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The Outer-Planar Anti-Ramsey Number of Matchings

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Abstract: A subgraph H of an edge-colored graph G is called rainbow if all of its edges have different colors. Let $ar(G, H)$ denote the maximum positive integer t, such that there is a t-edge-colored graph G without any rainbow subgraph H. We denote by kK_2 a matching of size k and $\mathcal{O}_n$ the class of all maximal outer-planar graphs on n vertices, respectively. The outer-planar anti-Ramsey number of graph H, denoted by $ar(\mathcal{O}_n, H)$, is defined as $\max\{ar(O_n, H) | O_n \in \mathcal{O}_n\}$. It seems nontrivial to determine the exact values for $ar(\mathcal{O}_n, H)$ because most maximal outer-planar graphs are asymmetry. In this paper, we obtain that $ar(\mathcal{O}_n, kK_2) \le n + 3k - 8$ for all $n \ge 2k$ and $k \ge 6$, which improves the existing upper bound for $ar(\mathcal{O}_n, kK_2)$, and prove that $ar(\mathcal{O}_n, kK_2) = n + 2k - 5$ for $n = 2k$ and $k \ge 5$. We also obtain that $ar(\mathcal{O}_n, 6K_2) = n + 6$ for all $n \ge 29$.

Keywords: maximal outer-planar graph; rainbow subgraph; matching; outer-planar anti-Ramsey number

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1. Introduction

In this paper, all graphs considered are finite, simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $e(G)$, $v(G)$ and $\delta(G)$ denote the number of edges, number of vertices and minimum degree of G, respectively. The circumference of graph G, denoted by $\ell(G)$, is the length of a longest cycle in G. Denote by $d_G(v)$ and $N_G(v)$ the degree and neighborhood of the vertex v in G respectively. For any subset $A \subseteq V(G)$, let $G[A]$ denote the subgraph of G induced by A, and $N_G(A) = \{v \in V(G) \backslash A | uv \in E(G), u \in A\}$. For a set B, we denote the cardinality of B by $|B|$. For two disjoint subsets A_1, A_2 of $V(G)$, let $e_G(A_1, A_2)$ denote the number of edges in G satisfying one end in A_1 and the other in A_2. A graph G is called a planar graph if it can be drawn in the plane such that its edges intersect only at their ends, and such a drawing is called a planar embedding of G. For convenience, a planar embedding of G is still represented by G. A graph G is outer-planar if it admits a planar embedding such that all vertices lie on the boundary of its outer face. An outer-planar graph G is maximal if $G + uv$ is not outer-planar for any two non-adjacent vertices u and v of G. A graph G is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and the other in Y. We denote a bipartite graph G with bipartition (X, Y) by $G[X, Y]$. If any two edges of M are not adjacent in G, where $M \subseteq E(G)$, then M is called a matching of graph G. The number of edges in a maximum matching of a graph G is called the matching number of G, denoted by $\alpha(G)$. Let M be a matching of graph G, if $v(G) = n$ and $|M| = \frac{n}{2}$, then M is called a perfect matching of G. A graph G is called factor-critical if $G - v$ contains a perfect matching for every vertex $v \in V(G)$. We call a graph G an H-minor if H may be obtained from G by means of a sequence of vertex deletions, edge deletions or edge contractions. A component of a graph G is odd component (even component) if the order of the component is odd (even). The number of odd components in G is denoted by $o(G)$. Let $G \cup H$ denote the vertex disjoint union of graphs G and H. Denote by $G + H$ the graph obtained from $G \cup H$ by adding all edges joining each vertex of G and each vertex of H. For a positive integer k and a graph G, denote by kG the vertex disjoint union of k copies of G. For any positive integer t, let $[t] := \{1, 2, \ldots, t\}$. The terminology and notation used but undefined in this paper can be found in [1].

If a subgraph H of an edge-colored graph G contains no two edges of the same color, then we say that G contains a rainbow H. Let K_n, P_n and C_n be the complete graph, path and cycle on n vertices, respectively. The anti-Ramsey number of H, denoted by $ar(K_n, H)$, is the maximum positive integer t such that there is a t-edge-colored K_n without any rainbow H. In 1975, Erdős et al. [2] introduced anti-Ramsey numbers, and showed that these are closely related to Turán numbers. In the following discussion, the subgraph induced by a matching is still called a matching, and let kK_2 denote a matching of size k. In 2004, Schiermeyer [3] considered the anti-Ramsey number of matchings and determined the exact values of $ar(K_n, kK_2)$ for all $k \geq 2$ and $n \geq 3k + 3$. Chen et al. [4] also studied $ar(K_n, kK_2)$ and completely determined the exact values of the anti-Ramsey number of matchings. When replacing K_n by other graph G, let $ar(G, H)$ denote the maximum positive integer t such that there is a t-edge-colored G without any rainbow H. The researchers studied $ar(G, kK_2)$ when G is a bipartite graph [5–7], complete split graph [8], hypergraph [9] and so on. For more results on anti-Ramsey numbers, we refer the readers to [10–17].

Let $\mathcal{T}_n$ be the family of all plane triangulations on n vertices. The planar anti-Ramsey number of H is denoted by $ar(\mathcal{T}_n, H) = \max\{ar(T_n, H) \mid T_n \in \mathcal{T}_n\}$. In 2014, Jendrol' et al. [18] investigated the planar anti-Ramsey number of kK_2, in which the upper and lower bounds of $ar(\mathcal{T}_n, kK_2)$ for all $k \geq 5$ and $n \geq 2k$ were established, and the exact values of $ar(\mathcal{T}_n, kK_2)$ for $2 \leq k \leq 4$ and $n \geq 2k$ were determined. Qin et al. [19] improved the upper bound of $ar(\mathcal{T}_n, kK_2)$ in [18] and determined the exact value of $ar(\mathcal{T}_n, 5K_2)$ for all $n \geq 11$. Later, Chen et al. [20] improved the upper and lower bounds of $ar(\mathcal{T}_n, kK_2)$ for $k \geq 6$ and $n \geq 3k - 6$ existing in [18,19], and determined the exact value of $ar(\mathcal{T}_n, 6K_2)$ for all $n \geq 30$. Recently, Qin et al. [21] determined the exact values of $ar(\mathcal{T}_n, kK_2)$ for all $k \geq 7$ and $n \geq 9k + 3$.

Let $\mathcal{O}_n$ be the family of all maximal outer-planar graphs on n vertices. For $n \geq 3$, let $\mathcal{O}_n^-$ ($\mathcal{O}_n^=$) denote the family of all outer-planar graphs with n vertices and $2n - 4$ ($2n - 5$) edges. The outer-planar anti-Ramsey number of H is denoted by $ar(\mathcal{O}_n, H) = \max\{ar(O_n, H) \mid O_n \in \mathcal{O}_n\}$. It seems non-trivial to determine the exact values for $ar(\mathcal{O}_n, H)$ because most maximal outer-planar graphs are asymmetry. There are two lemmas about the properties of maximal outer-planar graphs as follows.

Lemma 1 ([22])**.** *Let O_n be a maximal outer-planar graphs on n vertices. If $n \geq 3$, then $e(O_n) = 2n - 3$ and $\delta(O_n) \geq 2$.*

Lemma 2 ([22])**.** *Any maximal outer-planar graph contains neither a $K_{2,3}$-minor nor a K_4-minor.*

In 2018, Jin et al. [23] studied the outer-planar anti-Ramsey numbers of kK_2, which were further studied by Pei et al. [24] in 2022. We summarize their results as follows.

Theorem 1 ([23])**.** *Let n and k be positive integers. Then*

(1) $ar(\mathcal{O}_n, 2K_2) = \begin{cases} 3, & n = 4; \\ 1, & n \geq 5. \end{cases}$

(2) $ar(\mathcal{O}_n, 3K_2) = \begin{cases} 7, & n = 6; \\ n, & n \geq 7. \end{cases}$

(3) $ar(\mathcal{O}_n, 4K_2) = \begin{cases} 11, & n = 8; \\ n + 2, & n \geq 9. \end{cases}$

(4) *for all $k \geq 5$ and $n \geq 2k$, we have $n + 2k - 6 \leq ar(\mathcal{O}_n, kK_2) \leq n + 14k - 25$.*

Theorem 2 ([24])**.** *Let n and k be positive integers. Then*

(1) *for all $k \geq 2$ and $n \geq 3k - 3$, we have $ar(\mathcal{O}_n, kK_2) \leq n + 4k - 9$.*

(2) *for all $n \geq 15$, we have $ar(\mathcal{O}_n, 5K_2) = n + 4$.*

By Theorem 1, when $3 \leq k \leq 4$, if $n = 2k$, then $ar(\mathcal{O}_n, kK_2)$ is the lower bound given by Theorem 1(4) plus 1; if $n \geq 2k+1$, then $ar(\mathcal{O}_n, kK_2)$ is exactly the lower bound given by Theorem 1(4). By Theorem 2, $ar(\mathcal{O}_n, kK_2)$ is exactly the lower bound given by Theorem 1(4) when $k = 5$ and $n \geq 2k+5$.

2. Main Results

It is non-trivial to determine the exact values for $ar(\mathcal{O}_n, kK_2)$ for all $n \geq 2k$. The previous best upper bound for $ar(\mathcal{O}_n, kK_2)$ is $n + 4k - 9$. Here, we improve the existing upper bound of $ar(\mathcal{O}_n, kK_2)$ to $n + 3k - 8$.

Theorem 3. *For all $n \geq 2k$ and $k \geq 6$, we have $ar(\mathcal{O}_n, kK_2) \leq n + 3k - 8$.*

Also, we obtain that the exact value of $ar(\mathcal{O}_n, kK_2)$ when $n = 2k$, which is equal to the lower bound given by Theorem 1(4) plus 1.

Theorem 4. *For all $k \geq 5$ and $n = 2k$, we have $ar(\mathcal{O}_n, kK_2) = n + 2k - 5$.*

Finally, we attain that the exact value of $ar(\mathcal{O}_n, kK_2)$ for $k = 6$ and $n \geq 2k+17$, which is exactly the lower bound given by Theorem 1(4).

Theorem 5. *For all $n \geq 29$, we have $ar(\mathcal{O}_n, 6K_2) = n + 6$.*

The following two lemmas are useful in the proofs of Theorems 3 and 5.

Lemma 3. (**Tutte-Berge Lemma** [25]). *If G is a graph with n vertices, then there exists a subset $S \subset V(G)$ satisfying $|S| \leq \alpha(G)$, such that $\alpha(G) = \frac{1}{2}(n - o(G-S) + |S|)$. Furthermore, each odd component of $G - S$ is factor-critical and each even component of $G - S$ has a perfect matching.*

Lemma 4 ([24]). *Let $G = G[X, Y]$ be a bipartite outer-planar graph on n vertices. If $|Y| \geq |X| \geq 1$, then $e(G) \leq n + |X| - 2$.*

3. Proof of Theorem 3

The outer-planar anti-Ramsey number is closely related to the outer-planar Turán number of graphs. The outer-planar Turán number of H, denoted by $ex_{op}(n, H)$, is the maximum number of edges of an outer-planar graph on n vertices that does not contain H as a subgraph. To get Theorem 3, we first prove the following two lemmas.

Lemma 5. *For all $n \geq v(H)$, $ar(\mathcal{O}_n, H) \leq ex_{op}(n, H)$.*

Proof. Let $ar(\mathcal{O}_n, H) = t$. Then there exists an $O_n \in \mathcal{O}_n$, such that O_n does not contain any rainbow H under a given t-edge-coloring. Let $G \subset O_n$ be a rainbow spanning subgraph with t edges. Thus G is an outer-planar graph on n vertices that does not contain H as a subgraph. It follows that $ex_{op}(n, H) \geq t$. Therefore, $ar(\mathcal{O}_n, H) \leq ex_{op}(n, H)$ for all $n \geq v(H)$. □

Lemma 6. *For all $n \geq 2k$ and $k \geq 6$, $ex_{op}(n, kK_2) \leq \min\{2n-3, n+3k-8\}$.*

Proof. The proof will be conducted by induction on n. Since $n \geq 12$, then $ex_{op}(n, kK_2) \leq 2n - 3$ by Lemma 1. Thus $ex_{op}(n, kK_2) \leq 2n - 3 = \min\{2n-3, n+3k-8\}$ when $2k \leq n \leq 3k - 6$. Now we assume that $n \geq 3k - 5$. Next we will prove that $ex_{op}(n, kK_2) \leq n + 3k - 8$ for $k \geq 6$ and $n \geq 3k - 5$ by contradiction. Suppose $ex_{op}(n, kK_2) \geq n + 3k - 7$. Then there exists an outer-planar graph G such that $v(G) = n$ and $e(G) \geq n + 3k - 7$, and G does not contain kK_2 as a subgraph. Notice that $\alpha(G) \leq k - 1$. By Lemma 3, there exists a subset $S \subset V(G)$ satisfying $|S| \leq \alpha(G) \leq k - 1$, such that $o(G - S) = n + |S| - 2\alpha(G)$. Let $s = |S|$

and $p = o(G - S)$. Then $s \leq k - 1$ and $p \geq n + s + 2 - 2k$. Denote by $B_1, B_2, \ldots, B_p$ all the odd components of $G - S$. We may assume that $v(B_1) \geq v(B_2) \geq \cdots \geq v(B_p)$. Let $w = 0$ when $v(B_1) = 1$, otherwise let $w = \max\{i | v(B_i) > 1\}$. Let $V(B_j) = \{v_j\}$ for any $j > w$. Let $I = \{v_{w+1}, v_{w+2}, \ldots, v_p\}$. Since $n = v(G) \geq |S| + v(B_1) + v(B_2) + \cdots + v(B_p) \geq s + 3w + p - w = 2w + s + p \geq 2w + s + (n + s + 2 - 2k) = n + 2s + 2w - 2k + 2$, then $w \leq k - s - 1$.

We first prove that $s \leq 1$. Suppose $s \geq 2$. Then $|I| = p - w \geq (n + s + 2 - 2k) - (k - s - 1) = n + 2s - 3k + 3 \geq (3k - 5) + 2s - 3k + 3 = 2s - 2 \geq s = |S|$. Therefore, $e_G(S, I) \leq (|S| + |I|) + |S| - 2 = 2s + p - w - 2$ by Lemma 4. Since $s \geq 2$, then $v(G - I) \geq 2$. So $e(G - I) \leq 2(n - (p - w)) - 3 = 2n - 2p + 2w - 3$. Therefore, $e(G) = e_G(S, I) + e(G - I) \leq (2s + p - w - 2) + (2n - 2p + 2w - 3) = 2n + 2s - p + w - 5 \leq 2n + 2s - (n + s + 2 - 2k) + (k - s - 1) - 5 = n + 3k - 8$. But $e(G) \geq n + 3k - 7$, a contradiction. Thus $s \leq 1$.

Let $H_1, H_2, \ldots, H_\ell$ be all components of $G - S$, where $\ell \geq 1$. Then $v(H_i) \geq 1$ for any $i \in [\ell]$. We next prove that $\ell = 1$. Suppose $\ell \geq 2$. If there exists $j \in [\ell]$, such that $v(H_j) = 1$, then $e_G(S, V(H_j)) \leq 1$ since $s \leq 1$. Therefore, $G - V(H_j)$ is an outer-planar graph with $n - 1$ vertices containing no kK_2, and $e(G - V(H_j)) = e(G) - e_G(S, V(H_j)) \geq n + 3k - 7 - 1 = n + 3k - 8 > \min\{2(n-1) - 3, (n-1) + 3k - 8\}$. But $ex_{op}(n - 1, kK_2) \leq \min\{2(n-1) - 3, (n-1) + 3k - 8\}$ by induction hypothesis, a contradiction. Therefore, we have $v(H_i) \geq 2$ for any $i \in [\ell]$. Then $p = w$, and $e(G[S \cup V(H_i)]) \leq 2(s + v(H_i)) - 3$ for any $i \in [\ell]$. Thus, $e(G) = e(G[S \cup V(H_1)]) + \cdots + e(G[S \cup V(H_\ell)]) \leq 2(s + v(H_1)) - 3 + \cdots + 2(s + v(H_\ell)) - 3 = 2(n - s) + 2\ell s - 3\ell$. Since $\ell \geq 2$ and $s \leq 1$, then $e(G) \leq 2(n - s) + 2\ell s - 3\ell = 2n - 3 + (2s - 3)(\ell - 1) \leq 2n - 3 + (2s - 3) = 2n + 2s - 6$. On the other hand, we have $n \leq 3k - 2s - 3$ since $n + s + 2 - 2k \leq p = w \leq k - s - 1$. Thus $e(G) \leq 2n + 2s - 6 \leq n + 2s - 6 + (3k - 2s - 3) = n + 3k - 9$. But $e(G) \geq n + 3k - 7$, a contradiction. Therefore, $\ell = 1$. Then $G - S$ has only one component H_1. Since $k \geq 6$, then $n \geq 3k - 5 > 2k$. Combining $s \leq 1$, we have $v(H_1) \geq 2k$. Thus, H_1 must contain a kK_2 by Lemma 3, which contradicts to the fact that G contains no kK_2. Thus, $ex_{op}(n, kK_2) \leq n + 3k - 8 = \min\{2n - 3, n + 3k - 8\}$ when $n \geq 3k - 5$ and $k \geq 6$. Therefore, $ex_{op}(n, kK_2) \leq \min\{2n - 3, n + 3k - 8\}$ for all $n \geq 2k$ and $k \geq 6$. □

Now we prove Theorem 3.

Proof of Theorem 3. Since $n \geq 2k$, then $ar(\mathcal{O}_n, kK_2) \leq ex_{op}(n, kK_2)$ by Lemma 5. By Lemma 6, $ex_{op}(n, kK_2) \leq n + 3k - 8$ for all $n \geq 2k$ and $k \geq 6$. Therefore, $ar(\mathcal{O}_n, kK_2) \leq n + 3k - 8$ for all $n \geq 2k$ and $k \geq 6$. □

4. Proof of Theorem 4

By Theorem 1(1–3), we observe that the outer-planar anti-Ramsey number of kK_2 when $n = 2k$ is different from the case when $n \geq 2k + 1$. It is not hard to see that the outer-planar anti-Ramsey number of kK_2 when $n = 2k$ is equal to the lower bound given in Theorem 1(4) plus 1 for $2 \leq k \leq 4$. In this section, we will prove that it is also equal to the lower bound given in Theorem 1(4) plus 1 when $k \geq 5$.

Now we are ready to prove Theorem 4.

Proof of Theorem 4. We will first prove $ar(\mathcal{O}_n, kK_2) \geq n + 2k - 5$ for $k \geq 5$ and $n = 2k$. Construct a graph G^* as follows: choose a maximal outer-planar graph G on k vertices, and the vertices of the outer face in a planar embedding of G are $v_1, v_2, \ldots, v_k$ in order; add vertex set $\{u_1, u_2, \ldots, u_k\}$ such that u_i is only adjacent to v_i and v_{i+1} for each $i \in [k]$ (here v_{k+1} is identified as v_1). Then G^* is an outer-planar graph with $2k$ vertices, and $e(G^*) = e(G) + 2k = (2k - 3) + 2k = 4k - 3$ combining Lemma 1. Therefore, by the definition of maximal outer-planar graphs, G^* is a maximal outer-planar graph on n vertices, where $n = 2k$.

Suppose that H is any matching kK_2 of G^*. Then we have $v \in V(H)$ for any $v \in V(G^*)$ since $v(G^*) = 2k$. Note that $N_{G^*}(u_i) = \{v_i, v_{i+1}\}$ for each $i \in [k]$. Then for $u_k \in V(G^*)$, we have $u_k v_k \in E(H)$ or $u_k v_1 \in E(H)$. If $u_k v_k \in E(H)$, then $u_i v_i \in E(H)$, $i \in [k-1]$; If $u_k v_1 \in$

$E(H)$, then $u_iv_{i+1} \in E(H)$, $i \in [k-1]$. Therefore, either $E(H) = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ or $E(H) = \{u_1v_2, u_2v_3, \ldots, u_{k-1}v_k, u_kv_1\}$.

Let φ be an edge-coloring of G^* as follows: $\varphi(u_1v_1) = \varphi(u_2v_2) = 1$, $\varphi(u_1v_2) = \varphi(u_2v_3) = 2$, and color all the remaining edges of G^* with different new colors. Then the number of colors used for φ is $(4k-3)-2 = 4k-5 = n+2k-5$. Since H is not a rainbow kK_2 under the $(n+2k-5)$-edge-coloring φ, then G^* does not contain any rainbow kK_2. Therefore, $ar(\mathcal{O}_n, kK_2) \geq n+2k-5$. The graph G^* and the edges of coloring 1 and 2 under its $(n+2k-5)$-edge-coloring φ when $k=6$ are depicted in Figure 1.

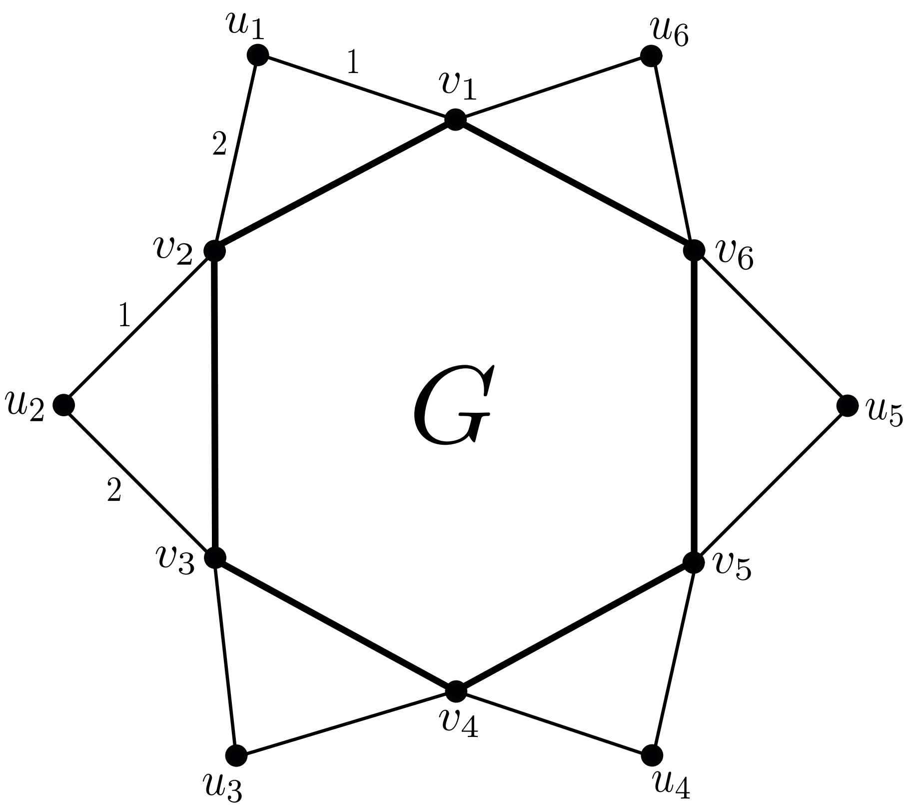

Figure 1. The graph G^* and the edges of coloring 1 and 2 under its $(n+2k-5)$-edge-coloring φ when $k=6$.

Next we will prove $ar(\mathcal{O}_n, kK_2) \leq n+2k-5$ for $k \geq 5$ and $n = 2k$ by contradiction. Suppose $ar(\mathcal{O}_n, kK_2) \geq n+2k-4$. Then there exists an $O_n \in \mathcal{O}_n$, such that O_n does not contain any rainbow kK_2 under a given t-edge-coloring, where $t \geq n+2k-4 = 2n-4$. Let G' be a rainbow spanning subgraph of O_n and $e(G') = t$. By Lemma 1, $e(O_n) = 2n-3$. Then $2n-4 \leq t \leq 2n-3$. Thus either $G' \in \mathcal{O}_n^-$ or $G' \in \mathcal{O}_n$. Note that O_n contains a cycle C_n, which means that G' contains a P_n. It follows that O_n must contain a rainbow kK_2, a contradiction.

This completes the proof of Theorem 4. □

5. Proof of Theorem 5

It is easy to see, from the previous results, that the exact value of the outer-planar anti-Ramsey number of kK_2 is equal to the lower bound given in Theorem 1(4) when n is large and $3 \leq k \leq 5$. It is natural to ask for whether it is also equal to the lower bound given in Theorem 1(4) when $k \geq 6$. We verify it is true when $k = 6$.

Now we shall prove Theorem 5.

By Theorem 1(4), $ar(\mathcal{O}_n, 6K_2) \geq n+6$. Next it suffices to prove that $ar(\mathcal{O}_n, 6K_2) \leq n+6$. By contradiction, suppose that $ar(\mathcal{O}_n, 6K_2) \geq n+7$. Then there exists an $O_n \in \mathcal{O}_n$, such that O_n contains no rainbow $6K_2$ under an edge-coloring c with k colors, where $k \geq n+7$. It follows from Theorem 2(2) that O_n contains a rainbow $5K_2$. Now let $G \subset O_n$ be a rainbow spanning subgraph with k edges which contains a $5K_2$. Then $\alpha(G) = 5$. Thus by Lemma 3, there exists a subset $S \subset V(G)$ satisfying $|S| \leq 5$ such that $o(G-S) = n + |S| - 10$. Let $s = |S|$ and $p = o(G-S)$, we have $0 \leq s \leq 5$ and $p = n + s - 10$. Denote by $B_1, B_2, \ldots, B_p$ all the odd components of $G - S$. We may assume that $v(B_1) \geq v(B_2) \geq \cdots \geq v(B_p)$. Let $w = 0$ when $v(B_1) = 1$, otherwise let $w = \max\{i| \, v(B_i) > 1\}$. Let $V(B_j) = \{v_j\}$ for any $j > w$. Without loss of generality, we assume that $d_G(v_{w+1}) \geq \cdots \geq d_G(v_p)$. Let $I = \{v_{w+1}, v_{w+2}, \ldots, v_p\}$. Let $Q = V(B_1) \cup V(B_2) \cup \cdots \cup V(B_w)$, where $w \geq 1$; otherwise let $Q = \emptyset$. Let $R = V(G) - (S \cup I \cup Q)$. Let $n_I^q = |\{v \in I : d_G(v) = q\}|$ and $n_I^{q^+} = |\{v \in I : d_G(v) \geq q\}|$. For convenience, we replace $e_G(A_1, A_2)$ by $e(A_1, A_2)$ in the following proof. We first present several useful claims, which shall be proved in Section 6.

Claim 1. If M_1 and M_2 are two edge-disjoint $5K_2$ of G, then

$$E(O_n[V(G) - V(M_1 \cup M_2)]) = \emptyset.$$

Claim 2. $s \geq 1$. Especially, $s \geq 2$ when $R \neq \emptyset$.

Claim 3. $R = \emptyset$.

Claim 4. If $G[S]$ is a connected graph, $e(O_n[I']) = 0$ and $|I'| > 2b$, then O_n contains a $K_{2,3}$-minor.

Claim 5. $s \geq 2$.

Claim 6. $v(B_1) \leq 5$.

Claim 7. $v(B_1) \geq 3$.

By Claim 3, we get $n = v(G) = v(B_1) + \cdots + v(B_p) + s = v(B_1) + \cdots + v(B_w) + n + 2s - w - 10$ when $w \geq 1$. So the following result holds for any $w \geq 1$:

$$v(B_1) + \cdots + v(B_w) = w + 10 - 2s. \tag{1}$$

Let $b = w + n_I^{2^+}$, and $I' = \{v_{b+1}, v_{b+2}, \ldots, v_p\}$. By Claims 6 and 7, $3 \leq v(B_1) \leq 5$. If $v(B_i) = 3$, then $B_i \cong K_3$ by Lemma 3. Since $w \geq 1$, then by (1), $v(B_1) + \cdots + v(B_w) = w + 10 - 2s$. Combining $3w \leq v(B_1) + \cdots + v(B_w) \leq 5w$, we have $(5-s)/2 \leq w \leq 5 - s$. It follows from Claims 5 and 7 that $2 \leq s \leq 4$. We next distinguish the following three cases to finish the proof of Theorem 5.

5.1. $s = 2$

In this case, $p = n - 8$ and $n_I^{3^+} = 0$. Since $(5-s)/2 \leq w \leq 5 - s$, we see $2 \leq w \leq 3$. We consider the following two situations according to w.

Case A.1. $w = 2$.

Then $v(B_1) = 5$, $v(B_2) = 3$ and $|I| = p - 2 = n - 10$. Since $n_I^2 \leq 2$, then $b \leq 4$. By Lemma 1, we have $e(G - I) \leq 2 \times 10 - 3 = 17$. By Lemma 4, $e(S, I) \leq (|I| + |S|) + |S| - 2 = n - 8$, which implies that $e(G - I) = e(G) - e(S, I) \geq 15$. Therefore, $15 \leq e(G - I) \leq 17$.

If $e(G - I) = 17$, then $G - I \in \mathcal{O}_{10}$. Thus $N_G(V(B_1)) = N_G(V(B_2)) = S$. Therefore, $G[S]$ is connected and $G - I$ contains two edge-disjoint $5K_2$.

If $e(G - I) = 16$, then $e(S, I) = e(G) - e(G - I) \geq n - 9$. Thus $d_G(v_3) = 2$. Then $G[S \cup Q \cup \{v_3\}] \in \mathcal{O}_{11}^-$, which implies that $G[S \cup Q \cup \{v_3\}]$ contains P_{11}. Therefore, $G - I'$ contains two edge-disjoint $5K_2$. On the other hand, combining $d_G(v_3) = 2$, we get $|N_G(V(B_i))| \leq 1$ for some $i \in [2]$, then $G[S]$ is connected.

If $e(G - I) = 15$, then $e(S, I) \geq n - 8$. Thus $d_G(v_3) = d_G(v_4) = 2$. Then $G[S \cup Q \cup \{v_3, v_4\}] \in \mathcal{O}_{12}^=$, which implies that $G - I'$ contains two edge-disjoint $5K_2$. On the other hand, combining $d_G(v_3) = d_G(v_4) = 2$, we get $|N_G(V(B_i))| \leq 1$ for each $i \in [2]$, then $G[S]$ is connected.

From the above discussion of $e(G - I)$ and combining Claim 1, we have $e(O_n[I']) = 0$. Since $n \geq 29$ and $b \leq 4$, then $|I'| = p - b \geq (n - 8) - 4 > 2b$. Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2.

Case A.2. $w = 3$.

Then $v(B_1) = v(B_2) = v(B_3) = 3$ and $|I| = p - 3 = n - 11$. Since $n_I^2 \leq 2$, then $b \leq 5$. By Lemma 2, there exists at least one $i \in [3]$ such that $|N_G(V(B_i))| \leq 1$. Thus combining Lemma 1, we have $e(G - I) \leq 2(|Q| + s) - 4 = 2 \times 11 - 4 = 18$. By Lemma 4, we have $e(S, I) \leq (|I| + |S|) + |S| - 2 = n - 9$, which implies that $e(G - I) \geq 16$. Therefore, $16 \leq e(G - I) \leq 18$.

If $e(G - I) = 18$, then $G - I \in \mathcal{O}_{11}^-$, which implies that $G - I$ contains P_{11}. Therefore, $G - I$ contains two edge-disjoint $5K_2$. On the other hand, since there exists at least one $i \in [3]$ such that $|N_G(V(B_i))| \leq 1$, we have $G[S]$ is connected.

If $e(G - I) = 17$, then $e(S, I) = e(G) - e(G - I) \geq n - 10$. Thus $d_G(v_4) = 2$. Therefore, $G[S \cup Q \cup \{v_4\}] \in \mathcal{O}_{12}^=$, which implies that $G - I'$ contains two edge-disjoint $5K_2$. On the other hand, combining $d_G(v_4) = 2$, we get that there exist at least two $i \in [3]$ such that $|N_G(V(B_i))| \leq 1$, thus $G[S]$ is connected.

If $e(G - I) = 16$, then $e(S, I) \geq n - 9$. Thus $d_G(v_4) = d_G(v_5) = 2$. Therefore, $G[S \cup Q \cup \{v_4, v_5\}]$ is the graph obtained from a maximal outer-planar graph with 13 vertices by deleting 3 edges, then $G - I'$ contains two edge-disjoint $5K_2$. On the other hand, combining $d_G(v_4) = d_G(v_5) = 2$, we get $|N_G(V(B_i))| \leq 1$ for each $i \in [3]$, thus $G[S]$ is connected.

From the above discussion of $e(G - I)$ and combining Claim 1, we have $e(O_n[I']) = 0$. Since $n \geq 29$ and $b \leq 5$, then $|I'| = p - b \geq (n - 8) - 5 > 2b$. Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2.

5.2. $s = 3$

In this case, $p = n - 7$ and $n_I^{4^+} = 0$. Since $(5 - s)/2 \leq w \leq 5 - s$, then $1 \leq w \leq 2$. We consider the following two situations according to w.

Case B.1. $w = 1$.

Then $v(B_1) = 5$ and $|I| = p - 1 = n - 8$. By Lemma 1, we have $e(G - I) \leq 2 \times 8 - 3 = 13$. By Lemma 4, $e(S, I) \leq (|I| + |S|) + |S| - 2 = n - 4$, which implies that $e(G - I) \geq 11$. Therefore, $11 \leq e(G - I) \leq 13$.

If $e(G - I) = 13$, then $G - I \in \mathcal{O}_8$. Thus $d_G(v_2) \leq 2$ and $|N_G(V(B_1))| \geq 2$. Since $e(S, I) = e(G) - e(G - I) \geq n - 6$, then $d_G(v_2) = d_G(v_3) = 2$. Thus $G[S \cup V(B_1) \cup \{v_2, v_3\}] \in \mathcal{O}_{10}$, which means that $G - I'$ contains two edge-disjoint $5K_2$. On the other hand, we have $e(S, V(B_1)) \leq 4$ because $d_G(v_2) = d_G(v_3) = 2$. So $e(G[S]) = e(G - I) - e(S, V(B_1)) - e(B_1) \geq 13 - 4 - 7 = 2$, which implies that $G[S]$ is connected.

If $e(G - I) = 12$, then $e(S, I) \geq n - 5$. Thus either $d_G(v_2) = 3$ and $d_G(v_3) = 2$ or $d_G(v_2) = d_G(v_3) = d_G(v_4) = 2$. Therefore, we have $G[S \cup V(B_1) \cup \{v_2, v_3\}] \in \mathcal{O}_{10}$; or $G[S \cup V(B_1) \cup \{v_2, v_3, v_4\}] \in \mathcal{O}_{11}^-$. Then we always get that $G - I'$ contains two edge-disjoint $5K_2$. From the degree situation of the vertices of I in G, we have $e(S, V(B_1)) \leq 3$. So $e(G[S]) = e(G - I) - e(S, V(B_1)) - e(B_1) \geq 12 - 3 - 7 = 2$, which implies that $G[S]$ is connected.

If $e(G - I) = 11$, then $e(S, I) \geq n - 4$. Thus either $d_G(v_2) = 3$ and $d_G(v_3) = d_G(v_4) = 2$ or $d_G(v_2) = d_G(v_3) = \cdots = d_G(v_5) = 2$. Therefore, we have $G[S \cup V(B_1) \cup \{v_2, v_3, v_4\}] \in \mathcal{O}_{11}^-$; or $G[S \cup V(B_1) \cup \{v_2, v_3, \ldots, v_5\}] \in \mathcal{O}_{12}^=$. Then we always get that $G - I'$ contains two edge-disjoint $5K_2$. From the degree situation of the vertices of I in G, we have $e(S, V(B_1)) \leq 2$. So $e(G[S]) = e(G - I) - e(S, V(B_1)) - e(B_1) \geq 11 - 2 - 7 = 2$, which implies that $G[S]$ is connected.

From the above discussion of $e(G - I)$ and combining Claim 1, we have $e(O_n[I']) = 0$. Since $s = 3$, then $n_I^2 \leq 4$. So $b \leq 5$. Then $|I'| = p - b \geq (n - 7) - 5 > 2b$ because $n \geq 29$ and $b \leq 5$. Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2.

Case B.2. $w = 2$.

Then $v(B_1) = v(B_2) = 3$ and $|I| = p - 2 = n - 9$. By Lemma 1, we have $e(G - I) \leq 2 \times 9 - 3 = 15$. By Lemma 4, $e(S, I) \leq (|I| + |S|) + |S| - 2 = n - 5$, which implies that $e(G - I) = e(G) - e(S, I) \geq 12$. Therefore, $12 \leq e(G - I) \leq 15$.

If $e(G - I) = 15$, then $G - I \in \mathcal{O}_9$. Thus $d_G(v_3) \leq 2$, $|N_G(V(B_1))| \geq 2$ and $|N_G(V(B_2))| \geq 2$. Since $e(S, I) = e(G) - e(G - I) \geq n - 8$, then $d_G(v_3) = 2$. Thus

$G[S \cup Q \cup \{v_3\}] \in \mathcal{O}_{10}$, which implies that $G - I'$ contains two edge-disjoint $5K_2$. On the other hand, we have $e(S,Q) \leq 7$ because $d_G(v_3) = 2$. Therefore, $e(G[S]) = e(G-I) - e(S,Q) - e(B_1) - e(B_2) \geq 15 - 7 - 3 - 3 = 2$, which means that $G[S]$ is connected.

If $e(G-I) = 14$, then $e(S,I) \geq n-7$. Thus either $d_G(v_3) = 3$ or $d_G(v_3) = d_G(v_4) = 2$. Therefore, we have $G[S \cup Q \cup \{v_3\}] \in \mathcal{O}_{10}$; or $G[S \cup Q \cup \{v_3, v_4\}] \in \mathcal{O}_{11}^-$. Then we always get that $G - I'$ contains two edge-disjoint $5K_2$. From the degree situation of the vertices of I in G, we have $e(S,Q) \leq 6$. So $e(G[S]) = e(G-I) - e(S,Q) - e(B_1) - e(B_2) \geq 14 - 6 - 3 - 3 = 2$, which means that $G[S]$ is connected.

If $e(G-I) = 13$, then $e(S,I) \geq n-6$. Thus either $d_G(v_3) = 3$ and $d_G(v_4) = 2$ or $d_G(v_3) = d_G(v_4) = d_G(v_5) = 2$. Therefore, we have $G[S \cup Q \cup \{v_3, v_4\}] \in \mathcal{O}_{11}^-$; or $G[S \cup Q \cup \{v_3, v_4, v_5\}] \in \mathcal{O}_{12}^=$. Then we always get that $G - I'$ contains two edge-disjoint $5K_2$. From the degree situation of the vertices of I in G, we have $e(S,Q) \leq 5$. So $e(G[S]) = e(G-I) - e(S,Q) - e(B_1) - e(B_2) \geq 13 - 5 - 3 - 3 = 2$, which means that $G[S]$ is connected.

If $e(G-I) = 12$, then $e(S,I) \geq n-5$. Thus either $d_G(v_3) = 3$ and $d_G(v_4) = d_G(v_5) = 2$ or $d_G(v_3) = d_G(v_4) = \cdots = d_G(v_6) = 2$. Therefore, we have $G[S \cup Q \cup \{v_3, v_4, v_5\}] \in \mathcal{O}_{12}^=$; or $G[S \cup Q \cup \{v_3, v_4, \ldots, v_6\}]$ is the graph obtained from a maximal outer-planar graph with 13 vertices by deleting 3 edges. Then we always get that $G - I'$ contains two edge-disjoint $5K_2$. From the degree situation of the vertices of I in G, we have $e(S,Q) \leq 4$. So $e(G[S]) = e(G-I) - e(S,Q) - e(B_1) - e(B_2) \geq 12 - 4 - 3 - 3 = 2$, which means that $G[S]$ is connected.

From the above discussion of $e(G-I)$ and combining Claim 1, we have $e(O_n[I']) = 0$. Since $n_I^2 \leq 4$, we have $b \leq 6$. Then $|I'| = p - b \geq (n-7) - 6 > 2b$ because $n \geq 29$ and $b \leq 6$. Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2.

5.3. $s = 4$

In this case, $p = n-6$, $w = 1$ and $n_I^{5^+} = 0$. Then $v(B_1) = 3$ and $|I| = p - 1 = n - 7$. Since $s = 4$, we see $n_I^2 \leq 6$. So $b \leq 7$. Then we get $|I'| = p - b \geq (n-6) - 7 > 2b$ because $n \geq 29$ and $b \leq 7$. By Lemma 1, $e(G-I) \leq 2 \times 7 - 3 = 11$. By Lemma 4, $e(S,I) \leq (|I| + |S|) + |S| - 2 = n - 1$, which implies that $e(G-I) = e(G) - e(S,I) \geq 8$. Therefore, $8 \leq e(G-I) \leq 11$. We consider the following four situations according to $e(G-I)$.

Case C.1. $e(G-I) = 11$.

Then $G - I \in \mathcal{O}_7$. Thus $d_G(v_2) \leq 2$. Since $e(S,I) = e(G) - e(G-I) \geq n-4$, then $d_G(v_2) = d_G(v_3) = d_G(v_4) = 2$. Thus $G[S \cup V(B_1) \cup \{v_2, v_3, v_4\}] \in \mathcal{O}_{10}$, which implies that $G - I'$ contains two edge-disjoint $5K_2$. Therefore, by Claim 1, $e(O_n[I']) = 0$. On the other hand, we have $e(S, V(B_1)) \leq 5$ because $d_G(v_2) = d_G(v_3) = d_G(v_4) = 2$. Thus $e(G[S]) = e(G-I) - e(S, V(B_1)) - e(B_1) \geq 11 - 5 - 3 = 3$. We next prove that $G[S]$ is connected. If $e(G[S]) = 3$ and $G[S]$ contains a cycle, then $n_I^{3^+} = 0$ and $n_I^2 \leq 2$ because $S \cup V(B_1) \in \mathcal{O}_7$. So $e(S,I) \leq n-5$, which contradicts to $e(S,I) \geq n-4$. Therefore, either $e(G[S]) \geq 4$ or $e(G[S]) = 3$ and $G[S]$ contains no cycle. Then we clearly get that $G[S]$ is connected. Thus, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2.

Case C.2. $e(G-I) = 10$.

Then $d_G(v_2) \leq 3$ and $e(S,I) = e(G) - e(G-I) \geq n-3$. Thus either $d_G(v_2) = 3$ and $d_G(v_3) = d_G(v_4) = 2$ or $d_G(v_2) = d_G(v_3) = \cdots = d_G(v_5) = 2$. Therefore, we have $G[S \cup V(B_1) \cup \{v_2, v_3, v_4\}] \in \mathcal{O}_{10}$; or $G[S \cup V(B_1) \cup \{v_2, v_3, \ldots, v_5\}] \in \mathcal{O}_{11}^-$. Then we always get that $G - I'$ contains two edge-disjoint $5K_2$. Thus, by Claim 1, $e(O_n[I']) = 0$. We next prove that $G[S]$ is connected. If $\ell(G[S]) = 4$, then it is obvious that $G[S]$ is connected. If $\ell(G[S]) = 3$, then we have $|N_G(V(B_1))| \leq 2$ combining Lemma 2, thus $e(S, V(B_1)) \leq 3$. Then $e(G[S]) = e(G-I) - e(S, V(B_1)) - e(B_1) \geq 10 - 3 - 3 = 4$, which implies that $G[S]$ is connected. If $\ell(G[S]) \leq 2$, that is, $G[S]$ contains no cycle, then we get $|N_G(V(B_1))| \leq 3$ by Lemma 2. Thus $e(S, V(B_1)) \leq 4$. So $e(G[S]) = e(G-I) - e(S, V(B_1)) - e(B_1) \geq 10 - 4 - 3 = 3$, which means that $G[S]$ is connected.

Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2.

Case C.3. $e(G-I) = 9$.

Then $e(S, I) \geq n-2$. Combining Lemma 2, the degree situation of the vertices of I in G satisfies one of the following: (1) $d_G(v_2) = 4$, $d_G(v_3) = d_G(v_4) = 2$; (2) $d_G(v_2) = d_G(v_3) = 3$, $d_G(v_4) = 2$; (3) $d_G(v_2) = 3$, $d_G(v_3) = d_G(v_4) = d_G(v_5) = 2$; (4) $d_G(v_2) = d_G(v_3) = \cdots = d_G(v_6) = 2$. Thus, we have $G[S \cup V(B_1) \cup \{v_2, v_3, v_4\}] \in \mathcal{O}_{10}$; or $G[S \cup V(B_1) \cup \{v_2, v_3, \ldots, v_5\}] \in \mathcal{O}_{11}^-$; or $G[S \cup V(B_1) \cup \{v_2, v_3, \ldots, v_6\}] \in \mathcal{O}_{12}^=$. Obviously, we always get that $G - I'$ contains two edge-disjoint $5K_2$. Thus, by Claim 1, we have $e(O_n[I']) = 0$.

We first claim that $\ell(G[S]) \leq 3$. Since otherwise combining Lemma 2, we have $n_I^{3^+} = 0$ and $n_I^2 \leq 4$, which means $e(S, I) \leq n-3$. Next we will prove $G[S]$ is connected. If $\ell(G[S]) = 3$, then one of (3)–(4) above is satisfied. By Lemma 2, we have $|N_G(V(B_1))| \leq 1$. Thus $e(S, V(B_1)) \leq 2$. So $e(G[S]) = e(G-I) - e(S, V(B_1)) - e(B_1) \geq 9-2-3 = 4$, which implies that $G[S]$ is connected. If $\ell(G[S]) \leq 2$, that is, $G[S]$ contains no cycle, then one of (1)–(4) above is satisfied. Thus by Lemma 2, we have $|N_G(V(B_1))| \leq 2$. Then $e(S, V(B_1)) \leq 3$. So $e(G[S]) = e(G-I) - e(S, V(B_1)) - e(B_1) \geq 9-3-3 = 3$, which means that $G[S]$ is connected.

Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2.

Case C.4. $e(G-I) = 8$.

Then $e(S, I) \geq n-1$. Combining Lemma 2, the degree situation of the vertices of I in G satisfies one of the following: (1) $d_G(v_2) = 4, d_G(v_3) = d_G(v_4) = d_G(v_5) = 2$; (2) $d_G(v_2) = d_G(v_3) = 3$, $d_G(v_4) = d_G(v_5) = 2$; (3) $d_G(v_2) = 3$, $d_G(v_3) = d_G(v_4) = \cdots = d_G(v_6) = 2$; (4) $d_G(v_2) = d_G(v_3) = \cdots = d_G(v_7) = 2$. Therefore, we have $G[S \cup V(B_1) \cup \{v_2, v_3, \ldots, v_5\}] \in \mathcal{O}_{11}^-$; or $G[S \cup V(B_1) \cup \{v_2, v_3, \ldots, v_6\}] \in \mathcal{O}_{12}^=$; or $G[S \cup V(B_1) \cup \{v_2, v_3, \ldots, v_7\}]$ is the graph obtained from a maximal outer-planar graph with 13 vertices by deleting 3 edges. Then we always get that $G - I'$ contains two edge-disjoint $5K_2$. Thus, by Claim 1, $e(O_n[I']) = 0$.

We claim that $G[S]$ is connected. If $G[S]$ contains a cycle, then $e(S, I) \leq n-7+5 = n-2$, a contradiction. Thus $G[S]$ does not contain any cycle. Then one of (1)–(4) above is satisfied. Thus $e(S, V(B_1)) \leq 2$. So $e(G[S]) = e(G-I) - e(S, V(B_1)) - e(B_1) \geq 8-2-3 = 3$, which means that $G[S]$ is connected.

Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2.

This completes the proof of Theorem 5.

6. Proof of Claims 1–7

In this section, we shall prove the seven claims used in the proof of Theorem 5.

6.1. Proof of Claim 1

Suppose that there exists some $e \in E(O_n[V(G) - V(M_1 \cup M_2)])$. If there exists an $e' \in E(M_1)$ such that $c(e) = c(e')$, then $M_2 \cup \{e\}$ is a rainbow $6K_2$ in O_n, a contradiction. Thus, $c(e) \neq c(e')$ for any $e' \in E(M_1)$. But then $M_1 \cup \{e\}$ is a rainbow $6K_2$ in O_n, a contradiction. This completes the proof of Claim 1.

6.2. Proof of Claim 2

We first prove $s \geq 1$. Suppose that $s = 0$. Then $p = n-10$. Hence, combining Lemma 1, we have $e(G) = \sum_{i=1}^{w} e(B_i) + e(G[R]) \leq \sum_{i=1}^{w}(2v(B_i) - 3) + (2|R| - 3) = 2(\sum_{i=1}^{w} v(B_i) + |R|) - 3(w+1) = 2(n - (p-w)) - 3(w+1) \leq 17 - w$. Therefore, we have $e(G) < n+7$ because $w \geq 0$ and $n \geq 29$, a contradiction.

We next prove $s \geq 2$ when $R \neq \emptyset$. Suppose that $s \leq 1$ when $R \neq \emptyset$. Since $s \geq 1$, then $s = 1$. So $p = n-9$. Hence, combining Lemma 1, we have $e(G) = \sum_{i=1}^{w} e(G[S \cup V(B_i)]) + e(G[S \cup R]) + e(S, I) \leq \sum_{i=1}^{w}(2(1 + v(B_i)) - 3) + (2(1 + |R|) - 3) + (p-w) = 2(\sum_{i=1}^{w} v(B_i) + |R|) + p - 2w - 1 = 2(n - (p-w) - 1) + p - 2w - 1 = n+6 < n+7$, a contradiction. This completes the proof of Claim 2.

6.3. Proof of Claim 3

Suppose that $R \neq \emptyset$. Since $|R| \leq n - s - p$ and $p = n + s - 10$, we have $2 \leq |R| \leq 10 - 2s$. Then $s \leq 4$. Thus combining Claim 2, we have $2 \leq s \leq 4$. We distinguish the following three cases to finish the proof of this claim.

Case 3.1. $s = 4$.

In this case, $p = n - 6$ and $|R| = 2$. Then $|I| = p$. By Lemma 1, $e(G - I) \leq 2(v(G) - |I|) - 3 = 9$. Since $|I \cup R| = n - 4 > |S|$, then $e(S, I \cup R) \leq n + 4 - 2 = n + 2$ by Lemma 4. Note that $e(G[R]) = 1$. Thus, $e(G[S]) = e(G) - e(S, I \cup R) - e(G[R]) \geq n + 7 - (n + 2) - 1 = 4$. Therefore, $G[S]$ must contain a cycle, that is, $3 \leq \ell(G[S]) \leq 4$. We consider the following two subcases.

Subcase 3.1.1. $\ell(G[S]) = 4$.

By Lemma 2, $n_I^{3^+} = 0$ and $n_I^2 \leq 4$. Thus, $e(S, I) \leq n - 6 + 4 = n - 2$. If $e(S, I) < n - 2$, then $e(G) = e(S, I) + e(G - I) < n - 2 + 9 = n + 7$, a contradiction. If $e(S, I) = n - 2$, then $n_I^2 = 4$, which implies that $e(S, R) \leq 2$. Therefore, $e(G - I) = e(G[S]) + e(S, R) + e(G[R]) \leq 5 + 2 + 1 = 8$. But $e(G - I) = e(G) - e(S, I) \geq 9$, a contradiction.

Subcase 3.1.2. $\ell(G[S]) = 3$.

Then $e(G[S]) = 4$. By Lemma 2, $n_I^{4^+} = 0$, $n_I^3 = 1$ and $n_I^2 \leq 3$; or $n_I^{3^+} = 0$ and $n_I^2 \leq 5$. Therefore, $e(S, I) \leq n - 6 + 5 = n - 1$. If $e(S, I) = n - 1$, then $n_I^{4^+} = 0$, $n_I^3 = 1$ and $n_I^2 = 3$; or $n_I^{3^+} = 0$ and $n_I^2 = 5$. So $e(S, R) \leq 2$, which implies that $e(G - I) = e(G[S]) + e(S, R) + e(G[R]) \leq 4 + 2 + 1 = 7$. But $e(G - I) = e(G) - e(S, I) \geq 8$, a contradiction. If $e(S, I) \leq n - 2$, because $e(S, I) = e(G) - e(G - I) \geq n + 7 - 9 = n - 2$, then $e(S, I) = n - 2$. Thus combining Lemma 2, we have $n_I^{4^+} = 0$, $n_I^3 = 1$ and $n_I^2 \geq 2$; or $n_I^{3^+} = 0$ and $n_I^2 \geq 4$. Then $e(S, R) \leq 3$. Therefore, $e(G - I) = e(G[S]) + e(S, R) + e(G[R]) \leq 4 + 3 + 1 = 8$. But $e(G - I) = e(G) - e(S, I) \geq 9$, a contradiction.

Case 3.2. $s = 3$.

In this case, $p = n - 7$ and $n_I^{4^+} = 0$. Then $w \leq 1$ because $|R| \geq 2$. We consider the following two subcases based on w.

Subcase 3.2.1. $w = 0$.

Then $|I| = p$ and $|R| = 4$. By Lemma 1, we have $e(G - I) \leq 11$.

If $e(G - I) = 11$, then $G - I \in \mathcal{O}_7$. So $n_I^3 = 0$ and $|N_G(R)| \geq 2$. When $|N_G(R)| = 2$, we have $G[S] \cong K_3$. Combining Lemma 2, we have $n_I^2 \leq 2$. Thus $e(S, I) \leq n - 5$. But $e(S, I) = e(G) - e(G - I) \geq n - 4$, a contradiction.

If $e(G - I) = 10$, then $G - I \in \mathcal{O}_7^-$ and $|N_G(R)| \geq 1$. When $|N_G(R)| = 1$, we have $G[S] \cong K_3$. Combining Lemma 2, we have $n_I^3 = 0$ and $n_I^2 \leq 3$. Thus $e(S, I) \leq n - 4$. But $e(S, I) = e(G) - e(G - I) \geq n - 3$, a contradiction.

If $e(G - I) \leq 9$, then $e(S, I) = e(G) - e(G - I) \geq n - 2$. But by Lemma 4, we have $e(S, I) \leq (|I| + |S|) + |S| - 2 = n - 3$, a contradiction.

Subcase 3.2.2. $w = 1$.

Then $v(B_1) = 3$, $|R| = 2$ and $|I| = p - 1 = n - 8$. Clearly, we have $B_1 \cong K_3$ by Lemma 3. By Lemma 1, we have $e(G - I) \leq 13$.

If $e(G - I) = 13$, then $G - I \in \mathcal{O}_8$. Then $n_I^3 = 0$, $|N_G(V(B_1))| \geq 2$ and $|N_G(R)| \geq 2$. When both of the above equalities hold, we have $G[S] \cong K_3$. Combining Lemma 2, we have $n_I^2 \leq 1$. Therefore, $e(S, I) \leq n - 7$. But $e(S, I) = e(G) - e(G - I) \geq n - 6$, a contradiction.

If $e(G - I) = 12$, then $G - I \in \mathcal{O}_8^-$. Combining Lemma 2, if $e(G[S]) = 3$, then $|N_G(V(B_1))| + |N_G(R)| \geq 3$, $n_I^3 = 0$ and $n_I^2 \leq 2$; if $e(G[S]) \leq 2$, then $|N_G(V(B_1))| + |N_G(R)| \geq 4$, and either $n_I^3 = 1$ and $n_I^2 = 0$, or $n_I^3 = 0$ and $n_I^2 \leq 2$. In both cases of $e(G[S])$, we have $e(S, I) \leq n - 6$. But $e(S, I) = e(G) - e(G - I) \geq n - 5$, a contradiction.

If $e(G - I) = 11$, then $G - I \in \mathcal{O}_8^=$. If $e(G[S]) = 3$, then $|N_G(V(B_1))| + |N_G(R)| \geq 2$. If $e(G[S]) \leq 2$, then $|N_G(V(B_1))| + |N_G(R)| \geq 3$. In both cases of $e(G[S])$ and combining Lemma 2, we have $n_I^3 = 1$ and $n_I^2 \leq 1$; or $n_I^3 = 0$ and $n_I^2 \leq 3$. Thus $e(S, I) \leq n - 5$. But $e(S, I) = e(G) - e(G - I) \geq n - 4$, a contradiction.

If $e(G - I) \leq 10$, then $e(S, I) = e(G) - e(G - I) \geq n - 3$. But by Lemma 4, we have $e(S, I) \leq (|I| + |S|) + |S| - 2 = n - 4$, a contradiction.

Case 3.3. $s = 2$.

In this case, $p = n - 8$ and $n_I^{3^+} = 0$. Since $|R| \geq 2$, we see $w \leq 2$. We consider the following two subcases based on w.

Subcase 3.3.1. $w = 0$.

Then $|I| = p$ and $|R| = 6$. By Lemma 1, we have $e(G - I) \leq 13$. If $e(G - I) = 13$, then $G - I \in \mathcal{O}_8$. Combining Lemma 2, we have $n_I^2 \leq 1$. Therefore, $e(S, I) \leq n - 7$. But $e(S, I) = e(G) - e(G - I) \geq n - 6$, a contradiction. If $e(G - I) \leq 12$, then $e(S, I) = e(G) - e(G - I) \geq n - 5$. But $e(S, I) \leq (|I| + |S|) + |S| - 2 = n - 6$ by Lemma 4, a contradiction.

Subcase 3.3.2. $w = 1$.

Then $|I| = p - 1 = n - 9$. By Lemma 1, $e(G - I) \leq 15$.

If $14 \leq e(G - I) \leq 15$, then $G - I$ is the graph obtained from a maximal outer-planar graph with 9 vertices by deleting $15 - e(G - I)$ edges. Hence, when $e(G - I) = 14$, we have at least one of $N_G(V(B_1))$ and $N_G(R)$ is S; when $e(G - I) = 15$, we have $N_G(V(B_1)) = N_G(R) = S$. Then $n_I^2 \leq 15 - e(G - I)$. Therefore, $e(S, I) \leq n - 9 + 15 - e(G - I)$. Then $e(G) = e(G - I) + e(S, I) \leq n + 6$, a contradiction.

If $e(G - I) \leq 13$, then $e(S, I) = e(G) - e(G - I) \geq n - 6$. But $e(S, I) \leq (|I| + |S|) + |S| - 2 = n - 7$ by Lemma 4, a contradiction.

Subcase 3.3.3. $w = 2$.

Then $v(B_1) = v(B_2) = 3$, $|R| = 2$ and $|I| = p - 2 = n - 10$. Obviously, we have $B_1 \cong B_2 \cong K_3$ by Lemma 3. By Lemma 1, $e(G - I) \leq 17$.

If $e(G - I) = 17$, then $G - I \in \mathcal{O}_{10}$. Thus, $N_G(V(B_1)) = N_G(V(B_2)) = N_G(R) = S$. But then G contains a $K_{2,3}$-minor, one part of which is S, and the other part is $\{V(B_1), V(B_2), R\}$. This contradicts to Lemma 2.

If $15 \leq e(G - I) \leq 16$, then $G - I$ is the graph obtained from a maximal outer-planar graph with 10 vertices by deleting $17 - e(G - I)$ edges. Hence, when $e(G - I) = 15$, we have at least one of $N_G(V(B_1))$, $N_G(V(B_2))$ and $N_G(R)$ is S; when $e(G - I) = 16$, we have at least two of $N_G(V(B_1))$, $N_G(V(B_2))$ and $N_G(R)$ are S. Then $n_I^2 \leq 16 - e(G - I)$. Therefore, $e(S, I) \leq n - 10 + 16 - e(G - I)$. Then $e(G) = e(G - I) + e(S, I) \leq n + 6$, a contradiction.

If $e(G - I) \leq 14$, then $e(S, I) = e(G) - e(G - I) \geq n - 7$. But $e(S, I) \leq (|I| + |S|) + |S| - 2 = n - 8$ by Lemma 4, a contradiction. This completes the proof of Claim 3.

6.4. *Proof of Claim 4*

Clearly, $d_G(v) \leq 1$ for any $v \in I'$. Since $e(O_n[I']) = 0$, $\delta(O_n) \geq 2$, and combining Claim 3, we have $N_{O_n}(v) \cap V(B_i) \neq \emptyset$ for some $i \in [b]$ and any $v \in I'$. Since $|I'| > 2b$, then I' contains at least three vertices, say v_{b+1}, v_{b+2} and v_{b+3}, such that $d_G(v_{b+j}) = 1$, $N_{O_n}(v_{b+j}) \cap V(B_\ell) \neq \emptyset$ for any $j \in [3]$ and some $\ell \in [b]$. Since $G[S]$ is connected, then O_n contains a $K_{2,3}$-minor, one part of which is $\{S, V(B_\ell)\}$, and the other part is $\{v_{b+1}, v_{b+2}, v_{b+3}\}$. This contradicts to Lemma 2. This completes the proof of Claim 4.

6.5. *Proof of Claim 5*

By Claim 2, we just need to prove that $s \geq 2$ when $R = \emptyset$. Suppose that $s \leq 1$ when $R = \emptyset$. Again combining Claim 2, we have $s = 1$. Then $p = n - 9$ and $n_I^{2^+} = 0$. Thus $b = w$. If $w = 0$, then $n = v(G) = v(B_1) + \cdots + v(B_p) + s = p + s = n - 8$ by Claim 3, a contradiction. So $w \geq 1$. Hence, by Claim 3 and Lemma 1, $n + 7 \leq e(G) = \sum_{i=1}^{w} e(G[S \cup V(B_i)]) + e(S, I) \leq \sum_{i=1}^{w}(2(1 + v(B_i)) - 3) + (p - w) = 2(n - (p - w) - 1) + p - 2w = n + 7$, which implies that $G[S \cup V(B_i)]$ is a maximal outer-planar graph for each $i \in [w]$. By (1), we get $v(B_1) + \cdots + v(B_w) = w + 8$. Then we have $w \leq 4$ because $v(B_1) + \cdots + v(B_w) \geq 3w$. So $1 \leq w \leq 4$. If $w = 1$, then $v(B_1) = 9$. If $w = 2$, then either $v(B_1) = 7$ and $v(B_2) = 3$, or $v(B_1) = v(B_2) = 5$. If $w = 3$, then $v(B_1) = 5$ and $v(B_2) = v(B_3) = 3$. If $w = 4$, then $v(B_1) = v(B_2) = \cdots = v(B_4) = 3$. From the above four cases of w, we always get that $G[S \cup Q]$ contains two edge-disjoint $5K_2$. Thus by Claim 1, $e(O_n[I']) = 0$. Since $n \geq 29$ and $b = w \leq 4$, then $|I'| = p - b \geq (n - 9) - 4 > 2b$. Therefore,

O_n contains a $K_{2,3}$-minor by Claim 4, which contradicts to Lemma 2. This completes the proof of Claim 5.

6.6. Proof of Claim 6

Suppose that $v(B_1) \geq 7$. Then $w \geq 1$. Thus by (1) and $v(B_1) + \cdots + v(B_w) \geq 7 + 3(w-1) = 3w+4$, we have $s + w \leq 3$. Then $w = 1$ and $s = 2$ by Claim 5. Therefore, $v(B_1) = 7$, $p = n-8$ and $|I| = p - 1 = n - 9$. By Lemma 2, $n_I^{3^+} = 0$ and $n_I^2 \leq 2$. Then $e(S, I) \leq n - 9 + 2 = n - 7$. Thus $e(G - I) \geq 14$. On the other hand, we have $e(G - I) \leq 15$ by Lemma 1. So $14 \leq e(G - I) \leq 15$.

If $e(G - I) = 14$, then $e(S, I) = e(G) - e(G - I) \geq n - 7$. Thus $d_G(v_2) = d_G(v_3) = 2$. Then $|N_G(V(B_1))| \leq 1$. Since $e(S \cup V(B_1)) = 14$, then $G[S \cup V(B_1) \cup \{v_2, v_3\}] \in \mathcal{O}_{11}^-$. On the other hand, combining $|N_G(V(B_1))| \leq 1$, we can obtain that $G[S]$ is connected.

If $e(G - I) = 15$, then $S \cup V(B_1) \in \mathcal{O}_9$. Thus $N_G(V(B_1)) = S$. Since $e(S, I) = e(G) - e(G - I) \geq n - 8$, then $d_G(v_2) = 2$. Therefore, $G[S \cup V(B_1) \cup \{v_2\}] \in \mathcal{O}_{10}$ and $G[S]$ is connected.

From the above two cases of $e(G - I)$, we always get that $G - I'$ contains two edge-disjoint $5K_2$. Thus by Claim 1, $e(O_n[I']) = 0$. Since $n_I^2 \leq 2$, we see $b \leq 3$. Then $|I'| = p - b \geq (n-8) - 3 > 2b$ because $n \geq 29$ and $b \leq 3$. Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2. This completes the proof of Claim 6.

6.7. Proof of Claim 7

Suppose that $v(B_1) < 3$. Then $|I| = p$. Combining Claim 3, we have $p + s = n$. Then $s = 5$ because $p = n + s - 10$. Thus $n_I^{6^+} = 0$ and $p = n - 5$. In the following proof, let U_1 denote the graph obtained by hanging a path of length 2 at one vertex of C_3; let U_2 denote the graph obtained by hanging an edge at each of two vertices of C_3; let T_1 denote the tree with 5 vertices and diameter 3.

We will prove that $G[S]$ does not contain any cycle. Suppose not, that is, $3 \leq \ell(G[S]) \leq 5$. We consider the following three cases according to $\ell(G[S])$.

If $\ell(G[S]) = 5$, then $G[S]$ is clearly connected, and we have $n_I^{3^+} = 0$ and $n_I^2 \leq 5$ by Lemma 2. Thus $e(S, I) \leq n$. By Lemma 1, we have $e(G[S]) \leq 7$, which implies that $e(S, I) = e(G) - e(G[S]) \geq n$. Therefore, $e(S, I) = n$. Then $n_I^2 = 5$ and $e(G[S]) = 7$. It follows that $b = 5$ and $G[S] \in \mathcal{O}_5$. Thus, $G[S \cup \{v_1, v_2, \ldots, v_5\}] \in \mathcal{O}_{10}$, which means that $G[S \cup \{v_1, v_2, \ldots, v_5\}]$ contains two edge-disjoint $5K_2$. Then we obtain $e(O_n[I']) = 0$ by Claim 1. Then $|I'| = p - b = (n-5) - 5 > 2b$ because $n \geq 29$ and $b = 5$. Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2.

If $\ell(G[S]) = 4$, then $e(G[S]) \leq 6$. By Lemma 2, $n_I^{4^+} = 0$, $n_I^3 = 1$ and $n_I^2 \leq 4$; or $n_I^{3^+} = 0$ and $n_I^2 \leq 6$. So $e(S, I) \leq n + 1$. Since $n + 7 \leq e(G) = e(S, I) + e(G[S]) \leq n + 7$, then $e(G[S]) = 6$ and $e(S, I) = n + 1$. Therefore, $G[S]$ is connected, and either $n_I^3 = 1$ and $n_I^2 = 4$ or $n_I^2 = 6$. Thus, $5 \leq b \leq 6$ and $G[S \cup \{v_1, v_2, \ldots, v_b\}]$ contains two edge-disjoint $5K_2$. Then by Claim 1, $e(O_n[I']) = 0$. Since $n \geq 29$ and $b \leq 6$, then $|I'| = p - b \geq (n-5) - 6 > 2b$. Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2.

If $\ell(G[S]) = 3$, then we also have $e(G[S]) \leq 6$. If $e(G[S]) = 6$, then $e(S, I) = e(G) - e(G[S]) \geq n + 1$ and $G[S] \cong K_1 + 2K_2$. By Lemma 2, $n_I^{4^+} = 0$, $n_I^3 = 1$ and $n_I^2 = 4$; or $n_I^{3^+} = 0$ and $n_I^2 = 6$. It follows that $5 \leq b \leq 6$ and $G[S \cup \{v_1, v_2, \ldots, v_b\}]$ contains two edge-disjoint $5K_2$. Then by Claim 1, $e(O_n[I']) = 0$. We have $|I'| = p - b \geq (n-5) - 6 > 2b$ because $n \geq 29$ and $b \leq 6$. Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2. Therefore, $e(G[S]) \leq 5$. By Lemma 2, $n_I^5 = 0$, $n_I^4 = 1$ and $n_I^2 \leq 4$; or $n_I^{4^+} = 0$, $n_I^3 = 2$ and $n_I^2 \leq 3$; or $n_I^{4^+} = 0$, $n_I^3 = 1$ and $n_I^2 \leq 5$; or $n_I^{3^+} = 0$ and $n_I^2 \leq 7$. From the degree situation of the vertices of I in G, we can obtain $e(S, I) \leq n - 5 + 7 = n + 2$. Since $n + 7 \leq e(G) = e(S, I) + e(G[S]) \leq n + 7$, $e(G[S]) = 5$ and $e(S, I) = n + 2$. It follows that $G[S]$ is connected, and $G[S] \in \{U_1, U_2, (K_2 \cup 2K_1) + K_1\}$. On the other hand, we can also get that the degree situation of the vertices of I in G satisfies one of the following: (1) $n_I^5 = 0$, $n_I^4 = 1$ and $n_I^2 = 4$; (2) $n_I^{4^+} = 0$, $n_I^3 = 2$ and $n_I^2 = 3$; (3) $n_I^{4^+} = 0$, $n_I^3 = 1$

and $n_I^2 = 5$; (4) $n_I^{3^+} = 0$ and $n_I^2 = 7$. If $G[S] \in \{U_1, U_2\}$, then one of (1)–(4) is satisfied. If $G[S] \cong (K_2 \cup 2K_1) + K_1$, then one of (2)–(4) is satisfied. Thus, from the above three structures of $G[S]$, we always find that $b \leq 7$ and $G[S \cup \{v_1, v_2, \ldots, v_b\}]$ contains two edge-disjoint $5K_2$. Then by Claim 1, $e(O_n[I']) = 0$. Then $|I'| = p - b \geq (n-5) - 7 > 2b$ because $n \geq 29$ and $b \leq 7$. Therefore, by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2.

From the above discussion of $\ell(G[S])$, we get that $G[S]$ contains no cycle. Then $e(G[S]) \leq 4$. By Lemma 4, we have $e(S, I) \leq (|I| + |S|) + |S| - 2 = n + 3$ because $|I| > |S|$, which implies that $e(G[S]) \geq 4$. Thus $e(G[S]) = 4$. Then $G[S]$ is connected, and $G[S] \in \{P_5, T_1, K_{1,4}\}$. Since $e(G) \geq n + 7$, we have $e(S, I) = e(G) - e(G[S]) \geq n + 3$. Combining $e(S, I) \leq n + 3$, we have $e(S, I) = n + 3$ and $e(G) = n + 7$. Thus by Lemma 4, we have $d_G(v_i) \geq 1$ for each $v \in I'$. Then the degree situation of the vertices of I in G satisfies one of the following: (1) $n_I^5 = 1$ and $n_I^2 = 4$; (2) $n_I^5 = 0$, $n_I^4 = 1$, $n_I^3 = 1$ and $n_I^2 = 3$; (3) $n_I^5 = 0$, $n_I^4 = 1$ and $n_I^2 = 5$; (4) $n_I^{4^+} = 0$, $n_I^3 = 3$ and $n_I^2 = 2$; (5) $n_I^{4^+} = 0$, $n_I^3 = 2$ and $n_I^2 = 4$; (6) $n_I^{4^+} = 0$, $n_I^3 = 1$ and $n_I^2 = 6$; (7) $n_I^{3^+} = 0$ and $n_I^2 = 8$. If $G[S] \cong P_5$, then one of (1)–(7) is satisfied. If $G[S] \cong T_1$, then one of (2)–(7) is satisfied. If $G[S] \cong K_{1,4}$, then one of (4)–(7) is satisfied. From the above three structures of $G[S]$, we can get that $b \leq 8$ and $G[S \cup \{v_1, v_2, \ldots, v_b\}]$ contains two edge-disjoint $5K_2$. Then by Claim 1, $e(O_n[I']) = 0$.

Note that $b \leq 8$ and $n \geq 29$. If $b < 8$ and $n \geq 29$, or $b = 8$ and $n > 29$, then it can be found that $|I'| = p - b \geq (n-5) - 8 > 2b$. Then by Claim 4, O_n contains a $K_{2,3}$-minor, which contradicts to Lemma 2. So we only need to consider the case of $b = 8$ and $n = 29$. If there exists some $i \in [b]$ such that $|N_{O_n}(v_i) \cap I'| \geq 3$, without loss of generality, we assume that $N_{O_n}(v_i) \cap I' = \{v_{b+1}, v_{b+2}, v_{b+3}\}$. Then O_n contains a $K_{2,3}$-minor, one part of which is $\{S, \{v_i\}\}$, and the other part is $\{v_{b+1}, v_{b+2}, v_{b+3}\}$. It contradicts to Lemma 2. If $|N_{O_n}(v_i) \cap I'| \leq 2$ for each $i \in [b]$, then $e_{O_n}(\{v_1, v_2, \ldots, v_b\}, I') \leq 2b$. It follows that $e(O_n) = e(G) + e_{O_n}(\{v_1, v_2, \ldots, v_b\}, I') \leq n + 7 + 2b < 2n - 3$, which contradicts to Lemma 1. This completes the proof of Claim 7.

7. Concluding Remarks

Theorem 4 determines the exact value of $ar(\mathcal{O}_n, kK_2)$ for $n = 2k$. It seems non-trivial to determine the exact value of $ar(\mathcal{O}_n, kK_2)$ when $n \geq 2k + 1$. Theorem 3 gives a better upper bound of $ar(\mathcal{O}_n, kK_2)$ for all $n \geq 2k + 1$. However, we conjecture that the exact value of $ar(\mathcal{O}_n, kK_2)$ for $n \geq 2k + 1$ is equal to the lower bound given in Theorem 1(4). In [23,24], the authors proved that the conjecture holds when $3 \leq k \leq 4$, and $k = 5$ and $n \geq 2k + 5$. Theorem 5 verifies the conjecture holds when $n \geq 2k + 17$ and $k = 6$. The conjecture is wide open when $k \geq 7$.

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References

1. Bondy J.A.; Murty U.S.R. *Graph Theory*; Springer: Berlin/Heidelberg, Germany, 2008.
2. Erdős, P.; Simonovits, M.; Sós, V.T. Anti-Ramsey theorems. *Colloq. Math. Soc. János Bolyai.* **1975**, *10*, 633–643.
3. Schiermeyer, I. Rainbow numbers for matchings and complete graphs. *Discret. Math.* **2004**, *286*, 157–162. [CrossRef]

4. Chen, H.; Li, X.L.; Tu, J.H. Complete solution for the rainbow numbers of matchings. *Discret. Math.* **2009**, *309*, 3370–3380. [CrossRef]
5. Li, X.L.; Tu, J.H.; Jin, Z.M. Bipartite rainbow numbers of matchings. *Discret. Math.* **2009**, *309*, 2575–2578. [CrossRef]
6. Li, X.L.; Xu, Z.X. The rainbow number of matchings in regular bipartite graphs. *Appl. Math. Lett.* **2009**, *22*, 1525–1528. [CrossRef]
7. Jin, Z.M. Anti-Ramsey numbers for matchings in 3-regular bipartite graphs. *Appl. Math. Comput.* **2017**, *292*, 114–119.
8. Jin, Z.M.; Ye, K.C.; Sun, Y.F.; Chen, H. Rainbow matchings in edge-colored complete split graphs. *Eur. J. Comb.* **2018**, *70*, 297–316. [CrossRef]
9. Özkahya, L.; Young, M. Anti-Ramsey number of matchings in hypergraphs. *Discret. Math.* **2013**, *313*, 2359–2364. [CrossRef]
10. Axenovich, M.; Jiang, T.; Kündgen, A. Bipartite anti-Ramsey numbers of cycles. *J. Graph Theory* **2010**, *47*, 9–28. [CrossRef]
11. Horňák, M.; Jendrol', S.; Schiermeyer, I.; Soták, R. Rainbow numbers for cycles in plane triangulations. *J. Graph Theory* **2015**, *78*, 248–257. [CrossRef]
12. Lan, Y.X.; Shi, Y.T.; Song, Z.X. Planar anti-Ramsey numbers of paths and cycles. *Discret. Math.* **2019**, *342*, 3216–3224. [CrossRef]
13. Fujita, S.; Magnant, C.; Ozeki, K. Rainbow generalizations of Ramsey theory: A survey. *Graphs Comb.* **2010**, *26*, 1–30. [CrossRef]
14. Ding, J.L.; Bian, H.; Yu, H.Z. Anti-Ramsey numbers in complete *k*-partite graphs. *Math. Probl. Eng.* **2020**, 5136104. [CrossRef]
15. Lv, B.H.; Ye, K.C.; Wang, H.P. Rainbow numbers for small cycles. *J. Math. Comput. Sci.* **2018**, *8*, 297–303. [CrossRef]
16. Jin, Z.M.; Li, X.L. Anti-Ramsey numbers for graphs with independent cycles. *Electron. J. Comb.* **2009**, *16*, R85. [CrossRef]
17. Lan, Y.X.; Shi, Y.T.; Song, Z.X. Planar Turán number and planar anti-Ramsey number of graphs. *Oper. Res. Trans.* **2021**, *25*, 200–216. [CrossRef]
18. Jendrol', S.; Schiermeyer, I.; Tu, J.H. Rainbow numbers for matchings in plane triangulations. *Discret. Math.* **2014**, *331*, 158–164.
19. Qin, Z.M.; Lan, Y.X.; Shi, Y.T. Improved bounds for rainbow numbers of matchings in plane triangulations. *Discret. Math.* **2019**, *342*, 221–225. [CrossRef]
20. Chen, G.; Lan, Y.X.; Song, Z.X. Planar anti-Ramsey numbers of matchings. *Discret. Math.* **2019**, *342*, 2106–2111.
21. Qin, Z.M.; Lan, Y.X.; Shi, Y.T.; Yue, J. Exact rainbow numbers for matchings in plane triangulations. *Discret. Math.* **2021**, *344*, 112301.
22. West, D.B. *Introduction to Graph Theory*; Prentice Hall: Hoboken, NJ, USA, 2001. [CrossRef]
23. Jin, Z.M.; Ye, K. Rainbow number of matchings in planar graphs. *Discret. Math.* **2018**, *341*, 2846–2858. [CrossRef]
24. Pei, Y.F.; Lan, Y.X.; He, H. Improved bounds for anti-Ramsey numbers of matchings in outer-planar graphs. *Appl. Math. Comput.* **2022**, *418*, 126843. [CrossRef]
25. Lovász, L.; Plummer, M.D. *Matching Theory*; Elsevier Science Publishers B.V.: Amsterdam, The Netherlands, 1991.