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The Metallic Ratio of Pulsating Fibonacci Sequences

Kittipong Laipaporn, Kiattiyot Phibul and Prathomjit Khachorncharoenkul *

School of Science, Walailak University, Nakhon Si Thammarat 80160, Thailand; lkittipo@wu.ac.th (K.L.); kiattiyot.ph@st.wu.ac.th (K.P.)
* Correspondence: prathomjit.kh@mail.wu.ac.th

Abstract: The golden ratio and the Fibonacci sequence (F_n) are well known, as is the fact that the ratio $\frac{F_{n+1}}{F_n}$ converges to the golden ratio for sufficiently large n. In this paper, we investigate the metallic ratio—a generalized version of the golden ratio—of pulsating Fibonacci sequences in three forms. Two of these forms are considered in the sense of pulsating recurrence relations, and their diagrams can be represented by symmetry, which is one of their distinguishing characteristics. The third form is the Fibonacci sequence in bipolar quantum linear algebra (BQLA), which also pulsates.

Keywords: pulsating Fibonacci sequence; bipolar Fibonacci sequence; golden ratio; metallic ratio; bipolar quantum linear algebra; mathematical induction

MSC: 11B39; 11B83

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1. Introduction

One of the most well-known sequences in the world is the Fibonacci sequence, $F_{n+2} = F_{n+1} + F_n$, for any $n \geqslant 0$ with initial conditions $F_0 = 0$ and $F_1 = 1$. This sequence has been studied for a long time and has been explored in many fields. It can be found in a variety of mathematical fields, such as abstract algebra and number theory. For example, in 1986, H.J. Wilcox [1] established the Fibonacci sequence in a finite abelian group with inspiration from D.D. Wall [2] and A.P. Shah [3]. Both of them studied the Fibonacci sequence together with modulo some fixed integer m. In 1990, S.W. Knox [4] (in the spirit of [1–3]) considered the k-nacci sequence of a finite group. The development along this route has continued. In 2003, E. Özkan et al. [5] provided some results with the Wall number of the ordinary three-step Fibonacci sequence (s_n) in a nilpotent group with nilpotency class 4 and exponent p for a prime number $p > 3$, defined by

$$s_{n+3} = s_{n+2} + s_{n+1} + s_n, \tag{1}$$

where $s_0 = s_1 = 0$, and $s_2 = 1$. In the same year, R. Dikici and E. Özkan [6] also studied a similar sequence (1) with the same initial data in a 3-generator relatively free group in the variety of nilpotent groups of class 2 and exponent p but in a generalized version as $s_{n+3} = as_{n+2} + bs_{n+1} + cs_n$ for fixed $a, b, c \in \mathbb{N}$. In 2020, the largest Fibonacci number, whose decimal expansion is of the form $ab \ldots bc \ldots c$, was found by P. Trojovský [7]. Moreover, we can see applications of Fibonacci numbers in applied mathematics and computer science, as follows. A.F. Nematollahi et al. [8] proposed a new metaheuristic optimization algorithm known as the golden ratio optimization method (GROM) that uses the golden ratio of the Fibonacci series to update the solutions in two different phases. This method is a parameter-free and simple implementation. Furthermore, GROM is very robust, and almost similar results have been obtained in different trials. F. Caldarola et al. [9] showed that all the Carboncettus words thus defined are Sturmian words, except in the case of $n = 5$, and the limit of the sequence of Carboncettus words is the Carboncettus limit word itself. These results originate from the Carboncettus octagon, a new geometric structure based on Fibonacci numbers which is similar to a regular octagon; see [10]. Furthermore, the Fibonacci sequence has numerous essential applications in diverse fields, including aesthetic applications, as shown in [11–13] or applications related to cross-branch testing, including probability theory [14], statistical physics [15], and education [16–18].

In this paper, we combine "the metallic ratio" and "the pulsated Fibonacci sequence", which generalize the concepts of the golden ratio and the Fibonacci sequence, respectively. To the best of our knowledge, this is the first work that provides a study on this topic. We give the definitions that we use below. *The metallic ratio* was defined by D. Passoja [12], in 2015, in the form of a continued fraction expansion. In addition, in 2020, R. Sivaraman [19] tried to generalize the recurrence relations to produce a more general ratio from which golden, silver, and bronze ratios follow:

For a given positive integer k as $\rho_k = \frac{k+\sqrt{k^2+4}}{2} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ where

$$x_{n+2} = kx_{n+1} + x_n, \tag{2}$$

and $x_1 = x_2 = 1$ for each $n \geqslant 1$. Indeed, in 2011, O. Yayenie [20] proposed a new type of generalized Fibonacci sequence $\{x_n\}_{n=0}^{\infty}$, which is defined recursively by $x_0 = 0, x_1 = 1$, and $x_n = \begin{cases} ax_{n-1} + x_{n-2} & \text{if } n \text{ is even,} \\ bx_{n-1} + x_{n-2} & \text{if } n \text{ is odd,} \end{cases}$ where $a, b \in \mathbb{R}\backslash\{0\}$ and $n \geqslant 2$. Notice that in the case $a = b = k$, the above sequence is Sequence (2). For special cases, we have *the golden ratio* $\Phi = \rho_1 = \frac{1}{2}(1+\sqrt{5})$, *the silver ratio* $\rho_2 = 1+\sqrt{2}$, and *the bronze ratio* $\rho_3 = \frac{1}{2}(3+\sqrt{13})$. In addition, in 1985, the origin of the pulsated Fibonacci sequence was shown by K.T. Atanassov et al. [21] who introduced a new perspective on the generalization of the Fibonacci sequence. After this, the generalization of pulsated Fibonacci sequences has been expressed. For example, in 2013, Atanassov [22] constructed *the* (a, b)*-pulsated Fibonacci sequence* as follows:

$$\begin{aligned} &\alpha_0 = a, \beta_0 = b, \\ &\alpha_{2n+1} = \beta_{2n+1} = \alpha_{2n} + \beta_{2n}, \\ &\alpha_{2n+2} = \alpha_{2n+1} + \beta_{2n}, \\ &\beta_{2n+2} = \beta_{2n+1} + \alpha_{2n} \end{aligned} \tag{3}$$

where $n \in \mathbb{N}_0$ and $a, b \in \mathbb{R}$. In the same year, the above sequence was modified by Atanassov [23], which was called *the* (a, b, c)*-pulsated Fibonacci sequence*. The following year, *the* $(a_1, a_2, \ldots, a_m)$*-pulsated Fibonacci sequence* [24] was introduced and is described as follows:

$$\begin{gathered} \alpha_{1,0} = a_1, \alpha_{2,0} = a_2, \ldots, \alpha_{m,0} = a_m, \\ \alpha_{1,2k+1} = \alpha_{2,2k+1} = \ldots = \alpha_{m,2k+1} = \sum_{i=1}^{m} \alpha_{i,2k}, \\ \alpha_{j,2k+2} = \alpha_{j,2k+1} + \alpha_{m-j+1,2k} \end{gathered} \tag{4}$$

where $j, k, m \in \mathbb{N}_0$ such that $1 \leqslant j \leqslant m$ and $a_1, a_2, \ldots, a_m \in \mathbb{R}$. In 2019, *the complex pulsating Fibonacci sequence* was introduced by S. Halici and A. Karatas [25] and is described as follows:

$$\begin{aligned} P_0 &= a + ci, Q_0 = b + ci, \\ Re(P_{n+1}) &= Im(P_n), \\ Re(Q_{n+1}) &= Im(Q_n), \\ Im(P_{2n+1}) &= Re(Q_{2n}) + Im(P_{2n}), \\ Im(Q_{2n+1}) &= Re(P_{2n}) + Im(Q_{2n}), \\ Im(P_{2n+2}) &= Im(Q_{2n+2}) = Im(P_{2n+1} + Q_{2n+1}) \end{aligned}$$

where $n \in \mathbb{N}_0$ and $a, b, c \in \mathbb{R}$. Recent types of pulsating Fibonacci sequences were published in 2021 and 2022. One is referred to as the pulsating $(m, c)-$Fibonacci sequence [26]. For real numbers $a_1, a_2, \ldots, a_m$ and c, *the pulsating* $(m, c)-$*Fibonacci sequence* is defined by

$$\begin{aligned}
\alpha_{1,0} &= a_1, \alpha_{2,0} = a_2, \ldots, \alpha_{m,0} = a_m, \\
\alpha_{1,2k+1} &= \alpha_{2,2k+1} = \cdots = \alpha_{m,2k+1} = c \sum_{i=1}^{m} \alpha_{i,2k}, \\
\alpha_{j,2k+2} &= \alpha_{j,2k+1} + \sum_{\substack{i=1 \\ i \neq j}}^{m} \alpha_{i,2k}
\end{aligned}$$

where j, k, and m are non-negative integers such that $1 \leqslant j \leqslant m$, and $c \neq 0$. Another one is referred to as *the complex pulsating* $(a_1, a_2, \ldots, a_m, c)$*-Fibonacci sequence* [27], which is given as follows. Let $a_1, a_2, \ldots, a_m$ and c be real numbers. Then,

$$\begin{aligned}
P_{1,0} = a_1 + ci, P_{2,0} &= a_2 + ci, \ldots, P_{m,0} = a_m + ci, \\
\mathrm{Re}(P_{j,k+1}) &= \mathrm{Im}(P_{j,k}), \\
\mathrm{Im}(P_{j,2k+1}) &= \mathrm{Re}(P_{m-j+1,2k}) + \mathrm{Im}(P_{j,2k}), \\
\mathrm{Im}(P_{1,2k+2}) = \mathrm{Im}(P_{2,2k+2}) = \cdots = \mathrm{Im}(P_{m,2k+2}) &= \mathrm{Im}\left(\sum_{i=1}^{m} P_{i,2k+1}\right)
\end{aligned}$$

for any non-negative integers j, k, and m such that $1 \leqslant j \leqslant m$.

However, the related problem of finding the metallic ratio, particularly the golden ratio, remains. The aim of this paper is to study the ratio of the consecutive terms of the following sequences. The first pulsating Fibonacci sequence to merge Sequences (2) and (3) is given by

$$\begin{aligned}
\alpha_0 &= a, \beta_0 = b, \\
\alpha_{2n+1} &= \beta_{2n+1} = \alpha_{2n} + \beta_{2n}, \\
\alpha_{2n+2} &= k\alpha_{2n+1} + \beta_{2n}, \\
\beta_{2n+2} &= k\beta_{2n+1} + \alpha_{2n}
\end{aligned} \tag{5}$$

where $n \in \mathbb{N}_0$, $k > 0$, and $a, b \geqslant 0$, such that a, b are not both zero simultaneously. By the pattern of pulsating of Sequence (5), in even subscript, the green line represents sequence α, and the yellow line represents sequence β, which are symmetrical with each other. Moreover, while $\alpha_2 = \alpha_1 + \beta_0$, $\beta_4 = \beta_3 + \alpha_2$ and so on are shown in solid lines, $\beta_2 = \beta_1 + \alpha_0$, $\alpha_4 = \alpha_3 + \beta_2$ and so on are shown in dashed lines, where both types of lines are symmetrical; see Figure 1.

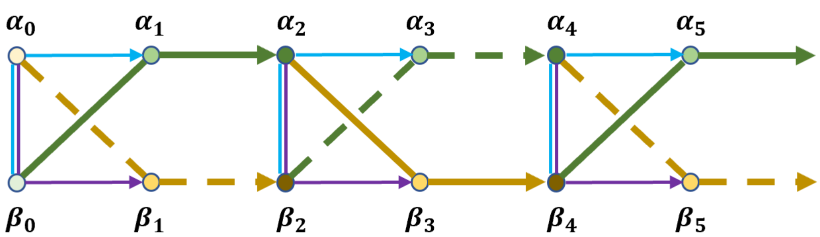

Figure 1. The (a, b)-pulsating Fibonacci sequence when $k = 1$ in Sequence (5).

Another sequence is in the same trace, but we consider the sequence (4) in the case of $m = 3$, shown as follows.

$$\begin{aligned}
\alpha_0 &= a, \beta_0 = b, \gamma_0 = c, \\
\alpha_{2n+1} &= \beta_{2n+1} = \gamma_{2n+1} = \alpha_{2n} + \beta_{2n} + \gamma_{2n}, \\
\alpha_{2n+2} &= k\alpha_{2n+1} + \gamma_{2n}, \\
\beta_{2n+2} &= k\beta_{2n+1} + \beta_{2n}, \\
\gamma_{2n+2} &= k\gamma_{2n+1} + \alpha_{2n}
\end{aligned} \tag{6}$$

where $n \in \mathbb{N}_0$, $k > 0$, and $a, b, c \geqslant 0$, such that a, b, and c are not all zero simultaneously; see Figure 2.

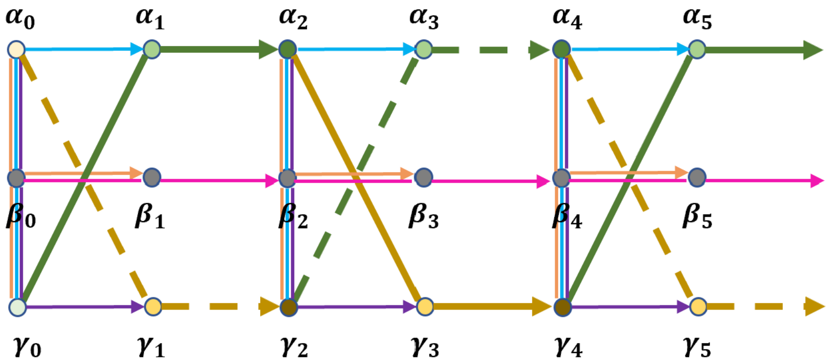

Figure 2. The (a, b, c)-pulsating Fibonacci sequence when $k = 1$ in Sequence (6). As seen in Figure 1, the green and yellow lines are symmetrical to each other, as are the solid and dashed lines.

Outline of the paper: In this paper, the main results are separated into two sections. In Section 2, the metallic ratios of pulsated Fibonacci sequences are presented in Theorems 1 and 2. In order to pave the way for the main results, the auxiliary result is found for α, β, and γ in Sequence (6), which is shown in the first part of this section. In Section 3, a new type of Fibonacci sequence introduced in 2016 by [28]—namely, the bipolar Fibonacci sequence—is presented, and we extend some concepts of Section 2 to this sequence; see Theorem 3. Both results in Sections 2 and 3 are equivalent. In Section 4, a discussion of the results and future work is presented. Lastly, in Section 5, we summarize our results and suggest some conjectures.

2. Pulsating Fibonacci Sequence

The following lemma is used to obtain the result in Theorem 2, which is one of the main results related to Sequence (6).

Lemma 1. *Let α, β, γ be the sequence (6). Then, for each $n \in \mathbb{N}$, the formulas for α and γ are*

$$\alpha_{2n} = \begin{cases} \sum_{i=0}^{n-1} k\alpha_{2i+1} + \alpha_0 & \text{if } 2 \mid n \\ \sum_{i=0}^{n-1} k\alpha_{2i+1} + \gamma_0 & \text{if } 2 \nmid n \end{cases} \quad \text{and} \quad \gamma_{2n} = \begin{cases} \sum_{i=0}^{n-1} k\gamma_{2i+1} + \gamma_0 & \text{if } 2 \mid n \\ \sum_{i=0}^{n-1} k\gamma_{2i+1} + \alpha_0 & \text{if } 2 \nmid n, \end{cases}$$

and the formula for β is $\beta_{2n} = \sum_{i=0}^{n-1} k\beta_{2i+1} + \beta_0$.

Proof. First, we will prove by mathematical induction the case of sequences α and γ. Clearly, for $n = 1$, we obtain $\alpha_2 = k\alpha_1 + \gamma_0$, and $\gamma_2 = k\gamma_1 + \alpha_0$. For $n = 2$, we have $\alpha_4 = k\alpha_3 + \gamma_2 = k\alpha_3 + k\alpha_1 + \alpha_0$, because α_1 and γ_1 are the same value. Similarly, $\gamma_4 = k\gamma_3 + \alpha_2 = k\gamma_3 + k\gamma_1 + \gamma_0$. Then, using the inductive hypothesis, we consider the two cases when n is even and odd. If n is even, then $\gamma_{2n} = \sum_{i=0}^{n-1} k\gamma_{2i+1} + \gamma_0$. Since $\alpha_{2n+1} = \gamma_{2n+1}$, we obtain $\alpha_{2n+2} = k\alpha_{2n+1} + \gamma_{2n} = k\alpha_{2n+1} + \sum_{i=0}^{n-1} k\alpha_{2i+1} + \gamma_0$. Thus, $\alpha_{2n+2} = \sum_{i=0}^{n} k\alpha_{2i+1} + \gamma_0$ if $2 \nmid n+1$. If n is odd, then $\gamma_{2n} = \sum_{i=0}^{n-1} k\gamma_{2i+1} + \alpha_0$. Since $\alpha_{2n+1} = \gamma_{2n+1}$, we obtain $\alpha_{2n+2} = k\alpha_{2n+1} + \sum_{i=0}^{n-1} k\alpha_{2i+1} + \alpha_0$. Then, $\alpha_{2n+2} = \sum_{i=0}^{n} k\alpha_{2i+1} + \alpha_0$ if $2 \mid n+1$. In a similar manner, we obtain $\gamma_{2n+2} = \sum_{i=0}^{n} k\gamma_{2i+1} + \alpha_0$ if $2 \nmid n+1$, and $\gamma_{2n+2} = \sum_{i=0}^{n} k\gamma_{2i+1} + \gamma_0$ if $2 \mid n+1$. Hence,

$$\alpha_{2n+2} = \begin{cases} \sum_{i=0}^{n} k\alpha_{2i+1} + \alpha_0 & \text{if } 2 \mid n+1 \\ \sum_{i=0}^{n} k\alpha_{2i+1} + \gamma_0 & \text{if } 2 \nmid n+1 \end{cases}$$

and

$$\gamma_{2n+2} = \begin{cases} \sum_{i=0}^{n} k\gamma_{2i+1} + \gamma_0 & \text{if } 2 \mid n+1 \\ \sum_{i=0}^{n} k\gamma_{2i+1} + \alpha_0 & \text{if } 2 \nmid n+1. \end{cases}$$

Next, based on mathematical induction and the fact that $\beta_{2n+2} = \beta_{2n+1} + \beta_{2n}$, the sequence β follows. □

The ratios of the consecutive terms for Sequences (5) and (6) are presented in Theorems 1 and 2, respectively.

Theorem 1. *Let (α_n) and (β_n) be an (a, b)-pulsating Fibonacci sequence as Sequence (5). Then, the pulsating metallic ratio is*

- $\lim_{n\to\infty} \frac{\alpha_{2n+1}}{\alpha_{2n}} = \lim_{n\to\infty} \frac{\beta_{2n+1}}{\beta_{2n}} = 2,$
- $\lim_{n\to\infty} \frac{\alpha_{2n+2}}{\alpha_{2n+1}} = \lim_{n\to\infty} \frac{\beta_{2n+2}}{\beta_{2n+1}} = \frac{2k+1}{2}.$

Proof. From Sequence (5), we obtain

$$\lim_{n\to\infty} \frac{\alpha_{2n+1}}{\alpha_{2n}} = \lim_{n\to\infty} \left(1 + \frac{\beta_{2n}}{\alpha_{2n}}\right). \tag{7}$$

Then, the limit of the ratio of β_{2n} and α_{2n} as n approaches infinity must be found. From the formulas of α and β, we have that $\alpha_{2n+2} = k\alpha_{2n+1} + \beta_{2n} = k\alpha_{2n} + (k+1)\beta_{2n}$, and $\beta_{2n+2} = k\beta_{2n+1} + \alpha_{2n} = k\beta_{2n} + (k+1)\alpha_{2n}$. Now, for $n \in \mathbb{N}$, we obtain

$$\frac{\alpha_{2n+2}}{\alpha_{2n}} = k + \frac{(k+1)\beta_{2n}}{\alpha_{2n}}, \tag{8}$$

$$\frac{\beta_{2n+2}}{\beta_{2n}} = k + \frac{(k+1)\alpha_{2n}}{\beta_{2n}}, \tag{9}$$

and $|\alpha_{2n} - \beta_{2n}| = |a - b|$ by using mathematical induction. It is obvious that (β_{2n}) is a strictly increasing sequence and unbounded. Therefore, $\lim_{n\to\infty} \beta_{2n} = \infty$, which forces $\lim_{n\to\infty} \frac{\alpha_{2n} - \beta_{2n}}{\beta_{2n}} = 0$. Thus, we obtain $\lim_{n\to\infty} \frac{\alpha_{2n}}{\beta_{2n}} = 1$. From Equation (7), we conclude that $\lim\limits_{n\to\infty}\dfrac{\alpha_{2n+1}}{\alpha_{2n}} = 2$. Similarly, we obtain $\lim\limits_{n\to\infty}\dfrac{\beta_{2n+1}}{\beta_{2n}} = \lim\limits_{n\to\infty}\left(1+\dfrac{\alpha_{2n}}{\beta_{2n}}\right) = 2$. In addition, from Equations (8) and (9), we obtain $\lim\limits_{n\to\infty}\dfrac{\alpha_{2n+2}}{\alpha_{2n}} = \lim\limits_{n\to\infty}\dfrac{\beta_{2n+2}}{\beta_{2n}} = 2k+1$. Thus, $\lim\limits_{n\to\infty}\dfrac{\alpha_{2n+2}}{\alpha_{2n+1}} = \lim\limits_{n\to\infty}\dfrac{\alpha_{2n+2}}{\alpha_{2n}}\cdot\dfrac{\alpha_{2n}}{\alpha_{2n+1}} = \dfrac{2k+1}{2}$. Similarly, we have $\lim\limits_{n\to\infty}\dfrac{\beta_{2n+2}}{\beta_{2n+1}} = \dfrac{2k+1}{2}$. □

Corollary 1. *For Sequence (5), the pulsating golden ratio is*

- $\lim\limits_{n\to\infty}\dfrac{\alpha_{2n+1}}{\alpha_{2n}} = \lim\limits_{n\to\infty}\dfrac{\beta_{2n+1}}{\beta_{2n}} = 2,$
- $\lim\limits_{n\to\infty}\dfrac{\alpha_{2n+2}}{\alpha_{2n+1}} = \lim\limits_{n\to\infty}\dfrac{\beta_{2n+2}}{\beta_{2n+1}} = \dfrac{3}{2}.$

Proof. This follows directly by substituting $k = 1$ in Theorem 1. □

Theorem 2. *Let (α_n), (β_n) and (γ_n) be an (a,b,c)-pulsating Fibonacci sequence as Sequence (6). Then, the pulsating metallic ratio is*

- $\lim\limits_{n\to\infty}\dfrac{\alpha_{2n+1}}{\alpha_{2n}} = \lim\limits_{n\to\infty}\dfrac{\beta_{2n+1}}{\beta_{2n}} = \lim\limits_{n\to\infty}\dfrac{\gamma_{2n+1}}{\gamma_{2n}} = 3,$
- $\lim\limits_{n\to\infty}\dfrac{\alpha_{2n+2}}{\alpha_{2n+1}} = \lim\limits_{n\to\infty}\dfrac{\beta_{2n+2}}{\beta_{2n+1}} = \lim\limits_{n\to\infty}\dfrac{\gamma_{2n+2}}{\gamma_{2n+1}} = \dfrac{3k+1}{3}.$

Proof. Since

$$\begin{aligned}\frac{\alpha_{2n+3}}{\alpha_{2n+1}} &= \frac{\alpha_{2n+2}+\beta_{2n+2}+\gamma_{2n+2}}{\alpha_{2n+1}} = \frac{k\alpha_{2n+1}+\gamma_{2n}+k\beta_{2n+1}+\beta_{2n}+k\gamma_{2n+1}+\alpha_{2n}}{\alpha_{2n+1}}\\ &= \frac{3k\alpha_{2n+1}+\alpha_{2n}+\beta_{2n}+\gamma_{2n}}{\alpha_{2n+1}} = \frac{(3k+1)\alpha_{2n+1}}{\alpha_{2n+1}},\end{aligned}$$

we obtain $\lim\limits_{n\to\infty}\dfrac{\alpha_{2n+3}}{\alpha_{2n+1}} = 3k+1$. Similarly, $\lim\limits_{n\to\infty}\dfrac{\beta_{2n+3}}{\beta_{2n+1}} = 3k+1 = \lim\limits_{n\to\infty}\dfrac{\gamma_{2n+3}}{\gamma_{2n+1}}$. Moreover, from the fact that $\alpha_m = \beta_m = \gamma_m$ for any positive odd number m, we have

$$\lim_{n\to\infty}\frac{\alpha_{2n+3}}{\beta_{2n+1}} = \lim_{n\to\infty}\frac{\alpha_{2n+3}}{\gamma_{2n+1}} = \lim_{n\to\infty}\frac{\beta_{2n+3}}{\gamma_{2n+1}} = \lim_{n\to\infty}\frac{\alpha_{2n+3}}{\alpha_{2n+1}} = 3k+1.$$

Using Lemma 1, we know that

$$\alpha_{2n} = \begin{cases}\sum\limits_{i=0}^{n-1} k\alpha_{2i+1}+\alpha_0 & \text{if } 2 \mid n\\ \sum\limits_{i=0}^{n-1} k\alpha_{2i+1}+\gamma_0 & \text{if } 2 \nmid n\end{cases},$$

and it implies

$$\alpha_{2n+2} = \begin{cases}\sum\limits_{i=0}^{n} k\alpha_{2i+1}+\alpha_0 & \text{if } 2 \mid n+1\\ \sum\limits_{i=0}^{n} k\alpha_{2i+1}+\gamma_0 & \text{if } 2 \nmid n+1\end{cases} = \begin{cases}\sum\limits_{i=0}^{n} k\alpha_{2i+1}+\alpha_0 & \text{if } 2 \nmid n\\ \sum\limits_{i=0}^{n} k\alpha_{2i+1}+\gamma_0 & \text{if } 2 \mid n.\end{cases}$$

Hence, we obtain

$$\frac{\alpha_{2n+2}}{\alpha_{2n}} = \begin{cases} \dfrac{\sum_{i=0}^{n} k\alpha_{2i+1} + \gamma_0}{\sum_{i=0}^{n-1} k\alpha_{2i+1} + \alpha_0} & \text{if } 2 \mid n \\ \dfrac{\sum_{i=0}^{n} k\alpha_{2i+1} + \alpha_0}{\sum_{i=0}^{n-1} k\alpha_{2i+1} + \gamma_0} & \text{if } 2 \nmid n \end{cases} = \begin{cases} \dfrac{k\alpha_{2n+1} + \sum_{i=0}^{n-1} k\alpha_{2i+1} + \gamma_0}{\sum_{i=0}^{n-1} k\alpha_{2i+1} + \alpha_0} & \text{if } 2 \mid n \\ \dfrac{k\alpha_{2n+1} + \sum_{i=0}^{n-1} k\alpha_{2i+1} + \alpha_0}{\sum_{i=0}^{n-1} k\alpha_{2i+1} + \gamma_0} & \text{if } 2 \nmid n \end{cases}$$

$$= \begin{cases} \dfrac{k\alpha_{2n} + k\beta_{2n} + k\gamma_{2n} + \sum_{i=0}^{n-1} k\alpha_{2i+1} + \gamma_0}{\sum_{i=0}^{n-1} k\alpha_{2i+1} + \alpha_0} & \text{if } 2 \mid n \\ \dfrac{k\alpha_{2n} + k\beta_{2n} + k\gamma_{2n} + \sum_{i=0}^{n-1} k\alpha_{2i+1} + \alpha_0}{\sum_{i=0}^{n-1} k\alpha_{2i+1} + \gamma_0} & \text{if } 2 \nmid n. \end{cases}$$

By Lemma 1, we conclude that

$$\frac{\alpha_{2n+2}}{\alpha_{2n}} = \begin{cases} \dfrac{k\alpha_0 + k\beta_0 + k\gamma_0 + 3\sum_{i=0}^{n-1} k^2\alpha_{2i+1} + \sum_{i=0}^{n-1} k\alpha_{2i+1} + \gamma_0}{\sum_{i=0}^{n-1} k\alpha_{2i+1} + \alpha_0} & \text{if } 2 \mid n \\ \dfrac{k\gamma_0 + k\beta_0 + k\alpha_0 + 3\sum_{i=0}^{n-1} k^2\alpha_{2i+1} + \sum_{i=0}^{n-1} k\alpha_{2i+1} + \alpha_0}{\sum_{i=0}^{n-1} k\alpha_{2i+1} + \gamma_0} & \text{if } 2 \nmid n. \end{cases}$$

Since $a, b, c \in \mathbb{R}^+$ and $(\alpha_n), (\beta_n), (\gamma_n)$ are strictly increasing sequences, we obtain $\lim_{n\to\infty} \alpha_n > 0$, $\lim_{n\to\infty} \beta_n > 0$, and $\lim_{n\to\infty} \gamma_n > 0$. Therefore, we have $\sum_{i=0}^{\infty} \alpha_{2i+1} = \infty$, $\sum_{i=0}^{\infty} \beta_{2i+1} = \infty$, and $\sum_{i=0}^{\infty} \gamma_{2i+1} = \infty$. Thus,

$$\lim_{n\to\infty} \frac{1}{\sum_{i=0}^{n-1} \alpha_{2i+1}} = \lim_{n\to\infty} \frac{1}{\sum_{i=0}^{n-1} \beta_{2i+1}} = \lim_{n\to\infty} \frac{1}{\sum_{i=0}^{n-1} \gamma_{2i+1}} = 0.$$

Hence, $\lim_{n\to\infty} \dfrac{\alpha_{2n+2}}{\alpha_{2n}} = 3k+1$. Similarly, $\lim_{n\to\infty} \dfrac{\beta_{2n+2}}{\beta_{2n}} = 3k+1$, and $\lim_{n\to\infty} \dfrac{\gamma_{2n+2}}{\gamma_{2n}} = 3k+1$. Next, Lemma 1 and the fact that $\alpha_{2n+1} = \beta_{2n+1} = \gamma_{2n+1} = \alpha_{2n} + \beta_{2n} + \gamma_{2n}$ imply that

$$\alpha_{2n} = \begin{cases} \sum_{i=0}^{n-1} k\gamma_{2i+1} + \alpha_0 & \text{if } 2 \mid n \\ \sum_{i=0}^{n-1} k\gamma_{2i+1} + \gamma_0 & \text{if } 2 \nmid n. \end{cases}$$

Then, for each $n \in \mathbb{N}$ we have

$$A_n \leqslant \frac{\gamma_{2n}}{\alpha_{2n}} \leqslant B_n$$

where

$$A_n = \min\left\{ \frac{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \gamma_0}{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \alpha_0}, \frac{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \alpha_0}{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \gamma_0} \right\}$$

and

$$B_n = \max\left\{ \frac{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \gamma_0}{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \alpha_0}, \frac{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \alpha_0}{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \gamma_0} \right\}.$$

From the fact that $\sum_{i=0}^{\infty} \gamma_{2i+1} = \infty$, it implies

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \gamma_0}{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \alpha_0} = \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \alpha_0}{\sum_{i=0}^{n-1} k\gamma_{2i+1} + \gamma_0} = 1$$

and then we get $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = 1$. Thus $\left(\frac{\gamma_{2n}}{\alpha_{2n}}\right)$ converges to 1 as $n \to \infty$. We obtain $\lim_{n\to\infty} \frac{\beta_{2n}}{\alpha_{2n}} = 1$; hence,

$$\lim_{n\to\infty} \frac{\alpha_{2n+1}}{\alpha_{2n}} = \lim_{n\to\infty} \frac{\alpha_{2n} + \beta_{2n} + \gamma_{2n}}{\alpha_{2n}} = 3.$$

Similarly, $\lim_{n\to\infty} \frac{\beta_{2n+1}}{\beta_{2n}} = 3$, and $\lim_{n\to\infty} \frac{\gamma_{2n+1}}{\gamma_{2n}} = 3$. As a result,

$$\lim_{n\to\infty} \frac{\alpha_{2n+2}}{\alpha_{2n+1}} = \lim_{n\to\infty} \frac{\alpha_{2n+2}}{\alpha_{2n}} \cdot \frac{\alpha_{2n}}{\alpha_{2n+1}} = \frac{3k+1}{3}.$$

By the same argument, we obtain $\lim_{n\to\infty} \frac{\beta_{2n+2}}{\beta_{2n+1}} = \frac{3k+1}{3}$, and $\lim_{n\to\infty} \frac{\gamma_{2n+2}}{\gamma_{2n+1}} = \frac{3k+1}{3}$. □

Corollary 2. *For Sequence (6), the pulsating golden ratio is*

- $\lim_{n\to\infty} \frac{\alpha_{2n+1}}{\alpha_{2n}} = \lim_{n\to\infty} \frac{\beta_{2n+1}}{\beta_{2n}} = \lim_{n\to\infty} \frac{\gamma_{2n+1}}{\gamma_{2n}} = 3,$
- $\lim_{n\to\infty} \frac{\alpha_{2n+2}}{\alpha_{2n+1}} = \lim_{n\to\infty} \frac{\beta_{2n+2}}{\beta_{2n+1}} = \lim_{n\to\infty} \frac{\gamma_{2n+2}}{\gamma_{2n+1}} = \frac{4}{3}.$

Proof. This follows directly by substituting $k = 1$ in Theorem 2. □

3. Bipolar Pulsating Fibonacci Sequence

In this section, our aim is to provide a smooth connection between bipolar quantum linear algebra (BQLA), which was first introduced by W.R. Zhang, and a new type of recurrence relation, as in the pulsating Fibonacci sequence. The concept of bipolar and its applications—for example, bioeconomics, bipolar disorder, bipolar cognitive mapping, and metal square—are described in the monograph [29]. Particularly, in chapter 8 of this monograph, Zhang used BQLA and bipolar quantum cellular automata (BQCA) to prove many laws, such as the symmetry law (or elementary energy equilibrium), energy transfer equilibrium law [30], the law of energy symmetry (or YinYang-n-element system nonequilibrium strengthening law) [31]. Moreover, he delivered some conjectures related to symmetry. One of them is that antimatter–matter bipolar symmetry or broken symmetry is bipolar equivalent to contraction–expansion bipolar symmetry or broken symmetry.

For simplicity, we present the terminology that will be used in this section as follows. *A bipolar dynamic equilibrium* is a process of bipolar interaction and state change among bipolar equilibrium, non-equilibrium and eternal equilibrium states of any action–reaction pair or any collection of such pairs. *A bipolar quantum agent* (BQA) is a bipolar dynamic equilibrium. The set of all bipolar agents is *the bipolar set* $B = \{(-a, b) \mid a, b \in \mathbb{R}_0^+\}$. The *norm* of $(-a, b) \in B$ is $|(-a, b)| = |a| + |b|$. For $(-a, b), (-c, d) \in B$, *the addition of the bipolar set* is defined as $(-a, b) + (-c, d) = (-a - c, b + d)$, and *the multiplication of the bipolar set* is defined as $(-a, b)(-c, d) = (-bc - ad, ac + bd)$. Both operations have commutative and associative properties with the identities $(0, 0)$ and $(0, 1)$, respectively. Moreover, $(0, a)$ in B is equivalent to $a \in \mathbb{R}_0^+$ in the sense that if we consider $(-x, y)$ as a vector in $\mathbb{R}^2$ space, the result $(0, 3)(-x, y) = (-3x, 3y)$ shows that the vector $(-x, y)$ is triply stretched. Hence, $(0, a)$ behaves as a constant in B, similar to how a behaves in $\mathbb{R}_0^+$.

Next, we introduce *the bipolar Fibonacci sequence*, created by F. Marchetti [28] in 2016, $F_n = (-f_n, f_{n+2})$ for $n \geqslant 0$, where (f_n) is the Fibonacci sequence, and $f_0 = f_1 = 1$. To consider the golden ratio of (F_n), Marchetti defined a new operation for BQLA as follows. *The division of the bipolar set* is a defined set for $(-a, b), (-c, d) \in B$ such that $c^2 - d^2 \neq 0$,

$$\frac{(-a, b)}{(-c, d)} = \left(-\frac{bc - ad}{c^2 - d^2}, \frac{ac - bd}{c^2 - d^2}\right).$$

This operation has a few points to be aware of, which are described in Remark 2.

In addition, we provide some properties of this division that contribute to our proof: $\dfrac{(-a, b) + (-c, d)}{(-a, b)} = (0, 1) + \dfrac{(-c, d)}{(-a, b)}$, and $\dfrac{(-a, b)(-c, d)}{(-a, b)} = (-c, d)$ for each $(-a, b), (-c, d) \in B$. Furthermore, as $n \to \infty$, Marchetti showed that $\dfrac{F_{n+1}}{F_n}$ converges to $(0, \Phi)$, where $\Phi = \dfrac{1 + \sqrt{5}}{2}$, and this limit is sensible because $(0, \Phi)$ is the constant Φ in $\mathbb{R}$. However, the definition of the convergent sequence in B was not given.

In this paper, a bipolar agent $(-L, M)$ is said to be *the limit of a sequence* $(-a_n, b_n)$ *in* B or *a sequence* $(-a_n, b_n)$ *converges to* $(-L, M)$, denoted by $\lim_{n\to\infty} (-a_n, b_n) = (-L, M)$, if for every number $\epsilon > 0$, there exists a natural number N such that for any $n \in \mathbb{N}$, if $n \geqslant N$, then

$$|\operatorname{Yin}((-a_n, b_n)) - L| + |\operatorname{Yang}((-a_n, b_n)) - M| < \epsilon$$

where $\operatorname{Yin}((-a, b)) = a$, and $\operatorname{Yang}((-a, b)) = b$, for any $(-a, b) \in B$. Consequently, for any sequence $(-a_n, b_n)$ in B, we have that $\lim_{n\to\infty} (-a_n, b_n) = (-L, M)$ if and only if

$\lim_{n\to\infty} \operatorname{Yin}((-a_n, b_n)) = L$, and $\lim_{n\to\infty} \operatorname{Yang}((-a_n, b_n)) = M$. Moreover, if $\lim_{n\to\infty} \dfrac{(-a_n, b_n)}{(-c_n, d_n)} = (0, 1)$, then $\lim_{n\to\infty} \dfrac{(-c_n, d_n)}{(-a_n, b_n)} = (0, 1)$.

Next, we aim to find the ratio of Sequences (10) and (11), which are inspired by sequence (5) and a bipolar sequence from Marchetti.

For (α_n) and (β_n) satisfying Sequence (5), we define the sequences (F_n) and (G_n) as follows:

$$\begin{aligned} F_{2n+1} &= G_{2n+1} = F_{2n} + G_{2n}, \\ F_{2n+2} &= (0,k)F_{2n+1} + G_{2n}, \\ G_{2n+2} &= (0,k)G_{2n+1} + F_{2n} \end{aligned} \tag{10}$$

where $F_0 = (-\alpha_0, \alpha_2)$, $F_1 = (-\alpha_1, \alpha_3)$, $G_0 = (-\beta_0, \beta_2)$, $G_1 = (-\beta_1, \beta_3)$, $k \in \mathbb{R}^+$, and $n \in \mathbb{N}_0$. Then, the sequence in (10) is called *a bipolar pulsating Fibonacci sequence*, which is depicted in Figure 3.

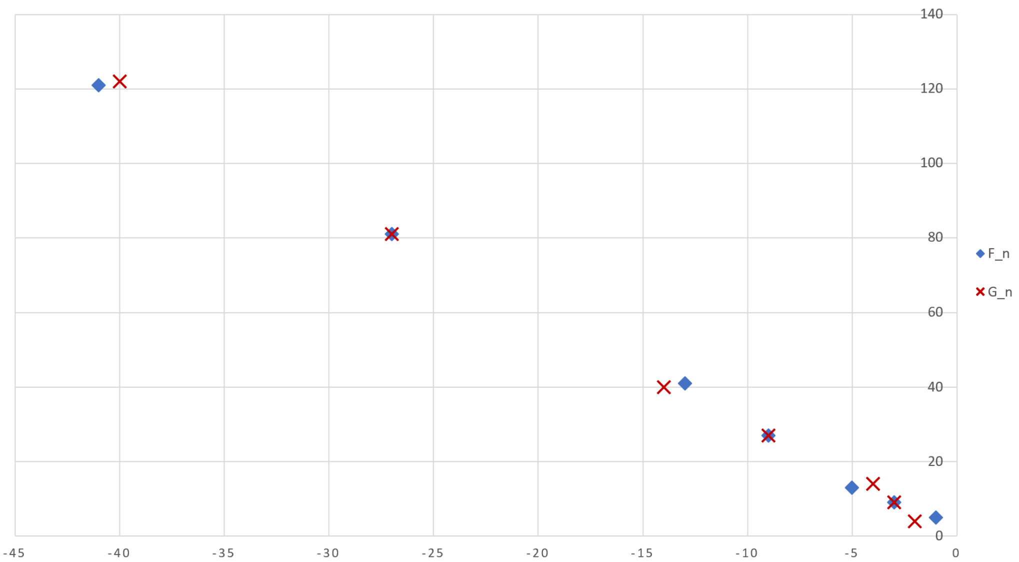

Figure 3. A bipolar pulsating Fibonacci sequence in the case where $k = 1, \alpha_0 = 1$, and $\beta_0 = 2$ in Sequence (10). This implies that $F_0 = (-1,5)$, $F_1 = (-3,9)$, $G_0 = (-2,4)$, and $G_1 = (-3,9)$. The blue diamonds and the red crosses represent the sequences (F_n) and (G_n), respectively.

Note that it is easy to show that $F_n = (-\alpha_n, \alpha_{n+2})$, and $G_n = (-\beta_n, \beta_{n+2})$ for any $n \in \mathbb{N}_0$ using mathematical induction. Now, we are ready to investigate the metallic ratio of this sequence.

Theorem 3. *Let (F_n) and (G_n) be a bipolar pulsating Fibonacci sequence as Sequence (10). Then, the bipolar pulsating metallic ratio is*

- $\displaystyle\lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n}} = \lim_{n\to\infty} \frac{G_{2n+1}}{G_{2n}} = (0,2),$
- $\displaystyle\lim_{n\to\infty} \frac{F_{2n+2}}{F_{2n+1}} = \lim_{n\to\infty} \frac{G_{2n+2}}{G_{2n+1}} = \left(0, \frac{2k+1}{2}\right).$

Remark 1. *Although (F_n) and (G_n) are in the bipolar set, and the ratios are computed by a more complicated division operation, their results $(0,2)$ and $\left(0, \frac{2k+1}{2}\right)$ still associate with the results in Theorem 1, which are 2 and $\frac{2k+1}{2}$, respectively.*

Proof. First, we consider for any $n \in \mathbb{N}_0$,

$$
\begin{aligned}
\frac{F_{2n+1}}{F_{2n}} = \frac{F_{2n}+G_{2n}}{F_{2n}} &= (0,1) + \frac{(-\beta_{2n}, \beta_{2n+2})}{(-\alpha_{2n}, \alpha_{2n+2})} \\
&= (0,1) + \left(-\frac{\beta_{2n+2}\alpha_{2n} - \beta_{2n}\alpha_{2n+2}}{\alpha_{2n}^2 - \alpha_{2n+2}^2}, \frac{\beta_{2n}\alpha_{2n} - \beta_{2n+2}\alpha_{2n+2}}{\alpha_{2n}^2 - \alpha_{2n+2}^2} \right) \\
&= \left(-\frac{\beta_{2n+2}\alpha_{2n} - \beta_{2n}\alpha_{2n+2}}{\alpha_{2n}^2 - \alpha_{2n+2}^2}, \frac{\beta_{2n}\alpha_{2n} - \beta_{2n+2}\alpha_{2n+2}}{\alpha_{2n}^2 - \alpha_{2n+2}^2} + 1 \right).
\end{aligned}
$$

From the proof of Theorem 1, we recall that $\lim\limits_{n\to\infty} \frac{\alpha_{2n}}{\beta_{2n}} = 1$, $\lim_{n\to\infty} \frac{\alpha_{2n}}{\alpha_{2n+1}} = \frac{1}{2}$, and $\lim\limits_{n\to\infty} \frac{\alpha_{2n+2}}{\alpha_{2n+1}} = \lim\limits_{n\to\infty} \frac{\beta_{2n+2}}{\beta_{2n+1}} = \frac{2k+1}{2}$. Next, we consider part of Yin and part of Yang as follows. In part of Yin, we obtain

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\beta_{2n+2}\alpha_{2n} - \beta_{2n}\alpha_{2n+2}}{\alpha_{2n}^2 - \alpha_{2n+2}^2} &= \lim_{n\to\infty} \frac{1 - \left(\frac{\beta_{2n}}{\alpha_{2n}}\right)\left(\frac{\alpha_{2n+2}}{\beta_{2n+2}}\right)}{\left(\frac{\alpha_{2n}}{\alpha_{2n+1}}\right)\left(\frac{\beta_{2n+1}}{\beta_{2n+2}}\right) - \left(\frac{\alpha_{2n+1}}{\alpha_{2n}}\right)\left(\frac{\alpha_{2n+2}}{\alpha_{2n+1}}\right)\left(\frac{\alpha_{2n+2}}{\beta_{2n+2}}\right)} \\
&= \frac{1-(1)(1)}{\left(\frac{1}{2}\right)\left(\frac{2}{2k+1}\right) - (2)\left(\frac{2k+1}{2}\right)(1)} = 0.
\end{aligned}
$$

Since $\lim\limits_{n\to\infty} \frac{\beta_{2n}}{\alpha_{2n}} = 1$, and $\lim\limits_{n\to\infty} \frac{\alpha_{2n+2}}{\alpha_{2n}} = \lim\limits_{n\to\infty} \frac{\beta_{2n+2}}{\beta_{2n}} = 2k+1$, we obtain

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\beta_{2n}\alpha_{2n} - \beta_{2n+2}\alpha_{2n+2}}{\alpha_{2n}^2 - \alpha_{2n+2}^2} &= \lim_{n\to\infty} \frac{\frac{\beta_{2n}}{\alpha_{2n}} - \left(\frac{\beta_{2n+2}}{\beta_{2n}}\right)\left(\frac{\beta_{2n}}{\alpha_{2n}}\right)\left(\frac{\alpha_{2n+2}}{\alpha_{2n}}\right)}{1 - \left(\frac{\alpha_{2n+2}}{\alpha_{2n}}\right)^2} \\
&= \frac{1-(2k+1)(1)(2k+1)}{1-(2k+1)^2} = 1.
\end{aligned}
$$

As a result, in part of Yang, the limit tends to 2. Hence, we have $\lim\limits_{n\to\infty} \frac{F_{2n+1}}{F_{2n}} = (0,2)$. Next, for the sequence (G_n), we have

$$
\begin{aligned}
\frac{G_{2n+1}}{G_{2n}} &= \frac{F_{2n}+G_{2n}}{G_{2n}} = (0,1) + \frac{(-\alpha_{2n}, \alpha_{2n+2})}{(-\beta_{2n}, \beta_{2n+2})} \\
&= (0,1) + \left(-\frac{\alpha_{2n+2}\beta_{2n} - \alpha_{2n}\beta_{2n+2}}{\beta_{2n}^2 - \beta_{2n+2}^2}, \frac{\alpha_{2n}\beta_{2n} - \alpha_{2n+2}\beta_{2n+2}}{\beta_{2n}^2 - \beta_{2n+2}^2} \right) \text{ for any } n \geqslant 0.
\end{aligned}
$$

Since $\lim\limits_{n\to\infty} \frac{\alpha_{2n}}{\beta_{2n}} = 1$, and $\lim\limits_{n\to\infty} \frac{\alpha_{2n+2}}{\alpha_{2n}} = \lim\limits_{n\to\infty} \frac{\beta_{2n+2}}{\beta_{2n}} = 2k+1$, we can see that

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\alpha_{2n+2}\beta_{2n} - \alpha_{2n}\beta_{2n+2}}{\beta_{2n}^2 - \beta_{2n+2}^2} &= \lim_{n\to\infty} \frac{\left(\frac{\alpha_{2n+2}}{\alpha_{2n}}\right)\left(\frac{\alpha_{2n}}{\beta_{2n}}\right) - \left(\frac{\alpha_{2n}}{\beta_{2n}}\right)\left(\frac{\beta_{2n+2}}{\beta_{2n}}\right)}{1 - \left(\frac{\beta_{2n+2}}{\beta_{2n}}\right)^2} \\
&= \frac{(2k+1)(1) - (1)(2k+1)}{1-(2k+1)^2} = 0,
\end{aligned}
$$

and

$$\lim_{n\to\infty}\frac{\alpha_{2n}\beta_{2n}-\alpha_{2n+2}\beta_{2n+2}}{\beta_{2n}^2-\beta_{2n+2}^2}=\lim_{n\to\infty}\frac{\left(\frac{\alpha_{2n}}{\beta_{2n}}\right)-\left(\frac{\alpha_{2n+2}}{\alpha_{2n}}\right)\left(\frac{\alpha_{2n}}{\beta_{2n}}\right)\left(\frac{\beta_{2n+2}}{\beta_{2n}}\right)}{1-\left(\frac{\beta_{2n+2}}{\beta_{2n}}\right)^2}$$
$$=\frac{(1)-(2k+1)(1)(2k+1)}{1-(2k+1)^2}=1.$$

Thus, $\lim_{n\to\infty}\frac{G_{2n+1}}{G_{2n}}=(0,2)$. In addition, we have $\frac{F_{2n+2}}{F_{2n+1}}=\frac{F_{2n+1}+G_{2n}}{F_{2n+1}}=(0,k)+\frac{G_{2n}}{F_{2n+1}}$, and

$$\frac{G_{2n}}{F_{2n+1}}=\frac{(-\beta_{2n},\beta_{2n+2})}{(-\alpha_{2n+1},\alpha_{2n+3})}=\left(-\frac{\alpha_{2n+1}\beta_{2n+2}-\alpha_{2n+3}\beta_{2n}}{\alpha_{2n+1}^2-\alpha_{2n+3}^2},\frac{\alpha_{2n+1}\beta_{2n}-\alpha_{2n+3}\beta_{2n+2}}{\alpha_{2n+1}^2-\alpha_{2n+3}^2}\right).$$

From the fact that $\alpha_{2n+1}=\beta_{2n+1}$ for all $n\geqslant 0$ and Theorem 1, it follows that

$$\lim_{n\to\infty}\frac{\alpha_{2n+1}\beta_{2n+2}-\alpha_{2n+3}\beta_{2n}}{\alpha_{2n+1}^2-\alpha_{2n+3}^2}=\lim_{n\to\infty}\frac{\left(\frac{\beta_{2n+2}}{\alpha_{2n+1}}\right)-\left(\frac{\alpha_{2n+3}}{\alpha_{2n+1}}\right)\left(\frac{\beta_{2n}}{\alpha_{2n+1}}\right)}{1-\left(\frac{\alpha_{2n+3}}{\alpha_{2n+1}}\right)^2}$$
$$=\frac{\left(\frac{2k+1}{2}\right)-(2k+1)\left(\frac{1}{2}\right)}{1-(2k+1)^2}=0,$$

and

$$\lim_{n\to\infty}\frac{\alpha_{2n+1}\beta_{2n}-\alpha_{2n+3}\beta_{2n+2}}{\alpha_{2n+1}^2-\alpha_{2n+3}^2}=\lim_{n\to\infty}\frac{\left(\frac{\beta_{2n}}{\alpha_{2n+1}}\right)-\left(\frac{\alpha_{2n+3}}{\alpha_{2n+1}}\right)\left(\frac{\beta_{2n+2}}{\alpha_{2n+1}}\right)}{1-\left(\frac{\alpha_{2n+3}}{\alpha_{2n+1}}\right)^2}$$
$$=\frac{\left(\frac{1}{2}\right)-(2k+1)\left(\frac{2k+1}{2}\right)}{1-(2k+1)^2}=\frac{1}{2};$$

hence, $\lim_{n\to\infty}\frac{G_{2n}}{F_{2n+1}}=\left(0,\frac{1}{2}\right)$. Consequently, $\lim_{n\to\infty}\frac{F_{2n+2}}{F_{2n+1}}=\left(0,\frac{2k+1}{2}\right)$. In the same manner, since $\frac{G_{2n+2}}{G_{2n+1}}=\frac{G_{2n+1}+F_{2n}}{G_{2n+1}}=(0,k)+\frac{F_{2n}}{G_{2n+1}}$, and

$$\frac{F_{2n}}{G_{2n+1}}=\frac{(-\alpha_{2n},\alpha_{2n+2})}{(-\beta_{2n+1},\beta_{2n+3})}=\left(-\frac{\alpha_{2n+2}\beta_{2n+1}-\alpha_{2n}\beta_{2n+3}}{\beta_{2n+1}^2-\beta_{2n+3}^2},\frac{\alpha_{2n}\beta_{2n+1}-\alpha_{2n+2}\beta_{2n+3}}{\beta_{2n+1}^2-\beta_{2n+3}^2}\right),$$

it implies that

$$\lim_{n\to\infty}\frac{F_{2n}}{G_{2n+1}}=\left(-\frac{\left(\frac{2k+1}{2}\right)-\left(\frac{1}{2}\right)(2k+1)}{1-(2k+1)^2},\frac{\left(\frac{1}{2}\right)-\left(\frac{2k+1}{2}\right)(2k+1)}{1-(2k+1)^2}\right)=\left(0,\frac{1}{2}\right).$$

Hence, we obtain $\lim_{n\to\infty}\frac{G_{2n+2}}{G_{2n+1}}=\left(0,\frac{2k+1}{2}\right)$. □

In the rest of this section, we consider Sequence (10) as a special case to the following sequence.

Let (α_n) and (β_n) be sequences obtained from Sequence (5) in the case of $k = 1$. Then, we define the sequences (F_n) and (G_n) as follows:

$$\begin{aligned} F_{2n+1} &= G_{2n+1} = F_{2n} + G_{2n}, \\ F_{2n+2} &= F_{2n+1} + G_{2n}, \\ G_{2n+2} &= G_{2n+1} + F_{2n} \end{aligned} \tag{11}$$

where $F_0 = (-\alpha_0, \alpha_2)$, $F_1 = (-\alpha_1, \alpha_3)$, $G_0 = (-\beta_0, \beta_2)$, $G_1 = (-\beta_1, \beta_3)$, and $n \in \mathbb{N}_0$.

Corollary 3. *For Sequence (11), the bipolar pulsating golden ratio is*

- $\displaystyle\lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n}} = \lim_{n\to\infty} \frac{G_{2n+1}}{G_{2n}} = (0, 2),$
- $\displaystyle\lim_{n\to\infty} \frac{F_{2n+2}}{F_{2n+1}} = \lim_{n\to\infty} \frac{G_{2n+2}}{G_{2n+1}} = \left(0, \frac{3}{2}\right).$

Proof. The proof follows directly from Theorem 3 and the fact that $(0, 1)$ is the identity of the multiplication. □

Finally, the following remark shows some points to be aware of when dividing in the bipolar set. This remark was adjusted from the comments in [32].

Remark 2. *Let* $(-a, b), (-c, d) \in B$. *Then,* $\dfrac{(-a, b)}{(-c, d)} \in B$ *if and only if it satisfies one of the following conditions.*

- $c > d, a \leqslant b, c \geqslant \dfrac{bd}{a}$ *if* $a \neq 0$
- $c > d, a \geqslant b, c \geqslant \dfrac{ad}{b}$ *if* $b \neq 0$
- $c < d, a \leqslant b, c \leqslant \dfrac{ad}{b}$ *if* $b \neq 0$
- $c < d, a \geqslant b, c \leqslant \dfrac{bd}{a}$ *if* $a \neq 0$

4. Discussion

There are other forms of the pulsating sequence (5) that appeared in the last part of [21]. The following recurrence relations are other pulsating sequences in the same spirit as Sequence (3).

$$\begin{aligned} \alpha_0 &= a, \beta_0 = b, \\ \alpha_{2n+1} &= \beta_{2n+1} = \alpha_{2n} + \beta_{2n}, \\ \alpha_{2n+2} &= k\beta_{2n+1} + \beta_{2n}, \\ \beta_{2n+2} &= k\alpha_{2n+1} + \alpha_{2n} \end{aligned} \tag{12}$$

where $n \in \mathbb{N}_0$, $k > 0$, and $a, b \geqslant 0$ such that a, b are not both zero simultaneously.

$$\begin{aligned} \alpha_0 &= a, \beta_0 = b, \\ \alpha_{2n+1} &= \beta_{2n+1} = \alpha_{2n} + \beta_{2n}, \\ \alpha_{2n+2} &= k\alpha_{2n+1} + \alpha_{2n}, \\ \beta_{2n+2} &= k\beta_{2n+1} + \beta_{2n} \end{aligned} \tag{13}$$

where $n \in \mathbb{N}_0$, $k > 0$, and $a, b \geqslant 0$ such that a, b are not both zero simultaneously.

$$\begin{aligned} \alpha_0 &= a, \beta_0 = b, \\ \alpha_{2n+1} &= \beta_{2n+1} = \alpha_{2n} + \beta_{2n}, \\ \alpha_{2n+2} &= k\beta_{2n+1} + \alpha_{2n}, \\ \beta_{2n+2} &= k\alpha_{2n+1} + \beta_{2n} \end{aligned} \tag{14}$$

where $n \in \mathbb{N}_0$, $k > 0$, and $a, b \geqslant 0$ such that a, b are not both zero simultaneously. Under the condition $\alpha_{2n+1} = \beta_{2n+1} = \alpha_{2n} + \beta_{2n}$, it implies that the pulsating sequences (5) and (12) are the same sequence, and it also occurs in Sequences (13) and (14). So, the results of Theorem 1 can be applied to the pulsating sequence (12). Furthermore, Sequence (13) is similar to the origin sequence (2), and in the same way as the proof of Theorem 1, we can reach forward the limit of $\frac{\alpha_{2n+2}}{\alpha_{2n+1}}$ and $\frac{\beta_{2n+2}}{\beta_{2n+1}}$. As a result, the sequence (14) outcomes will appear right away. That is why we only considered the form of the pulsating sequence (5).

Returning to the original version of the bipolar Fibonacci sequence $F_n = (-f_n, f_{n+2})$ for $n \geqslant 0$, where (f_n) is the Fibonacci sequence, and $f_0 = f_1 = 1$, for a fixed $k \geqslant 1$, the standard form of the metallic ratio should be from the generalized bipolar Fibonacci sequence as follows. For $n \geqslant 0$,

$$F_{n+2} = (0, k)F_{n+1} + F_n \tag{15}$$

where $F_n \in B$, $F_0 = (-1, k+1)$, and $F_1 = (-1, k(k+1)+1)$. If we let $F_n = (-f_n, f_{n+2})$ for all $n \geqslant 0$, we automatically have a recurrence relation $f_{n+2} = kf_{n+1} + f_n$, where $n \geqslant 0$, $f_0 = f_1 = 1$; then, $\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$ is the ordinary metallic ratio $\rho_k = \frac{k + \sqrt{k^2+4}}{2}$. By the rule of the division of the bipolar set and the fact that $\rho_k^2 = k\rho_k + 1$, the following ratio of Sequence (15) is presented immediately

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = (0, k) + \lim_{n\to\infty} \frac{F_{n-1}}{F_n} = (0, k) + \left(0, \frac{1}{\rho_k}\right) = (0, \rho_k).$$

Hence, we see $(0, \rho_k)$ plays a role as the metallic ratio of the sequence $F_{n+2} = (0, k)F_{n+1} + F_n$, where $F_0 = (-1, k+1)$, and $F_1 = (-1, k(k+1)+1)$. This is very similar to the original sequence (2), $x_{n+2} = kx_{n+1} + x_n$. So, instead, we examined a bipolar set and a pulsating sequence, which can be interwoven with concepts of the metallic means.

Notice that, in Section 3, even though the algebraic operation of addition in the bipolar set is quite straightforward, it is different for the multiplication and the division of the bipolar set. To illustrate these operations, we examine two agents $(-2, 3)$ and $(-a, b)$ in B as vectors. From the results of $(-2, 0)(-a, b) = (-2b, 2a)$ and $(0, 3)(-a, b) = (-3a, 3b)$, we can see that $(-2, 3)(-a, b) = (-2, 0)(-a, b) + (0, 3)(-a, b)$ is a sum of two vectors, where one $(-2b, 2a)$ is a vector twice the length of vector $(-a, b)$ in the opposite direction with respect to the line $y = -x$ in the XY-plane, and another one is a triple stretch of a vector $(-a, b)$.

The division operation of bipolar is defined from the inverse operation of multiplication under some conditions. It contains the same trend of multiplication in some cases, such as $\frac{(-a,b)}{(-2,0)} = \left(-\frac{b}{2}, \frac{a}{2}\right)$, and $\frac{(-a,b)}{(0,3)} = \left(-\frac{a}{3}, \frac{b}{3}\right)$, but $\frac{(-a,b)}{(-2,3)} \neq \frac{(-a,0)}{(-2,3)} + \frac{(0,b)}{(-2,3)}$. Indeed, $\frac{(-a,b)}{(-2,3)} = \left(-\frac{2b-3a}{-5}, \frac{2a-3b}{-5}\right) = \left(-\frac{3a-2b}{5}, \frac{3b-2a}{5}\right)$ for any nonnegative real numbers a and b satisfied in Remark 2; then, it is a difference of two vectors $\left(-\frac{3a}{5}, \frac{3b}{5}\right)$ and $\left(-\frac{2b}{5}, \frac{2a}{5}\right)$. So, we can see that the result from the division operation is more complicated. Surprisingly, this operation does not effect the results of the ratios in Theorem 3.

As mentioned previously in Section 3, the behavior of $(0, a)$ is that of a constant in B, similar to how a constant a performs in $\mathbb{R}$. So, the results of the ratio in Theorem 3, i.e., $(0, 2)$ and $\left(0, \frac{2k+1}{2}\right)$ should be equivalent to 2 and $\frac{2k+1}{2}$, which are the results in Theorem 3. From these facts, we assert that, together with the same structure of recurrence relations in Sequences (5) and (10) and the intrinsic nature of the metallic ratio, this may dominate the novelty of the division operations of a bipolar set. In other words, if we look at these characteristics as if they were human genes, the novelty of the division has to be the recessive genes but the others are the dominant genes. Moreover, the phenomenon of the equivalent results between Sections 2 and 3 is one of the indications that emphasize the celebrity number, the golden ratio. This number almost appears in everything (see [33]), including arts, architecture, music and even bipolar concepts, which still did not seclude from Φ and its partisans. This is another reason that we proposed Section 3 in this paper.

Finally, the elementary tools for solving the problems in this paper have prompted us to choose this concept for our students to work on in the active learning classroom to follow in the footsteps of S. Abramovich et al. [34] in one of our future works. The others are Conjectures 1 and 2 at the end of Section 5.

5. Conclusions

For Sequence (5),

$$\begin{aligned}
\alpha_0 &= a, \beta_0 = b, \\
\alpha_{2n+1} &= \beta_{2n+1} = \alpha_{2n} + \beta_{2n}, \\
\alpha_{2n+2} &= k\alpha_{2n+1} + \beta_{2n}, \\
\beta_{2n+2} &= k\beta_{2n+1} + \alpha_{2n}
\end{aligned}$$

where $n \in \mathbb{N}_0$, $k > 0$, and $a, b \geqslant 0$, such that a, b are not both zero simultaneously. The pulsating metallic ratio is

- $\lim\limits_{n\to\infty} \frac{\alpha_{2n+1}}{\alpha_{2n}} = \lim\limits_{n\to\infty} \frac{\beta_{2n+1}}{\beta_{2n}} = 2,$
- $\lim\limits_{n\to\infty} \frac{\alpha_{2n+2}}{\alpha_{2n+1}} = \lim\limits_{n\to\infty} \frac{\beta_{2n+2}}{\beta_{2n+1}} = \frac{2k+1}{2}.$

For Sequence (6),

$$\begin{aligned}
\alpha_0 &= a, \beta_0 = b, \gamma_0 = c, \\
\alpha_{2n+1} &= \beta_{2n+1} = \gamma_{2n+1} = \alpha_{2n} + \beta_{2n} + \gamma_{2n}, \\
\alpha_{2n+2} &= k\alpha_{2n+1} + \gamma_{2n}, \\
\beta_{2n+2} &= k\beta_{2n+1} + \beta_{2n}, \\
\gamma_{2n+2} &= k\gamma_{2n+1} + \alpha_{2n}
\end{aligned}$$

where $n \in \mathbb{N}_0$, $k > 0$, and $a, b, c \geqslant 0$, such that a, b, and c are not all zero simultaneously. The pulsating metallic ratio is

- $\lim\limits_{n\to\infty} \frac{\alpha_{2n+1}}{\alpha_{2n}} = \lim\limits_{n\to\infty} \frac{\beta_{2n+1}}{\beta_{2n}} = \lim\limits_{n\to\infty} \frac{\gamma_{2n+1}}{\gamma_{2n}} = 3,$
- $\lim\limits_{n\to\infty} \frac{\alpha_{2n+2}}{\alpha_{2n+1}} = \lim\limits_{n\to\infty} \frac{\beta_{2n+2}}{\beta_{2n+1}} = \lim\limits_{n\to\infty} \frac{\gamma_{2n+2}}{\gamma_{2n+1}} = \frac{3k+1}{3}.$

And, for Sequence (10),

$$\begin{aligned}
F_{2n+1} &= G_{2n+1} = F_{2n} + G_{2n}, \\
F_{2n+2} &= (0, k)F_{2n+1} + G_{2n}, \\
G_{2n+2} &= (0, k)G_{2n+1} + F_{2n}
\end{aligned}$$

where $F_0 = (-\alpha_0, \alpha_2)$, $F_1 = (-\alpha_1, \alpha_3)$, $G_0 = (-\beta_0, \beta_2)$, $G_1 = (-\beta_1, \beta_3)$, $k \in \mathbb{R}^+$, and $n \in \mathbb{N}_0$. The bipolar pulsating metallic ratio is

- $\lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n}} = \lim_{n\to\infty} \frac{G_{2n+1}}{G_{2n}} = (0,2)$,
- $\lim_{n\to\infty} \frac{F_{2n+2}}{F_{2n+1}} = \lim_{n\to\infty} \frac{G_{2n+2}}{G_{2n+1}} = \left(0, \frac{2k+1}{2}\right)$.

In summary, we investigated the behavior of the ratio of the sequences shown in (5) and (6) that are paired with the metallic ratio by the limits in the theorems and corollaries in Section 2. We also showed that there is a bridge between the pulsating Fibonacci sequence and the bipolar Fibonacci sequence through the limits of the ratio of the consecutive terms of Sequences (5) and (10) in Section 3.

Moreover, we obtained the other ratios

$$\lim_{n\to\infty} \frac{\alpha_{2n+3}}{\alpha_{2n+1}} = \lim_{n\to\infty} \frac{\beta_{2n+3}}{\beta_{2n+1}} = \lim_{n\to\infty} \frac{\alpha_{2n+2}+\beta_{2n+2}}{\alpha_{2n+1}} = \lim_{n\to\infty} \left(\frac{\frac{\alpha_{2n+2}}{\alpha_{2n}} + \frac{\beta_{2n+2}}{\beta_{2n}} \frac{\beta_{2n}}{\alpha_{2n}}}{1+\frac{\beta_{2n}}{\alpha_{2n}}} \right) = 2k+1$$

in the case of Sequence (5) and

$$\lim_{n\to\infty} \frac{\alpha_{2n+3}}{\alpha_{2n+1}} = \lim_{n\to\infty} \frac{\beta_{2n+3}}{\beta_{2n+1}} = 3k+1$$

in the case of Sequence (6). The change in the value from $2k+1$ to $3k+1$ persuaded us to suggest that if we consider the $(a_1, a_2, \ldots, a_m)$-pulsating Fibonacci sequence, then the limit should be $mk+1$. In the same manner as in Theorems 1 and 2, the limits should be changed to m and $\frac{mk+1}{m}$ with respect to the evenness and oddness of the subscripts in the $(a_1, a_2, \ldots, a_m)$-pulsating Fibonacci sequence. We end our conclusion with some conjectures.

Conjecture 1. *Let $(\alpha_{1,n}), (\alpha_{2,n}), \ldots, (\alpha_{m,n})$ be an $(a_1, a_2, \ldots, a_m)$-pulsating Fibonacci sequence as Sequence (4). Then, the ratio*

$$\lim_{n\to\infty} \frac{\alpha_{1,2n+3}}{\alpha_{1,2n+1}} = \lim_{n\to\infty} \frac{\alpha_{2,2n+3}}{\alpha_{2,2n+1}} = \cdots = \lim_{n\to\infty} \frac{\alpha_{m,2n+3}}{\alpha_{m,2n+1}} = mk+1.$$

Conjecture 2. *Let $(\alpha_{1,n}), (\alpha_{2,n}), \ldots, (\alpha_{m,n})$ be an $(a_1, a_2, \ldots, a_m)$-pulsating Fibonacci sequence as Sequence (4). Then, the pulsating metallic ratio is*

- $\lim_{n\to\infty} \frac{\alpha_{1,2n+1}}{\alpha_{1,2n}} = \lim_{n\to\infty} \frac{\alpha_{2,2n+1}}{\alpha_{2,2n}} = \cdots = \lim_{n\to\infty} \frac{\alpha_{m,2n+1}}{\alpha_{m,2n}} = m,$
- $\lim_{n\to\infty} \frac{\alpha_{1,2n+2}}{\alpha_{1,2n+1}} = \lim_{n\to\infty} \frac{\alpha_{2,2n+2}}{\alpha_{2,2n+1}} = \cdots = \lim_{n\to\infty} \frac{\alpha_{m,2n+2}}{\alpha_{m,2n+1}} = \frac{mk+1}{m}.$

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