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On Fractional Hybrid Non-Linear Differential Equations Involving Three Mixed Fractional Orders with Boundary Conditions

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Abstract: In this paper, we study a class of non-linear fractional hybrid differential equations involving three mixed fractional orders with boundary conditions. Under weak assumptions, a formula of solutions is constructed and the existence results of the solutions for the problem are established. The results can be used to solve more general fractional hybrid equations, such as the general variable coefficient fractional hybrid Langevin equations. Moreover, the form of the solution for this kind of equation can provide a theoretical basis for the further study of the positive solution and its symmetry. We provide an example to support our main result.

Keywords: mixed fractional derivatives; fractional differential equations; hybrid differential equations; boundary value problem

MSC: 34A08; 34A37; 34B10

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1. Introduction

In conventional calculus, fractional calculus is a very popular tool for the modeling of many phenomena in science and engineering, such as in the fields of physics, chemistry, image processing, the electrodynamics of complex media and polymer rheology [1–11]. The non-local property of the fractional derivatives (and integrals) enables more effective representation of the reality of nature. For this reason, many researchers have focused on various types of fractional differential equations with more general boundary value conditions [5,6,9,10].

Hybrid differential equations can be considered to be quadratic perturbations of non-linear differential equations. As special cases of dynamical systems, they are of widespread interest to researchers. In 2010, Dhage and Lakshmikantham initiated an investigation of a new category of non-linear differential equations whereby they introduced ordinary hybrid differential equations, and showed the existence of extremal solutions for this boundary value problem with the help of some fundamental differential inequalities [12]. In the last few decades, fractional hybrid differential equations with boundary value conditions have attracted the interest and the attention of many researchers. In 2012, Zhao et al. extended Dhage's work to fractional orders and investigated an initial value problem of fractional hybrid differential equations [13]:

$$\begin{cases} {}^{L}D^{q}_{0^+}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J = (0,1], \\ x(0) = 0, \end{cases}$$

where $q \in (0,1)$. ${}^{L}D^{q}_{0^+}$ is the standard Riemann–Liouville fractional derivative. Later, the topics of different fractional hybrid initial and boundary value problems were discussed by many researchers [14–16].

In [14], the authors considered the following initial value problem for fractional hybrid integro-differential equations:

$$\begin{cases} {}^{L}D^{\alpha}_{0^+}\left(\dfrac{x(t)-\sum\limits_{i=1}^{m} I^{\beta_i}_{0^+}h_i(t,x(t))}{g(t,x(t))}\right)=f(t,x(t)), \quad t\in J=(0,1], \\ x(0)=0, \end{cases}$$

where $\alpha \in (0,1)$ and $I^{\beta_i}_{0^+}$ denotes the Riemann–Liouville fractional sequential integrals of order $\beta_i (i=1,2,\cdots,m)$. $f,g,h_i \in C(J\times\mathbb{R},\mathbb{R})$ and $h_i(0,0)=0 (i=1,2,\cdots,m)$.

In [15], the authors considered the following fractional hybrid differential equations with boundary conditions involving Caputo's derivative

$$\begin{cases} {}^{c}D^{q}_{0^+}\left(\dfrac{x(t)}{f(t,x(t))}\right)=g(t,x(t)), \quad t\in J=(0,T], \\ a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=c, \end{cases}$$

where $q\in(0,1)$. ${}^{c}D^{q}_{0^+}$ is the standard Caputo fractional derivative. $f,g\in C(J\times\mathbb{R},\mathbb{R})$.

In [16], the authors considered the existence result for a fractional hybrid differential equation with boundary conditions given by

$$\begin{cases} {}^{c}D^{\alpha}_{0^+}\left(\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right)=h(t,x(t)), \quad t\in J=(0,1], \\ \left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]|_{t=0}=0, \ \left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]|_{t=1}=0, \end{cases}$$

where $\alpha\in(1,2]$, $f,g,h\in C(J\times\mathbb{R},\mathbb{R})$.

Motivated by the studies mentioned above, the main objective of our present investigation was to study the following boundary value problem of the fractional hybrid differential equations:

$$\begin{cases} {}^{L}D^{\alpha}_{0^+}\left(\dfrac{{}^{c}D^{\beta}_{0^+}x(t)-\sum\limits_{i=1}^{m} I^{\eta_i}_{0^+}h_i(t,x(t))}{g(t,x(t))}\right)=f(t,x(t),{}^{L}D^{\delta}_{0^+}x(t)), \quad t\in J=(0,1], & (1) \\ x'(0)=0, \ x(0)=d, \ I^{1-\alpha}_{0^+}\left(\frac{{}^{c}D^{\beta}_{0^+}x(t)}{g(t,x(t))}\right)(0+)=0, & (2) \end{cases}$$

where $\alpha\in[0,1]$, $\beta\in(1,2]$, $\alpha+\beta>2$, $\delta\in[0,\alpha)$, $\eta_i\in[0,1]$ and $\beta+\eta_i-\delta\in(1,2)$, $d\in\mathbb{R}$. f, g, $h_i(i=1,2,\cdots,m)$ are Lebesgue integrable, and they are suitable functions to be specified later.

The problem (1)–(2) considered here is general in the sense that it includes the following well-known classes of fractional differential equations:

Case I: Let $\eta_i=0(i=1,2,\cdots,m)$, the problem (1)–(2) is reduced to the following boundary value problem of the fractional hybrid differential equations

$$\begin{cases} {}^{L}D^{\alpha}_{0^+}\left(\dfrac{{}^{c}D^{\beta}_{0^+}x(t)-\sum\limits_{i=1}^{m} h_i(t,x(t))}{g(t,x(t))}\right)=f(t,x(t),{}^{L}D^{\delta}_{0^+}x(t)), \quad t\in J=(0,1], & (3) \\ x'(0)=0, \ x(0)=d_0, \ \left(\frac{{}^{c}D^{\beta}_{0^+}x(t)}{g(t,x(t))}\right)(0+)=0. \end{cases}$$

Case II: Let $m = 1$, $\eta_1 = \delta = 0$ and $h_1(t, x(t)) = -\lambda x(t)$, λ is a constant, then (1) can be written as the fractional hybrid Langevin equation:

$$ {}^{L}D^{\alpha}_{0^+}\left(\frac{{}^{c}D^{\beta}_{0^+}x(t) + \lambda x(t)}{g(t, x(t))}\right) = f(t, x(t)), \tag{4} $$

which generalizes the well-known results in [17].

If $g(t, x(t)) \equiv 1$, then (4) is reduced to the fractional Langevin equation of the form

$$ {}^{L}D^{\alpha}_{0^+}\left({}^{c}D^{\beta}_{0^+}x(t) + \lambda x(t)\right) = f(t, x(t)), $$

which has been studied by many researchers [18–20].

Case III: If $m = 1$, $\eta_1 = 0$ and $h_1(t, x(t)) = -\lambda(t)x(t)$, $\lambda(t)$ is Lebesgue integrable and $g(t, x(t)) \equiv 1$, then (1) is reduced to the variable coefficient fractional Langevin equation of the form

$$ {}^{L}D^{\alpha}_{0^+}\left({}^{c}D^{\beta}_{0^+}x(t) + \lambda(t)x(t)\right) = f(t, x(t), {}^{L}D^{\delta}_{0^+}x(t)). $$

To our knowledge, the above equation with the boundary value conditions has rarely been studied. Our study can advance this field and make a theoretical contribution to the further study of the positive solution and its symmetry.

In comparison to previous research, the problem (1)–(2) considered by us is more general. In the problem (1)–(2), without the hybrid boundary value conditions (cf., e.g., [16]), a mixed-type fractional equation with boundary value conditions is discussed, which can inform more practical problems. We consider the problem (1)–(2) without the assumptions of continuity on f, g, h_i and apply the existence result to more fractional differential equations, for example, the general fractional non-homogeneous differential Equation (12) with variable coefficient, the variable coefficient fractional hybrid Langevin Equation (13) and the variable coefficient fractional Langevin Equation (14) in Section 4. There are few results about the above equations with a variable coefficient.

The rest of this paper is organized as follows. In Section 2, we consider some concepts and results of fractional calculus. In Section 3, we present a formula for solutions to the linear case of (1)–(2). In Section 4, using Schauder's fixed point theorem, we obtain the existence results of solutions for the problem (1)–(2). In Section 5, we provide an example to demonstrate application of our result.

2. Preliminaries

In this paper, let $L^p(J, \mathbb{R})$ be the Banach space of all Lebesgue measurable functions $l : J \to \mathbb{R}$ with the norm $\|l\|_{L^p} = \left(\int_J |l(t)|^p dt\right)^{\frac{1}{p}} < \infty$, and $AC([a, b], \mathbb{R})$ be the space of all the absolutely continuous functions defined on $[a, b]$. We use the following notation:

$$ AC^n([a, b], \mathbb{R}) = \{f : f \in C^{n-1}([a, b], \mathbb{R}) \text{ and } f^{(n-1)} \in AC([a, b], \mathbb{R})\}. $$

In particular, $AC^1([a, b], \mathbb{R}) = AC([a, b], \mathbb{R})$.

First, some concepts and results are presented. $\Gamma(\cdot)$ and $B(\cdot, \cdot)$ are the Gamma and Beta functions, respectively.

Definition 1 ([3,4])**.** *The left-sided fractional integral of order q for a function $x(t) \in L^1$ is defined by*

$$ \begin{aligned} (I^q_{a^+}x)(t) &= \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}x(s)ds, \quad t > a, \quad q > 0, \\ (I^0_{a^+}x)(t) &= x(t). \end{aligned} $$

Definition 2 ([3,4]). *If $k(t) \in AC^n([a,b],\mathbb{R})$, then the Riemann–Liouville fractional derivative $({}^L D^q_{a^+} k)(t)$ of order q exists almost everywhere on $[a,\ b]$ and can be written as*

$$\begin{aligned}({}^L D^q_{a^+} k)(t) &= \frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-q-1}k(s)ds, \quad t>a,\ n-1<q\le n,\\ ({}^L D^0_{a^+} k)(t) &= k(t).\end{aligned}$$

Definition 3 ([3,4]). *If $k(t) \in AC^n([a,b],\mathbb{R})$, then the Caputo fractional derivative $({}^c D^q_{a^+} k)(t)$ of order q exists almost everywhere on $[a,\ b]$ and can be written as*

$$({}^c D^q_{a^+} k)(t) = \left({}^L D^q_{a^+}\left[k(s) - \sum_{j=0}^{n-1}\frac{k^{(j)}(a)}{j!}(s-a)^j\right]\right)(t), \quad t>a,\ n-1<q<n,$$

moreover, if $k(a) = k'(a) = \cdots = k^{(n-1)}(a) = 0$, then $({}^c D^q_{a^+} k)(t) = ({}^L D^q_{a^+} k)(t)$.

Lemma 1 ([4]). *If $\mu > 0$ and $\theta > 0$, then*

$$\begin{aligned}[I^\mu_{a^+}(s-a)^{\theta-1}](t) &= \frac{\Gamma(\theta)}{\Gamma(\theta+\mu)}(t-a)^{\theta+\mu-1},\\ \left({}^L D^\mu_{a^+} 1\right)(t) &= \frac{(t-a)^{-\mu}}{\Gamma(1-\mu)}, \quad 0<\mu<1,\\ \left({}^L D^\mu_{a^+}(s-a)^{\mu-j}\right)(t) &= 0, \quad j = 1,2,\cdots,[\mu]+1, \end{aligned} \tag{5}$$

where $[\mu]$ denotes the integer part of the real number μ.

Lemma 2 ([4]). *If $k(t) \in L^p([a,b])(1 \le p \le \infty)$ and $\theta_1, \theta_2 > 0$, then the following relations hold:*

(1) $({}^L D^{\theta_1}_{a^+} I^{\theta_1}_{a^+} k)(t) = k(t)$ *a.e.* $t \in [a,b]$;
(2) $({}^c D^{\theta_1}_{a^+} I^{\theta_1}_{a^+} k)(t) = k(t)$;
(3) *For* $\theta_1 > \theta_2 > 0$, $({}^L D^{\theta_2}_{a^+} I^{\theta_1}_{a^+} k)(t) = I^{\theta_1-\theta_2}_{a^+} k(t)$, *a.e.* $t \in [a,b]$;
(4) *For* $\theta_2 > \theta_1 > 0$, $({}^L D^{\theta_2}_{a^+} I^{\theta_1}_{a^+} k)(t) = {}^L D^{\theta_2-\theta_1}_{a^+} k(t)$, *a.e.* $t \in [a,b]$;
(5) *For* $\theta_1 > 1$, $(\frac{d}{dt} I^{\theta_1}_{a^+})k(t) = I^{\theta_1-1}_{a^+} k(t)$;
(6) *For* $\theta_1, \theta_2 > 0$, $(I^{\theta_1}_{a^+} I^{\theta_2}_{a^+} k)(t) = I^{\theta_1+\theta_2}_{a^+} k(t)$.

Lemma 3 ([4]). *For $\theta > 0$, a general solution of the fractional differential equation $({}^L D^\theta_{0^+} x)(t) = 0$ is given by*

$$x(t) = \sum_{i=1}^n c_i t^{\theta-i}, \quad t>0,$$

where $c_i \in \mathbb{R}$, $i = 1,2,\cdots,n(n = [\theta]+1)$.

Lemma 4 ([4]). *For $\theta > 0$, a general solution of the fractional differential equation $({}^c D^\theta_{0^+} x)(t) = 0$ is given by*

$$x(t) = \sum_{i=0}^{n-1} c_i t^i, \quad t>0,$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\cdots,n-1(n = [\theta]+1)$.

Next, we present and prove the following lemmas.

Lemma 5. *If $\omega > 0$, $-1 < \tau < 0$, then for $\psi \in L^{\frac{1}{p}}(0 < p < 1)$,* *(i)* $\int_a^t (t-s)^{\omega-1}\psi(s)ds \le \left(\frac{1-p}{\omega-p}\right)^{1-p} (t-a)^{\omega-p}\|\psi\|_{L^{\frac{1}{p}}}$, *for* $\omega > p, t > a > 0$;

(ii) $\int_0^t (t-s)^{\omega-1}s^\tau\psi(s)ds \le \left(B\left(\frac{\omega-p}{1-p}, \frac{1+\tau-p}{1-p}\right)\right)^{1-p} t^{\omega+\tau-p}\|\psi\|_{L^{\frac{1}{p}}}$, *for* $p < \min\{\omega, 1+\tau\}$, $t > 0$.

Proof. Using Hölder's inequality, it follows that

$$\begin{aligned}
\int_a^t (t-s)^{\omega-1}\psi(s)ds &\le \left(\int_a^t (t-s)^{\frac{\omega-1}{1-p}}ds\right)^{1-p}\|\psi\|_{L^{\frac{1}{p}}} = \left(\frac{1-p}{\omega-p}\right)^{1-p}(t-a)^{\omega-p}\|\psi\|_{L^{\frac{1}{p}}}, \quad \text{for} \quad \omega > p, \\
\int_0^t (t-s)^{\omega-1}s^\tau\psi(s)ds &\le \left(\int_0^t (t-s)^{\frac{\omega-1}{1-p}}s^{\frac{\tau}{1-p}}d\tau\right)^{1-p}\|\psi\|_{L^{\frac{1}{p}}} \\
&= \left(B\left(\frac{\omega-p}{1-p}, \frac{1+\tau-p}{1-p}\right)\right)^{1-p} t^{\omega+\tau-p}\|\psi\|_{L^{\frac{1}{p}}}, \quad \text{for} \quad p < \min\{\omega, 1+\tau\},\ t > 0.
\end{aligned}$$

□

Lemma 6. *If* $\gamma \in (1,2)$ *and* $y \in L^{\frac{1}{p}}(0 < p < 1)$, *then*

(i) *for* $0 < p < \gamma - 1$, $(I^\gamma_{0+}y(s))(t) \in C(J,\mathbb{R})$;

(ii) *for* $0 < p < \frac{\gamma-1}{2}$, $(I^{\gamma-1}_{0+}y(s))(t) \in AC(J,\mathbb{R})$ *and* $(I^\gamma_{0+}y(s))(t) \in AC^2(J,\mathbb{R})$.

Proof. (i) For any t_1, $t_2 \in J$ and $t_1 < t_2$, from Hölder's inequality and Lagrange's mean value theorem, we have

$$\begin{aligned}
&\left|(I^\gamma_{0+}y(s))(t_2) - (I^\gamma_{0+}y(s))(t_1)\right| \\
= \ & \frac{1}{\Gamma(\gamma)}\left|\int_0^{t_1}\left[(t_2-s)^{\gamma-1} - (t_1-s)^{\gamma-1}\right]y(s)ds + \int_{t_1}^{t_2}(t_2-s)^{\gamma-1}y(s)ds\right| \\
\le \ & \frac{(t_2-t_1)}{\Gamma(\gamma-1)}\int_0^{t_1}(\xi-s)^{\gamma-2}|y(s)|ds + \left(\int_{t_1}^{t_2}(t_2-s)^{\frac{\gamma-1}{1-p}}ds\right)^{1-p}\frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\gamma)} \\
\le \ & \frac{(t_2-t_1)}{\Gamma(\gamma-1)}\int_0^{t_1}(\xi-s)^{\gamma-2}|y(s)|ds + (\frac{1-p}{\gamma-p})^{1-p}\frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\gamma)}(t_2-t_1)^{\gamma-p} \\
\le \ & \frac{(t_2-t_1)}{\Gamma(\gamma-1)}\left(\int_0^{t_1}(\xi-s)^{\frac{\gamma-2}{1-p}}ds\right)^{1-p}\|y\|_{L^{\frac{1}{p}}} + (\frac{1-p}{\gamma-p})^{1-p}\frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\gamma)}(t_2-t_1)^{\gamma-p} \qquad (6) \\
\le \ & \frac{(t_2-t_1)}{\Gamma(\gamma-1)}(\frac{1-p}{\gamma-1-p})^{1-p}[\xi^{\frac{\gamma-1-p}{1-p}} - (\xi-t_1)^{\frac{\gamma-1-p}{1-p}}]^{1-p}\|y\|_{L^{\frac{1}{p}}} + (\frac{1-p}{\gamma-p})^{1-p}\frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\gamma)}(t_2-t_1)^{\gamma-p} \\
\to \ & 0, \quad as \quad t_2 \to t_1,
\end{aligned}$$

that is, $(I^\gamma_{0+}y)(t) \in C(J,\mathbb{R})$.

(ii) It follows from (6) and the definition of the derivative for Lebesgue integration that $\frac{d}{dt}\big((I^\gamma_{0+}y(s))(t)\big)$ exists and $\frac{d}{dt}\big((I^\gamma_{0+}y(s))(t)\big) = (I^{\gamma-1}_{0+}y(s))(t)$. Next, we show that $(I^{\gamma-1}_{0+}y(s))(t) \in AC(J,\mathbb{R})$. In fact, for every finite collection $\{(a_i, b_i)\}_{1\le i\le n}$ on J with $\sum_{i=1}^{n}(b_i - a_i) \to 0$, by applying Hölder's inequality, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n}\left|(I_{0+}^{\gamma-1}y(s))(b_i)-(I_{0+}^{\gamma-1}y(s))(a_i)\right| \\
\leq\ & \frac{1}{\Gamma(\gamma-1)}\left\{\sum_{i=1}^{n}\left|\int_0^{a_i}[(b_i-s)^{\gamma-2}-(a_i-s)^{\gamma-2}]y(s)ds\right|+\sum_{i=1}^{n}\int_{a_i}^{b_i}(b_i-s)^{\gamma-2}|y(s)|ds\right\} \\
\leq\ & \frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\gamma-1)}\left\{\sum_{i=1}^{n}\left[\int_0^{a_i}|(b_i-s)^{\gamma-2}-(a_i-s)^{\gamma-2}|^{\frac{1}{1-p}}ds\right]^{1-p}+\sum_{i=1}^{n}\left[\int_{a_i}^{b_i}(b_i-s)^{\frac{\gamma-2}{1-p}}ds\right]^{1-p}\right\} \\
=\ & \frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\gamma-1)}\left\{(2-\gamma)\sum_{i=1}^{n}\left(\int_0^{a_i}\left|\int_{a_i}^{b_i}(\zeta-s)^{\gamma-3}d\zeta\right|^{\frac{1}{1-p}}ds\right)^{1-p}+\Big(\frac{1-p}{\gamma-1-p}\Big)^{1-p}\sum_{i=1}^{n}(b_i-a_i)^{\gamma-1-p}\right\} \\
\leq\ & \overline{M}\sum_{i=1}^{n}\left\{\int_0^{a_i}\left[(a_i-s)^{\theta}-(b_i-s)^{\theta}\right]ds\right\}^{1-p}+\frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\gamma-1)}\Big(\frac{1-p}{\gamma-1-p}\Big)^{1-p}\sum_{i=1}^{n}(b_i-a_i)^{\gamma-1-p} \\
=\ & \overline{M}\sum_{i=1}^{n}\Big[\frac{a_i^{1+\theta}-b_i^{1+\theta}+(b_i-a_i)^{1+\theta}}{1+\theta}\Big]^{1-p}+\frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\gamma-1)}\Big(\frac{1-p}{\gamma-1-p}\Big)^{1-p}\sum_{i=1}^{n}(b_i-a_i)^{\gamma-1-p} \\
\to\ & 0,
\end{aligned}
$$

where $\overline{M}>0$ is a constant and $\theta=\dfrac{\gamma-2-p}{1-p}\in(-1,0)$. Now, $(I_{0+}^{\gamma-1}y(s))(t)\in AC(J,\mathbb{R})$, i.e., $(I_{0+}^{\gamma}y(s))(t)\in AC^2(J,\mathbb{R})$. □

The proof of our main result is based on the following fixed point theorem.

Theorem 1 ([21] (Schauder Fixed Point Theorem)). *If U is a non-empty, closed, bounded convex subset of a Banach space X and $T:U\to U$ is completely continuous, then T has a fixed point in U.*

3. The Linear Case

We consider the following linear fractional hybrid differential equations with boundary value conditions

$$
\begin{cases}
{}^{L}D_{0+}^{\alpha}\left(\dfrac{{}^{c}D_{0+}^{\beta}x(t)-\sum\limits_{i=1}^{m}I_{0+}^{\eta_i}h_i(t)}{g(t)}\right)=f(t),\quad t\in J=(0,1],\\
x'(0)=0,\ x(0)=d,\ I_{0+}^{1-\alpha}\Big(\frac{{}^{c}D_{0+}^{\beta}x(t)}{g(t)}\Big)(0+)=0,
\end{cases}
\tag{7}
$$

where $f\in L^{\frac{1}{p_1}}(J,\mathbb{R})(0<p_1<\frac{\alpha}{2})$, $g\in L^{\frac{1}{p_2}}(J,\mathbb{R})(0<p_2<\frac{\beta-1}{2})$ and $h_i\in L^{\frac{1}{p_3}}(J,\mathbb{R})(0<p_3<\frac{\beta+\eta_i-1}{2})(i=1,2,\cdots,m)$.

Theorem 2. *A function $x\in AC^2(J,\mathbb{R})$ is a solution of (7) if and only if x is a solution of the following equation*

$$
x(t)=d+\sum_{i=1}^{m}I_{0+}^{\beta+\eta_i}h_i(t)+I_{0+}^{\beta}\Big(g(\cdot)(I_{0+}^{\alpha}f)(\cdot)\Big)(t). \tag{8}
$$

Proof. (Necessity) For $t\in J$, it follows from Lemmas 2(1) and 3 that

$$
{}^{L}D_{0+}^{\alpha}\left(\frac{{}^{c}D_{0+}^{\beta}x(t)-\sum\limits_{i=1}^{m}I_{0+}^{\eta_i}h_i(t)-g(t)(I_{0+}^{\alpha}f)(t)}{g(t)}\right)=0,
$$

then

$$ {}^{c}D_{0^+}^{\beta}x(t) = c_0 t^{\alpha-1}g(t) + \sum_{i=1}^{m} I_{0^+}^{\eta_i}h_i(t) + g(t)(I_{0^+}^{\alpha}f)(t), $$

furthermore

$$ {}^{c}D_{0^+}^{\beta}\left[x(t) - I_{0^+}^{\beta}\left(c_0 t^{\alpha-1}g(t) + \sum_{i=1}^{m} I_{0^+}^{\eta_i}h_i(t) + g(t)(I_{0^+}^{\alpha}f)(t)\right)\right] = 0. $$

Using Lemmas 2 and 4, we arrive at

$$ \begin{aligned} x(t) &= c_1 + c_2 t + c_0 I_{0^+}^{\beta}\left(s^{\alpha-1}g(s)\right)(t) + \sum_{i=1}^{m} I_{0^+}^{\beta+\eta_i}h_i(t) + I_{0^+}^{\beta}\left(g(\cdot)(I_{0^+}^{\alpha}f)(\cdot)\right)(t), \\ x'(t) &= c_2 + c_0 I_{0^+}^{\beta-1}\left(s^{\alpha-1}g(s)\right)(t) + \sum_{i=1}^{m} I_{0^+}^{\beta+\eta_i-1}h_i(t) + I_{0^+}^{\beta-1}\left(g(\cdot)(I_{0^+}^{\alpha}f)(\cdot)\right)(t). \end{aligned} $$

The boundary value conditions imply that

$$ c_0 = 0,\ c_1 = x(0) = d,\ c_2 = x'(0) = 0. $$

(Sufficiency) Let $x(t)$ satisfy (8). It follows from Lemmas 2 and 6 that ${}^{c}D_{0^+}^{\beta}x(t)$ exists on J. It is not difficult to verify that $x(t)$ satisfies (7). The proof is completed. □

4. Main Results

To prove the existence of solutions to the problem (1)–(2), we need the following hypotheses on the functions $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $h_i : J \times \mathbb{R} \to \mathbb{R} (i = 1, 2, \cdots, m)$, respectively.

Hypothesis 1 (H1). *$f(\cdot, u, v) : J \to \mathbb{R}$ is measurable for all $u, v \in \mathbb{R}$, $f(t, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in J$, and there exist the functions φ, $a \in L^{\frac{1}{p_1}}(J, \mathbb{R}^+)(0 < p_1 < \min\{\alpha, 1 - \alpha\})$ such that*

$$ |f(t, u, v)| \le \varphi(t)(|u|^{\lambda_1} + |v|^{\lambda_2}) + a(t), $$

where $0 < \lambda_1 < \lambda_2 < 1$.

Hypothesis 2 (H2). *$g(\cdot, u) : J \to \mathbb{R} \setminus \{0\}$ is measurable for all $u \in \mathbb{R}$, $g(t, \cdot) : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous for a.e. $t \in J$, and there exists a function $\mu \in L^{\frac{1}{p_2}}(J, \mathbb{R}^+)(0 < p_2 < \frac{\beta-1}{2})$ such that $|g(t, x(t))| \le \mu(t)$.*

Hypothesis 3 (H3). *$h_i(\cdot, u) : J \to \mathbb{R}$ is measurable for all $u \in \mathbb{R}$, $h_i(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in J$, and there exist functions ϕ_i, $b \in L^{\frac{1}{p_3}}(J, \mathbb{R}^+)(0 < p_3 < \frac{\beta+\eta_i-1}{2})$ and a continuous non-decreasing function $\Psi : [0, \infty) \to [0, \infty)$ such that*

$$ |h_i(t, u(t))| \le \phi_i(t)\Psi(|u(t)|) + b(t) $$

and

$$ \liminf_{r\to\infty} \frac{\Psi(r)}{r} = \sigma < \infty. $$

We define

$$ C_\delta = \{x : x(t) \in C([0, 1], \mathbb{R}),\ t^{\delta}\, {}^{L}D_{0^+}^{\delta}x(t) \in C([0, 1], \mathbb{R})\} $$

with the norm
$$\|x\|_\delta = \max\{\max_{t\in[0,1]} |x(t)|,\ \max_{t\in[0,1]} t^\delta |{}^L D^\delta_{0^+} x(t)|\}.$$

Clearly, C_δ is a Banach space.

According to Theorem 2, we have the following result.

Theorem 3. *Assume that (H1)–(H3) are satisfied, then a function x is a solution of the problem (1)–(2) if and only if x is a solution of the following equation*

$$x(t) = d + \sum_{i=1}^{m} \big(I^{\beta+\eta_i}_{0^+} h_i\big)(t) + (I^{\beta}_{0^+} \widetilde{G})(t),$$

where

$$\begin{aligned}
\big(I^{\beta+\eta_i}_{0^+} h_i\big)(t) &:= \int_0^t \frac{(t-s)^{\beta+\eta_i-1}}{\Gamma(\beta+\eta_i)} h_i(s, x(s))ds,\\
\widetilde{G}(t, x(t)) &= g(t, x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), {}^L D^\delta_{0^+} x(s))ds.
\end{aligned}$$

The proof of Theorem 3 is similar to that of Theorem 2, and is thus omitted.

For convenience, we set constants

$$\begin{aligned}
\Lambda_1 &:= \left(B\left(\frac{\alpha-p_1}{1-p_1}, \frac{1-\lambda_2\delta-p_1}{1-p_1}\right)\right)^{1-p_1} \frac{1}{\Gamma(\alpha)};\\
\Lambda_2 &:= \left(\frac{1-p_2}{\beta-\delta-p_2}\right)^{1-p_2} \frac{\|\mu\|_{L^{\frac{1}{p_2}}}}{\Gamma(\beta-\delta)};\\
M_i &:= \left(\frac{1-p_3}{\beta+\eta_i-\delta-p_3}\right)^{1-p_3} \frac{1}{\Gamma(\beta+\eta_i-\delta)},\ \ i = 1, 2, \cdots, m;\\
\widehat{M} &:= |d| \max\left\{\frac{1}{\Gamma(1-\delta)}, 1\right\} + \sum_{i=1}^{m} \max\left\{\frac{\Gamma(\beta+\eta_i-\delta)}{\Gamma(\beta+\eta_i)}, 1\right\} M_i \|b\|_{L^{\frac{1}{p_3}}}\\
&\quad + \Lambda_2\Lambda_1 \|a\|_{L^{\frac{1}{p_1}}} \max\left\{\frac{\Gamma(\beta-\delta)}{\Gamma(\beta)}, 1\right\}.
\end{aligned}$$

Theorem 4. *If assumptions (H1)–(H3) hold, then the problem (1)–(2) has a solution $x \in C_\delta$, provided that,*

$$\sigma \sum_{i=1}^{m} \max\left\{\frac{1}{\Gamma(\beta+\eta_i)}, \frac{1}{\Gamma(\beta+\eta_i-\delta)}\right\} \left(\frac{1-p_3}{\beta+\eta_i-\delta-p_3}\right)^{1-p_3} \|\phi_i\|_{L^{\frac{1}{p_3}}} < 1.$$

Proof. We define an operator $\mathcal{F} : C_\delta \to C_\delta$ as

$$(\mathcal{F}x)(t) = d + \sum_{i=1}^{m} \big(I^{\beta+\eta_i}_{0^+} h_i\big)(t) + (I^{\beta}_{0^+} \widetilde{G})(t),$$

then
$$({}^L D^\delta_{0^+} \mathcal{F}x)(t) = \frac{dt^{-\delta}}{\Gamma(1-\delta)} + \sum_{i=1}^{m} I^{\beta+\eta_i-\delta}_{0^+} h_i(t, x(t)) + (I^{\beta-\delta}_{0^+} \widetilde{G})(t).$$

It is easy to see that, $\mathcal{F}$ is well defined, and the fixed point of $\mathcal{F}$ is the solution of the problem (1)–(2).

For $x \in C_\delta$, by Lemma 5, we have

$$\begin{aligned}
& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), {}^L D_{0^+}^{\delta} x(s)) ds \\
\leq\ & \int_0^t \frac{(t-s)^{\alpha-1} s^{-\lambda_2 \delta}}{\Gamma(\alpha)} \Big[\varphi(s) \cdot (\|x\|_\delta^{\lambda_1} + \|x\|_\delta^{\lambda_2}) + a(s)\Big] ds \\
\leq\ & \Lambda_1 \Big[\|\varphi\|_{L^{\frac{1}{p_1}}} (\|x\|_\delta^{\lambda_1} + \|x\|_\delta^{\lambda_2}) + \|a\|_{L^{\frac{1}{p_1}}}\Big], \qquad (9)
\end{aligned}$$

$$\begin{aligned}
\big|(I_{0^+}^{\beta-\delta} \widetilde{G})(t)\big| \leq\ & \Lambda_1 \int_0^t \frac{(t-s)^{\beta-\delta-1} \mu(s)}{\Gamma(\beta-\delta)} ds \cdot \Big[\|\varphi\|_{L^{\frac{1}{p_1}}} (\|x\|_\delta^{\lambda_1} + \|x\|_\delta^{\lambda_2}) + \|a\|_{L^{\frac{1}{p_1}}}\Big] \\
\leq\ & \Lambda_2 \cdot \Lambda_1 \Big[\|\varphi\|_{L^{\frac{1}{p_1}}} (\|x\|_\delta^{\lambda_1} + \|x\|_\delta^{\lambda_2}) + \|a\|_{L^{\frac{1}{p_1}}}\Big], \qquad (10)
\end{aligned}$$

$$\begin{aligned}
\Big|\Big(I_{0^+}^{\beta+\eta_i-\delta} h_i(s, x(s))\Big)(t)\Big| \leq\ & \int_0^t \frac{(t-s)^{\beta+\eta_i-\delta-1} \phi_i(s)}{\Gamma(\beta+\eta_i-\delta)} ds \cdot \Psi(\|x\|_\delta) + \int_0^t \frac{(t-s)^{\beta+\eta_i-\delta-1} b(s)}{\Gamma(\beta+\eta_i-\delta)} ds \\
\leq\ & M_i \cdot \Big[\|\phi_i\|_{L^{\frac{1}{p_3}}} \Psi(\|x\|_\delta) + \|b\|_{L^{\frac{1}{p_3}}}\Big]. \qquad (11)
\end{aligned}$$

We prove that $\mathcal{F}$ has a fixed point in C_δ by splitting three steps.

Step I. For an $r > 0$, we define a ball $B_r \subset C_\delta$ as $B_r = \{x \in C_\delta : \|x\|_\delta \leq r\}$. We claim that $\mathcal{F} B_r \subseteq B_r$. If this is not true, then, for each positive number r, there exists a function $\widetilde{x}(\cdot) \in B_r$, for some $t_0 \in J$,

$$\|(\mathcal{F}\widetilde{x})\|_\delta := \max\{|(\mathcal{F}\widetilde{x})(t_0)|,\ t_0^\delta |{}^L D_{0^+}^\delta (\mathcal{F}\widetilde{x})(t_0)|\} > r.$$

Moreover, noting that (10) and (12), we have

$$\begin{aligned}
|(\mathcal{F}\widetilde{x})(t)| \leq\ & |d| + \sum_{i=1}^m \frac{\Gamma(\beta+\eta_i-\delta)}{\Gamma(\beta+\eta_i)} \big|I_{0^+}^{\beta+\eta_i-\delta} h_i(t, \widetilde{x}(t))\big| + \frac{\Gamma(\beta-\delta)}{\Gamma(\beta)} \big|(I_{0^+}^{\beta-\delta} \widetilde{G}(s, \widetilde{x}(s)))(t)\big| \\
\leq\ & \widehat{M} + \sum_{i=1}^m \frac{\Gamma(\beta+\eta_i-\delta)}{\Gamma(\beta+\eta_i)} M_i \|\phi_i\|_{L^{\frac{1}{p_3}}} \Psi(r) + \frac{\Gamma(\beta-\delta)}{\Gamma(\beta)} \Lambda_1 \Lambda_2 \|\varphi\|_{L^{\frac{1}{p_1}}} \cdot (r^{\lambda_1} + r^{\lambda_2}), \\
t^\delta |{}^L D_{0^+}^\delta (\mathcal{F}\widetilde{x})(t)| \leq\ & \frac{|d|}{\Gamma(1-\delta)} + \sum_{i=1}^m |I_{0^+}^{\beta+\eta_i-\delta} h_i(t, \widetilde{x}(t))| + |(I_{0^+}^{\beta-\delta} \widetilde{G})(t)| \\
\leq\ & \widehat{M} + \sum_{i=1}^m M_i \|\phi_i\|_{L^{\frac{1}{p_3}}} \Psi(r) + \Lambda_1 \Lambda_2 \|\varphi\|_{L^{\frac{1}{p_1}}} \cdot (r^{\lambda_1} + r^{\lambda_2}),
\end{aligned}$$

this yields

$$\begin{aligned}
r < \|(\mathcal{F}\widetilde{x})\|_\delta \leq\ & \widehat{M} + \Lambda_1 \Lambda_2 \|\varphi\|_{L^{\frac{1}{p_1}}} \cdot (r^{\lambda_1} + r^{\lambda_2}) \max\Big\{\frac{\Gamma(\beta-\delta)}{\Gamma(\beta)}, 1\Big\} \\
& + \sum_{i=1}^m \max\Big\{\frac{\Gamma(\beta+\eta_i-\delta)}{\Gamma(\beta+\eta_i)}, 1\Big\} M_i \|\phi_i\|_{L^{\frac{1}{p_3}}} \Psi(r),
\end{aligned}$$

then we deduce that

$$\sigma \sum_{i=1}^m \max\Big\{\frac{1}{\Gamma(\beta+\eta_i)}, \frac{1}{\Gamma(\beta+\eta_i-\delta)}\Big\} \Big(\frac{1-p_3}{\beta+\eta_i-\delta-p_3}\Big)^{1-p_3} \|\phi_i\|_{L^{\frac{1}{p_3}}} \geq 1,$$

which is a contradiction. Therefore, $\mathcal{F} B_r \subseteq B_r$.

Step II. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in B_r, then there exists $\varepsilon > 0$, such that $\|x_n - x\|_\delta < \varepsilon$ for n sufficiently large. By (H1)–(H3) and (9), we have

$$\begin{aligned}
|f(t,x_n(t),{}^LD^{\delta}_{0+}x_n(t)) - f(t,x(t),{}^LD^{\delta}_{0+}x(t))| &\leq t^{-\lambda_2\delta}\varphi(t)(\varepsilon^{\lambda_1}+\varepsilon^{\lambda_2}+2r^{\lambda_1}+2r^{\lambda_2})+2a(t),\\
|h_i(t,x_n(t)) - h_i(t,x(t))| &\leq 2\phi_i(t)\Psi(r)+2b(t),\\
|g(t,x_n(t)) - g(t,x(t))| &\leq 2\mu(t),
\end{aligned}$$

and

$$\begin{aligned}
&|\widetilde{G}(t,\, x_n(t)) - \widetilde{G}(t,\, x(t))|\\
\leq\ & \frac{|g(t,x_n(t)) - g(t,x(t))|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x_n(s),{}^LD^{\delta}_{0+}x_n(s))|ds\\
&+\frac{|g(t,x(t))|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x_n(s),{}^LD^{\delta}_{0+}x_n(s)) - f(s,x(s),{}^LD^{\delta}_{0+}x(s))|ds\\
\leq\ & 4\mu(t)\left[\Lambda_1(\varepsilon^{\lambda_1}+\varepsilon^{\lambda_2}+r^{\lambda_1}+r^{\lambda_2})+\left(\frac{1-p_1}{\alpha-p_1}\right)^{1-p_1}\frac{\|a\|_{L^{\frac{1}{p_1}}}}{\Gamma(\alpha)}\right].
\end{aligned}$$

Moreover, using (H1)–(H3), for almost every $t \in J$, we arrive at

$$\begin{aligned}
|f(t,x_n(t),{}^LD^{\delta}_{0+}x_n(t)) - f(t,x(t),{}^LD^{\delta}_{0+}x(t))| &\to 0, \text{ as } n\to\infty,\\
|h_i(t,x_n(t)) - h_i(t,x(t))| &\to 0, \text{ as } n\to\infty,\\
|g(t,x_n(t)) - g(t,x(t))| &\to 0, \text{ as } n\to\infty,
\end{aligned}$$

hence

$$\begin{aligned}
&|\widetilde{G}(t,\, x_n(t)) - \widetilde{G}(t,\, x(t))|\\
\leq\ & \frac{|g(t,x_n(t)) - g(t,x(t))|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x_n(s),{}^LD^{\delta}_{0+}x_n(s))|ds\\
&+\frac{|g(t,x(t))|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x_n(s),{}^LD^{\delta}_{0+}x_n(s)) - f(s,x(s),{}^LD^{\delta}_{0+}x(s))|ds\\
\to\ & 0, \text{ as } n\to\infty.
\end{aligned}$$

From Lebesgue's dominated convergence theorem, (10) and (11), it follows that

$$\begin{aligned}
&\left|(I^{\beta+\eta_i}_{0+}h_i(s,x_n(s)))(t) - (I^{\beta+\eta_i}_{0+}h_i(s,x(s)))(t)\right| \to 0, \quad \text{as } n\to\infty,\\
&\left|(I^{\beta}_{0+}\widetilde{G}(s,x_n(s)))(t) - (I^{\beta}_{0+}\widetilde{G}(s,x(s)))(t)\right| \to 0, \quad \text{as } n\to\infty,
\end{aligned}$$

which implies that $|(\mathcal{F}x_n)(t) - (\mathcal{F}x)(t)| \to 0$ as $n\to\infty$. Similarly, we have $|({}^LD^{\delta}_{0+}\mathcal{F}x_n)(t) - ({}^LD^{\delta}_{0+}\mathcal{F}x)(t)| \to 0$ as $n\to\infty$. This yields the continuity of $\mathcal{F}$.

Step III. Arguments similar to those in Lemma 6, yield

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \to 0 \quad \text{and} \quad |t_2({}^LD^{\delta}_{0+}\mathcal{F}x)(t_2) - t_1({}^LD^{\delta}_{0+}\mathcal{F}x)(t_1)| \to 0,$$

therefore $\mathcal{F}B_r \subset B_r$ is equicontinuous. $\mathcal{F}$ is completely continuous which follows from the Arzela–Ascoli theorem. Now, Theorem 1 shows that $\mathcal{F}$ has a fixed point in B_r which is a solution of (1)–(2). □

When $\alpha = 1$, $m = 1$, $\eta_1 = \alpha$, $g(t,x(t)) \equiv 1$, $h(t,x(t)) = -\lambda(t)x(t)$ and $f(t,x(t),{}^LD^{\delta}_{0+}x(t)) = -\mu{}^LD^{\delta}_{0+}x(t) + h(t)$, the Equation (1) reduces to the general fractional non-homogeneous differential equation with a variable coefficient of the form

$${}^cD^{\beta+1}_{0+}x(t) + \mu{}^LD^{\delta}_{0+}x(t) + \lambda(t)x(t) = h(t).$$

The existence result of the solution to the above case can be stated below.

Corollary 1. *Assume that (H1), (H3) and $\lambda(t) \in L^{\frac{1}{p}}(0 < p < \frac{\beta-1}{2})$ are satisfied, then the problem*

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\beta+1}x(t) + \mu^{L}\mathcal{D}_{0^+}^{\delta}x(t) + \lambda(t)x(t) = h(t), \quad t \in J = (0, 1], \\ x'(0) = 0,\ x(0) = d,\ \lim\limits_{t\to0^+} t^{1-\alpha}\,({}^{c}\mathcal{D}_{0^+}^{\beta}x)(t) = 0, \end{cases} \tag{12}$$

has a solution $x \in C_\delta$ if

$$\max\left\{\frac{1}{\Gamma(\beta)}, \frac{1}{\Gamma(\beta-\delta)}\right\} \cdot \left(\frac{1-p}{\beta-p}\right)^{1-p} \cdot \frac{\|\lambda\|_{L^{\frac{1}{p}}}}{\Gamma(\beta)} < 1.$$

When $m = 1$, $\eta_1 = 0$ and $h(t, x(t)) = -\lambda(t)x(t)$, the Equation (1) reduces to the variable coefficient fractional hybrid Langevin equation of the following form

$${}^{L}\mathcal{D}_{0^+}^{\alpha}\left(\frac{{}^{c}\mathcal{D}_{0^+}^{\beta}x(t) + \lambda(t)x(t)}{g(t, x(t))}\right) = f(t, x(t), {}^{L}\mathcal{D}_{0^+}^{\delta}x(t)).$$

Obviously, the result below follows immediately.

Corollary 2. *Assume that (H1)–(H3) and $\lambda(t) \in L^{\frac{1}{p}}(0 < p < \frac{\beta-1}{2})$ are satisfied, then the problem*

$$\begin{cases} {}^{L}\mathcal{D}_{0^+}^{\alpha}\left(\dfrac{{}^{c}\mathcal{D}_{0^+}^{\beta}x(t) + \lambda(t)x(t)}{g(t, x(t))}\right) = f(t, x(t), {}^{L}\mathcal{D}_{0^+}^{\delta}x(t)), \quad t \in J = (0, 1], \\ x'(0) = 0,\ x(0) = d,\ \lim\limits_{t\to0^+} t^{1-\alpha}\,({}^{c}\mathcal{D}_{0^+}^{\beta}x)(t) = 0, \end{cases} \tag{13}$$

has a solution $x \in C_\delta$ if

$$\max\left\{\frac{1}{\Gamma(\beta)}, \frac{1}{\Gamma(\beta-\delta)}\right\} \cdot \left(\frac{1-p}{\beta-p}\right)^{1-p} \cdot \frac{\|\lambda\|_{L^{\frac{1}{p}}}}{\Gamma(\beta)} < 1.$$

For $g(t, x(t)) \equiv 1, \delta = 0$, we can determine the existence of solutions for the fractional Langevin equation with a variable coefficient.

Corollary 3. *Let $\lambda(\cdot) \in L^{\frac{1}{p}}(0 < p < \frac{\beta-1}{2})$ and $f : J \times \mathbb{R} \to \mathbb{R}$. If $f(\cdot, u) : J \to \mathbb{R}$ is measurable for all $u \in \mathbb{R}$, $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in J$, and there exist the functions φ, $a \in L^{\frac{1}{p}}(J, \mathbb{R}^+)(0 < p < \alpha)$ such that*

$$|f(t, u)| \le \varphi(t)|u|^\sigma + a(t), \quad 0 < \sigma < 1.$$

Then the problem

$$\begin{cases} {}^{L}\mathcal{D}_{0^+}^{\alpha}\left({}^{c}\mathcal{D}_{0^+}^{\beta}x(t) + \lambda(t)x(t)\right) = f(t, x(t)), \quad t \in J = (0, 1], \\ x'(0) = 0,\ x(0) = d,\ \lim\limits_{t\to0^+} t^{1-\alpha}\,({}^{c}\mathcal{D}_{0^+}^{\beta}x)(t) = 0, \end{cases} \tag{14}$$

has a solution $x \in C(J, \mathbb{R})$ if

$$\left(\frac{1-p}{\beta-p}\right)^{1-p} \cdot \frac{\|\lambda\|_{L^{\frac{1}{p}}}}{\Gamma(\beta)} < 1.$$

5. Example

Example 1. *We consider the following boundary value problem for the fractional hybrid differential equation:*

$$\begin{cases} {}^{L}D_{0^+}^{\frac{4}{5}}\left(\dfrac{{}^{c}D_{0^+}^{\frac{7}{5}}x(t)+t^{\frac{1}{12}}|x(t)|^{\frac{1}{2}}}{g(t,x(t))}\right)=\dfrac{1}{\sqrt[7]{t}}\left(|x(t)|^{\frac{1}{2}}+|{}^{L}D_{0^+}^{\frac{1}{6}}x(t)|^{\frac{3}{5}}\right)+\dfrac{1}{\sqrt[10]{t}}, \quad t\in J=(0,1], \\ x'(0)=0,\ x(0)=d,\ \lim\limits_{t\to 0^+} t^{\frac{1}{5}}\,({}^{c}D_{0^+}^{\frac{7}{5}}x)(t)=0. \end{cases} \tag{15}$$

Putting $\alpha=\frac{4}{5}$, $\beta=\frac{7}{5}$, $m=1$, $\delta=\frac{1}{6}$, $\eta_1=0$, $g(t,x(t))=\frac{x(t)}{\sqrt[8]{t}(1+|x(t)|)}$, $h(t,x(t))=-t^{\frac{1}{12}}|x(t)|^{\frac{1}{2}}$, $f(t,\,x(t),\,{}^{L}D_{0^+}^{\frac{1}{6}}x(t))=\frac{1}{\sqrt[7]{t}}\left(|x(t)|^{\frac{1}{2}}+|{}^{L}D_{0^+}^{\frac{1}{6}}x(t)|^{\frac{3}{5}}\right)+\frac{1}{\sqrt[10]{t}}$, *we can see*

$$\begin{aligned} |f(t,\,x(t),\,{}^{L}D_{0^+}^{\frac{1}{6}}x(t))| &\le \varphi(t)\left(|x(t)|^{\frac{1}{2}}+|{}^{L}D_{0^+}^{\frac{1}{6}}x(t)|^{\frac{3}{5}}\right)+a(t), \quad \varphi(t)=\frac{1}{\sqrt[7]{t}}\in L^6,\ a(t)=\frac{1}{\sqrt[10]{t}}\in L^6; \\ |g(t,x(t))| &= \frac{|x(t)|}{\sqrt[8]{t}(1+|x(t)|)}\le\frac{1}{\sqrt[8]{t}}=\mu(t)\in L^7; \\ |h(t,x(t))| &\le \Phi(t^{\frac{1}{6}}|x(t)|), \quad \Phi(u)=\sqrt{u}. \end{aligned}$$

Clearly, $\liminf\limits_{r\to\infty}\dfrac{\Psi(r)}{r}=0$. *By Theorem 4, the problem (15) has at least one solution* $x\in C_{\frac{1}{6}}$.

6. Conclusions and Outlook

Using a new technique which does not need the assumptions $f\in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $g,h_i\in C(J\times\mathbb{R},\mathbb{R})$, we obtain results that have practical applications for a couple of problems. The results for some important fractional differential equations, for example, the general fractional non-homogeneous differential Equation (12) with variable coefficient, can be obtained directly by our results. The proposed technique can be extended to other fractional hybrid differential equations. Furthermore, the technique can be employed to solve other types of equations. In future work, we will study the positive solution and its symmetry for the following boundary value problem of the fractional hybrid differential equations:

$${}^{H}D_{0^+}^{\alpha_1,\alpha_2}\left(\frac{{}^{c}D_{0^+}^{\beta}x(t)-\sum\limits_{i=1}^{m}I_{0^+}^{\eta_i}h_i(t,x(t))}{g(t,x(t))}\right)=f(t,x(t),{}^{L}D_{0^+}^{\delta}x(t)), \quad t\in J=(0,1],$$

where $\alpha_1,\alpha_2\in[0,1]$, $\beta\in(1,2]$, $\delta\in[0,\alpha_1)$, $\eta_i\in[0,1]$ and $\beta+\eta_i-\delta\in(1,2)$. ${}^{H}D_{0^+}^{\alpha_1,\alpha_2}$ is the standard Hilfer fractional derivative.

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