is a linear bijection that preserves the norm (or p-norm). This concludes that $\ell_p(\nabla_q^2)$ is linearly isomorphic to the space ℓ_p. □

We emphasize here that the q-difference sequence spaces $\ell_p(\nabla_q^2)$ and $\ell_\infty(\nabla_q^2)$ reduce to $\ell_p(\nabla^2)$ and $\ell_\infty(\nabla^2)$, respectively, as $q \to 1^-$. Thus the relations $\ell_p \subseteq \ell_p(\nabla^2) \subseteq \ell_p(\nabla_q^2)$ and $\ell_\infty \subseteq \ell_\infty(\nabla^2) \subseteq \ell_\infty(\nabla_q^2)$ are trivial. Additionally, we consider the sequence $(\mathfrak{f}_r) = (r)$ mentioned in p.4. We observe that $(f_r) \in \ell_\infty(\nabla_q^2)$ but $(\mathfrak{f}_r) \notin \ell_\infty$.

Proposition 3. $\ell_p(\nabla_q^2) \subset \ell_\infty(\nabla_q^2)$ *strictly holds.*

Proof. Since $\ell_p \subset \ell_\infty$ holds, the inclusion part is obvious.

The relation $\ell_p \subset \ell_\infty$ is strict, so we take a sequence $h = (h_r) \in \ell_\infty \setminus \ell_p$. Let us define a sequence $h' = (h'_r)$ by $h'_r = \sum_{v=0}^{r} \binom{r-v+1}{r-v}_q h_v$ for each $r \in \mathbb{N}$. Then, $\nabla_q^2 h' = h \in \ell_\infty \setminus \ell_p$. This implies the fact that $h' \in \ell_\infty(\nabla_q^2) \setminus \ell_p(\nabla_q^2)$, as desired. □

Proposition 4. $\ell_p(\nabla_q^2) \subset \ell_{p'}(\nabla_q^2)$ *strictly holds, where* $1 \leq p < p' < \infty$.

Proof. We utilize the similar method applied in the proof of Theorem 3 to establish this result. □

A Schauder basis for $\mathfrak{X}$(normed linear space) $\subset \omega$ is a sequence $(u_r)_{r\in\mathbb{N}}$ such that for each $\mathfrak{f} \in \mathfrak{X}$, there corresponds a unique sequence, say (y_r), of scalars,

$$\mathfrak{f} = \sum_r y_r u_r \quad \forall r \in \mathbb{N}.$$

It is known that, for a triangle H, the matrix domain $\mathfrak{X}_H$ has a basis if $\mathfrak{X}$ has a basis. As a result of this fact along with with Theorem 2, we deduce the following result:

Theorem 1. *Define the sequence $\xi^{(t)}(q) = (\xi^{(t)}_r(q))$ by*

$$\xi^{(t)}_r(q) = \begin{cases} \binom{r-t+1}{r-t}_q & , \quad t \leq r, \\ 0 & , \quad t > r. \end{cases}$$

Then

(a) *The basis of the space $\ell_p(\nabla^2_q)$ is given by the set $\left\{\xi^{(0)}(q), \xi^{(1)}(q), \xi^{(2)}(q), \dots\right\}$ and every $\mathfrak{f} \in \ell_p(\nabla^2_q)$ has a unique representation of the form $\mathfrak{f} = \sum_t \mathfrak{g}_t \xi^{(t)}(q)$, where $\mathfrak{g}_r = (\nabla^2_q \mathfrak{f})_r$.*

(b) *The sequence space $\ell_\infty(\nabla^2_q)$ has no Schauder basis.*

3. Duals of the Spaces $\ell_p(\nabla^2_q)$ and $\ell_\infty(\nabla^2_q)$

For $\mathfrak{X} \subset \omega$, the α-, β- and γ-dual of $\mathfrak{X}$ are the sets

$$\begin{aligned} \mathfrak{X}^\alpha &:= \left\{x = (x_r) \in \omega : x\mathfrak{f} = (x_r \mathfrak{f}_r) \in \ell_1 \ \forall \mathfrak{f} = (\mathfrak{f}_r) \in \mathfrak{X}\right\}, \\ \mathfrak{X}^\beta &:= \left\{x = (x_r) \in \omega : x\mathfrak{f} = (x_r \mathfrak{f}_r) \in cs \ \forall \mathfrak{f} = (\mathfrak{f}_r) \in \mathfrak{X}\right\}, \\ \mathfrak{X}^\gamma &:= \left\{x = (x_r) \in \omega : x\mathfrak{f} = (x_r \mathfrak{f}_r) \in bs \ \forall \mathfrak{f} = (\mathfrak{f}_r) \in \mathfrak{X}\right\}, \end{aligned}$$

respectively.

In this section, we obtain $\mathfrak{X}^\alpha$, $\mathfrak{X}^\beta$ and $\mathfrak{X}^\gamma$ for $\mathfrak{X} \in \left\{\ell_p(\nabla^2_q), \ell_\infty(\nabla^2_q)\right\}$. Before proceeding further, we list the following lemmas which are required to obtain the duals of these spaces. Here onward, the family of all finite subsets of $\mathbb{N}$ is denoted by $\mathcal{N}$ and $1/p* + 1/p = 1$.

Lemma 1 ([31])**.** *These results are well known:*

(i) *$H = (h_{rt}) \in (\ell_\infty, \ell_1)$ iff*

$$\sup_{R \in \mathcal{N}} \sum_t \left| \sum_{r \in R} h_{rt} \right| < \infty. \tag{3}$$

(ii) *$H = (h_{rt}) \in (\ell_\infty, c)$ iff*

$$\exists h_t \in \mathbb{C} \ni \lim_{r \to \infty} h_{rt} = h_t \text{ for each } t \in \mathbb{N}, \tag{4}$$

$$\lim_{r \to \infty} \sum_t |h_{rt}| = \sum_t \left| \lim_{r \to \infty} h_{rt} \right|. \tag{5}$$

(iii) *$H = (h_{rt}) \in (\ell_\infty, \ell_\infty)$ iff*

$$\sup_r \sum_t |h_{rt}| < \infty. \tag{6}$$

(iv) *Let $1 < p < \infty$. Then, $H = (h_{rt}) \in (\ell_p, \ell_\infty)$ iff*

$$\sup_r \sum_t |h_{rt}|^{p*} < \infty. \tag{7}$$

(v) *Let $1 < p < \infty$. Then, $H = (h_{rt}) \in (\ell_p, c)$ iff (4) and (7) hold.*

Lemma 2. *These results hold:*

(i) *(ref. [32], Theorem 5.1.0 with $p_r = p$ for all r) $H = (h_{rt}) \in (\ell_p, \ell_1)$ if*

$$\sup_{R \in \mathcal{N}} \sup_t \left| \sum_{r \in R} h_{rt} \right|^p < \infty, \quad (0 < p \leq 1). \tag{8}$$

$$\sup_{R\in\mathcal{N}}\sum_t\left|\sum_{r\in R}h_{rt}\right|^{p*}<\infty,\quad (1<p<\infty). \tag{9}$$

(ii) *(ref. [33], Theorem 1 (i) with $p_r = p$ for all r) $H = (h_{rt}) \in (\ell_p, \ell_\infty)$ if*

$$\sup_{r,t}|h_{rt}|^p<\infty,\ \ (0<p\le 1). \tag{10}$$

(iii) *(ref. [33], Corollary for Theorem 1 with $p_r = p$ for all r) $H = (h_{rt}) \in (\ell_p, c)$ if (4) and (10) hold.*

Theorem 2. *Let*

$$\begin{aligned}
\nu_1 &:= \left\{d=(d_r)\in\omega : \sup_{R\in\mathcal{N}}\sup_t\left|\sum_{r\in R}\binom{r-t+1}{r-t}_q d_j\right|^p<\infty\right\},\ (0<p\le 1),\\
\nu_2 &:= \left\{d=(d_r)\in\omega : \sup_{R\in\mathcal{N}}\sum_t\left|\sum_{r\in R}\binom{r-t+1}{r-t}_q d_r\right|^{p*}<\infty\right\},\ (1<p<\infty),\\
\nu_3 &:= \left\{d=(d_r)\in\omega : \sup_{R\in\mathcal{N}}\sum_t\left|\sum_{r\in R}\binom{r-t+1}{r-t}_q d_r\right|<\infty\right\}.
\end{aligned}$$

Then,

(i) $[\ell_p(\nabla_q^2)]^\alpha = \begin{cases} \nu_1 & , \quad 0<p\le 1, \\ \nu_2 & , \quad 1<p<\infty. \end{cases}$

(ii) $\left[\ell_\infty(\nabla_q^2)\right]^\alpha = \nu_3$.

Proof. For $d = (d_r) \in \omega$, define the matrix $\Lambda(q) = (\lambda_{rt}^q)$ defined for all $r, t \in \mathbb{N}$ by

$$\lambda_{rt}^q = \begin{cases} \binom{r-t+1}{r-t}_q d_j & , \quad 0\le t\le r, \\ 0 & , \quad t>r \end{cases}$$

This leads to the equality:

$$d_r\mathfrak{f}_r=\sum_{t=0}^{r}\binom{r-t+1}{r-t}_q d_r\mathfrak{g}_t=(\Lambda(q)\mathfrak{g})_r, \tag{11}$$

where the sequence $\mathfrak{g} = (\mathfrak{g}_t)$ is given $\mathfrak{g} = \nabla_q^2\mathfrak{f}$. Thus $d\mathfrak{f} = (d_r\mathfrak{f}_r) \in \ell_1$ whenever $\mathfrak{f} \in \ell_p(\nabla_q^2)$ iff $\Lambda(q)\mathfrak{g} \in \ell_1$ whenever $\mathfrak{g} \in \ell_p$. Thus, $d = (d_r) \in \left[\ell_p(\nabla_q^2)\right]^\alpha$ if $\Lambda(q) \in (\ell_p, \ell_1)$. Therefore, by applying Lemma 2(i), we obtain that

$$[\ell_p(\nabla_q^2)]^\alpha=\begin{cases} \nu_1 & , \quad 0<p\le 1, \\ \nu_2 & , \quad 1<p<\infty. \end{cases}$$

In a similar way, the proof of Part (ii) is established by utilizing Lemma 1(i) in place of Lemma 2(i) in the above statements. We omit the details here to avoid repetition of the same lines. □

Theorem 3. *Let*

$$\begin{aligned}
\nu_4 &:= \left\{d=(d_r)\in\omega : \lim_{r\to\infty}\sum_{z=t}^{r}\binom{z-t+1}{z-t}_q d_z \text{ exists}\right\},\\
\nu_5 &:= \left\{d=(d_r)\in\omega : \sup_{r,t}\left|\sum_{z=t}^{r}\binom{z-t+1}{z-t}_q d_z\right|^p<\infty\right\},\ (0<p\le 1),
\end{aligned}$$

$$\nu_6 := \left\{ d = (d_r) \in \omega : \sup_r \sum_t \left| \sum_{z=t}^{r} \binom{z-t+1}{z-t}_q d_z \right|^{p*} < \infty \right\},$$
$$\nu_7 := \left\{ d = (d_r) \in \omega : \lim_{r\to\infty} \sum_t \left| \sum_{z=t}^{r} \binom{z-t+1}{z-t}_q d_z \right| = \sum_t \left| \lim_{r\to\infty} \sum_{z=t}^{r} \binom{z-t+1}{z-t}_q d_z \right| \text{ exists} \right\}.$$

Then,

(i) $[\ell_p(\nabla_q^2)]^\beta = \begin{cases} \nu_4 \cap \nu_5 & , \quad 0 < p \le 1, \\ \nu_4 \cap \nu_6 & , \quad 1 < p < \infty. \end{cases}$

(ii) $\left[\ell_\infty(\nabla_q^2)\right]^\beta = \nu_3 \cap \nu_7.$

Proof. For $d = (d_r) \in \omega$, define the matrix $\Theta(q) = (\theta_{rt}^q)$ for all $r, t \in \mathbb{N}$ by

$$\theta_{rt}^q = \begin{cases} \sum_{z=t}^{r} \binom{z-t+1}{z-t}_q d_z & , \quad 0 \le t \le r, \\ 0 & , \quad t > r \end{cases}$$

This leads to the equality:

$$\begin{aligned} \sum_{t=0}^{r} d_t \mathfrak{f}_t &= \sum_{t=0}^{r} \left[\sum_{z=0}^{t} \binom{t-z+1}{t-z}_q \mathfrak{g}_z \right] d_t \\ &= \sum_{t=0}^{r} \left[\sum_{z=t}^{r} \binom{z-t+1}{z-t}_q d_z \right] \mathfrak{g}_t \\ &= (\Theta(q)\mathfrak{g})_r \ (r \in \mathbb{N}), \end{aligned} \tag{12}$$

where the sequence $\mathfrak{g} = (\mathfrak{g}_t)$ is given $\mathfrak{g} = \nabla_q^2 \mathfrak{f}$. We see that $\left(\sum_{t=0}^{r} d_t \mathfrak{f}_t\right)$ converges whenever $\mathfrak{f} \in \ell_p(\nabla_q^2)$ if $\Theta(q)\mathfrak{g} \in c$ whenever $\mathfrak{g} \in \ell_p$. This means that $d = (d_t) \in \left[\ell_p(\nabla_q^2)\right]^\beta$ if $\Theta(q) \in (\ell_p, c)$. Hence, by utilizing Lemma 1(v) and Lemma 2(iii), we conclude that

$$[\ell_p(\nabla_q^2)]^\beta = \begin{cases} \nu_4 \cap \nu_5 & , \quad 0 < p \le 1, \\ \nu_4 \cap \nu_6 & , \quad 1 < p < \infty. \end{cases}$$

The β-dual of $\ell_\infty(\nabla_q^2)$ is obtained in the similar fashion by utilizing Lemma 1(ii), respectively, in place of Lemma 1(v) and Lemma 2(iii) in the above statements. To avoid repetition of similar statements, we omit the details. □

Theorem 4. *We have*

(i) $[\ell_p(\nabla_q^2)]^\gamma = \begin{cases} \nu_5 & , \quad 0 < p \le 1, \\ \nu_6 & , \quad 1 < p < \infty. \end{cases}$

(ii) $\left[\ell_\infty(\nabla_q^2)\right]^\gamma = \nu_6$ *with* $p* = 1.$

Proof. To obtain the γ-dual of the space $\ell_p(\nabla_q^2)$, we utilize Lemma 2(ii) and Lemma 1(iv) in place of Lemma 2(iii) and Lemma 1(v) in the proof of Theorem 3, respectively. For obtaining the γ-dual of $\ell_\infty(\nabla_q^2)$, we utilize Lemma 1(iii) instead of Lemma 1(ii). Details are omitted. □

4. Matrix Mappings

We here characterize the matrix classes $(\mathfrak{X}, \mathfrak{Y})$, where $\mathfrak{X} \in \{\ell_p(\nabla_q^2), \ell_\infty(\nabla_q^2)\}$ and $\mathfrak{Y} \in \{\ell_\infty, c, c_0, \ell_1\}$. A very useful and interesting proceeding result follows from [5].

Theorem 5. *Let $\mathfrak{X} = \ell_p$ or ℓ_∞ and $\mathfrak{Y} \subset \omega$. Let $C^{(r)} = (c_{vt}^{(r)})$ and $C = (c_{rt})$ be defined by*

$$c_{vt}^{(r)} = \begin{cases} 0 & (t > v), \\ \sum_{z=t}^{v} \binom{z-t+1}{z-t}_q h_{rz} & (0 \le t \le v), \end{cases}$$
$$c_{rt} = \sum_{z=t}^{v} \binom{z-t+1}{z-t}_q h_{rz}$$

for all $r, t \in \mathbb{N}$. Then, $H = (h_{rt}) \in (\mathfrak{X}(\nabla_q^2), \mathfrak{Y})$ if $C^{(r)} = (c_{vt}^{(r)}) \in (\mathfrak{X}, c)$ for each $r \in \mathbb{N}$, and $C = (c_{rt}) \in (\mathfrak{X}, \mathfrak{Y})$.

Proof. Let $H \in (\mathfrak{X}(\nabla_q^2), \mathfrak{Y})$ and $\mathfrak{f} \in \mathfrak{X}(\nabla_q^2)$. Then, we have the following equality

$$\sum_{t=0}^{v} h_{rt}\mathfrak{f}_t = \sum_{t=0}^{v}\sum_{z=0}^{t} \binom{t-z+1}{t-z}_q \mathfrak{g}_z h_{rt} = \sum_{t=0}^{v}\sum_{z=t}^{v} \binom{z-t+1}{z-t}_q h_{rz}\mathfrak{g}_t = \sum_{t=0}^{v} c_{vt}^{(r)}\mathfrak{g}_t \tag{13}$$

for all $v, r \in \mathbb{N}$. Since $H\mathfrak{f}$ exists, $C^{(r)} \in (\mathfrak{X}, c)$. Letting $v \to \infty$ in (13), we obtain $H\mathfrak{f} = C\mathfrak{g}$. As $H\mathfrak{f} \in \mathfrak{Y}$, $C\mathfrak{g} \in \mathfrak{Y}$. Therefore, $C \in (\mathfrak{X}, \mathfrak{Y})$.

Conversely, let $C^{(r)} = (c_{vt}^{(r)}) \in (\mathfrak{X}, c)$ $(r \in \mathbb{N})$, and $C = (c_{rt}) \in (\mathfrak{X}, \mathfrak{Y})$. Let $\mathfrak{f} \in \mathfrak{X}(\nabla_q^2)$. Then, $H_r \in \mathfrak{X}^\beta$ $(r \in \mathbb{N})$ which leads us to the fact that $H_r \in \left[\mathfrak{X}(\nabla_q^2)\right]^\beta$ for each $r \in \mathbb{N}$. By using (13), $H\mathfrak{f} = C\mathfrak{g}$ as $v \to \infty$. Thus $H \in (\mathfrak{X}(\nabla_q^2), \mathfrak{Y})$. □

Now, we utilize (5) to characterize some matrix classes from $\mathfrak{X} \in \{\ell_1(\nabla_q^2), \ell_p(\nabla_q^2), \ell_\infty(\nabla_q^2)\}$ to $\mathfrak{Y} \in \{\ell_1, c_0, c, \ell_\infty\}$. We give below some conditions which are necessary for deducing our results:

$$\sup_{v\in\mathbb{N}} \sum_t \left|c_{vt}^{(r)}\right| < \infty; \tag{14}$$
$$\lim_{v\to\infty} c_{vt}^{(r)} \text{ exists for all } t \in \mathbb{N}; \tag{15}$$
$$\sup_{r\in\mathbb{N}} \sum_t |c_{rt}|^{p*} < \infty; \tag{16}$$
$$\lim_{r\to\infty} c_{rt} \text{ exists for all } t \in \mathbb{N}. \tag{17}$$
$$\lim_{r\to\infty} c_{rt} = 0 \text{ for all } t \in \mathbb{N}; \tag{18}$$
$$\sup_{R\in\mathcal{N}} \sum_t \left|\sum_{r\in R} c_{rt}\right|^{p*} < \infty; \tag{19}$$
$$\sup_{r,t\in\mathbb{N}} |c_{rt}| < \infty; \tag{20}$$
$$\sup_{t\in\mathbb{N}} \sum_r |c_{rt}| < \infty; \tag{21}$$
$$\lim_{r\to\infty} \sum_t |c_{rt}| = \sum_t |\lim_{r\to\infty} c_{rt}|; \tag{22}$$
$$\lim_{r\to\infty} \sum_t |c_{rt}| = 0; \tag{23}$$
$$\sup_{R\in\mathcal{N}} \sum_t \left|\sum_{r\in R} c_{rt}\right| < \infty. \tag{24}$$

Lemma 3. *One can see the necessary and sufficient condition from Table 1 for $H = (h_{rt}) \in (\mathfrak{X}, \mathfrak{Y})$, where $\mathfrak{X} \in \left\{\ell_1(\nabla_q^2), \ell_p(\nabla_q^2), \ell_\infty(\nabla_q^2)\right\}$ $(1 < p < \infty)$ and $\mathfrak{Y} \in \{\ell_1, c_0, c, \ell_\infty\}$.*

Table 1. Characterization of the matrix class $(\mathfrak{X}, \mathfrak{Y})$, where $\mathfrak{X} \in \left\{\ell_1(\nabla_q^2), \ell_p(\nabla_q^2), \ell_\infty(\nabla_q^2)\right\}$ and $\mathfrak{Y} \in \{\ell_1, c_0, c, \ell_\infty\}$.

From\To	ℓ_1	c_0	c	ℓ_∞
$\ell_1(\nabla_q^2)$	(14), (15), (21)	(14), (15), (18), (20)	(14), (15), (17), (20)	(14), (15), (20)
$\ell_p(\nabla_q^2)$	(14), (15), (19)	(14), (15), (16), (18)	(14), (15), (16), (17)	(14), (15), (16)
$\ell_\infty(\nabla_q^2)$	(14), (15), (24)	(14), (15), (23)	(14), (15), (17), (22)	(14), (15), (16), (with $p* = 1$)

5. Spectrum of ∇_q^2 Over the Space ℓ_1

Finally, some spectral analyses of ∇_q^2 over ℓ_1 are examined.

Consider a complex normed space $\mathfrak{X} \neq \{\theta\}$ and any linear operator $\phi : D(\phi) \to \mathfrak{X}$ ($D(\phi)$:=domain of ϕ). We use the following notations for the proceeding work:

$$\begin{aligned}
\phi^* &:= \text{The adjoint of the operator } \phi; \\
R(\phi) &:= \text{The range of the operator } \phi; \\
B(\mathfrak{X}) &:= \text{The set of all bounded linear operators from } \mathfrak{X} \text{ into itself}; \\
I &:= \text{Identity operator in } D(\phi); \\
\phi_\varsigma &:= \phi - \varsigma I \ (\varsigma \in \mathbb{C}).
\end{aligned}$$

For any $\varsigma \in \mathbb{C}$, the inverse ϕ_ς^{-1} of the operator ϕ_ς is called the resolvent operator of ϕ, provided that ϕ_ς is invertible. Further, ς is a regular value of ϕ if

(A1) ϕ_ς^{-1} exists;

(A2) ϕ_ς^{-1} is bounded;

(A3) ϕ_ς^{-1} is defined on a set which is dense in $\mathfrak{X}$.

Define the set $r(\phi, \mathfrak{X}) = \{\varsigma \in \mathbb{C} : \varsigma \text{ as a regular value of } \phi\}$. Then, $r(\phi, \mathfrak{X})$ is called the resolvent set of ϕ. The spectrum ϕ is the set $s(\phi, \mathfrak{X}) = \mathbb{C} \setminus r(\phi, \mathfrak{X})$. It is further classified into three disjoint sets:

(1) The set $s_p(\phi, \mathfrak{X}) = \{\varsigma \in \mathbb{C} : \text{(A1) does not hold}\}$ is called *point spectrum* of ϕ over the space $\mathfrak{X}$.

(2) The set $s_c(\phi, \mathfrak{X}) = \{\varsigma \in \mathbb{C} : \text{(A1) and (A3) hold but (A2) does not hold}\}$ is called *continuous spectrum* of ϕ over the space $\mathfrak{X}$.

(3) $s_r(\phi, \mathfrak{X}) = \{\varsigma \in \mathbb{C} : \text{(A1) holds but (A3) does not hold, (A2) may or may not hold}\}$ is called *residual spectrum* of ϕ over the space $\mathfrak{X}$.

In the literature, several studies can be traced concerning spectral analysis of special matrices in different sequence spaces. We do, however, briefly mention those studies in the literature that deal with the determination and the classification of spectrum involving only difference operators. The readers may refer Table 2 for studies concerning spectral analysis of difference operators over various sequence spaces.

Table 2. Related work by some authors.

Difference Operators	Studied over the Opace (s)	References
Δ	ℓ_p, bv_p	[34,35]
Δ	ℓ_p, c, c_0 $(0 < p < 1)$	[36,37]
Δ	bv, ℓ_1	[38]
∇^2	c_0	[2]
$\mathfrak{B}(a, b)$	$bv_p, \ell_p,$	[4]
$\mathfrak{B}(a, b)$	bv, ℓ_1	[39]
$\mathfrak{B}(a, b, c)$	bv_p, ℓ_p, c, c_0	[40,41]
∇^r	c	[42]

Table 2. *Cont.*

Difference Operators	Studied over the Opace (s)	References
∇_v^r	c_0, ℓ_1	[8,43]
$\mathfrak{B}_v^{(m)}$	c_0	[44]
$\mathfrak{B}_v^{(m)}$	ℓ_1	[10]

Lemma 4 ([45]). *The matrix $H = (h_{rt})$ gives rise to a bounded linear operator $\phi \in B(\ell_1)$ if the supremum of ℓ_1 norms of the columns of H is bounded.*

Lemma 5 ([46]). *The adjoint operator ϕ^* is one–one if ϕ has a dense range.*

Theorem 6. $\nabla_q^2 : \ell_1 \to \ell_1$ *is a linear operator and* $\left\| \nabla_q^2 \right\|_{(\ell_1, \ell_1)} = 2(1+q)$.

Proof. This is immediate from Lemma 4 together with

$$\binom{2}{0}_q + \binom{2}{1}_q + q\binom{2}{2}_q = 2(1+q).$$

□

Theorem 7. $s_p(\nabla_q^2, \ell_1) = \emptyset$.

Proof. Consider the equality $\nabla_q^2 \mathfrak{f} = \varsigma \mathfrak{f}$ for $\theta \neq \mathfrak{f} \in \ell_1$. This yields the following system of equations:

$$\begin{aligned}
\mathfrak{f}_0 &= \varsigma\, f_0 \\
-(1+q)\mathfrak{f}_0 + \mathfrak{f}_1 &= \varsigma\, f_1 \\
q\mathfrak{f}_0 - (1+q)\mathfrak{f}_1 + \mathfrak{f}_2 &= \varsigma\, f_2 \\
q\mathfrak{f}_1 - (1+q)\mathfrak{f}_2 + \mathfrak{f}_3 &= \varsigma\, f_3 \\
&\vdots \\
q\mathfrak{f}_{m-2} - (1+q)\mathfrak{f}_{m-1} + \mathfrak{f}_m &= \varsigma\, f_m \\
&\vdots \quad .
\end{aligned}$$

For a fixed $m \in \mathbb{N}$, let $\mathfrak{f}_i = 0$ for all $i < m$ and $f_m \neq 0$. Then, we obtain that $\varsigma = 1$. Taking $\varsigma = 1$ in the proceeding equation yields $\mathfrak{f}_m = 0$, which is a contradiction to the fact $\mathfrak{f}_m \neq 0$. Thus $s_p(\nabla_q^2, \ell_1) = \emptyset$. □

Theorem 8. *Let $0 < q < 1$ and assume that the series $\sum_{r=0}^{\infty} \eta_r \left| \frac{1+q}{1-\varsigma} \right|^r$ converges for an increasing sequence of real numbers and $\eta_r^{1/r} \left| \frac{1+q}{1-\varsigma} \right| < 1$ for large r. Then, $s(\nabla_q^2, c_0) = \{\varsigma \in \mathbb{C} : \left| \frac{1-\varsigma}{1+q} \right| \leq 1\}$.*

Proof. Consider $\varsigma \in \mathbb{C}$ with $\left| \frac{1-\varsigma}{1+q} \right| > 1$. The operator $(\nabla_q^2 - \varsigma I) = (\lambda_{rt})$ being a triangle has an inverse $(\nabla_q^2 - \varsigma I)^{-1} = (\mu_{rt})$ given by

$$\mu_{rt}=\begin{bmatrix} \frac{1}{1-\varsigma} & 0 & 0 & 0 & \cdots \\ \frac{1+q}{(1-\varsigma)^2} & \frac{1}{1-\varsigma} & 0 & 0 & \cdots \\ \frac{(1+q)^2}{(1-\varsigma)^3}-\frac{q}{(1-\varsigma)^2} & \frac{1+q}{(1-\varsigma)^2} & \frac{1}{1-\varsigma} & 0 & \cdots \\ \frac{(1+q)^3}{(1-\varsigma)^4}-\frac{2q(1+q)}{(1-\varsigma)^3} & \frac{(1+q)^2}{(1-\varsigma)^3}-\frac{q}{(1-\varsigma)^2} & \frac{1+q}{(1-\varsigma)^2} & \frac{1}{1-\varsigma} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Equivalently, for each $t \in \mathbb{N}$, entries μ_{rt} can be written as

$$\begin{aligned} \mu_{tt} &= \frac{1}{1-\varsigma}; \\ \mu_{t+1,t} &= \frac{q+1}{(1-\varsigma)^2}; \\ \mu_{t+2,t} &= \frac{(q+1)^2}{(1-\varsigma)^3}-\frac{q}{(1-\varsigma)^2}; \\ \mu_{t+3,t} &= \frac{(q+1)^3}{(1-\varsigma)^4}-\frac{2q(q+1)}{(1-\varsigma)^3}; \\ &\vdots \end{aligned}$$

and so on. Clearly for $r,t \in \mathbb{N}$, $|\mu_{rt}| < \infty$.

Now we proceed to show that $(\mu_{rt}) \in B(\ell_1)$, i.e., $\sup_t \sum_r |\mu_{rt}| < \infty$. We first prove that the series $\sum_r |\mu_{rt}|$ converges for each $t \in \mathbb{N}$.

Let

$$\begin{aligned} S_t &= \sum_r |\mu_{rt}| \\ &= |\mu_{tt}| + |\mu_{t+1,t}| + |\mu_{t+2,t}| + |\mu_{t+3,t}| + \dots \end{aligned}$$

which gives

$$\begin{aligned} S_t &= \Big|(1-\varsigma)^{-1}\Big| + \Big|(1-\varsigma)^{-2}(1+q)\Big| + \Big|(1-\varsigma)^{-3}(1+q)^2-(1-\varsigma)^{-2}q\Big| \\ &\quad + \Big|(1-\varsigma)^{-4}(1+q)^3-q(1-\varsigma)^{-3}2(1+q)\Big| + \dots. \\ &\le (1+q)^{-1}\Big\{\Big|(1-\varsigma)^{-1}(1+q)\Big| + \Big|(1-\varsigma)^{-1}(1+q)\Big|^2 + \Big|(1-\varsigma)^{-1}(1+q)\Big|^3 \\ &\quad + q(1+q)^{-1}\Big|(1-\varsigma)^{-1}(1+q)\Big|^2 + \Big|(1-\varsigma)^{-1}(1+q)\Big|^4 \\ &\quad + 2q(1+q)^{-1}\Big|(1-\varsigma)^{-1}(1+q)\Big|^3 + \Big|(1-\varsigma)^{-1}(1+q)\Big|^5 \\ &\quad + 3q(1+q)^{-1}\Big|(1-\varsigma)^{-1}(1+q)\Big|^4 + q^2(1+q)^{-2}\Big|(1-\varsigma)^{-1}(1+q)\Big|^3 + \dots\Big\} \\ &= (1+q)^{-1}\Big\{|\xi| + |\xi|^2 + |\xi|^3 + \dots\Big\} + (1+q)^{-1}\Big\{q(1+q)^{-1}|\xi|^2 + 2q(1+q)^{-1}|\xi|^3 \\ &\quad + 3q(1+q)^{-1}|\xi|^4 + (1+q)^{-2}q^2|\xi|^3 + \dots\Big\} \\ &= (1+q)^{-1}|\xi|(1-|\xi|)^{-1} + (1+q)^{-1}\Big\{|\xi|^2(1+q)^{-1}q \\ &\quad + |\xi|^3\Big(2(1+q)^{-1}q + q^2(1+q)^{-2}\Big) + \dots\Big\} \\ &= (1+q)^{-1}|\xi|(1-|\xi|)^{-1} + (1+q)^{-1}\Big\{|\xi|^2\eta_2(q) + |\xi|^3\eta_3(q) + \dots\Big\}, \end{aligned} \tag{25}$$

where
$$|\xi| = \left|(1-\varsigma)^{-1}(1+q)\right| < 1,$$
and
$$\eta_2(q) = (1+q)^{-1}q, \eta_3(q) = 2(1+q)^{-1}q + q^2(1+q)^{-2}$$
are the coefficients of $|\xi|^2$, $|\xi|^3, \ldots,$ respectively. We fairly see that the sequence $(\eta_t(q))$ is increasing. It follows from our assumption that there exists a number v such that for $t > v$ the series $\sum_{t=v+1}^{\infty} \eta_t(q)|\xi|^t = 0$. Further, let $\eta = \max_{1<t\leq v} \eta_t(q)$. Then, by using inequality (25) together with the fact that $|\xi| = \left|(1-\varsigma)^{-1}(q+1)\right| < 1$, we obtain

$$\begin{aligned}
S_t &< (1+q)^{-1}(1-|\xi|)^{-1}|\xi| + \eta(1+q)^{-1}|\xi|^2\left\{|\xi|^v + \ldots + |\xi|^2 + |\xi| + 1\right\} \\
&= (1+q)^{-1}(1-|\xi|)^{-1}|\xi| + \eta(1+q)^{-1}|\xi|^2(1-|\xi|)^{-1}(1-|\xi|^{v-1}) \\
&= (1+q)^{-1}(1-|\xi|)^{-1}|\xi|\left\{1+\eta|\xi|(1-|\xi|^{v-1})\right\} \\
&< \infty.
\end{aligned}$$

Thus, (S_t) is a sequence of positive reals and $\lim_{t\to\infty} S_t < \infty$. Hence $\sup_t S_t < \infty$. Thus, $(\mu_{rt}) \in B(\ell_1)$, whenever $|1-\varsigma| \leq 1+q$. Moreover, the domain of $(\nabla_q^2 - \varsigma I)^{-1}$ is dense in ℓ_1, which is clear from the fact that $\nabla_q^2 - \varsigma I$. Hence

$$s(\nabla_q^2, c_0) \subseteq \{\varsigma \in \mathbb{C} : |1-\varsigma| \leq 1+q\}. \tag{26}$$

The converse part is two-fold:

Case 1: When $\varsigma = 1$, then (S_t) is unbounded. Thus, for $\varsigma = 1$, $(\nabla_q^2 - \varsigma I)^{-1} \notin B(\ell_1)$.
Case 2: When $\varsigma \neq 1$, $(\nabla_q^2 - \varsigma I)$ is a triangle, and so $(\nabla_q^2 - \varsigma I)^{-1}$ exists.

Let $\varsigma \in \mathbb{C}$ with $|1-\varsigma| < 1+q$. Then (S_t) is unbounded. Consequently, $(\nabla_q^2 - \varsigma I)^{-1} \notin B(\ell_1)$ with $|1-\varsigma| < 1+q$.

Again, let $\varsigma \in \mathbb{C}$ with $|1-\varsigma| = 1+q$ which again yields that $\lim_t S_t = \infty$. Hence (S_t) is unbounded. Hence $(\nabla_q^2 - \varsigma I)^{-1} \notin B(\ell_1)$ with $|1-\varsigma| = 1+q$. Thus

$$\{\varsigma \in \mathbb{C} : |1-\varsigma| \leq 1+q\} \subseteq s(\nabla_q^2, \ell_1). \tag{27}$$

Thus, by using (26) and (27), we conclude that

$$s(\nabla_q^2, \ell_1) = \{\varsigma \in \mathbb{C} : \left|\frac{1-\varsigma}{1+q}\right| \leq 1\}.$$

□

Theorem 9. $s_p(\nabla_q^{2,*}, \ell_\infty) = \{\varsigma \in \mathbb{C} : |\frac{1-\varsigma}{1+q}| \leq 1\}$.

Proof. Let $\mathfrak{f} = (\mathfrak{f}_k)$ be a non-zero sequence such that $\nabla_q^{2,*}\mathfrak{f} = \varsigma\mathfrak{f}$, which gives the system of linear equations as follows:

$$\begin{aligned}
\mathfrak{f}_0 - (1+q)\mathfrak{f}_1 + q\mathfrak{f}_2 &= \varsigma\mathfrak{f}_0, \\
\mathfrak{f}_1 - (1+q)\mathfrak{f}_2 + q\mathfrak{f}_3 &= \varsigma\mathfrak{f}_1, \\
\mathfrak{f}_2 - (1+q)\mathfrak{f}_3 + q\mathfrak{f}_4 &= \varsigma\mathfrak{f}_2, \\
&\vdots \quad .
\end{aligned}$$

One may observe that for $|1-\varsigma| \leq 1+q$, we obtain that

$$|\mathfrak{f}_0| = \left|(1-\varsigma)^{-1}((1+q)\mathfrak{f}_1 - q\mathfrak{f}_2)\right|$$

$$= \left|(1-\varsigma)^{-1}(1+q)\left(\mathfrak{f}_1-(1+q)^{-1}q\mathfrak{f}_2\right)\right|$$
$$\geq \left|\mathfrak{f}_1-(1+q)^{-1}q\mathfrak{f}_2\right|,$$

and $|\mathfrak{f}_1| \geq \left|\mathfrak{f}_2-(1+q)^{-1}q\mathfrak{f}_3\right|$. Thus, following the similar pattern, we obtain that $|\mathfrak{f}_0| \geq |\mathfrak{f}_1| \geq |\mathfrak{f}_2| \geq \ldots$. This implies that $\mathfrak{f} \in \ell_\infty$.

Conversely, it is trivial that if $\mathfrak{f} \in \ell_\infty$, then $|\frac{1-\varsigma}{1+q}| \leq 1$.
□

Theorem 10. $s_r(\nabla_q^2, \ell_1) = \{\varsigma \in \mathbb{C} : |\frac{1-\varsigma}{1+q}| \leq 1\}$.

Proof. Let $\varsigma \in \mathbb{C}$ with $|1-\varsigma| \leq 1+q$. Then, the operator $\nabla_q^2 - \varsigma I$ is a triangle and so is invertible, provided that $\varsigma \neq 1$. By using Theorem 6, the operator $\nabla_q^2 - \varsigma I$ is one–one for $\varsigma = 1$ and so is invertible. Again by Theorem 9, the operator $\nabla_q^{2,*} - \varsigma I$ is not one–one for $|\frac{1-\varsigma}{1+q}| \leq 1$. Thus with the help of Lemma 5, we conclude that $R(\nabla_q^2 - \varsigma I) \neq \ell_1$.

Hence $s_r(\nabla_q^2, \ell_1) = \{\varsigma \in \mathbb{C} : |\frac{1-\varsigma}{1+q}| \leq 1\}$. □

Theorem 11. $s_c(\nabla_q^2, \ell_1) = \phi$.

Proof. This is straightforward from Theorems 8, 7 and 10 together with the fact that $s(\nabla_q^2, \ell_1) = s_p(\nabla_q^2, \ell_1) \cup s_r(\nabla_q^2, \ell_1) \cup s_c(\nabla_q^2, \ell_1)$. □

6. Conclusions

This study is a natural continuation of the works investigated in [21]. The present literature contains various application of q-difference operators in different field of mathematics. But only a couple of studies [21,26] can be traced involving construction of sequence spaces by using q-difference operator. We constructed q-difference sequence spaces $\ell_p(\nabla_q^2) = (\ell_p)_{\nabla_q^2}$ and $\ell_\infty(\nabla_q^2) = (\ell_\infty)_{\nabla_q^2}$. This work is an exemplar that focusses on one of the many application of q-calculus in sequence spaces. Besides, we gave another application of q-difference operator by determining spectral analysis of the operator ∇_q^2 in the space ℓ_1.

It is evident that $\nabla_q^2 = \nabla^2$ as $q = 1$. Consequently, this study is a q-analog of difference sequence spaces of second order in ℓ_p and ℓ_∞. The investigated results advances the sequence spaces theory to a new level and paves the way for more research in this direction. For future scope, one may study the mth-generalization ($m \in \mathbb{N}$) of this study by following the theroy of mth order q-difference operator as studied by Bustoz and Gordillo [25]. Further, q-difference operators can be used in the study associated to medical diagnosis and decision making in the setting of spherical fuzzy sets [47].

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